MATHEMATICS - IB

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PREREQUISITES OF 2D-GEOMETRY

SYNOPSIS

Distance between two points :

✈ i) The distance between two points $A(x_1, y_1)$ & $B(x_2, y_2)$ is

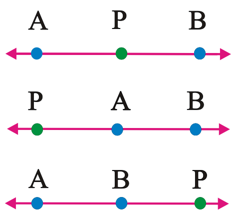
$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

ii) The distance of the point P (x, y) from the origin O is $OP = \sqrt{x^2 + y^2}$

iii) The distance of a point $P(x, y)$ from x-axis is $|y|$ and from y-axis is $|x|$

Section Formula :

✈ i) P is any point on the line passing through A and B. P divides AB in the ratio AP : PB.
If AP and PB are in the same sense (direction) then the division is internal, otherwise the division is external.



ii) The point 'P' which divides the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio m : n

a) internally then $P = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$; $(m + n \neq 0)$

b) externally then $P = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$; $(m - n \neq 0)$

iii) The mid point of the line segment joining (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

iv) If P (x, y) is any point on the line passing through A (x_1, y_1) and B (x_2, y_2) then the ratio in which P divides $\overline{AB}$, ie $AP : PB = x_1 - x : x - x_2$ or $y_1 - y : y - y_2$

Harmonic Conjugate :

✈ If P and Q divide AB internally and externally in the same ratio, then P is called as harmonic conjugate of Q and Q is called as harmonic conjugate of P, also P, Q are a pair of conjugate points w.r.t. A and B

i) Q is harmonic conjugate of P with respect to A, B then AP, AB, AQ are in H.P.

ii) If P, Q divide $\overline{AB}$ harmonically in the ratio m:n then A, B divide $\overline{PQ}$ harmonically in the ratio (m-n) : (m+n).

Points of trisection :

✈ If P and Q are points on the line segment joining A, B dividing $\overline{AB}$ in the ratio 1:2 or 2:1 then P and Q are called points of trisection of $\overline{AB}$.

i) If P and Q are points of trisection of $\overline{AB}$ then

a) mid point of $\overline{AB}$ is same as mid point of $\overline{PQ}$.

b) $PQ = \frac{AB}{3}$

Collinearity :

✈ Three or more points are said to be collinear iff they lie on a straight line.

i) The points A, B, C are collinear iff AB + BC = AC or AC+CB=AB or BA + AC = BC

ii) Points A,B,C are collinear iff Area of $\Delta ABC = 0$

iii) The condition for the three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) to be collinear is

$x_1 - x_2 : x_2 - x_3 = y_1 - y_2 : y_2 - y_3$

Area of the Triangle :

i) Area is non negative

ii) Area of the triangle formed by the vertices

$(x_1,y_1),(x_2,y_2)$ and (x_3,y_3) is $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

iii) Area of the triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$\frac{1}{2}\begin{vmatrix} x_1-x_2 & y_1-y_2 \\ x_1-x_3 & y_1-y_3 \end{vmatrix}$ sq.units

iv) Area of the triangle with vertices $(0, 0)$, (x_1, y_1) and (x_2, y_2) is

$\frac{1}{2}|x_1y_2 - x_2y_1|$ sq. units.

v) Area of the triangle formed by

$A\left(a,\frac{1}{a}\right), B\left(b,\frac{1}{b}\right)$ and $C\left(c,\frac{1}{c}\right)$

is $\left|\frac{(a-b)(b-c)(c-a)}{2abc}\right|$

vi) Area of an equilateral triangle is

a) $\frac{\sqrt{3}}{4}a^2$ where 'a' is length of the side of the triangle.

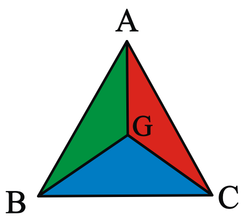
b) $\frac{h^2}{\sqrt{3}}$ where 'h' is length of the altitude of the triangle

✈ If G is centroid of Δ ABC then

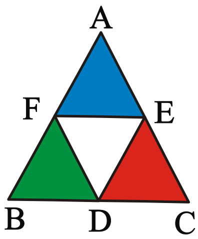
i) area of Δ ABC = 3 area of Δ ABG

= 3 area of Δ BCG

= 3 area of Δ ACG



ii) If D, E, F are mid points of sides BC, CA, AB of Δ ABC then



area of Δ ABC = 4 area of Δ AEF

= 4 area of Δ BDF

= 4 area of Δ DCE

= 4 area of Δ DEF

Area of Quadrilateral :

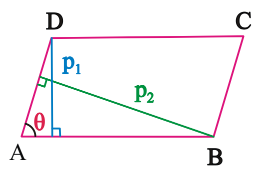
✈ i) Area of the quadrilateral formed by (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) is

$\frac{1}{2}\begin{vmatrix} x_1-x_3 & y_1-y_3 \\ x_2-x_4 & y_2-y_4 \end{vmatrix}$ sq. units

ii) Area of the pentagon formed by (x_k, y_k) $(k = 1, 2, 3, 4, 5)$ is

$\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_1 \end{vmatrix}$ sq. units

iii) If p_1, p_2 are the distances between two parallel sides and θ is the angle between two adjacent sides of a parallelogram then it's area is $\frac{p_1p_2}{\sin\theta}$

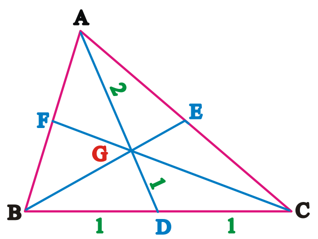


iv) In case of rhombus $p_1 = p_2 = p$ thus area of rhombus $= \frac{p^2}{\sin\theta}$

Centroid :

✈ In any triangle medians are concurrent and the point of concurrency is called centroid of the triangle.

i) Centroid divides each median from vertex in the ratio 2:1 internally.



ii) Centroid of the triangle formed by $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

iii) If D, E, F are midpoints of sides AB, BC, CA of Δ ABC then centroid of Δ ABC = centroid of Δ DEF.

iv) If G is centroid and D,E,F are midpoints of sides $\overline{BC}, \overline{CA}, \overline{AB}$ of Δ ABC then

(a) $AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2)$.

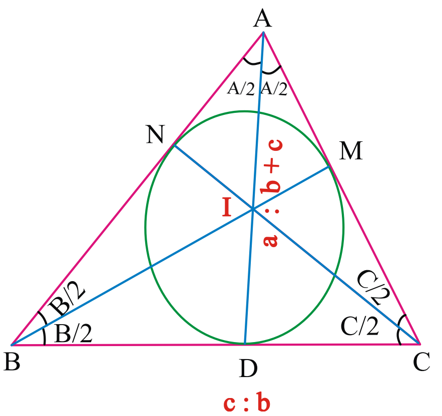
(b) $3\left(AB^2 + BC^2 + CA^2\right) = 4(AD^2 + BE^2 + CF^2)$

v) If G is centroid of Δ ABC and P is any point in the triangle then

$PA^2 + PB^2 + PC^2 = GA^2 + GB^2 + GC^2 + 3PG^2$

Incentre :

✈ The internal angular bisectors of a triangle are concurrent and the point of concurrency is called incentre of the triangle. Incentre is equidistant from all the three sides.



i) In a triangle ABC, if the internal angular bisector of A meets BC at D then BD : DC = AB : AC.

ii) If I is incentre of ΔABC then AI : ID = (AB+AC) : BC where AD is the internal angular bisector of $\angle$A.

iii) In ΔABC, if $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, BC = a, CA = b and AB = c then incentre of ΔABC is

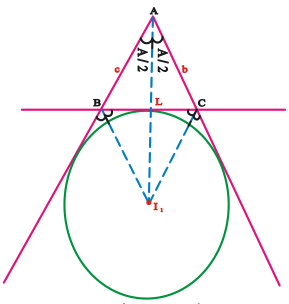
$$I = \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

iv) The incentre of a triangle formed by (0, 0), (a,0), (0,b) is

$$I = \left(\frac{a|b|}{|a|+|b|+\sqrt{a^2+b^2}}, \frac{b|a|}{|a|+|b|+\sqrt{a^2+b^2}}\right)$$

Ex-Centre :

✈ The internal angular bisector of one angle and external angular bisectors of other two angles of a triangle are concurrent and the point of concurrency is called Excentre.



i) The excentre opposite to the vertex A is

$$I_1 = \left(\frac{-ax_1+bx_2+cx_3}{-a+b+c}, \frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$$

ii) The excentre opposite to the vertex B is

$$I_2 = \left(\frac{ax_1-bx_2+cx_3}{a-b+c}, \frac{ay_1-by_2+cy_3}{a-b+c}\right)$$

iii) The excentre opposite to the vertex C is

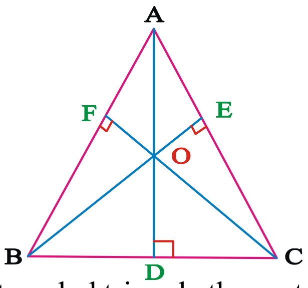
$$I_3 = \left(\frac{ax_1+bx_2-cx_3}{a+b-c}, \frac{ay_1+by_2-cy_3}{a+b-c}\right)$$

iv) In any triangle incentre I is orthocentre of the triangle formed by excentres I_1, I_2 & I_3.

If $a(PA)^2 + b(PB)^2 + c(PC)^2$ is minimum, then the point P with respect to ΔABC, is incentre.

Orthocentre :

✈ The altitudes of a triangle are concurrent and the point of concurrency is called orthocentre (O) of the triangle.



i) In a right angled triangle the vertex at the right angle is the orthocentre of the triangle.

ii) For acute angled triangle orthocentre lies inside the triangle.

iii) For obtuse angled triangle orthocentre lies outside the triangle.

iv) If 'O' is orthocentre of Δ ABC then the four points O, A, B and C are such that each point is orthocentre of the triangle formed by the remaining three points.

v) Orthocentre of the triangle formed by the points $\left(ct_1, \frac{c}{t_1}\right), \left(ct_2, \frac{c}{t_2}\right)$ and $\left(ct_3, \frac{c}{t_3}\right)$ is $\left(\frac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right)$

vi) The orthocentre of the triangle formed with $(0, 0)$, (x_1, y_1) and (x_2, y_2) as vertices is $(k(y_2-y_1), k(x_1-x_2))$ where $k = \frac{x_1x_2+y_1y_2}{x_1y_2-x_2y_1}$

vii) The triangle formed by the feet of altitudes in a triangle is called Orthic triangle or Pedal triangle. Here triangle DEF is the orthic triangle of triangle ABC .

Circum Centre:

- In any triangle perpendicular bisectors of sides are concurrent and the point of concurrence is called circum centre (S) of that triangle. Circum centre is at an equidistance from all the three vertices.

i) The circumcentre of a right angled triangle is mid point of its hypotenuse.

ii) For acute angled triangle circumcentre lies inside the triangle.

iii) For obtuse angled triangle circumcentre lies outside the triangle.

iv) The circum centre of the triangle formed by $(0, 0)$, (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{y_2(x_1^2+y_1^2)-y_1(x_2^2+y_2^2)}{2(x_1y_2-x_2y_1)}, \frac{x_2(x_1^2+y_1^2)-x_1(x_2^2+y_2^2)}{2(x_2y_1-x_1y_2)}\right)$$

- The co-ordinates of vertices of an equilateral triangle are not all rational.
- In an equilateral triangle orthocentre, circum centre, centroid, incentre coincide.

Nine Point Circle :

- In a triangle ABC, let D, E, F be the feet of the altitudes, and X, Y, Z be the mid point of the sides of triangle and P, Q, R are the mid points of AO, BO, CO where 'O' is the orthocentre then D, E, F, X, Y, Z, P, Q, R lie on a circle called nine point circle of the triangle.

i) The centre of the nine point circle, denoted by 'N', N is the mid point of orthocentre and circumcentre (ON=NS)

ii) Radius of the nine point circle = $\frac{1}{2}$ (circum radius)

iii) (a) OG : GS = 2 : 1 (3G=2S+O)

(b) ON : NG : GS = 3 : 1 : 2

Nature of Triangle Based on an Angle :

- i) If all the three angles in a triangle are acute, then the triangle is called an acute angled triangle.

ii) If any one of the three angles is greater than a right angle, then the triangle is called obtuse angled triangle.

iii) In a triangle ABC if BC is the largest side then

a) $AB^2 + AC^2 = BC^2 \Leftrightarrow$ triangle ABC is right angled

b) $AB^2 + AC^2 > BC^2 \Leftrightarrow$ triangle ABC is an acute angled triangle

c) $AB^2 + AC^2 < BC^2 \Leftrightarrow$ triangle ABC is an obtuse angled triangle.

Types of Quadrilaterals :

- i) The quadrilateral formed by A (x_1, y_1), B (x_2, y_2), C (x_3, y_3) and D (x_4, y_4) is a **Parallelogram** if mid point of AC = mid point of BD

ii) Parallelogram ABCD is a

a) Rhombus if AB = BC and AC $\neq$ BD

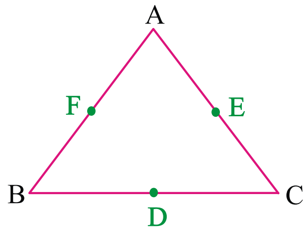
b) Rectangle if AB $\neq$ BC and AC = BD

c) Square if AB = BC and AC = BD

Missing Vertices :

- i) If G (x_o, y_o) is centroid of Δ ABC whose two vertices are (x_1, y_1) and (x_2, y_2), then third vertex $(x_3, y_3) = (3x_0 - x_1 - x_2, 3y_0 - y_1 - y_2)$

ii) If D, E, F are mid points of the sides



BC, CA, AB of Δ ABC then $A = E + F - D$, $B = F + D - E$, $C = D + E - F$

iii) If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are three consecutive vertices of a parallelogram, then its fourth vertex is $(x_1 + x_3 - x_2, y_1 + y_3 - y_2)$

iv) Two vertices of an equilateral triangle are (x_1,y_1) and (x_2,y_2) then the third vertex can be

$$\left(\frac{(x_1+x_2)\pm\sqrt{3}(y_1-y_2)}{2}, \frac{(y_1+y_2)\mp\sqrt{3}(x_1-x_2)}{2}\right)$$

v) If (x_1,y_1), (x_2,y_2) are two opposite vertices of a square then the other two vertices are

$$\left(\frac{(x_1+x_2)\pm(y_1-y_2)}{2}, \frac{(y_1+y_2)\mp(x_1-x_2)}{2}\right)$$

Length of the Medians :

✈ Length of the median through vertex

i) A is $\frac{1}{2}\sqrt{2b^2+2c^2-a^2}$

ii) B is $\frac{1}{2}\sqrt{2c^2+2a^2-b^2}$

iii) C is $\frac{1}{2}\sqrt{2a^2+2b^2-c^2}$

Where AB = c ; BC = a ; CA = b

Some standard results :

✈ The line segment joining the mid points of two sides of triangle is equal to half of the third side and parallel to the third side

✈ In a triangle ABC if AD is the median drawn to BC then $AB^2 + AC^2 = 2\left(AD^2 + BD^2\right)$

✈ A triangle is isosceles if any two of its medians are equal

✈ The diagonals in rhombus,square,rectangle and parallelogram bisect each other

✈ The figure obtained by joining the middle points of the quadrilateral in order is parallelogram

✈ In a parallelogram, if diagonals intersect at right angles, then parallelogram is rhombus

✈ Diagonals of a rhombus bisects the angles

✈ Let two straight lines meet at A and any line Parallel to angle bisector meet them in B and C then triangle ABC is isosceles triangle and AB = AC

✈ $\cos\underline{|POQ} = \frac{x_1x_2 + y_1y_2}{\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}}$

Where $P(x_1, y_1)$, $Q(x_2, y_2)$ and '0' be the origin.

✈ If P is the length of the diagonal of a square then

a) length of the side is $\frac{p}{\sqrt{2}}$ units.

b) Area of the square is $\frac{p^2}{2}$

Eg : 1

If the point $(x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1))$ divides the join of (x_1, y_1) and (x_2, y_2) internally, then $t \in$

Sol : ratio is $x_1 - x : x - x_2$

$= x_1 - x_1 - t(x_2 - x_1) : x_1 + t(x_2 - x_1) - x_2$

$= t(x_1 - x_2) : (x_1 - x_2) - t(x_1 - x_2)$

$= t : 1 - t > 0$ ($\because$ Division is internal)

$\Rightarrow t(1-t) > 0 \Rightarrow t \in (0,1)$

v) The ratio in which the line segment joining (x_1, y_1) and (x_2,y_2) is divided by
i) x-axis is $-y_1 : y_2$ ii) y-axis is $-x_1 : x_2$

Eg : 2

If Q is harmonic conjugate of P with respect to A, B and AP = 2, AQ = 6 then AB =

Sol : AP, AB, AQ are in H.P.

$\Rightarrow \frac{2}{AB} = \frac{1}{AP} + \frac{1}{AQ} \Rightarrow AB = 3$

Eg : 3

If P and Q are two points on the line joining A(-2,5), B(3,1) such that AP= PQ = QB then PQ=

Sol. $PQ = \frac{AB}{3} = \frac{\sqrt{25+16}}{3} = \frac{\sqrt{41}}{3}$

Eg : 4

In a triangle ABC, A(5,6), B(-1,4) and centroid is at (2, 4). Then area of triangle formed by the mid points of sides of ΔABC is

Sol : Area of $\Delta ABC = 3$ (Area of ΔGAB)

$= 4$ (Area of ΔDEF)

$\Rightarrow$ Area of $\Delta DEF = \frac{3}{4}$ area of ΔGAB

$= \frac{3}{4} \cdot \frac{1}{2} \begin{vmatrix} 5+1 & 6-4 \\ 5-2 & 6-4 \end{vmatrix} = \frac{9}{4}$ sq.units

Eg : 5

The area of the pentagon whose vertices are (4, 1), (3,6), (-5,1), (-3, -3) and (-3, 0) is

1) 30 Sq. Units 2) 60 Sq. Units

3) 120 Sq. Units 4) 75 Sq. Units

Sol : $\frac{1}{2}\begin{vmatrix} 4 & 3 & -5 & -3 & -3 & 4 \\ 1 & 6 & 1 & -3 & 0 & 1 \end{vmatrix} = \frac{60}{2} = 30$ Sq. Units

Eg : 6

The orthocentre of the triangle whose vertices are $\left(2, \frac{-1}{2}\right), \left(\frac{1}{2}, \frac{-1}{2}\right)$ and $\left(2, \frac{\sqrt{3}-1}{2}\right)$ is

Sol : Slope of AB is 0, slope of AC is not defined $\Rightarrow$ Triangle is right angled.

$\therefore$ Orthocentre $= A = \left(2, \frac{-1}{2}\right)$

Eg : 7

In ΔABC, the vertices are A=(2,3), B=(-2,-5), C=(-4,6). If P is a point on BC such that AP bisects the angle A, then P =

Sol : P divides $\overline{BC}$ in the ratio AB:AC =

$4\sqrt{5} : 3\sqrt{5} = 4 : 3$

$P = \left(\frac{-16-6}{7}, \frac{24-15}{7}\right) = \left(\frac{-22}{7}, \frac{9}{7}\right)$

Eg : 8

If A (3, -4), B(7, 2) are the ends of a diameter of a circle and C is a point on the circle then the circumcentre of ΔABC is

Sol : Angle in a semicircle is a right angle.

Circumcentre = mid point of AB = (5, -1)

Eg : 9

The radius of nine point circle of the triangle formed by (4,6), (0,4), (6,2) is

Sol : $AB^2 = 16 + 4 = 20$, $BC^2 = 36 + 4 = 40$,

$AC^2 = 4 + 16 = 20$. Triangle is right angled.

Circum radius $R = \frac{hyp}{2} = \frac{2\sqrt{10}}{2} = \sqrt{10}$

Radius of nine point circle is $\frac{R}{2} = \sqrt{\frac{10}{4}} = \sqrt{\frac{5}{2}}$

Eg : 10

The triangle with the vertices (-2,4), (0,0), (5,-1) is

Sol : $AB^2 = 20, BC^2 = 26, AC^2 = 49 + 25 = 74$

$AB^2 + BC^2 < CA^2, BC^2 + CA^2 > AB^2,$

$CA^2 + AB^2 > BC^2$

$\therefore$ Triangle is obtuse angled triangle.

Eg : 11

If a vertex of a triangle is (1,1) and the mid points of two sides through this vertex are (-1,2) and (3,2) then the centroid of the triangle is

Sol : D=E+F-A=(-1+3-1,2+2-1)=(1,3)

centroid of ΔABC = centroid of $\Delta DEF = \left(1, \frac{7}{3}\right)$

Eg : 12

If G is the centroid of ΔABC and BC=3, CA=4, AB=5 then BG=

Sol : Length of the median through B is

$\frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2} = \frac{1}{2}\sqrt{50 + 18 - 16} = \sqrt{13}$ $\therefore$

$BG = \frac{2}{3}\left(\sqrt{13}\right) = \frac{\sqrt{52}}{3}$

EXERCISE - I

1. **The distance between the points (sin θ, cos θ) and (cos θ, -sin θ) is**

   1) 1 2) $\sqrt{2}$ 3) 2 4) $\sqrt{6}$

2. **Distance between two points (2, cot θ) and (1, 0) is**

   1) cosec θ 2) sec θ

   3) | Sec θ | 4) | cosec θ |

3. P is a point on the line x=y. If the distance of P from (1,3) is 10 then x and y coordinates of P are both equal to

1) 9 or -5 2) -9 or 5 3) -9 or -5 4) 9 or 5

4. The coordinates of the point which divides the line segment joining (a+b, a-b) and (a-b, a+b) in the ratio of a:b externally is

1) $\left(\frac{a^2-2ab-b^2}{a-b},\frac{a^2+b^2}{a-b}\right)$

2) $\left(\frac{a^2-2ab-b^2}{a+b},\frac{a^2+b^2}{a+b}\right)$

3) $\left(\frac{a^2+2ab-b^2}{a+b},\frac{a^2-b^2}{ab}\right)$

4) $\left(\frac{a^2-ab-2b^2}{a+2b},\frac{a^2-ab-2b^2}{2a+b}\right)$

5. If the points A(a, b), B(−a, −b) and P(a^2, ab) are collinear then the ratio in which P divides $\overline{AB}$ is

1) 1 + a : 1 − a 2) 1 : a

3) a : 1 4) 1 − a : 1 + a

6. The ratio in which the y-axis divides the line segment joining (3,6), (12, −3) is

1) 1 : 4 internally 2) −2 : 1

3) 1 : 4 externally 4) 2 : 1

7. If A(−2, 5), B(3, 1) and P, Q are the points of trisection of $\overline{AB}$, then mid point of $\overline{PQ}$ is

1) (2, 3) 2) $\left(\frac{1}{2},3\right)$

3) $\left(-\frac{1}{2},4\right)$ 4) (1, 4)

8. The harmonic conjugate of (4, 1) with respect to the points (3, 2) and (−1, 6) is

1) (−4, 1) 2) (1, 4) 3) $\left(\frac{7}{3},\frac{8}{3}\right)$ 4) $\left(\frac{7}{6},\frac{8}{6}\right)$

9. The triangle with the vertices (4, 3), (−3,2), (1,−6) is

1) An obtuse angled triangle

2) An acute angled triangle

3) Right angled

4) Right angled isosceles

10. The points $\left(0,\frac{8}{3}\right),(1,3),(82,30)$ are vertices of

1) An obtuse angled triangle

2) An acute angled triangle

3) Right angled 4) Lies on a same line

11. The maximum area of the triangle formed by the points (0,0), (acos θ, bsin θ) and (acos θ, -b sin θ) (in square units)

1) $\frac{3}{4}$ab 2) ab 3) $\frac{ab}{2}$ 4) a^2b^2

12. An equilateral triangle has each side equal to 'a'. If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the vertices of the triangle then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 =$

1) $3a^4$ 2) $\frac{3a^4}{4}$ 3) $4a^4$ 4) a^4

13. The sides of a triangle are $\frac{y}{z}+\frac{z}{x},\frac{z}{x}+\frac{x}{y}$ and $\frac{x}{y}+\frac{y}{z}$ then its area in square units is

1) xyz 2) $\sqrt{\frac{x}{y}+\frac{y}{z}+\frac{z}{x}}$

3) $\sqrt{xyz}$ 4) $\frac{x^2y^2z^2}{2}$

14. The centroid and two vertices of a triangle are (4,−8), (−9, 7), (1,4) then the area of the triangle is

1) 333 sq.units 2) 166.5 sq.units

3) 111 sq.units 4) 55.5 sq.units

15. The points $(a,0),(0,b),(1,1)$ are collinear if

1) $\frac{1}{a}+\frac{1}{b}=1$ 2) $\frac{1}{a}+\frac{1}{b}=2$

3) $\frac{1}{a}+\frac{1}{b}=3$ 4) $\frac{1}{a}+\frac{1}{b}=4$

16. If 3, 5 be the distances between the parallel sides and 30^{o} is the angle between two adjacent sides of a parallelogram then its area

1) 15/2 2) 15 3) 30 4) 15/4

17. The vertices of a triangle are (2,1), (-2,-2), (1,0). Then sum of squares of the lengths of the medians of the triangle is

1) 25 2) 40 3) 30 4) 45

18. The lengths of the sides of a triangle ABC are AB=10, BC=7, CA=$\sqrt{37}$ then length of the median through the vertex C is

1) $3\sqrt{2}$ 2) $2\sqrt{3}$ 3) $3\sqrt{3}$ 4) $4\sqrt{2}$

19. If the sides of ΔABC are 5, 7, 8 units then $AG^2+BG^2+CG^2=$

1) 46 2) 138 3) 92 4) 69

20. The centroid of a triangle is (2,3) and two of its vertices are (5,6) and (-1,4) then the third vertex of the triangle is

1) (3,1) 2) (2,-1) 3) (4,-1) 4) (3,0)

21. If P (1,2), Q (4,6), R(5,7), S(a,b) are vertices of a parallelogram PQRS then

1) $a=2, b=4$ 2) $a=3, b=4$

3) $a=2, b=3$ 4) $a=3, b=5$

22. If $(2,4),(2,6)$ are two vertices of an equilateral triangle then the third vertex is

1) $(2+\sqrt{3},5)$ 2) $(\sqrt{3}-2,5)$

3) $(5,2+\sqrt{3})$ 4) $(5,2-\sqrt{3})$

23. If $(2,4),(4,2)$ are the extremities of the hypotenuse of a right angled isosceles triangle, then the third vertex is

1) $(2,2)\,or\,(4,4)$ 2) $(3,3)\,or\,(4,4)$

3) $(2,2)\,or\,(3,3)$ 4) $(2,3)\,or\,(3,2)$

24. The side of a square ABCD is 'a' units. A,B,C,D are in the anti-clockwise order. If AB and AD are coordinate axes. Then the coordinates of C are

1) (a, -a) 2) (-a,-a) 3) (-a,a) 4) (a,a)

25. If (1,a), (2,b), (c^2,-3) are VERTICES of a triangle then the condition for its centroid to lie on x-axis is

1) 3a + 3b = 1 2) a+b=3

3) ab=3 4) 2a+3b=7

26. If A(3, –4), B(7, 2) are the ends of a diameter of a circle and C(3, 2) is a point on the circle then the orthocentre of the Δ ABC is

1) (0, 0) 2) (3, –4) 3) (3, 2) 4) (7, 2)

27. Incentre of the triangle with vertices (4,-2), (5,5) (-2,4) is

1) (5/4, 3/4) 2) (3/2, 3/2)

3) (5/3, 5/3) 4) (5/2, 5/2)

KEY

01) 2	02) 4	3) 1	04) 1	5) 1	06) 3
07) 2	08) 3	09) 2	10) 4	11) 3	12) 2
13) 2	14) 2	15) 1	16) 3	17) 3	18) 1
19) 1	20) 2	21) 3	22) 1	23) 1	24) 4
25) 2	26) 3	27) 4			

SOLUTIONS

1. Put $\theta=0$ then $A(0,1)B(1,0)$

$$AB=\sqrt{(1-0)^2+(0-1)^2}=\sqrt{1+1}=\sqrt{2}$$

Let P(k, k) be any point on x = y

A(1,3) given PA=10

$PA^2=100$

$$(k-1)^2+(k-3)^2=100 \Rightarrow k^2+1-2k+k^2+9-6k=100$$

$k(k-9)+5(k-9)=0$

$k=-5(or)9$

2. distance $=\sqrt{1+\cot^2\theta}=|\operatorname{cosec}\theta|$

3. $P(k,k)$ $A(1,3)$, PA = 10

4. $\left(\frac{mx_2-nx_1}{m-n},\frac{my_2-ny_1}{m-n}\right)$

5. $m:n=x_1-x_2:x_2-x_3$

$a-a^2:a^2+a$

6. Y-axis divides (x_1y_1) and (x_2y_2) in the ratio

$-x_1 : x_2$

7. mid point of PQ = mid point of AB
8. $(4,1)$ divides $(3,2)$ and $(-1,6)$ in the ratio -1:5. The point that divides joining the line segment $(3,2)$ *and* $(-1,6)$ in the ratio 1:5 is $\left(\frac{7}{3},\frac{8}{3}\right)$
9. $AB^2 = 50, BC^2 = 80, AC^2 = 90$

   $AB^2 + BC^2 > AC^2, BC^2 + CA^2 > AB^2,$

   $CA^2 + AB^2 > BC^2$
10. $AB^2 = 26, BC^2 = 52, AC^2 = 26$

.

11. $\frac{1}{2}(x_1y_2 - x_2y_1)$

    $= \frac{1}{2}ab\sin 2\theta$ for maximum Δ $\sin 2\theta = 1$
12. $\frac{\sqrt{3}}{4}a^2 = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$
13. put $x = y = z = 1$, Area $= \frac{\sqrt{3}}{4}a^2$
14. Given $g(4,-8)\,A(-9,7),B(14)$ area of triangle

    $GAB = \frac{1}{2}\begin{vmatrix} x_1 - x_2 & y_1 - y_2 \\ x_1 - x_3 & y_1 - y_3 \end{vmatrix}$

    $= \frac{1}{2}\begin{vmatrix} 13 & -15 \\ 3 & -12 \end{vmatrix} = \frac{1}{2}|-156+45| = \frac{|1|}{2}$

    $\therefore$ Area of triangle ABC = 3(area of triangle GAB)$= \frac{3(111)}{2} = 166.5$ sq.unitsArea of the
15. Slopes are equal
16. $\Delta = \frac{p_1p_2}{\sin\theta}$
17. Given $A(2,1), B(-2,-2), C(1,0)$

    $AB = \sqrt{(-4)^2 + (-3)^2} = \sqrt{16+9} = 5$

    $BC = \sqrt{3^2 + 2^2} = \sqrt{9+4} = \sqrt{13}$

    $CA = \sqrt{1^2 + 1^2} = \sqrt{2}$

    now $AD^2 + BE^2 + CF^2 = \frac{3}{4}\left(AB^2 + BC^2 + CA^2\right)$
18. Use length of median through C

    $= \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$
19. $AB^2 + BC^2 + CA^2 = 3\left(GA^2 + GB^2 + GC^2\right)$
20. C = 3G-(A+B)
21. S=P+R-Q
22. Third vertex

    $\left(\frac{x_1 + x_2 \pm \sqrt{3}(y_1 - y_2)}{2}, \frac{y_1 + y_2 \mp \sqrt{3}(x_1 - x_2)}{2}\right)$

    $= \left(\frac{2+2 \pm \sqrt{3}(4-6)}{2}, \frac{4+6+\sqrt{3}(2-2)}{2}\right)$

    $= \left(2+\sqrt{3}, 5\right)$
23. Third Vertex $\left(\frac{2+4 \pm (4-2)}{2}, \frac{(4+2)+(2-4)}{2}\right)$

    $= (3 \pm 1, 3 \pm 1)$

    $= (4,4)$ or $(2,2)$
24. $A(0,0)B(a,0)D(0,a)$

    $\Rightarrow C(a,a)$
25. $G_y = 0$
26. $\angle C = 90^0$
27. Given $A = (4-2), B = (5,5), C = (-2,4)$

    $a = BC = \sqrt{(-7)^2 + (-1)^2} = \sqrt{49+1} = \sqrt{50} = 5\sqrt{2}$

    $b = CA = \sqrt{(-6)^2 + (6)^2} = \sqrt{36+36} = \sqrt{72} = 6\sqrt{2}$

    $c = AB = \sqrt{1^2 + 7^2} = \sqrt{1+49} = \sqrt{50} = 5\sqrt{2}$

    I n c e n t r e

    $= \left(\frac{20\sqrt{2}+30\sqrt{2}-10\sqrt{2}}{16\sqrt{2}}, \frac{-10\sqrt{2}+30\sqrt{2}+20\sqrt{2}}{16\sqrt{2}}\right) = \left(\frac{5}{2},\frac{5}{2}\right)$

EXERCISE - II

1. **The point $A(\sin\theta, \cos\theta)$ is 3 units away from the point $B(2\cos 75^0, 2\sin 75^0)$ if $0^0 \le \theta < 360^0$ then $\theta =$**
   1) 195^0 2) 105^0 3) 285^0 4) 270^0
2. **The abscissae of two points A and B are the roots of the equation $x^2+2ax-b^2=0$ and their ordinates are the roots of $y^2+2py-q^2=0$ then the distance AB in terms of a, b, p, q is**
   1) $\sqrt{a^2+b^2+p^2+q^2}$
   2) $2\sqrt{a^2+b^2+q^2+p^2}$
   3) $\sqrt{a^2+b^2+p^2}$
   4) $\sqrt{a^2+b^2+q^2}$
3. **The point P(x,y) is equidistant from the points Q(c+d,d-c) and R(c-d,c+d) then**
   1) cx = dy 2) cx + dy = 0
   3) dx = cy 4) dx + cy = 0
4. **The coordinates of the point that is two-thirds away from (-4,3) to (5,7) is**
   1) $\left(\frac{17}{2},3\right)$ 2) $\left(2,\frac{17}{3}\right)$ 3) $\left(2,\frac{3}{17}\right)$ 4) $\left(3,\frac{2}{17}\right)$
5. **The point whose coordinates are $x=x_1+t(x_2-x_1)$ and $y=y_1+t(y_2-y_1)$ divides the join of (x_1,y_1), (x_2,y_2) in the ratio**
   1) $\frac{t}{1+t}$ 2) $\frac{1+t}{t}$ 3) $\frac{t}{1-t}$ 4) $\frac{1-t}{t}$
6. **The area of triangle formed by the vertices (a, 1/a), (b, 1/b) and (c, 1/c) is**
   1) $\frac{a+b+c}{abc}$ 2) $\left|\frac{(a-b)(b-c)(c-a)}{2abc}\right|$
   3) $\frac{abc}{a+b+c}$ 4) $\frac{1}{2}(a^2+b^2+c^2)$
7. **Let $A(h,k), B(1,1), C(2,1)$ be the vertices of a right angle triangle with AC as its hypotenuse. If the area of the triangle is 1 then the set of values of K can be**
   1) $\{1,3\}$ 2) $\{0,2\}$
   3) $\{-1,3\}$ 4) $\{-3,-2\}$
8. **Area of the triangle with vertices (t,t-2), (t+3,t), (t+2, t+2) is**
   1) 4 2) 8 3) 6 4) 10
9. **The points with coordinates $(2a,3a)$, $(3b,2b)$ and (c,c) are collinear**
   1) for all values of a,b,c
   2) for no values of a,b,c
   3) iff a,c/5,b are in H.P.
   4) iff a,2c/5,b are in H.P.
10. **a,b,c are in A.P and x,y,z are in G.P. The points $(a,x),(b,y),(c,z)$ are collinear if**
    1) $x^2=y$ 2) $x=z^2$
    3) $y^2=z$ 4) $x=y=z$
11. **If 'O' is the origin and A (x_1, y_1), B (x_2, y_2) then the circum radius of Δ AOB is**
    1) $\frac{OA.OB.AB}{2|x_1y_2-x_2y_1|}$ 2) $\frac{OA.OB.AB}{|x_1y_2-x_2y_1|}$
    3) $\frac{2.OA.OB.AB}{|x_1y_2-x_2y_1|}$ 4) $\frac{OA.OB.AB}{2|x_1y_2+x_2y_1|}$
12. **If x_1,x_2,x_3 are in A.P. and y_1, y_2, y_3 are also in A.P. with same common difference then the points (x_1,y_1), (x_2,y_2) and (x_3,y_3) form**
    1) A scalene triangle 2) A right angled triangle
    3) An equilateral triangle 4) Collinear
13. **Area of the triangle formed by (0,0), $\left(a^{x^2},0\right),\left(0,a^{6x}\right)$ is $\frac{1}{2a^5}$ sq. units then x =**
    1) 1 or 5 2) -1 or 5 3) 1 or -5 4) -1 or -5
14. **If Δ_1, Δ_2 are the areas of incircle and circumcircle of a triangle with sides 3,4 and 5 then $\frac{\Delta_1}{\Delta_2}=$**
    1) $\frac{16}{25}$ 2) $\frac{4}{25}$ 3) $\frac{9}{25}$ 4) $\frac{9}{16}$
15. **Let A = (-4,0), B = (-1,4). C and D are points which are symmetric to points A and B respectively with respect to y-axis, then the**

area of the quadrilateral ABDC is
1) 8 sq.units 2) 12 sq.units
3) 20 sq.units 4) 10 sq.units

16. Instead of walking along two adjacent sides of a rectangular field , a boy took a short cut along the diagonal and saved the distance equal to half of the longer side. Then the ratio of the shorter side to the longer side is
1) 1:2 2) 2:3 3) 1:4 4) 3:4

17. Orthocentre of the triangle with vertices (4,1), (7,4), (5,-2) is
1) (0,0) 2) (1,2) 3) (3/2, 3/2) 4) (2,1)

18. O is the orthocentre of the triangle formed by A(1,-3), B(7,2), C(2,5) then the distance between the orthocentres of Δ BOC, Δ AOB is
1) $\sqrt{65}$ 2) $2\sqrt{65}$ 3) $\frac{1}{2}\sqrt{65}$ 4) 65

19. The circumcentre of the triangle formed by (-2,3), (2,-1) and (4,0) is
1) (3/2, 5/2) 2) (-3/2, 5/2)
3) (3/2, -5/2) 4) (-3/2, -5/2)

20. In a ΔABC, the sides $BC=5, CA=4, AB=3$. If A (0,0) and the internal bisector of angle A meets BC in $D\left(\frac{12}{7},\frac{12}{7}\right)$ then incentre of ΔABC is
1) (2,2) 2) (3,2) 3) (2,3) 4) (1,1)

21. (0,0), (20,15), (36,15) are the vertices of a triangle then the ex-centre opposite to vertex (0,0) is
1) (35,20) 2) (19,18) 3) (16,25) 4) (14,22)

22. The mid points of the sides of a triangle are (1/2, 0), (0, 1/2) and (1/2, 1/2) then its circumcentre is
1) (1,1) 2) (1,1/2) 3) (1/2,1) 4) (1/2,1/2)

23. If G be the centroid and I be the incentre of the triangle with vertices A(–36, 7), B(20, 7) and C(0, –8) and GI = $\frac{25}{3}\sqrt{205}\lambda$ then λ =
1) 1/25 2) 1/5 3) 25 4) 5

24. Orthocentre of the triangle is (2,1) and the circumcentre is $\left(\frac{7}{2},\frac{5}{2}\right)$ then its nine point circle centre is
1) $\left(\frac{7}{4},\frac{11}{4}\right)$ 2) $\left(\frac{7}{4},\frac{11}{2}\right)$ 3) $\left(\frac{11}{4},\frac{7}{4}\right)$ 4) $\left(\frac{7}{2},\frac{7}{4}\right)$

25. If $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ are vertices of equilateral triangle such that
$(x_1-2)^2+(y_1-3)^2=(x_2-2)^2+(y_2-3)^2=(x_3-2)^2+(y_3-3)^2$
then $x_1+x_2+x_3+2(y_1+y_2+y_3)=$
1) 18 2) 24 3) 6 4) 8

26. If (0, 0) is orthocentre of triangle formed by $A(\cos\alpha,\sin\alpha),B(\cos\beta,\sin\beta),C(\cos\gamma,\sin\gamma)$ **then** $\underline{|BAC}=$
1) 60^0 2) 30^0 3) 45^0 4) $22\frac{1}{2}^0$

27. Origin is the orthocentre of the triangle formed by the points (5, -1), (-2,3) and (-4, -7) then the nine point circle centre is
1) $\left(\frac{-1}{3},\frac{-5}{3}\right)$ 2) $\left(\frac{-1}{4},\frac{-5}{4}\right)$
3) (1, 1) 4) (5, 3)

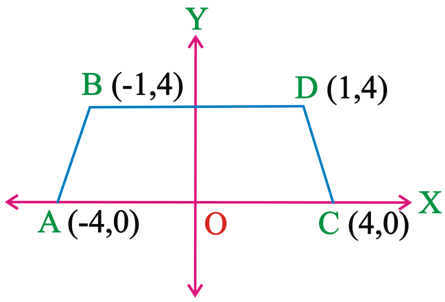
28. I,I_1,I_2,I_3 are incentre and excentres of ΔABC. If I(0, 0) I_1(2, 3), I_2(5, 7) then distance between orthocentres of Δ I I_1I_3 and Δ $I_1I_2I_3$
1) $\sqrt{13}$ 2) 5 3) $\sqrt{74}$ 4) $2\sqrt{37}$

29. If (a, b), (x, y), (p,q) are the coordinates of circumcentre, centroid, orthocentre of the triangle then
1) 3x = 2a + p, 3y = 2b + q
2) x = 3a + 2p, y = 3b + 2q
3) 3x = a + 2p, 3y = b + 2q
4) x = a + p, y = b + q

KEY

01) 1	02) 2	03) 3	04) 2	05) 3	06) 2
07) 3	08) 1	09) 4	10) 4	11) 1	12) 4
13) 4	14) 2	15) 3	16) 4	17) 2	18) 1
19) 1	20) 4	21) 1	22) 4	23) 1	24) 4
25) 2	26) 1	27) 2	28) 3	29) 1	

SOLUTIONS

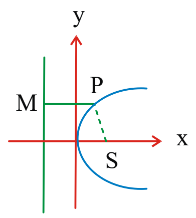
1. $\sqrt{\left(2\cos 75^0-\sin\theta\right)^2+\left(2\sin 75^0-\cos\theta\right)^2}=3$
2. Let x_1, x_2 are x-coordinates of A, B and $y_1,\ y_2$ are y-coordinates of A, B then
   $x_1+x_2=-2a$ $\quad x_1x_2=-b^2$
   $y_1+y_2=-2p$ $\quad y_1y_2=-q^2$
   $AB^2=(x_1-x_2)^2+(y_1-y_2)^2$
   $=(x_1+x_2)^2-4x_1x_2+(y_1+y_2)^2-4y_1y_2$
3. $PQ^2=PR^2$
4. $\frac{2}{3}:\frac{1}{3}=2:1$
5. $x_1-x:x-x_2$
6. Use area of the triangle formula
7. Slope of BC = 0 $\Rightarrow$ AB is vertical $\therefore h=1$
   Area of $\Delta ABC=1$
8. Put t = 0
9. Slopes are equal
10. a,b,c are in AP $\Rightarrow a-b=b-c$
    $(a,x),(b,y),(c,z)$ are collinear
    $\Leftrightarrow \frac{x-y}{a-b}=\frac{y-z}{b-c}$
    $\Rightarrow \frac{x-y}{y-z}=1$
    $\Rightarrow x-y=y-z\Rightarrow x,y,z$ are in A.P
    x,y,z are in A.P and also in G.P $\Rightarrow x=y=z$
11. $R=\frac{abc}{4\Delta}$
12. Put $x_1,x_2,x_3=1,2,3$
    $y_1,y_2,y_3=2,3,4$
    Slope of AB = Slope of BC
13. Area $=\frac{1}{2}\left|x_1y_2-x_2y_1\right|$
14. $r=\frac{\Delta}{s}, R=\frac{hyp}{2}$
    $r=1, R=\frac{5}{2}$
15. 

    Area $=\frac{1}{2}\begin{vmatrix}5 & 4\\ 5 & -4\end{vmatrix}=\frac{1}{2}\left|-20-20\right|=20$ sq.units
16. $\sqrt{a^2+b^2}=\frac{a}{2}+b$
17. Slope of BC is 3
    Altitude through A is $x+3y-7=0$, verify
18. Distance between the orthocentres=AC
19. $\perp^r$ bisector of AB is $x-y+1=0$.
20. I divides AD in the ratio $b+c:a$
21. $I_1=\left(\frac{-ax_1+bx_2+cx_3}{-a+b+c},\frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$
22. Vertices of the triangle are $(1,0),(0,1),(0,0)$
23. $G\left(\frac{-16}{3},2\right), I=(-1,0), GI=\frac{\sqrt{205}}{3}$
24. Nine point circle centre = mid point of orthocentre and circumcentre
25. SA = SB = SC
    S = G = (2, 3)
    $\therefore x_1+x_2+x_3+2\left(y_1+y_2+y_3\right)=6+18=24$
26. Let S=(0, 0)
    SA = SB = SC $\Rightarrow$ equilateral triangle
27. Nine point circle centre divides $\overline{OG}$ in the ratio 3 : 1
28. distance between I and $I_2=\sqrt{74}$
29. Centroid divides orthocentre,Circumcentre in the ratio 2 : 1

LOCUS

SYNOPSIS

- Locus is the set of points (and only those points) that satisfy the given consistant geometric condition(s).
  i.e i) Every point satisfying the given condition (s) is a point on the locus.
  ii) Every point on the locus satisfies the given condition(s).
- Locus is the path traced by the conditional point(s). It is a necessary condition, converse need not be true.
- Algebraic relation between x and y obtained by applying the geometrical conditions is called the equation of locus.
- The locus of a point which is equidistant from two fixed points A and B is the perpendicular bisector of the line segment AB.
- The locus of a point which is at a constant distance from a fixed point is a circle
- A and B are fixed points. P is the point moves such that $\frac{PA}{PB} = k$ is
  i) a straight line if k=1 ii) a circle if k $\neq$ 1 and k>0.
  iii) an empty set if $k < 0$.
- If the join of two fixed points A,B subtends a right angle at P, then the locus of P is a circle on AB as diameter.
- The locus of the third vertex of a right angled triangle when the ends of a hypotenuse are given as (x_1, y_1) and (x_2, y_2) is a circle whose equation is
  $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$
- Given A & B are two fixed points.The locus of a point P such that the area of Δ PAB is a constant is a pair of lines parallel to AB.
- If A, B,C are three points then the locus of a point P such that $PA^2 + PB^2 = K.PC^2$ is
  i) a straight line if K=2 ii) a circle if k $\neq$ 2 and K>0
  iii) an empty set if $k < 0$
- The locus of the point which moves equidistant from a fixed point and fixed st. line is a parabola.



- A,B are two fixed points and
  PA + PB = k then
  (i)If AB < k, locus of P is an ellipse
  (ii)If AB =k, locus of P is line segment AB
  (iii)If AB >k , locus of P does not exist
- A,B are two fixed points and |PA - PB| = k, then
  (i)If AB <k, locus of P does not exist
  (ii)If AB =k, locus of P is line through A and B except line segment AB
  (iii)If AB > k, locus of P is a hyperbola
- The curve represented by

  $S = ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$

  and $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$ is

  i) a circle if a=b, h=0, $g^2 + f^2 - ac \geq 0$, $\Delta \neq 0$

  ii) a pair of lines if $\Delta = 0$, $h^2 \geq ab, g^2 \geq ac$, $f^2 \geq bc$

  iii) a pair of parallel lines if $\Delta = 0$, $h^2 = ab$, $af^2 = bg^2$

  iv) a parabola if $\Delta \neq 0$, $h^2 = ab$.

  v) An ellipse if $\Delta \neq 0, h^2 < ab$.

  vi) a hyperbola if $\Delta \neq 0, h^2 > ab$

  vii) a rectangular hyperbola if $\Delta \neq 0$, a+b=0 and $h^2 > ab$

EXERCISE-I

1. **The locus of the point, for which the sum of the distances from the coordinate axes is 9 is**

1) $|x|+|y|=9$ 2) $|x|+|y|=3$
3) $|x|+|y|=0$ 4) $|x|+|y|=27$

2. **The equation of the locus of the point whose distance from the x-axis is twice that of from the y-axis is**

1) $y^2=4x^2$ 2) $4y^2 = x^2$
3) $y = 2x$ 4) $x = 2y$

3. **The equation to the locus of a point P for which the distance from P to (6, 5) is triple the distance from P to x-axis is**

1) $x^2 + 8y^2 - 12x - 10y + 51 = 0$
2) $x^2 + 8y^2 + 12x - 10y + 51 = 0$
3) $x^2 - 8y^2 - 12x - 10y + 61 = 0$
4) $3x^2 + y^2 - 10y - 25 = 0$

4. **If the distance from P to the points (5, -4), (7, 6) are in the ratio 2 : 3, then the locus of P is**

1) $5x^2 + 5y^2 - 12x - 86y + 17 = 0$
2) $5x^2 + 5y^2 - 34x + 120y + 29 = 0$
3) $5x^2 + 5y^2 - 5x + y + 14 = 0$
4) $3x^2 + 3y^2 - 20x + 38y + 87 = 0$

5. **The equation of the locus of the points equidistant from the points A(-2,3) and B(6, -5) is**

1) $x+y=3$ 2) $x-y=3$ 3) $2x+y=3$ 4) $2x-y=3$

6. **If A(a,0), B(-a,0) then the locus of the point P such that $PA^2+PB^2=2c^2$ is**

1) $x^2+y^2+a^2-c^2=0$ 2) $x^2+y^2+a^2+c^2=0$
3) $2x^2+y^2+3a^2-c^2=0$ 4) $x^2+y^2+a^2+2c^2=0$

7. **The ends of hypotenuse of a right angled triangle are (5, 0), (-5, 0) then the locus of third vertex is**

1) $x^2-y^2 = 25$ 2) $x^2+y^2=25$
3) $x^2+y^2=5$ 4) $x^2-y^2=5$

8. **A(0,0), B(1,2) are two points. If a point P moves such that the area of ΔPAB is 2 sq.units, then the locus of P is**

1) $4x^2+4xy-y^2 = 16$
2) $4x^2-4xy+y^2 = 16$
3) $x^2+4xy+y^2 = 16$
4) $x^2-4xy-4y^2 = 16$

9. **The locus of a point which is collinear with the points (1, 2) and (-2, 1) is**

1) $x+3y+5=0$ 2) $x+3y-5=0$
3) $x-3y-5=0$ 4) $x-3y+5 = 0$

10. **A straight line of length 3 units slides with its ends A, B always on x and y axes respectively. Locus of centroid of Δ OAB is**

1) $x^2+y^2 = 3$ 2) $x^2+y^2 = 9$
3) $x^2+y^2=1$ 4) $x^2+y^2=8$

11. **If θ is parameter, $A=(a\cos\theta, a\sin\theta)$ and $B=(b\sin\theta,-b\cos\theta)$ $C=(1,0)$ then the locus of the centroid of ΔABC is (EAM-2014)**

1) $(3x+1)^2+9y^2=a^2+b^2$
2) $(3x-1)^2+9y^2=a^2-b^2$
3) $(3x-1)^2+9y^2=a^2+b^2$
4) $(3x+1)^2+9y^2=a^2-b^2$

12. **If t is parameter, A = (aSec t, bTan t) and B = (-aTan t, bSec t), O = (0, 0) then the locus of the centroid of Δ OAB is**

1) $9xy = ab$ 2) $xy = 9ab$
3) $x^2-9y^2 = a^2-b^2$ 4) $x^2-y^2= \frac{1}{9}(a^2-b^2)$

13. **The Locus of the point ($\tan\theta+\sin\theta$, $\tan\theta-\sin\theta$) is**

1) $\left((x^2y)^{2/3}+(xy^2)^{2/3}\right)=1$ 2) $x^2-y^2 = xy$
3) $x^2-y^2= 12xy$ 4) $(x^2-y^2)^2= 16xy$

14. **The Locus of the point $(a + bt, b - \frac{a}{t})$ is**

1) $(x-a)(y-b)+ab=0$ 2) $(x-a)(y-b) = 0$
3) $(x-a)(y-b) = ab$ 4) $(x-a)(y+b) = ab$

15. **The sum of the distances of a point P from two perpendicular lines in a plane is 1. Then locus of P is (EAMCET 2008)**

1) Square 2) Circle
3) Straight line 4) Pair of Straight lines

16. **The locus of point of intersection of the lines $y+mx=\sqrt{a^2m^2+b^2}$ and my-x = $\sqrt{a^2+b^2m^2}$ is**

1) $x^2+y^2=\frac{1}{a^2}+\frac{1}{b^2}$ 2) $x^2+y^2=a^2+b^2$

3) $x^2-y^2=a^2-b^2$ 4) $\frac{1}{x^2}+\frac{1}{y^2}=a^2-b^2$

17. **The coordinates of the points A and B are (a,0) and (-a,0) respectively.If a point P moves so that $PA^2-PB^2=2k^2$,where K is constant, then the equation to the locus of the point P.**

1) $2ax+k^2=0$ 2) $2ax-k^2=0$

3) $ax+2k^2=0$ 4) $ax-2k^2=0$

18. **A point moves in the XY-plane such that the sum of it's distances from two mutually perpendicular lines is always equal to 5 units. The area enclosed by the locus of the point is (EAM-2020)**

1) $\frac{25}{4}$ 2) 25 3) 50 4) 100

19. **If A = (1,0), B= (-1,0) and C = (2,0) then the locus of the point P such that $PA^2+PB^2=2PC^2$ is a [EAM - 2019]**

1) straight line parallel to y-axis

2) circle with centre (0,0)

3) circle through (0,0)

4) straight line parallel to x-axis

20. **The curve represented by x=2(cost+sint) and y = 5(cos t - Sin t) is**

1) a circle 2) a parabola

3) an ellipse 4) a hyperbola

21. **Locus represented by x =a (cosh θ+sinh θ), y = b (cosh θ-sinh θ) is [EAM -2018]**

1) a hyperbola 2) a parabola

3) an ellipse 4) a straight line

22. **The curve represented by x=ct and y = $\frac{c}{t}$ is**

1) a circle 2) a parabola

3) an ellipse 4) a hyperbola

23. **Locus represented by x =a +b sec θ, y = b + a Tan θ is**

1) a hyperbola 2) a parabola

3) an ellipse 4) a straight line

24. **The equation**

$x^2y^2-2xy^2-3y^2-4x^2y+8xy+12y=0$

represents

1) Two Pairs of lines 2) a Parabola

3) an Ellipse 4) hyperbola

25. **From a point P perpendiculars PM, PN are drawn to x and y axes respectively. If MN passes through fixed point (a,b), locus of P is**

1) xy= ax+by 2) xy = ab

3) xy = bx+ay 4) x+y=xy

26. **The sum of the squares of the distances of a moving point from two fixed points (a,0) and (-a,0) is equal to a constant quantity $2c^2$ then the equation to its locus is**

1) $x^2+y^2=c^2+a^2$ 2) $x^2+y^2=c^2-a^2$

3) $x^2-y^2=c^2-a^2$ 4) $x^2-y^2=c^2+a^2$

KEY

1) 1	2) 1	3) 3	4) 2	5)2	6) 1
7) 2	8) 2	9) 4	10) 3	11) 3	12) 1
13) 4	14) 1	15) 1	16) 2	17) 1	18) 3
19) 1	20) 3	21) 1	22) 4	23) 1	24) 1
25) 3	26) 2				

SOLUTIONS

1. Perpendicular distance from P(x,y) to x-axis is $|y|$ and y -axis is $|x|$

$\therefore |x|+|y|=9$

2. $|y|=2|x|$

3. $PA=2|y|$ where A=(6,5)

4. *Let* $p(x.y)\,A(5,-4)\,B(7,6)$ *Given 3PA = 2PB S.O.B.S*

$9PA^2=4PB^2$

$9\left((x-5)^2+(y+4)^2\right)=4\left((x-7)^2+(y-6)^2\right)$

$\Rightarrow 9\left(x^2+y^2-10x+8y+41\right)=4\left(x^2+y^2-14x-12y+85\right)$

$$\Rightarrow 9x^2+9y^2-90x+72y+369-4x^2-4y^2+56x+48y-340=0$$

Locus of P is

$$5x^2+5y^2-34x+120y+29=0$$

5. $PA^2=PB^2$
(or)
$2(x_1-x_2)x+2(y_1-y_2)y=x_1^2+y_1^2-x_2^2-y_2^2$

6. $(x-a)^2+y^2+(x+a)^2+y^2=2c^2$

7. A = (5,0) , B = (-5,0)
$PA^2+PB^2=AB^2$
(or)
$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

8. A=(0,0) , B = (1,2) ; P = (x,y)
Area of $\Delta PAB=\frac{1}{2}|x_1y_2-x_2y_1|=2$

Given $A(0,0)B(1,2)p(x,y)$ area of $\Delta PAB=2$

$\frac{1}{2}|x_1y_1-x_2y_1|=2$

$|2x-y|=4$ S.O.B.S $(2x-y)^2=16$

$4x^2+y^2-4xy=16$

9. A = (1,2) , B = (-2,1)
Equation of AB is
$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$

10. $P(x,y)=\left(\frac{a}{3},\frac{b}{3}\right)$
$a^2+b^2=9$

11. $G(x,y)=\left(\frac{a\cos\theta+b\sin\theta+1}{3},\frac{a\sin\theta-b\cos\theta}{3}\right)$

$A=(a\cos\theta,a\sin\theta),B=(b\sin\theta,-b\cos\theta),C=(1,0)$

centroid of $\Delta ABC=$

$(x,y)=\left(\frac{a\cos\theta+b\sin\theta+1}{3},\frac{a\sin\theta-b\cos\theta+0}{3}\right)$

$3x-1=a\cos\theta+b\sin\theta$

$3y=a\sin\theta-b\cos\theta$

squaring and adding

$(3x-1)^2+(3y)^2=(a\cos\theta+b\sin\theta)^2+(a\sin\theta-b\cos\theta)^2$

$(3x-1)^2+9y^2=a^2\cos\theta+b^2\sin^2\theta$

$2ab\cos\theta\sin\theta+a^2\sin^2\theta+b^2\cos^2\theta-2ab\sin\theta\cos\theta$

$(3x-1)^2+9y^2=a^2+b^2$

12. $(x,y)=\left(\frac{a\sec t-a\tan t}{3},\frac{b\tan t+b\sec t}{3}\right)$
Eliminate t

13. Eliminate θ

14. $x=a+bt; y=b-\frac{a}{t}$
$(x-a)(y-b)=(bt)\left(-\frac{a}{t}\right)$

15. $|x|+|y|=1$

16. Squaring and adding the equations

17. $(x-a)^2+(y-0)^2-(x+a)^2-y^2=2k^2$

18. $|x|+|y|=5$
$Area=\frac{2c^2}{|ab|}=2(5)^2=50$

19. $(x-1)^2+y^2+(x+1)^2+y^2=2\left[(x-2)^2+y^2\right]$

20. $\frac{x}{2}=\cos t+\sin t,\frac{y}{5}=\cos t-\sin t$

21. $\frac{x}{a}=\cosh\theta+\sinh\theta,\frac{y}{b}=\cosh\theta-\sinh\theta$

22. $xy=ct.\frac{c}{t}$
$xy=c^2$ is a rectangular hyperbola

23. $x=a+b\sec\theta$; $y=b+a\tan\theta$
$\frac{x-a}{b}=\sec\theta$; $\frac{y-b}{a}=\tan\theta$
$\sec^2\theta-\tan^2\theta=1$

24. $y^2(x^2-2x-3)-4y(x^2-2x-3)=0$
$(y^2-4y)(x^2-2x-3)=0$

$y=0, y=4, x+1=0, x-3=0$

25. Let $P(\alpha,\beta)$

Equation of a line passing through M,N is $\beta x+\alpha y=\alpha\beta$ passing through (a,b)

26. Let $P(x,y)$ be the locus $PA^2+PB^2=2C^2$

$\therefore$ Locus of P is $x^2+y^2=c^2-a^2$

EXERCISE-II

1. A(0,4), B(0,-4) are two points. The locus of P which moves such that |PA-PB| =6 is

1) $9x^2-7y^2+63=0$ 2) $9x^2+7y^2-63=0$
3) $9x^2+7y^2+63=0$ 4) $9x^2-7y^2-63=0$

2. A = (1, -1), locus of B is $x^2+y^2=16$. If P divides AB in the ratio 3:2 then locus of P is

1) $(x-2)^2+(y-3)^2=4$ 2) $(x+1)^2+(y-2)^2=4$
3) $(x-3)^2+(y-2)^2=4$ 4) $(5x-2)^2+(5y+2)^2=144$

3. A line segment AB of length 'a' moves with its ends on the axes. The locus of the point P which divides the segment in the ratio 1 : 2 is

1) $9x^2+4y^2=a^2$ 2) $9(x^2+4y^2)=4a^2$
3) $9(x^2+4y^2)=8a^2$ 4) $9x^2+9y^2=4a^2$

4. If the roots of the equation $(x_1^2-16)m^2-2x_1y_1m+y_1^2+9=0$ are the slopes of two perpendicular lines intersecting at $P(x_1,y_1)$ then the locus of P is

1) $x^2+y^2=25$ 2) $x^2+y^2=7$
3) $x^2-y^2=25$ 4) $x^2-y^2=7$

5. The locus of foot of the perpendicular drawn from a fixed point (2, 3) to the variable line y = mx, m being variable is

1) $x^2+y^2-2x+3y=0$ 2) $x+y-5=0$
3) $x^2+y^2-2x-3y=0$ 4) $xy-3x-2y+6=0$

6. Vertices of a variable triangle are (5,12), $(13\cos\theta,13\sin\theta)$ and $(13\sin\theta,-13\cos\theta)$, where $\theta\in R$. Locus of it's orthocentre is :

1) $x^2+y^2+6x+8y-25=0$
2) $x^2+y^2-10x-24y-169=0$
3) $x^2+y^2+10x-24y-169=0$
4) $x^2+y^2+10x+24y+169=0$

7. The locus of foot of the perpendicular drawn from a fixed point (a, b) to the variable line y = mx, m being variable is

1) $x^2+y^2-ax+by=0$ 2) $x+y-(a+b)=0$
3) $x^2+y^2-ax-by=0$ 4) $xy-bx-ay+ab=0$

8. Vertices of a variable triangle are (3,4), $(5\cos\theta,5\sin\theta)$ and $(5\sin\theta,-5\cos\theta)$, where $\theta\in R$. Locus of it's orthocentre is

1) $x^2+y^2+6x+8y-25=0$
2) $x^2+y^2-6x-8y+25=0$
3) $x^2+y^2+6x-8y-25=0$
4) $x^2+y^2-6x-8y-25=0$

9. A = (2, 5), B = (4, -11) and the locus of 'C' is 9x + 7y + 4 = 0 then the locus of the centroid of $\triangle$ABC is [EAM -2017]

1) 27x+21y-8=0 2) 3x+4y-2=0
3) 24x+22y-6=0 4) 5x+3y-7=0

10. The base of a triangle lies along x=a and is of length a. The area of triangle is a^2. The locus of vertex is

1) (x+a)(x-3a) = 0 2) (x-a)(x+3a)=0
3) (x+a)(x+3a) = 0 4) (x+2a)(x-a)=0

11. If $a,x_1,x_2,x_3....$ and $b,y_1,y_2,.....$ form two infinite A.P's with common difference p and q respectively then the locus of $P(h,k)$ when $h=\frac{x_1+x_2+x_3.......+x_n}{n}$, $k=\frac{y_1+y_2+.....+y_n}{n}$ is

1) $q(x-a)=p(y-b)$
2) $b(x+p)=a(y+q)$
3) $p(x+a)=q(y+b)$
4) $p(y+a)=q(x+b)$

12. Given P = (1,0) and Q = (-1,0) and R is a variable point on one side of the line PQ such

that $\angle RPQ - \angle RQP = \frac{\pi}{4}$. The locus of the point R is

1) $x^2+y^2+2xy=1$ 2) $x^2+y^2-2xy=1$

3) $x^2-y^2-2xy=1$ 4) $x^2-y^2+2xy=1$

13. A variable circle passes through the fixed point (0,5) and touches x-axis. Then locus of centre of circle

1) a parabola 2) a circle

3) an ellipse 4) a hyperbola

14. The equation $x^3+x^2y+x+y=0$ represents

1) a straightline [EAM -2081]

2) a parabola and two lines

3) a hyparabola and two lines

4) a line and a circle

15. The graph represented by x = sint, y = cos^2**t is**

1) a parabola

2) a portion of parabola

3) a part of sine graph 4)a part of Hyperbola

16. If the equation of the locus of a point equidistant from the points (a_1,b_1) and (a_2,b_2) is $(a_1-a_2)x+(b_1-b_2)y+c=0$ then the value of c is [EAM -2019]

1) $\frac{1}{2}\left(a_2^2+b_2^2-a_1^2-b_1^2\right)$

2) $a_1^2-a_2^2+b_1^2-b_2^2$

3) $\frac{1}{2}\left(a_1^2+a_2^2+b_1^2+b_2^2\right)$

4) $\sqrt{a_1^2+b_1^2-a_2^2-b_2^2}$

17. A line L_1 cuts x and y axes at $P(a,0)\, and\, Q(0,b)$ respectively, another line L_2 perpendicular to L_1 cuts x and y axes at R and S respectively. The locus of the point of intersection of the lines PS and QR is

1) $x(x-a)+y(y-b)=0$

2) $x(x+a)+y(y+b)=0$

3) $x(x+a)+y(y-b)=0$

4) $x(x-a)+y(y+b)=0$

KEY

01) 1	02) 4	03) 2	04) 2	05) 3	06) 2
07) 3	08) 4	09) 1	10) 1	11) 1	12) 4
13) 1	14) 1	15) 2	16) 1	17) 1	

SOLUTIONS

1. $PA = PB \pm 6$

(or) $\frac{4(x-a)^2}{k^2-4b^2}+\frac{4y^2}{k^2}=1$

where k = 6, a = 0, b= 4

2. $A(1,-1),\ B(\alpha,\beta),\ P(x,y)$

$(x,y)=\left(\frac{3\alpha+2}{5},\frac{3\beta-2}{5}\right)$

Find α,β sub in $x^2+y^2=16$

3. $A(p,0)B(0,q)$ Use section formula

4. $m_1m_2=-1$

$\frac{y_1^2+9}{x_1^2-16}=-1$

5. Let P= (2,3), Q= (x,y)

PQ $\perp$ L

$\therefore$ Slope of PQ $\times$ m = -1

$\left(\frac{y-3}{x-2}\right)\left(\frac{y}{x}\right)=-1$

6. Circum Centre (S) = (0,0)

Orthocentre ; O (x,y) = 3G - 2S

= (5+13cosθ+13sinθ ; 12+13sinθ-13cosθ)

$(x-5)^2+(y-12)^2$

$=169\left[(\cos\theta+\sin\theta)^2+(\sin\theta-\cos\theta)^2\right]$

7. Let $P=(a,b),Q=(x,y)$

$\because y=mx \Rightarrow m=\frac{y}{x}$

$\because PQ \perp L \Rightarrow slope\ of\ PQ \times m=-1$

8. Circum centre (S)

O=3G-2S where O is orthocentre

$O(x,y)=(3+5\cos\theta+5\sin\theta, 4+5\sin\theta-5\cos\theta)$

9. Let $C(\alpha,\beta)$

$(x,y)=\left(\frac{6+\alpha}{3},\frac{-6+\beta}{3}\right)$

$(\alpha,\beta)=(3x-6,3y+6)$ sub

$9x+7y+4=0$

$(\alpha,\beta)=(3x-6,3y+6)$ lies on $9x+7y+4=0$

$9(3x-6)+7(3y+6)+4=0$

$27x+21y-8=0$

10. Consider $A(a,0),B(a,a)$ two points on a line $x=a$ and $P(x,y)$

Area of the triangle $=a^2$

(or) $\frac{1}{2}a|x-a|=a^2$

$|x-a|=2a$

$x-a=\pm 2a$

$(x+a)(x-3a)=0$

11. $x_1-a=x_2-x_1=\_\_\_\_=p$

$x_1=a+p$

$x_2=a+2p$

$x_n=a+np$

$\frac{\Sigma x_i}{n}=a+\frac{(n+1)p}{2}$

$x=a+\frac{(n+1)p}{2}$

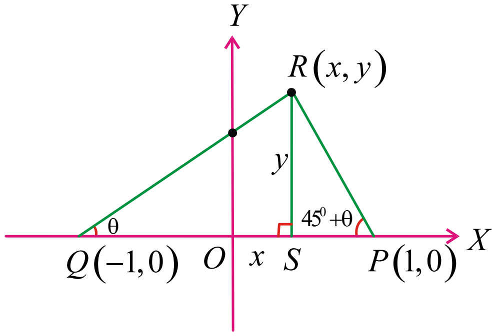
$\frac{x-a}{p}=\frac{n+1}{2}$

Similarly $\frac{y-b}{q}=\frac{n+1}{2}$

$\frac{x-a}{p}=\frac{y-b}{q}$

$q(x-a)=p(y-b)$

12.



$\tan\theta=\frac{y}{1+x}$

$\tan\left(\frac{\pi}{4}+\theta\right)=\frac{y}{1-x}$

$\frac{1+\tan\theta}{1-\tan\theta}=\frac{y}{1-x}$

$\frac{1+\frac{y}{1+x}}{1-\frac{y}{1+x}}=\frac{y}{1-x}$

13. Let center be C(h,k) ; r = k (Circle touches x-axis)

$(h-0)^2+(k-5)^2=k^2$

$h^2=10k-25$

Locus is $x^2=10y-25$ which represents a parabola.

14. $x^2(x+y)+1(x+y)=0$

$(x^2+1)(x+y)=0$

$x^2+1=0$ is not possible for all $x\in R$

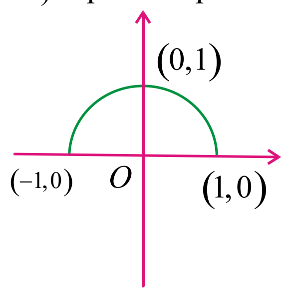
$\therefore x+y=0$ which represents a straight line.

15. $x=\sin t$; $y=\cos^2 t$

$-1\le x\le 1;\quad 0\le y\le 1$

$x^2+y=\sin^2 t+\cos^2 t=1$

$x^2=-(y-1)$ represents portion of a parabola



16. $A(a_1,b_1),B(a_2,b_2),P(x,y)$

PA = PB

(or) $2(a_1 - a_2)x + 2(b_1 - b_2)y = a_1^2 + b_1^2 - a_2^2 - b_2^2$

$c = \frac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2)$

17. Let $L_1 \equiv \frac{x}{a} + \frac{y}{b} = 1$

Now $L_2 \equiv \frac{x}{bk} - \frac{y}{ak} = 1$ PS equation is $\frac{x}{a} - \frac{y}{ak} = 1$, ...(1) $\frac{x}{bk} + \frac{y}{b} = 1$ ...(2)
eliminate K from (1) and (2)

EXERCISE-III

1. The line joining $(5,0)$ to $(10\cos\theta, 10\sin\theta)$ is divided internally in the ratio 2:3 at P, then the locus of P is

1) $x^2 + 2xy + y^2 - 6x = 0$ 2) $x + y - 3 = 0$
3) $(x-3)^2 + y^2 = 16$ 4) $x^2 = y - 3$

2. If the first point of trisection of AB is $(t, 2t)$ and the ends A,B moves on x and y axis respectively, then locus of mid point of AB is

1) $x = y$ 2) $2x = y$ 3) $4x = y$ 4) $x = 4y$

3. The variable line drawn through the point (1,3) meets the x-axis at A and y-axis at B. If the rectangle OAPB is completed. Where "O" is the origin, then locus of "P" is

1) $\frac{1}{y} + \frac{3}{x} = 1$ 2) $x + 3y = 1$

3) $\frac{1}{x} + \frac{3}{y} = 1$ 4) $3x + y = 1$

4. P and Q are two variable points on the axes of x and y respectively such that |OP| + |OQ|=a, then the locus of foot of perpendicular from origin on PQ is

1) $(x - y)(x^2 + y^2) = axy$
2) $(x + y)(x^2 + y^2) = axy$
3) $(x + y)(x^2 + y^2) = a(x - y)$
4) $(x + y)(x^2 - y^2) = axy$

5. The algebraic sum of the perpendicular distances from the points A (-2,0), B(0,2) and C(1,1) to a variable line be zero, then all such lines

1) are parallel
2) passes through a fixed point(0,0)
3) form a square
4) passes through the centroid of Δ ABC.

6. The straight line passing through the point (8,4) and cuts y-axis at B and x-axis at A.The locus of mid point of AB is

1) $xy + 2x + 4y = 64$
2) $xy - 2x - 4y = 0$
3) $xy - 4x - 2y + 8 = 0$
4) $xy + 4x + 2y = 72$

7. Sum of the distance of a point from two perpendicular lines is 3 the area enclosed by the locus of the point is

1) 18 2) 16 3) 4 4) 15

8. Locus of point of intersection of the lines $x\sin\theta - y\cos\theta = 0$ and

$ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2 - b^2$

1) $x^2 + y^2 = a^2$ 2) $x^2 + y^2 = b^2$

3) $x^2 + y^2 = a^2 + b^2$ 4) $x^2 + y^2 = (a+b)^2$

9. If $A(1,1), B(2,3), C(-1,1)$ are the points of P is a point such that the area of the quadrilateral. P A B and C is 3 sq units then locus of P is

1) $y^2 + 6y = 0$ 2) $y^2 - 6y = 0$
3) $x^2 + 6y = 0$ 4) $x^2 - 6y = 0$

10. The vertices of a triangle are $(1, \sqrt{3})$,

$(2\cos\theta, 2\sin\theta)$ *and* $(2\sin\theta, -2\cos\theta)$

where $\theta \in R$. The locus of orthocentre of the triangle is

1) $(x-1)^2 + (y-\sqrt{3})^2 = 4$

2) $(x-2)^2+\left(y-\sqrt{3}\right)^2=4$

3) $(x-1)^2+\left(y-\sqrt{3}\right)^2=8$

4) $(x-2)^2+\left(y-\sqrt{3}\right)^2=8$

11. A point moves such that the sum of the squares of its distance from the sides of a square of side unity is equal to 9, the locus of such point is

1) $x^2+y^2-x-y-\frac{7}{2}=0$

2) $x^2+y^2-2x-2y-7=0$

3) $x^2+y^2-x-y-\frac{5}{2}=0$

4) $x^2+y^2-2x-2y-5=0$

12. Variable straight lines $L_1: y=2x+c_1$ and $L_2: y=2x+c_2$ meet the x-axis in A_1 and A_2 respectively and y-axis in B_1 and B_2 respectively locus of intersection point of A_1B_2 and A_2B_1 is

1) $y+x=0$ 2) $y=x$

3) $y+2x=0$ 4) $y=2x$

13. Let a and b non zero real numbers. Then the equation $\left(ax^2+by^2+c\right)\left(x^2-5xy+6y^2\right)=0$ represents (IIT-08)

1) four straight lines, when c=0 and a,b are of the same sign

2) two straight lines and a circle, when a=b and c is of the sign opposite to that of a

3) a circle and an ellipse, when a and b are of the same sign and c is of the sign opposite to that of a

4) two straight lines and a circle,when a and b are of the same sign and c is of the sign opposite to that of a

14. If the distance of any point P(x,y) from the point $Q(x_1,y_1)$ is given by d(P,Q)=max. $\{|x-x_1|,|y-y_1|\}$. If Q is fixed point (1,2), and d(P,Q)=3, then the locus of P is

1) a circle 2) a stright line

3) a square 4) a triangle

15. A straight line passing through the point (x_1,y_1) meets the positive coordinate axis at A,B. The locus of the point P which divides AB in the ratio $l:m$ is

1) $\frac{lx_1}{x}+\frac{my_1}{y}=l+m$ 2) $\frac{mx_1}{x}+\frac{ly_1}{y}=l+m$

3) only 1 is true 4) both 1 and 2 are true

16. $A(0,ae)\,B(0,-ae)$ are two points.The equation to the locus of p such that $PA+PB=2a$ is

1) $\frac{x^2}{a^2\left(1-e^2\right)}+\frac{y^2}{a^2}=1$

2) $\frac{x^2}{a^2}-\frac{y^2}{a^2\left(1-e^2\right)}=1$

3) $\frac{x^2}{a^2}+\frac{y^2}{a^2\left(1+e^2\right)}=1$

4) $\frac{x^2}{a^2}-\frac{y^2}{a^2\left(1+e^2\right)}=1$

17. $A(ae,0),B(-ae,0)$ are two points. The equation to the locus of P such that $PA-PB=2a$ is

1) $\frac{x^2}{a^2}+\frac{y^2}{a^2\left(1-e^2\right)}=1$

2) $\frac{x^2}{a^2}-\frac{y^2}{a^2\left(1-e^2\right)}=1$

3) $\frac{x^2}{a^2}+\frac{y^2}{a^2\left(1+e^2\right)}=1$

4) $\frac{x^2}{a^2}-\frac{y^2}{a^2\left(1+e^2\right)}=1$

KEY

1) 3	2) 3	3) 3	4) 2	5) 4	6) 2
7) 1	8) 4	09) 2	10) 3	11) 1	12) 3
13) 2	14) 3	15) 4	16) 1	17) 1	

SOLUTIONS

1. $\left(\frac{15+20\cos\theta}{5}, \frac{20\sin\theta}{5}\right)=(x,y);$

   Eliminate 'θ'

2. Let $P(h,k)$ locus of mid point $A(a,0)$ $B(0,b)$

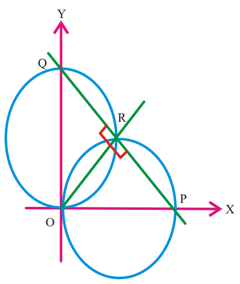
   $(t,2t)=\left[\frac{2a}{3},\frac{b}{3}\right]$ eliminate t we get $4h=k$

3. Let the line be $\frac{x}{a}+\frac{y}{b}=1$

   If passes through (1,3), $\because \frac{1}{a}+\frac{3}{b}=1$

   $A(a,0), B(0,b). \therefore P=(a,b)$

   $\therefore$ locus of P is $\frac{1}{x}+\frac{3}{y}=1$.

4. 

   Let $P(\alpha,0)$ $Q(0,\beta)$

   Equation of the circle passing through O,P,R is $x^2+y^2-\alpha x=0$

   $\alpha=\frac{x^2+y^2}{x}$

   III^{ly} $\beta=\frac{x^2+y^2}{y}$

   $|\alpha|+|\beta|=a$

5. Algebraic sum of the perpendicular distances from three non collinear points is zero, then the line passing through centroid of the triangle formed by these points.

6. Let Equation of AB $\frac{x}{a}+\frac{y}{b}=1$ ...(1)

   Let $P(h,k)$ locus of mid point of AB

   $a=2h, b=2k$ substitue in (1) we get

   $xy-2x-4y=0$

7. Let $P(x,y)$ be the locus

   $|x|+|y|=3 \Rightarrow$ area $=18$ sq.units

8. Eliminate θ

9. Let $P(x_1 y)$ be the locus of the point

   $\frac{1}{2}\begin{vmatrix} x-2 & 1+1 \\ y-3 & 1-1 \end{vmatrix}=3$

10. $\left(\frac{1+2\cos\theta+2\sin\theta}{3}, \frac{\sqrt{3}+2\sin\theta-2\cos\theta}{3}\right)$

    C(0,0) G H(x, y)

    1 : 2

    $\frac{x}{3}=\frac{1+2\cos\theta+2\sin\theta}{3}$

    $\Rightarrow x=1+2\cos\theta+2\sin\theta$

    $\frac{y}{3}=\frac{\sqrt{3}+2\sin\theta-2\cos\theta}{3}$

    $\Rightarrow y=\sqrt{3}+2\sin\theta-2\cos\theta$

    $(x-1)^2+\left(y-\sqrt{3}\right)^2=8$

11. Let (h,k) be the locus

    $h^2+(1-h)^2+k^2+(1-k)^2=9$

12. Equation A_1B_2 in $\frac{2x}{-c_1}+\frac{y}{c_2}=1$

    equation A_2B_2 in $\frac{-2x}{c_2}+\frac{y}{c_1}=1$ elimiate c_1 and c_2 from the above equations.

13. $x^2 - 5xy + 6y^2 = 0$ representes two straightlines if c<0, a=b then

$ax^2 + by^2 + c = 0 \Rightarrow x^2 + y^2 = \frac{c}{a}$; where $\frac{c}{a} > 0$

14. Locus of P consist of lines $|x-1| = 3, |y-2| = 3$

15. Let $A(a,0)$ $B(0,b)$ Let $P(x_1, y_1)$ divides AB in the ratio $l:m$ weget locus of P is

$$\frac{lx_1}{x} + \frac{my_1}{y} = l + m$$

$A(0,b)B(0,a)$ let $P(x,y)$ be the locus weget

locus of P is $\frac{mx_1}{x} + \frac{ly_1}{y} = l + m$.

16. $A(0,ae)B(0,-ae)P(x,y)$ $PA + PB = 2a$

locus of P is $\frac{x^2}{a^2(1-e^2)} + \frac{y^2}{a^2} = 1$

17. $PA^2 = (2a + PB)^2$

(or) $\frac{4x^2}{k^2} + \frac{4(y-b)^2}{k^2 - 4a^2} = 1$

$(k = 2a, a = ae, b = 0)$

TRANSFORMATION OF AXES

SYNOPSIS

- Change of axes or transformation of axes is of three types :
  i) Translation of axes
  ii) Rotation of axes
  iii) General Transformation

Translation of axes:

- i) Shifting the origin to some other point without changing the direction of axes.

ii) When the origin is translated to (h,k), the equations of transformation are
x = X+h, y =Y+k where (x, y) are the original coordinates and (X, Y) are the new coordinates of the point.

Rotation of axes:

- i) Rotating the system of coordinate axes through an angle 'θ' without changing the position of the origin.

ii) When the axes are rotated through an angle 'θ' in anticlockwise direction. The equations of transformation are given by

	X	Y
x	Cos θ	– Sin θ
y	Sin θ	Cos θ

Set-1 $x = X\cos\theta - Y\sin\theta$,
$y = X\sin\theta + Y\cos\theta$,

Set-2 $X = x\cos\theta + y\sin\theta$,
$Y = -x\sin\theta + y\cos\theta$,

- Transformation is used in reducing the general equation of any curve to the desired form. For example

i) To eliminate first degree terms, we apply translation.

ii) To eliminate the term containing 'xy', we apply rotation.

iii) The point to which the origin has to be shifted to eliminate first degree terms (x, y terms) in $S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is obtained by solving $\frac{\partial S}{\partial x} = 0, \frac{\partial S}{\partial y} = 0$

iv) To remove the first degree terms from the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ the origin is to be shifted to the point

$$(x_1, y_1) = \left[\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right], \; ab - h^2 \neq 0.$$

In this case, the transformed equation is
$aX^2 + 2hXY + bY^2 + (gx_1 + fy_1 + c) = 0$

v) To remove the first degree terms from the equation $ax^2 + by^2 + 2gx + 2fy + c = 0$, the origin is to be shifted to the point $\left(\frac{-g}{a}, \frac{-f}{b}\right)$. In this case, the transformed equation is

$$aX^2 + bY^2 + \left(\frac{-g^2}{a} + \frac{-f^2}{b} + c\right) = 0$$

vi) To remove the first degree terms from 2hxy + 2gx + 2fy + c = 0, the origin is to be shifted to the point $\left(\frac{-f}{h}, \frac{-g}{h}\right)$. In this case, the transformed equation is $2h^2XY - 2gf + ch = 0$

vii) The point to which the origin has to be shifted to eliminate x and y terms in the equation $a(x+\alpha)^2 + b(y+\beta)^2 = c$ is $(-\alpha, -\beta)$

viii) a) To remove xy term of $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ the angle of rotation of axes is

$$\theta = \frac{1}{2}\mathrm{Tan}^{-1}\left(\frac{2h}{a-b}\right), \text{if } a \neq b$$

$$= (2n+1)\frac{\Pi}{4}, n \in z \text{ if } a = b$$

b) If 'θ' is angle of rotation to eleminate XY term in $ax^2+2hxy+by^2+2gx+2fy+c=0$, then $n\frac{\pi}{2}+\theta$, n $\in$ Z is also an angle of rotation to eliminate XY term

ix) The angle of rotation of axes so that the equation ax + by + c = 0 is reduced as

a) X = constant is $\mathrm{Tan}^{-1}\left(\frac{b}{a}\right)$

b) Y = constant is $\mathrm{Tan}^{-1}\left(-\frac{a}{b}\right)$

x) The equation

$S \equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ has transformed to $AX^2+2HXY+BY^2+2GX+2FY+C=0$, when the origin is shifted to (l,m) then

A = a ; B = b ; H = h ;

$2G=\left(\frac{\partial S}{\partial x}\right)_{(l,m)}$ $2F=\left(\frac{\partial S}{\partial y}\right)_{(l,m)}$ C = S (l,m)

✈ The condition that the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ to take the form $aX^2+2hXY+bY^2=0$ when the axes are translated is $abc+2fgh-af^2-bg^2-ch^2=0$

General Transformation :

✈ i) Applying both translation and rotation.

ii) The equations of general transformation are given by

	X	Y
x - h	Cos θ	– Sin θ
y - k	Sin θ	Cos θ

Set-1: $x-h = X\cos\theta - Y\sin\theta$,

$y-k = X\sin\theta + Y\cos\theta$,

Set-2: $X=(x-h)\cos\theta+(y-k)\sin\theta$

$Y=-(x-h)\sin\theta+(y-k)\cos\theta$

Where (h, k) is the new origin and θ is the angle of rotation.

Note : 1) If the rotation is in clockwise direction then replace θ by $-\theta$.

2) On translation or rotation the position of the point, length of line segment, area, perimeter, angles are not changed. But the coordinates and equations will change.

EXERCISE-I

1. If (3,2) are coordinates of a point 'P' in the new system when origin is shifted to (3,7), then the original coordinates of 'P' are

1) (6,9) 2) (-6,9) 3) (6,-9) 4) (6,0)

2. The coordinates of the point (4,5) in the new system, when its origin is shifted to (3,7) are

1) (1, 2) 2) (-1, 2) 3) (-1, -2) 4) (1, -2)

3. If the point (5,7) is transformed to (-1,2) when the origin is shifted to A, then A=

1) (4,9) 2) (6,5) 3) (-6,-5) 4) (2,4)

4. If the origin is shifted to the point (-1,2) without changing the direction of axes, the equation $x^2-y^2+2x+4y=0$ becomes

1) $X^2+Y^2+3=0$ 2) $X^2+Y^2-3=0$
3) $X^2-Y^2+3=0$ 4) $X^2-Y^2-3=0$

5. If the transformed equation of a curve when the origin is translated to (1, 1) is $X^2+Y^2+2X-Y+2=0$ then the original equation of the curve is

1) $x^2+2y^2=1$ 2) $x^2+y^2+3y+3=0$
3) $x^2+y^2+3y-3=0$ 4) $x^2+y^2-3y+3=0$

6. When the axes are translated to the point (5,-2) then the transformed form of the equation xy + 2x - 5y - 11 = 0 is

1) $\frac{X}{Y}=1$ 2) $\frac{Y}{X}=1$ 3) XY = 1 4) $XY^2=2$

7. In order to make the first degree terms missing in the equation $2x^2+7y^2+8x-14y+15=0$, the origin should be shifted to the point

1) (1, -2) 2) (-2, -1) 3) (2, 1) 4) (-2, 1)

8. The point to which the origin should be shifted in order to remove the x and y terms in the equation [EAM -2018] $14x^2-4xy+11y^2-36x+48y+41=0$ is

1) (1, -2) 2) (-2, 1) 3) (-1, 2) 4) (2, -1)

9. If the distance between the two given points is 2 units and the points are transferred by shifting the origin to (2, 2), then the distance between the points in their new position is

1) 2 2) 5 3) 6 4) 7

10. When (0, 0) shifted to (3, -3) the coordinates of P(5, 5), Q(-2, 4) and R(7, -7) in the new system are A, B, C then area of triangle ABC in sq units is

1) 43 2) 23 3) 45 4) 50

11. When axes are rotated through an angle of 45^0 in positive direction without changing origin then the coordinates of $\left(\sqrt{2},4\right)$ in old system are [EAM -2019]

1) $\left(1-2\sqrt{2},1+2\sqrt{2}\right)$ 2) $\left(1+2\sqrt{2},1-2\sqrt{2}\right)$

3) $\left(2\sqrt{2},\sqrt{2}\right)$ 4) $\left(2,\sqrt{2}\right)$

12. If the axes are rotated through an angle 30^0 in the clockwise direction, the point $\left(4,2\sqrt{3}\right)$ in the new system is

1) (2,3) 2) $\left(2,\sqrt{3}\right)$ 3) $\left(\sqrt{3},2\right)$ 4) $\left(\sqrt{3},5\right)$

13. The transformed equation of $3x^2+3y^2+2xy=2$ when the coordinate axes are rotated through an angle of 45^0 is (EAMCET - 2008)

1) $X^2+2Y^2=1$ 2) $2X^2+Y^2=1$

3) $X^2+Y^2=1$ 4) $X^2+3Y^2=1$

14. If the transformed equation of a curve is $17X^2-16XY+17Y^2=225$ when the axes are rotated through an angle 45^o, then the original equation of the curve is

1) $25x^2+9y^2=225$ 2) $9x^2+25y^2=225$

3) $25x^2-9y^2=225$ 4) $9x^2-25y^2=225$

15. If the axes are rotated through an angle 180^o then the equation 2x - 3y + 4=0 becomes

1) 2X - 3Y - 4 = 0 2) 2X + 3Y - 4 = 0

3) 3X - 2Y + 4 = 0 4) 3X + 2Y + 4 = 0

16. When the axes are rotated through an angle 90^0 the equation $5x-2y+7=0$ transforms to

1) $2X-5Y+7=0$ 2) $2X+5Y-7=0$

3) $2X-5Y-7=0$ 4) $2X+5Y+7=0$

17. If the equation $4x^2+2\sqrt{3}xy+2y^2-1=0$ becomes $5X^2+Y^2=1$, when the axes are rotated through an angle θ, then θ is

1) 15^o 2) 30^o 3) 45^o 4) 60^o

18. The angle of rotation of axes in order to eliminate xy term in the equation $xy=c^2$ is

1) $\frac{\pi}{12}$ 2) $\frac{\pi}{6}$ 3) $\frac{\pi}{3}$ 4) $\frac{\pi}{4}$

19. The transformed equation of $x^2+y^2=r^2$, when the axes are rotated through an angle 36^0 is [EAM -2020]

1) $\sqrt{5}X^2-4XY+Y^2=r^2$

2) $X^2+2XY-\sqrt{5}Y^2=r^2$

3) $X^2-Y^2=r^2$ 4) $X^2+Y^2=r^2$

20. The transformed equation of $x\cos\alpha+y\sin\alpha=P$ when the axes are rotated through an angle α is

1) $X=P$ 2) $X+P=0$

3) $Y=P$ 4) $Y+P=0$

KEY

01) 1	02) 4	03)2	04) 3	05) 4	06) 3
07) 4	08) 1	09) 1	10) 1	11) 1	12) 4
13) 2	14)1	15) 1	16) 2	17) 2	18) 4
19) 4	20) 1				

SOLUTIONS

1. $(X,Y)=(3,2),(h,k)=(3,7),(x,y)=(X+h,Y+k)$

2. $(x,y)=(4,5)$

   (h,k) = (3,7)

   (X,Y) = (x-h,y-k)

3. $(x,y)=(5,7),(X,Y)=(-1,2)$

   $A=(x-X,y-Y)=(6,5)$

4. (h,k)=(-1,2) Put

   $x=X-1,\ y=Y+2$ trans formed equation is

   $X^2-Y^2+3=0$

5. Given $(h,k)=(1,1)$

   $x^2+y^2+2x-y+2=0$

   $x=x-h=x-1$

$y = y - k = y - 1$ original equation

$$\left(x-1\right)^2+\left(y-1\right)^2+2\left(x-1\right)-\left(y-1\right)+2=0$$

$$\Rightarrow x^2+1-2x+y^2+1-2y+2x-2-y+3=0$$

$$\Rightarrow x^2+y^2-3y+3=0$$

6. (h,k) =(5,-2)

   Put $x = X+5, y = Y-2$

   $\Rightarrow XY = 1$

7. $a=2, b=7, g=4, f=-7, c=15$

   New origin= $\left(\frac{-g}{a},\frac{-f}{b}\right)$=(-2,1)

8. $14x^2-4xy+11y^2-36x+48y+41=0$

$a=14,\ h=-2\ \ b=11,\ 2=-18,\ f=24,\ \ c=41$

$$=\left(\frac{hf-bg}{ab-h^2},\frac{gh-af}{ab-h^2}\right)=\left(\frac{-48+198}{154-4},\frac{36-336}{154-4}\right)=\left(1,-2\right)$$

9. Distance remains same

10. Area of triangle ABC = Area of triange PQR

11. Use $x = X\cos\theta - Y\sin\theta$

    $y = X\sin\theta + Y\cos\theta$

    $\theta = 45^0\ \left(x,y\right)=\left(\sqrt{2},4\right)$

    $x = x\cos\theta - y\sin\theta, y = x\sin\theta + y\cos\theta$

    $$x=\left(\sqrt{2}.\frac{1}{\sqrt{2}}-4.\frac{1}{\sqrt{2}}, y=\sqrt{2}.\frac{1}{\sqrt{2}}+4.\frac{1}{\sqrt{2}}\right.$$

    $$\left(x,y\right)=\left(1-2\sqrt{2},1+2\sqrt{2}\right)$$

12. $X = x\cos\theta + y\sin\theta$

    $Y = -x\sin\theta + y\cos\theta$

    Where $\theta = -30^0$

13. $x = X\cos\theta - Y\sin\theta$

    $y = X\sin\theta + Y\cos\theta$ G i v e n

    $\theta = 45^0$ $\qquad 3x^2+3y^2+2xy=2$

    $x = \cos\theta - y\sin\theta$

    $y = x\sin\theta + y\cos\theta$ $\qquad x=\frac{x-y}{\sqrt{2}}, y=\frac{x+y}{\sqrt{2}}$

    transformed equation

    $$3\left(\frac{x-y}{\sqrt{2}}\right)^2+3\left(\frac{x+y}{\sqrt{2}}\right)^2+2\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{x+y}{\sqrt{2}}\right)=2,$$

    $4x^2+2y^2=2,\quad 2x^2+y^2=1$

14. $\theta = 45^0$

    $17x^2-16xy+17y^2=225$

    $x = x\cos\theta + y\sin\theta$

    $y = -x\sin\theta + y\cos\theta$

    $$x=\frac{x+y}{\sqrt{2}}, y=\frac{-x+y}{\sqrt{2}}$$

    $$17\left(\frac{x+y}{\sqrt{2}}\right)^2-16\left(\frac{y+x}{\sqrt{2}}\right)\left(\frac{y-x}{\sqrt{2}}\right)+17\left(\frac{y-x}{\sqrt{2}}\right)^2=225$$

    $$17\left(\frac{2\left(x^2+y^2\right)}{2}\right)-16\frac{\left(y^2-x^2\right)}{2}=225$$

    $$17\left(x^2+y^2\right)-8\left(y^2-x^2\right)=225$$

    $$25x^2+9y^2=225$$

15. x = X cos180^0 - Y sin180^0,

    y = X sin180^0 + Ycos180^0,

16. $x = X\cos 90^0 - Y\sin 90^0$

    $y = X\sin 90^0 + Y\cos 90^0$

17. $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)=30^0$

18.Given equation $xy=c^2, a=0, b=0, h=\frac{1}{2},$

$$\theta=\frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)\Rightarrow\theta=\frac{1}{2}\tan^{-1}\left(\frac{1}{0}\right)=\frac{1}{2}\left(\frac{\pi}{2}\right)=\frac{\pi}{4}$$

19. $x = X\cos 36^0 - Y\sin 36^0,$

$y = X\sin 36^0 + Y\cos 36^0$

20. Use $x = X\cos\alpha - Y\sin\alpha$

$y = X\sin\alpha + Y\cos\alpha$

EXERCISE-II

1. The point to which the origin should be translated in order to make the first degree terms missing in the equation $3xy - 2x + y - 8 = 0$ is

1) $\left(-\frac{1}{3},\frac{2}{3}\right)$ 2) $\left(-\frac{1}{3},-\frac{2}{3}\right)$

3) $\left(\frac{2}{3},-\frac{1}{3}\right)$ 4) $\left(-\frac{2}{3},\frac{1}{3}\right)$

2. By translation of axes the equation $xy - x + 2y - 6 = 0$ changed as XY=c then c=

1) 4 2) 5 3) 6 4) 7

3. The origin is shifted to (1, 2), the equation $y^2 - 8x - 4y + 12=0$ changes to $Y^2 + 4aX = 0$ then a =

1) 2 2) -2 3) 1 4) -1

4. The transformed equation of $4x^2 + 9y^2 - 8x + 36y + 4 = 0$ when the axes are translated to (1,-2) is $aX^2 + bY^2 = c$. Then desending order of a,b,c

1) c,b,a 2) c,a,b 3) a,b,c 4) a,c,b

5. The condition that the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, can take the form $aX^2 + 2hXY + bY^2 = 0$ by translating the origin to a suitable point is

1) $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

2) $2fgh - af^2 - bg^2 - ch^2 = 0$

3) $abc - af^2 - bg^2 - ch^2 = 0$

4) abc+2fgh = 0

6. If $(\cos\alpha, \cos\beta)$ are the new co-ordinates of a point P when the axes are translated to the point $(1,1)$, then the original coordinates are

1) $(2\cos^2\alpha/2, 2\cos^2\beta/2)$

2) $(2\cos^2\alpha/2, 2\sin^2\beta/2)$

3) $(2\sin^2\alpha/2, 2\cos^2\beta/2)$

4) $(-2\cos^2\alpha/2, -2\cos^2\beta/2)$

7. The first degree terms of $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are removed by shifting origin to (α,β). The new equation is

1) $ax^2 + 2hxy + by^2 + 2y\alpha + 2b\beta + c = 0$

2) $ax^2 + 2hxy + by^2 + g\alpha + f\beta + c = 0$

3) $ax^2 + 2hxy + by^2 + h\alpha + b\beta + c = 0$

4) $ax^2 + 2hxy - by^2 - h\alpha - b\beta - c = 0$

8. When the angle of rotation of axes is $Tan^{-1}2$, the transformed equation of $4xy - 3x^2 = a^2$ is

1) $2XY + a^2 = 0$ 2) $XY - a^2 = 0$

3) $X^2 - 4Y^2 = a^2$ 4) $X^2 - 2Y^2 = a^2$

9. The angle of rotation of axes so that $\sqrt{3}x - y + 1 = 0$ transformed as y=k is

1) $\frac{\pi}{6}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{3}$ 4) $\frac{\pi}{2}$

10. The angle of rotation of the axes so that the equation $x + y - 6 = 0$ may be reduced to the form $X = 3\sqrt{2}$ is [EAM -2017]

1) $\pi/6$ 2) $\pi/4$ 3) $\pi/3$ 4) $\pi/2$

11. The coordinate axes are rotated about the origin 'O' in the counter clockwise direction through an angle 60°. If a and b are the intercepts made on the new axes by a straight line whose equation referred to the original axes is $3x + 4y-5=0$ then $\frac{1}{a^2}+\frac{1}{b^2}=$

1) 1/25 2) 1/9 3) 1/16 4) 1

12. The coordinate axes are rotated through an angle θ about the origin in

anticlock-wise sense.If the equation $2x^2+3xy-6x+2y-4=0$ changes to $ax^2+2hxy+by^2+2gx+2fy+c=0$ then a+b is equal to

1) $3\cos\theta-3\sin\theta$ 2) $3\cos\theta+2\sin\theta$
3) 1 4) 2

13. **Let L be the line 2x+y-2=0. The axes are rotated by 45^0 in clockwise direction then the intercepts made by the line L on the new axes are respectively [EAM -2016]**

1) $1,\sqrt{2}$ 2) $\sqrt{2}$, 1

3) $2\sqrt{2},\frac{2\sqrt{2}}{3}$ 4) $\frac{2\sqrt{2}}{3},2\sqrt{2}$

14. **The acute angle θ through which the coordinate axes should be rotated for the point A (2,4) to attain the new abscissa 4 is given by**

1) $\tan\theta=3/4$ 2) $\tan\theta=5/6$
3) $\tan\theta=7/8$ 4) $\tan\theta=\frac{3}{2}$

15. **A line has intercepts a,b on the axes when the axes are rotated through an angle α, the line makes equal intercepts on axes then** $\tan\alpha=$

1) $\left(\frac{a+b}{a-b}\right)$ 2) $\left(\frac{a-b}{a+b}\right)$

3) $\left(\frac{a}{b}\right)$ 4) $\left(\frac{b}{a}\right)$

16. **The new equation of the curve $4(x-2y+1)^2+9(2x+y+2)^2=25$, if the lines $2x+y+2=0$ and $x-2y+1=0$ are taken as the new x and y axes respectively is**

1) $4X^2+9Y^2=5$ 2) $4X^2+9Y^2=25$
3) $4X^2+9Y^2=7$ 4) $4X^2-9Y^2=7$

17. **The line joining the points A(2,0) and B(3,1) is rotated through an angle of 45^0, about A in the anticlock wise direction. the coordinates of B in the new position (EAMCET 2011)**

1) $\left(2,\sqrt{2}\right)$ 2) $\left(\sqrt{2},2\right)$ 3) (2,2) 4) $\left(\sqrt{2},\sqrt{2}\right)$

18. **If the axes are translated to the circumcentre of the triangle formed by $(9,3),(-1,7),(-1,3)$, then the centroid of the triangle in the new system is**

1) $\left(5,\frac{5}{3}\right)$ 2) $(4,3)$

3) $\left(\frac{-5}{3},\frac{-2}{3}\right)$ 4) $(0,0)$

19. **A point (2,2) undergoes reflection in the x-axis and then the coordinate axes are rotated through an angle of $\pi/4$ in anticlockwise direction .The final position of the point in the new coordinate system is**

1) $\left(0,2\sqrt{2}\right)$ 2) $\left(0,-2\sqrt{2}\right)$

3) $\left(2\sqrt{2},0\right)$ 4) $\left(-2\sqrt{2},0\right)$

20. **The coordinate axes are rotated through an angle 22^0 about the origin. If the equation $4x^2+12xy+9y^2+6x+9y+2=0$ changes to $aX^2+2hXY+bY^2+2gX+2fY+c=0$ then value of $\frac{g^2-ac}{a^2+h^2}=$**

1) $\frac{1}{52}$ 2) $\frac{1}{36}$ 3) $\frac{-27}{52}$ 4) $\frac{1}{40}$

KEY

1) 1	2) 1	3) 2	4) 1	5) 1	6) 1
7) 2	8) 3	9) 3	10) 2	11) 4	12) 4
13) 3	14) 1	15) 2	16) 1	17) 1	18) 3
19) 2	20) 1				

SOLUTIONS

1. New $origin=\left(-\frac{f}{h},-\frac{g}{h}\right)$

2. New $origin=\left(-\frac{f}{h},-\frac{g}{h}\right)=(-2,1)$

3. (h,k) =((1,2)

$x=X+1,y=Y+2$

$Y^2-8X=0$

$\therefore a=-2$

4. Given (h,k) = (1,-2) original equation

$4x^2+9y^2-8x+36y+4=0$

$x=x+1, y=y-2$ transformed equation

$\Rightarrow 4(x+1)^2+9(y-2)^2-8(x+1)+36(y-2)+4=0$

$4x^2+9y^2=36$ Given equation $ax^2+by^2=C$

$a=4,\ b=9,\ c=36$ desending order c,b,a

6. $(x,y)=(X+h,Y+k)=\left(2\cos^2\frac{\alpha}{2},2\cos^2\frac{\beta}{2}\right)$

7. x = X+α , y = Y+β

8. Given $\theta=\tan^{-1}2\Rightarrow\tan\theta=\frac{2}{1}$,

$\cos\theta=\frac{1}{\sqrt{5}},\sin\theta=\frac{2}{\sqrt{5}}$

$x=x\cos\theta-y\sin\theta, y=x\sin\theta+y\cos\theta$

$x=\frac{x-2y}{\sqrt{5}},\quad y=\frac{2x+y}{\sqrt{5}}$

Transformed equation

$4\left(\frac{x-2y}{\sqrt{5}}\right)\left(\frac{2x+y}{\sqrt{5}}\right)-3\left(\frac{x-2y}{\sqrt{5}}\right)^2=a^2$

$\Rightarrow\frac{4(2x^2-3xy-2y^2)}{5}\ \frac{-3(x^2+y^2-4xy)}{5}=a^2$

$\Rightarrow 5x^2-20y^2=5a^2\Rightarrow x^2-4y^2=a^2$

9. $a=\sqrt{3}, b=-1, \theta=\tan^{-1}\left(\frac{-a}{b}\right)$

10. a = 1, b = 1 , $\theta=\tan^{-1}\left(\frac{b}{a}\right)$

11. $p=\frac{5}{3},q=\frac{5}{4}$, $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}+\frac{1}{q^2}$

12. a+b= 2+0=2

13. $\theta=-45^0$

14. $(x,y)=(2,4), X=4$

Using $X=x\cos\theta+y\sin\theta$

$\Rightarrow 2\cos\theta+4\sin\theta=4$, dividing by $\sin\theta$

$\Rightarrow 2+4\tan\theta=4\sec\theta$,

s.b.s $\quad\therefore\tan\theta=\frac{3}{4}$

15. $\frac{x}{a}+\frac{y}{b}=1$

$\frac{(X\cos\alpha-Y\sin\alpha)}{a}+\frac{(X\sin\alpha+Y\cos\alpha)}{b}=1$

x.coefficient = y coefficient

$(a-b)\cos\alpha=(a+b)\sin\alpha$

$\Rightarrow\tan\alpha=\frac{a-b}{a+b}$

16. Take

$4\left(\frac{x-2y+1}{\sqrt{5}}\right)^2+9\left(\frac{2x+y+2}{\sqrt{5}}\right)^2=5$.

$\therefore 4X^2+9Y^2=5\cdot$

17. $AB=\sqrt{1+1}=\sqrt{2}$

$\therefore$ by distance, verification the new coordinates of B are $(2,\sqrt{2})$

18. Given points forms a right angle triangle.
circum centre = Mid point of AB = (4,5)

Centroid $G=\left(\frac{7}{3},\frac{13}{3}\right)$

$\therefore$ centroid in the new system $=\left(\frac{7}{3}-4,\frac{13}{3}-5\right)=\left(\frac{-5}{3},\frac{-2}{3}\right)$

19. Reflection of (2,2) in X-axis is (2,-2) = (x,y)

use $X=x\cos\theta+y\sin\theta$,

$Y=-x\sin\theta+y\cos\theta$

$x=x\cos\theta+y\sin\theta,\ y=-x\sin\theta+y\cos\theta$

$x=2.\frac{1}{\sqrt{2}}-2.\frac{1}{\sqrt{2}}=0,\ y=-2.\frac{1}{\sqrt{2}}-2.\frac{1}{\sqrt{2}}=-2\sqrt{2}$

$(x,y)=(0,-2\sqrt{2})$

20. $a=4, c=2, g=3, h=6$

$\frac{g^2-ac}{a^2+h^2}=\frac{1}{52}$

STRAIGHT LINES

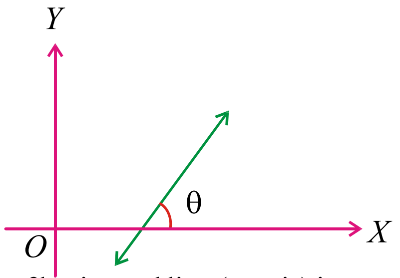
SYNOPSIS

Inclination of a line :

- If a line makes an angle $\theta(0 \le \theta < \pi)$ with x-axis measured in positive direction then θ is called inclination of the line.
  i) Inclination of horizontal line is zero
  ii) Inclination of vertical line is $\pi / 2$

Slope of a line :

- If the inclination of a non vertical line is θ then $tan\,\theta$ is called slope of the line and is usually denoted by m, thus $m = tan\,\theta$



i) Slope of horizontal line (x-axis) is zero $\left(\because \theta = 0^0\right)$

ii) Slope of vertical line (y-axis) is not defined $\left(\because \theta = 90^0\right)$

iii) $\theta = 0^0 \Leftrightarrow m = 0$

$0^0 < \theta < 90^0 \Leftrightarrow m > 0$

$\theta = 90^0 \Leftrightarrow m$ is not defined

$90^0 < \theta < 180^0 \Leftrightarrow m < 0$

- Slope of the line joining two points $A(x_1, y_1)$, $B\left(x_2, y_2\right)$ is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ $\left(x_1 \ne x_2\right)$

i) If $x_1 = x_2$ then the line $\overline{AB}$ is vertical and hence its slope is not defined

ii) If $y_1 = y_2$ then the line $\overline{AB}$ is horizontal and hence its slope is 0

- Two nonvertical lines are parallel if their slopes are equal.
- Two non vertical lines are perpendicular if product of their slopes is –1
- If θ is an angle between two nonvertical lines having slopes m_1, m_2 then

$$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1 m_2}, m_1 m_2 \ne -1$$

i) If θ is acute then $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

ii) If θ is one angle between two lines then the other angle is $\pi - \theta$. Usually the acute angle between two lines is taken as the angle between the lines

Intercept(s) of a line :

- If a line cuts x-axis at A(a, 0) and y-axis at B(0,b) then a and b are called x-intercept and y-intercept of that line respectively

i) Intercept of a line may be positive or negative or zero

ii) x-intercept of a horizontal line is not defined

iii) y-intercept of a vertical line is not defined

iv) Intercepts of a line passing through origin are zero.

Equation of a straight line in various forms :

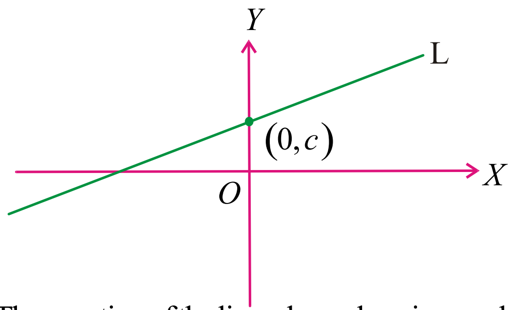
- **i) Line parallel to x-axis:** Equation of horizontal line passing through (a,b) is y = a

ii) Line parallel to y-axis: Equation of vertical line passing through (a, b) is x = b

iii) Slope - point form : The equation of the line with slope **m** and passing through the point $\left(x_1, y_1\right)$ is $y - y_1 = m\,(x - x_1)$

v) Slope - Intercept form :

a) The equation of the line whose slope is m and which cuts an intercept 'c' on the y-axis is y = mx+c



b) The equation of the line whose slope is m and which cuts an intercept 'a' on the x-axis is $y = m(x - a)$

c) The equation of the line passing through the origin and having slope m is $y=mx$

vi) Intercept Form : Suppose a line L makes intercept on x-axis is a and on y-axis is b then its equation is $\frac{x}{a}+\frac{y}{b}=1$

a) If the portion of the line intercepted between the axes is divided by the point (x_1, y_1) in the ratio m : n, then the equation of the line is $\frac{nx}{x_1}+\frac{my}{y_1}=m+n$

(or) $\frac{mx}{x_1}+\frac{ny}{y_1}=m+n$

b) Equation of the line whose intercept between the axes is bisected at the point (x_1, y_1) is

$\frac{x}{x_1}+\frac{y}{y_1}=2$

c) Equation of the line making equal intercepts on the axes and through the point (x_o, y_o) is $x + y = x_o + y_o$

d) Equation of the line making equal intercepts in magnitude but opposite in sign and passing through (x_o, y_o) is $x - y = x_o - y_o$

e) The equation of the line passing through the point (x_1, y_1) and whose intercepts are in the ratio m : n is $nx+my=nx_1+my_1$ (or) $mx+ny=mx_1+ny_1$

vii) General equation of line :

a) A linear equation in x and y always represents a line.

b) The equation of a line in general form is $ax+by+c=0$, where a, b, c are real numbers such that $a^2+b^2 \neq 0$ having slope $=-a/b$, x-intercept $=-c/a$, y-intercept $=-c/b$.

c) The equation of a line parallel to $ax+by+c=0$ is of the form $ax+by+k=0, k \in R$.

d) The equation of a line perpendicular to $ax+by+c=0$ is of the form $bx-ay+k=0$, $k \in R$

e) Equation of a line passing through (x_1, y_1) and (i) parallel to $ax+by+c=0$ is

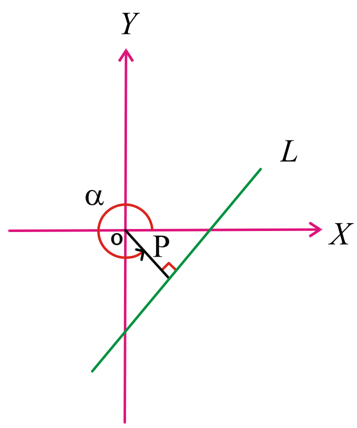
$$a(x-x_1)+b(y-y_1)=0$$

(ii) Perpendicular to $ax+by+c=0$ is

$$b(x-x_1)-a(y-y_1)=0$$

viii) Normal form :

a) The equation of the straight line upon which the length of the normal drawn from origin is 'p' and this perpendicular makes an angle $\alpha, (0 \le \alpha < 2\pi)$ with positive x-axis is $x\cos\alpha + y\sin\alpha = p$, $(p>0)$



b) The normal form of a line $ax+by+c=0$ is

$$\frac{(-a)}{\sqrt{a^2+b^2}}x+\frac{(-b)}{\sqrt{a^2+b^2}}y=\frac{c}{\sqrt{a^2+b^2}}, if\ c>0$$

and $$\frac{a}{\sqrt{a^2+b^2}}x+\frac{b}{\sqrt{a^2+b^2}}y=\frac{-c}{\sqrt{a^2+b^2}}, if\ c<0$$

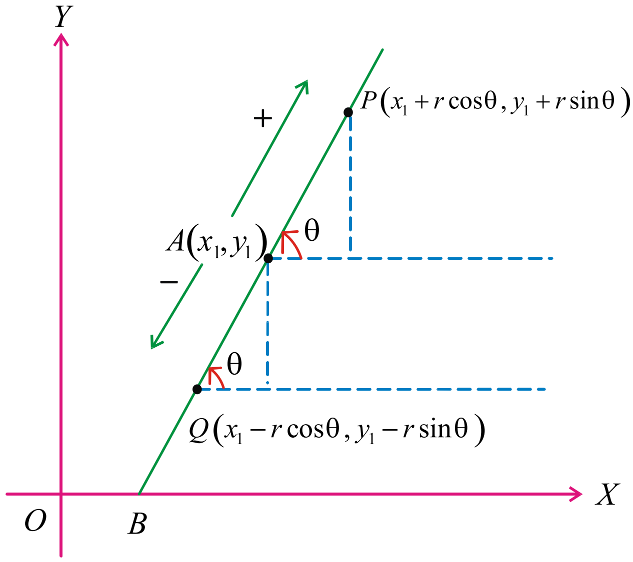
ix) Symmetric form and Parametric equations of a straight line :

a) The equation of the straight line passing through (x_1, y_1) and makes an angle θ with the positive direction of x-axis is $\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}$

Where $\theta \in (0, \pi/2) \cup (\pi/2, \pi)$

b) The co-ordinates (x, y) of any point P on the line at a distance 'r' units away from the point $A(x_1, y_1)$ can be taken as $(x_1+r\cos\theta,\ y_1+r\sin\theta)$ (or) $(x_1-r\cos\theta, y_1-r\sin\theta)$

c) The equations $x = x_1 + r\cos\theta$, $y = y_1 + r\sin\theta$ are called parametric equations of a line with parameter **'r'** of the line passing through the point (x_1, y_1) and having inclination θ.



$\cos\theta = \dfrac{x - x_1}{AP}$, $\sin\theta = \dfrac{y - y_1}{AP}$

or $x - x_1 = AP\cos\theta$, $y - y_1 = AP\sin\theta$.

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

Distances :

✈ i) The perpendicular distance to the line $ax + by + c = 0$

(a) from origin is $\dfrac{|c|}{\sqrt{a^2 + b^2}}$

(b) from the point (x_1, y_1) is $\dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

ii) The distance of a point (x_1, y_1) from the line $L \equiv ax+by+c=0$ measured along a line making an angle α with x-axis is $\left|\dfrac{ax_1 + by_1 + c}{a\cos\alpha + b\sin\alpha}\right|$

iii) The distance between parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $\dfrac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$.

iv) The distance between the parallel lines $ax+by+c_1=0$ and $ax + by + c_2 = 0$ measured along the line having inclination θ is $\left|\dfrac{c_1 - c_2}{a\cos\theta + b\sin\theta}\right|$

v) The equation of a line parallel and lying midway between the above two lines is $ax + by + \dfrac{c_1 + c_2}{2} = 0$

vi) Equation of the line parallel to ax+by+c=0 and at a distance d from the line is $ax + by + c \pm d\sqrt{a^2 + b^2} = 0$

Position of a point (s) w.r.to line (s) :

✈ i) The ratio in which the line $L \equiv ax + by + c = 0$ divides the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is $-L_{11} : L_{22}$ where $L_{11} = ax_1 + by_1 + c,\ L_{22} = ax_2 + by_2 + c$

ii) The points A , B lie on the same side or opposite side of the line L = 0 according as L_{11}, L_{22} have same sign or opposite sign that is $L_{11}.L_{22} > 0$ or $L_{11}.L_{22} < 0$

iii) A point $A(x_1, y_1)$ and origin lies on the same or opposite side of a line $L = ax + by + c = 0$ according as $c.L_{11} > 0$ or $c.L_{11} < 0$

iv) The point (x_1, y_1) lies between the parallel lines $ax_1 + by_1 + c = 0,\ ax_2 + by_2 + c = 0$ or does not lie between them according as $\dfrac{ax_1 + by_1 + c_1}{ax_1 + by_1 + c_2}$ is negative or positive

v) The point $A(x_1, y_1)$ lies above or below the line $L = ax + by + c = 0$ according as $\dfrac{L_{11}}{b} > 0$ or $\dfrac{L_{11}}{b} < 0$

Proof : The fig. Shows a point $P(x_1, y_1)$ lying above a given line. If an ordinate is dropped from P to meet the line L at N, then the x coordinate of N will be x_1.

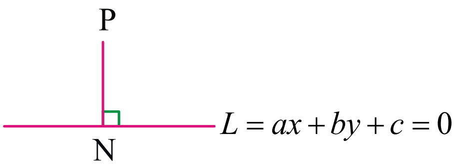
Putting $x = x_1$ in the equation $ax + by + c = 0$ gives

ordinate of $N = -\dfrac{(ax_1 + c)}{b}$

If $P(x_1, y_1)$ lies above the line, then we have

$y_1 > -\dfrac{(ax_1 + c)}{b}$ i.e. $y_1 + \dfrac{(ax_1 + c)}{b} > 0$

i.e. $\dfrac{(ax_1 + by_1 + c)}{b} > 0$, i.e. $\dfrac{L(x_1, y_1)}{b} > 0$



Hence, $P(x_1, y_1)$ lies above the line

$ax + by + c = 0$, and if $\dfrac{L(x_1, y_1)}{b} < 0$, it would mean that P lies below the line $ax + by + c = 0$.

✈ **If $P(x_1, y_1)$ lie between the parallel lines**

$ax + by + c_1 = 0, ax + by + c_2 = 0$ **then**

$(ax_1 + by_1 + c_1)(ax_1 + by_1 + c_2) < 0$.

✈ **If $P(x_1, y_1)$ does not lie between the parallel lines**

$ax + by + c_1 = 0, ax + by + c_2 = 0$ **then**

$(ax_1 + by_1 + c_1)(ax_1 + by_1 + c_2) > 0$

Proof :

Make c_1, c_2 having same sign.

(If necessary)

$\Rightarrow$ **(0,0) lie on same side of both the lines**

$\Rightarrow ax_1 + b_1y_1 + c_1,\ c_1$ **have opposite signs**

$ax_1 + b_1y_1 + c_2,\ c_2$ **have opposite signs**

since $c_1c_2 > 0$, we have

$(ax_1 + by_1 + c_1)(ax_1 + by_1 + c_2) > 0$

Point of intersection of lines and Concurrency of Straight Lines :

✈ i) Consider two lines $L_1 \equiv a_1x + b_1y + c_1 = 0$

and $L_2 \equiv a_2x + b_2y + c_2 = 0$ then

point of intersection is

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right) or$$

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

ii) Three or more lines are said to be concurrent, if they have a point in common. The common point is called the point of concurrence.

a) If $L_1 = 0, L_2 = 0$ are two interesecting lines, then the equation of any line other than

$L_1 = 0$ and $L_2 = 0$ passing through point of intersection can be taken as

$L_1 + \lambda L_2 = 0$. Where λ is a parameter

b) The three lines $L_i \equiv a_ix + b_iy + c_i = 0, i = 1,2,3$ are

concurrent $iff \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

(or) Point of intersection of any two lines lies on the third line

(or) there exist constants $\lambda_1, \lambda_2, \lambda_3$ not all zero

such that $\lambda_1L_1 + \lambda_2L_2 + \lambda_3L_3 = 0$

c) If $p_1x+q_1y = 1$, $p_2x+q_2y = 1$, $p_3x+q_3y = 1$ are concurrent lines then the points (p_1,q_1), (p_2,q_2), (p_3,q_3) are collinear

d) If ka+lb+mc=0, then the point of concurrency of the lines represented by ax+by+c=0 is $\left(\dfrac{k}{m}, \dfrac{l}{m}\right)$

Angle between lines :

✈ i) If 'θ' is an acute angle between the lines having slopes m_1 *and* m_2 then $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1m_2}\right|$

ii) If 'θ' is an acute angle between the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ then

$$\cos\theta=\left|\frac{a_1a_2+b_1b_2}{\sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}}\right| \text{ and } \tan\theta=\left|\frac{a_1b_2-a_2b_1}{a_1a_2+b_1b_2}\right|$$

other angle between the lines is $\pi-\theta$

iii) The slope m of a line which is equally inclined with two intersecting lines of slopes m_1 and m_2 is given by $\frac{m_1-m}{1+mm_1}=\frac{m-m_2}{1+mm_2}$

iv) The slopes of the lines making an angle α with a line having slope m are $\frac{m-\tan\alpha}{1+m\tan\alpha},\frac{m+\tan\alpha}{1-m\tan\alpha}$

v) Consider two lines $L_1=a_1x+b_1y+c_1=0$ and $L_2=a_2x+b_2y+c_2=0$

a) Lines are parallel if $\frac{a_1}{a_2}=\frac{b_1}{b_2}$

b) Lines are coincident if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

c) Lines are perpendicular if $a_1a_2+b_1b_2=0$

d) Lines are equally inclined with x-axis

$$if\ \frac{a_1}{a_2}=-\frac{b_1}{b_2}$$

Triangles and Quadrilaterals :

✈ i) Let d_1 be the distance between the parallel lines $ax+by+c_1=0$, $ax+by+c_2=0$ and d_2 be the distance between the parallel lines $a_1x+b_1y+k_1=0$, $a_1x+b_1y+k_2=0$ then the figure formed by four lines is

a) a square if $d_1=d_2$ and $aa_1+bb_1=0$,

b) Rhombus if $d_1=d_2$ and $aa_1+bb_1\neq 0$,

c) Rectangle if $d_1\neq d_2$ and $aa_1+bb_1=0$,

d) Parallelogram if $d_1\neq d_2$ and $aa_1+bb_1\neq 0$

✈ ii)The area of triangle formed by the line $\frac{x}{a}+\frac{y}{b}=1$ with the co- ordinate axis is $\frac{1}{2}|ab|$

iii) The area of triangle formed by line $ax+by+c=0$ with the co - ordinate axes is $\frac{c^2}{2|ab|}$

iv) Area of the rhombus $a|x|+b|y|+c=0$ is $4(area\ of\ \Delta)=\frac{2c^2}{|ab|}$

v) If p_1,p_2 are distances between parallel sides and 'θ' is angle between adjacent sides of parallelogram then its area is $\left|\frac{p_1p_2}{\sin\theta}\right|$

vi) Area of parallelogram whose sides are $a_1x+b_1y+c_1=0$, $a_1x+b_1y+c_2=0$, $a_2x+b_2y+d_1=0$ and $a_2x+b_2y+d_2=0$ is $\left|\frac{(c_1-c_2)(d_1-d_2)}{a_1b_2-a_2b_1}\right|$

vii) Area of rhombus $=\frac{1}{2}d_1d_2$ where d_1,d_2 are lengths of the diagonals

Foot and Image :

✈ i) If (h,k) is the foot of the perpendicular from (x_1,y_1) to the line $ax+by+c=0$ then $\frac{h-x_1}{a}=\frac{k-y_1}{b}=\frac{-(ax_1+by_1+c)}{a^2+b^2}$ or

$(h,k)=(x_1+a\lambda,y_1+b\lambda)$ where

$$\lambda=\frac{-(ax_1+by_1+c)}{a^2+b^2}$$

ii) If (h,k) is the image (reflection) of the point (x_1,y_1) w.r.t the line $ax+by+c=0$ then $\frac{h-x_1}{a}=\frac{k-y_1}{b}=\frac{-2(ax_1+by_1+c)}{a^2+b^2}$ or

$(h,k)=(x_1+a\lambda,y_1+b\lambda)$ where

$$\lambda=\frac{-2(ax_1+by_1+c)}{a^2+b^2}$$

iii) Image of (a , b) w.r.to $y = x$ is (b , a)

iv) Image of (a , b) w.r.to $x + y = 0$ is $(-b,-a)$

v) Reflection of $f(x,y)=0$ in x-axis is

$f(x,-y)=0$

vi) Reflection of $f(x,y)=0$ in y-axis is

$f(-x,y)=0$

vii) Reflection of $f(x,y)=0$ in $x=y$ is

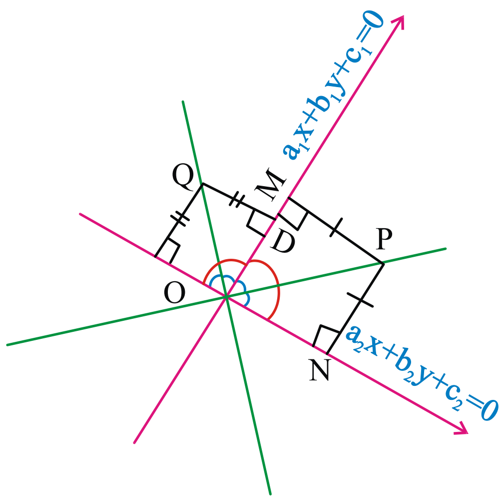
$f(y,x)=0$

Angular bisectors of two straight lines :

✈ Angular bisector is the locus of a point which moves in such a way so that its distance from two intersecting lines remains same.

The equations of the two bisectors of the angles between the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$



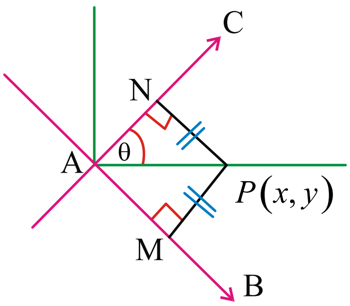
i) If the two given lines are not perpendicular i.e. $a_1 a_2 + b_1b_2 \neq 0$ and not parallel i.e. $a_1 b_2 \neq a_2b_1$ then one of these equations is the equation of the bisector of the acute angle between two given lines and the other that of the obtuse angle between two given lines.

ii) Whether both given lines are perpendicular or not, but the angular bisectors of these lines will always be mutually perpendicular.

iii) **The bisectors of the acute and the obtuse angles:**

Take one of the lines and let its slope be m_1 and take one of the bisectors and let its slope be m_2. If θ be the acute angle between them, then find tan θ

$$= \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$$



If $\tan\theta > 1$ then the bisector taken is the bisector of the obtuse angle and the other one will be the bisector of the acute angle.

If $0 < \tan\theta < 1$ then the bisector taken is the bisector of the acute angle and the other one will be the bisector of the obtuse angles.

iv) consider the lines are $a_1 x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, where $c_1 > 0,\ c_2 > 0$ then,

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

will represent the equation of the bisector of the acute or obtuse angle between the lines according as $a_1a_2 + b_1b_2$ is negative or positive.

v) **The equation of the bisector of the angle which contains a given point :**

The equation of the bisector of the angle between the two lines containing the point (x_1, y_1) is

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

or $$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = -\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

according as $a_1x_1 + b_1y_1 + c_1$ and $a_2x_1 + b_2y_1 + c_2$ are of the same signs or of opposite signs.

vi) For example the equation of the bisector of the angle containing the origin is given by

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=+\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$ for same sign

of c_1 and c_2 (for opposite sign take –ve sign in place of +ve sign)

vii) If $c_1c_2(a_1a_2+b_1b_2)<0$, then the origin will lie in the acute angle and if $c_1c_2(a_1a_2+b_1b_2)>0$, then origin will lie in the obtuse angle.

viii) Equation of straight lines passing through $P(x_1, y_1)$ and equally inclined with the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are those which are parallel to the bisectors between these two lines and passing through the point P.

Eg. 1:

The medians AD and BE of the triangle with vertices A(0,b), B(0,0) and C(a,0) are mutually perpendicular if

Sol: $AD\perp BE\Rightarrow\left(\frac{-2b}{a}\right)\left(\frac{b}{a}\right)=-1$

$$\Rightarrow 2b^2=a^2$$

Eg. 2:

If (3,-1),(2,4),(-5,7) are the mid points of the sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ of triangle ABC. Then the equation of the side $\overline{CA}$ is

Sol : Here $m=-1$ and given point (x_1, y_1) is (2, 4).
By point slope form equation of the line is
$y-4=-1(x-2)$

iv) Two - point form : The equation of a line passing through two points

$A(x_1,y_1)$ *and* $B(x_2,y_2)$ is

$$(y-y_1)(x_2-x_1)=(x-x_1)(y_2-y_1)$$

(or) $$\begin{vmatrix} x & y & 1\\ x_1 & y_1 & 1\\ x_2 & y_2 & 1\end{vmatrix}=0$$

Eg. 3:

Equation of the diagonal (through the origin) of the quadrilateral formed by the lines x = 0, y = 0, x + y = 1 and 6x + y = 3 is

Sol : Here $(x_1,y_1)=(0,0)$, $(x_2,y_2)=\left(\frac{2}{5},\frac{3}{5}\right)$

Using two-point form, the equation of the line is $3x-2y=0$

Eg. 4:

Equation to the straight line cutting off an intercept 2 from negative y axis and inclined at 30° to the positive direction of axis of x, is

Sol : Equation of line passing through (0,-2) and having slope $\frac{1}{\sqrt{3}}$ is $\sqrt{3}y-x+2\sqrt{3}=0$

Eg. 5:

The sum of x,y intercepts made by the lines x+y=a, x+y=ar, x+y=ar² ...... on coordinate axes when r=1/2, $a\neq 0$

Sol: required sum

$=2a+2ar+2ar^2+......$(infinite $G.P$)

= 2a/1-r = 4a

Eg. 6:

Normal form of the equation x+y+1=0 is

Sol: The given equation is $x+y+1=0\Rightarrow -x-y=1$

$$\Rightarrow\frac{(-1)x}{\sqrt{2}}+\frac{(-1)y}{\sqrt{2}}=\frac{1}{\sqrt{2}}$$

$$\Rightarrow x\cos\left(\pi+\frac{\pi}{4}\right)+y\sin\left(\pi+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

$$\Rightarrow x\cos\frac{5\pi}{4}+y\sin\frac{5\pi}{4}=\frac{1}{\sqrt{2}}$$

Eg. 7:

(1,2),(3,6)are two opposite vertices of a rectangle and if the other two vertices lie on the line 2y = x + c, then c and other two vertices are

Sol: Mid point of given vertices is $P(x_1,y_1)=(2,4)$ which lies on $2y=x+c$ then *c=6.*

Now r=BP=AP=$\sqrt{5}$, $\tan\theta=\frac{1}{2}$

Hence B=$(x_1+r\cos\theta,\ y_1+r\sin\theta)$=(4,5)

C=$(x_1-r\cos\theta, y_1-r\sin\theta)$ =(0,3)

Eg. 8:

The distance between A(2, 3) on the line of gradient 3/4 and the point of intersection P of this line with 5x + 7y + 40 = 0 is

Sol : Since m = 3/4, then $\cos\theta$ = 4/5 and $\sin\theta$ = 3/5.

$$r=\left|\frac{5\times2+7\times3+40}{5\left(\frac{4}{5}\right)+7\left(\frac{3}{5}\right)}\right|=\frac{355}{41}$$

Eg. 9:

The range of θ in the interval $(0,\pi)$ such that the points (3, 5) and $(\sin\theta, \cos\theta)$ lie on the same side of the line x + y – 1 = 0 is

Sol : Since $(3+5-1)\,(\sin\theta+\cos\theta-1)>0$

$$\Rightarrow \quad \sin\left(\frac{\pi}{4}+\theta\right)>\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \frac{\pi}{4}<\frac{\pi}{4}+\theta<\frac{3\pi}{4}$$

$$\Rightarrow \quad 0<\theta<\frac{\pi}{2}$$

Eg. 10:

The range of α, if (α, α^2) lies inside the triangle having sides along the lines 2x + 3y = 1, x + 2y - 3 = 0, 6y = 5x - 1

Sol : Let A, B, C be vertices of the triangle.

$A=(-7, 5), B=\left(\frac{5}{4},\frac{7}{8}\right)$

$C=\left(\frac{1}{3},\frac{1}{9}\right)$. Sign of A w.r.t. BC to -ve.

If P lies inside the triangle ABC, then sign of P will be the same as sign of A w.r.t. the line BC

$\Rightarrow \quad 5\alpha-6\alpha^2-1<0$ .....(i)

similarly $2\alpha+3\alpha^2-1>0$ .....(ii)

And $\quad \alpha+2\alpha^2-3<0$ .....(iii)

Solving (i), (ii) and (iii) for α and then taking intersection,

we get $\quad \alpha\in\left(\frac{1}{2},1\right)\cup\left(-\frac{3}{2},-1\right)$

Eg. 11:

The line $x+\lambda y-4=0$ passes through the point of intersection of $4x-y+1=0$ and x + y + 1 = 0. Then the value of λ is

Sol : The three lines are concurrent

$$\Rightarrow \quad \begin{vmatrix}1 & \lambda & -4\\ 4 & -1 & 1\\ 1 & 1 & 1\end{vmatrix}=0$$

$$\Rightarrow \quad -2-3\lambda-20=0 \Rightarrow \quad \lambda=-\frac{22}{3}$$

Eg. 12:

In ΔABC A is (1,2) if the internal angle bisector of B is 2x-y+10=0 and perpendicular bisector of AC is y=x then the equation of BC is

Sol: Image of A w.r.to bisector of B is (-7,6) lies on BC and image of A in the perpendicular bisector of AC is C(2,1)

$\therefore$ equation of BC is 5x+9y-19=0

Eg. 13:

For the straight lines 4x + 3y – 6 = 0 and 5x + 12y + 9 = 0, find the equation of the -

(i) Bisector of the obtuse angle between them is

ii) Bisector of the acute angle between them is

(iii) Bisector of the angle which contains origin is

(iv) Bisector of the angle which contains (1, 2) is

Sol: after making $c_1>0$ and $c_2>0$;

$a_1a_2+b_1b_2$ = (-4)(5)+(-3)(12) = -56 <0

i) The bisector of the acute angle is

$$\frac{-4x-3y+6}{\sqrt{(-4)^2+(-3)^2}}=\frac{5x+12y+9}{\sqrt{5^2+12^2}}$$

$7x + 9y – 3 = 0$

ii) The bisector of the obtuse angle is

$$\frac{-4x-3y+6}{\sqrt{(-4)^2+(-3)^2}}=-\frac{5x+12y+9}{\sqrt{5^2+12^2}}$$

$9x – 7y – 41 = 0$

(iii) The bisector of the angle containing the origin

$$\frac{-4x-3y+6}{\sqrt{(-4)^2+(-3)^2}}=\frac{5x+12y+9}{\sqrt{5^2+12^2}}$$

$7x + 9y – 3 = 0$

(iv) For the point (1, 2),

$4x + 3y - 6 = 4 \times 1 + 3 \times 2 - 6 > 0$

$5x + 12y + 9 = 12 \times 2 + 9 > 0$

Hence equation of the bisector of the angle containing the point (1, 2) is

$$\frac{4x+3y-6}{5} = \frac{5x+12y+9}{13}$$

$9x - 7y - 41 = 0$

Eg. 14:

A light ray emerging from the point source placed at P(2, 3) is reflected at a point 'Q' on the y-axis and then passes through the point R(5, 10). Coordinate of 'Q' is -

Sol: Image of point P(2,3) in Y-axis is $P^1(-2,3)$

Equation of $P^1R \equiv y-3=1(x+2)$

$x - y + 5 = 0$

P^1R meets the Y-axis at Q(0,5)

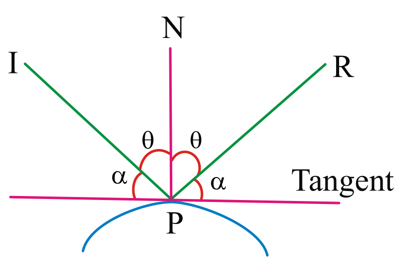
Optimization:

✈ Let A and B are two points on same side of line $L \equiv ax+by+c=0$

i) The point **P** such that PA + PB is minimum, is intersection of L = 0 and the line joining A to image of B or line joining B to image of A w.r.to L = 0

ii) The point is P such that $|PA - PB|$ is Maximum, is point of intersection of line $L = 0$ and line joining A and B.

Reflection in surface :



IP = incident ray

PN = normal to the surface

PR = reflected ray

$\angle$ IPN = $\angle$ NPR

$\therefore$ Angle of incident = Angle of reflection

No. of lines, no. of triangles and no. of circles :

✈ No. of lines drawn through the point A which are at a distance d from the point B

a) If AB = d then the no. of lines through A at a distance d from B is 1

b) If AB>d then the no. of lines through A at a distance d from B is 2

c) If AB<d then the no. of lines through A at a distance d from B is 0

✈ No of right angled traingles in a circle depends on height h of the traingle and radius r of the circle

a) If h = r, no. of right angled traingles = 2

b) If h < r, no. of right angled traingles = 4

c) If h > r, no. of right angled traingles = 0

✈ No. of circles touching three lines

a) No circle if the lines are parallel

b) one circle if the lines are concurrent

c) 2 circles if two lines are parallel and third cuts them

d) 4 circles if the lines are not concurrent and no two of them are parallel.

EXERCISE - I

1. If the lines y = – 3x + 4, ay = x + 10 and 2y + bx + 9 = 0 represent three consecutive sides of a rectangle then ab =

1) 18 2) –3 3) $\frac{1}{2}$ 4) $\frac{-1}{3}$

2. If the straight line (3x+4y+5)+k(x+2y-3)=0 is parallel to x-axis then the value of k is

1) 1 2) -3 3) 4 4) 2

3. The equation of the stratight line cutting off an intercept 8 on x–axis and making an angle of 60^0 with the positive direction of y –axis is

1) $x-\sqrt{3}y-8=0$ 2) $x+\sqrt{3}y=8$

3) $y-\sqrt{3}x=8$ 4) $y+\sqrt{3}x=8$

4. If (-4,5) is one vertex and 7x-y+8=0 is one diagonal of a square, then the equation of the other diagonal is

1) x+7y-31=0 2) x+7y-15=0

3) x+7y+8=0 4) x+7y-35=0

5. The number of lines that are parallel to 2x + 6y– 7 = 0 and have an intercept 10

between the co-ordinate axes is

1) 1 2) 2 3) 4 4) infinitely many

6. **If the line (x-y+1) + k (y-2x+4) = 0 makes equal intercept on the axes then the value of k is**

1) $1/3$ 2) $3/4$ 3) $1/2$ 4) $2/3$

7. **Equation of the line on which the length of the perpendicular from origin is 5 and the angle which this perpendicular makes with the x axis is 60^{O}**

1) $x+\sqrt{3}y=12$ 2) $\sqrt{3}\,x+y=10$

3) $x+\sqrt{3}y=8$ 4) $x+\sqrt{3}\,y=10$

8. **The slope of a straight line through A(3,2) is 3/4 then the coordinates of the two points on the line that are 5 units away from A are**

1) (-7,5) (1,-1) 2) (7,5) (-1,-1)
3) (6,9) (-2,4) 4) (7,3) (-2,1)

9. **Radius of the circle touching the lines 3x+4y-14=0, 6x+8y+7=0 is (EAM- 2011)**

1) 7 2) $\frac{7}{2}$ 3) $\frac{7}{4}$ 4) $\frac{7}{6}$

10. **The distance between the parallel lines given by $\left(x+7y\right)^{2}+4\sqrt{2}\left(x+7y\right)-42=0$ is**

(EAM- 2012)

1)1 2)5 3)6 4)2

11. **If the straight line drawn through the point $P\left(\sqrt{3},2\right)$ making an angle $\frac{\pi}{6}$ with x-axis meets the line $\sqrt{3}$ x-4y+8=0 at Q. Then PQ is**

1) 4 2) 5 3) 6 4) 9

12. **If the line 3x+4y-8=0 is denoted by L, then the points (2,-5),(-5,2)**

1) lie on L

2) lie on same side of L

3) lie on opposite sides of L

4) equidistant from L

13. **If the lines ax+by+c = 0, bx+cy+a = 0 and cx+ay+b=0 $a\neq b\neq c$ are concurrent then the point of concurrency is**

1) (0,0) 2) (1,1) 3) (2,2) 4) (-1,-1)

14. **The line segment joining the points (1,2) and (k,1) is divides by the line $3x+4y-7=0$ in the ratio 4:9 then k is**

1) 2 2) -2 3) 3 4) -3

15. **If the point of intersection of kx+4y+2=0, x-3y+5=0 lies on 2x+7y-3=0 then k=**

1) 2 2) 3 3)-2 4) -3

16. **If 4a+5b+6c=0 then the set of lines ax+by+c=0 are concurrent at the point**

1) $\left(\frac{2}{3},\frac{5}{6}\right)$ 2) $\left(\frac{1}{3},\frac{1}{2}\right)$ 3) $\left(\frac{1}{2},\frac{4}{3}\right)$ 4) $\left(\frac{1}{3},\frac{7}{3}\right)$

17. **Equation of the line passing through the point of intersection of the lines 2x+3y-1=0, 3x+4y-6=0 and perpendicular to 5x-2y-7=0 is (EAM- 2009)**

1) 2x+5y-19=0 2) 2x+5y+17=0
3) 2x+5y-16=0 4) 2x+5y-22=0

18. **Let a and b be nonzero reals . Then the equation of the line passing through the origin and the point of inter section of x/a + y/b =1 and x/b + y/a=1**

1) ax+by=0 2) bx+ay=0
3) y-x=0 4) x+y=0

19. **The angle between the lines kx+y+9=0, y-3x=4 is 45^{O} then the value of k is(EAM-2007)**

1) 2 or ½ 2) 2 or -1/2
3) -2 of ½ 4) -2 or -1/2

20. **If a, c, b are three terms of a G.P., then the line ax + by +c =0**

1) has a fixed direction
2) always passes through a fixed point
3) forms a triangle with the axes whose area is constant
4) always cuts intercepts on the axes such that their sum is zero

21. **If a straight line perpendicular to 3x-4y-6=0 forms a triangle with the coordinate axes whose area is 6sq. units, then the equation of the straight line (s) is**

(EAM- 2019)

1) x-2y=6 2) 4x+3y=12
3) 4x+3y+24=0 4) 3x-4y=12

22. **The equation of base of an equilateral triangle**

is x+y=2 and the vertex is (2, -1). Then area of triangle is

1) $2\sqrt{3}$ 2) $\sqrt{3}/6$ 3) $1\sqrt{3}$ 4) $2\sqrt{3}$

23. The quadrilateral formed by the lines 2x-5y+7=0, 5x+2y-1=0, 2x-5y+2=0, 5x+2y+3=0 is

1) Rectangle 2) Square
3) Parallelogram 4) Rhombus

24. Foot of the perpendicular of origin on the line joining the points

$(a\cos\theta, a\sin\theta)$, $(a\cos\phi, a\sin\phi)$ **is**

1) $(\cos\theta+\cos\phi, \sin\theta+\sin\phi)$

2) $(\cos\theta-\cos\phi, \sin\theta-\sin\phi)$

3) $\left(\frac{a(\cos\theta+\cos\phi)}{2}, \frac{a(\sin\theta+\sin\phi)}{2}\right)$

4) $(\cos\theta\cos\phi, \sin\theta\sin\phi)$

25. If 2x+3y=5 is the perpendicular bisector of the line segment joining the points A $(1, \frac{1}{3})$ and B then B= (EAM- 2018)

1) $\left(\frac{21}{13}, \frac{49}{39}\right)$ 2) $\left(\frac{17}{13}, \frac{31}{39}\right)$

3) $\left(\frac{7}{13}, \frac{49}{39}\right)$ 4) $\left(\frac{21}{13}, \frac{31}{39}\right)$

26. Image of the curve $x^2+y^2=1$ in the line $x+y=1$ is [EAM -2020]

1) $x^2+y^2+2x+2y+1=0$

2) $x^2+y^2-2x+2y+1=0$

3) $x^2+y^2+2x-2y+1=0$

4) $x^2+y^2-2x-2y+1=0$

27. A line passing through the points $(7,2),(-3,2)$ then the image of the line in x-axis is

1) y = 4 2) y = 9 3) y = –1 4) y = –2

28. One vertex of a square ABCD is A(-1,1) and the equation of one diagonal BD is 3x+y-8=0 then C=

1) (-5,3) 2) (5,3) 3) (-5,-3) 4) (2,5)

29. If the algebraic sum of the perpendicular distances from the points (2,0) (0,2) and (4,4) to a variable line is 'O', then the line passes through the fixed point

1) (1,1) 2) (3,3) 3) (2,2) 4) (0,0)

30. The vertices of a triangle are (2,0) (0,2) (4,6) then the equation of the median through the vertex (2,0) is [EAM -2016]

1) x+y-2=0 2)x=2
3) x+2y-2=0 4) 2x+y-4=0

31. A(1,-1) B(4,-1) C(4,3) are the vertices of a triangle. Then the equation of the altitude through the vertex 'A' is

1) x = 4 2) y = 4 3) y + 1= 0 4) x =1

32. Equation of a diameter of the circum circle of the triangle formed by the lines 3x+4y-7=0, 3x-y+5=0 and 8x-6y+1=0 is

1) 3x-y-5=0 2) 3x+y+5=0
3) 3x-y+5=0 4) 3x+y-5=0

33. The incentre of the triangle formed by the lines $x\cos\alpha+y\sin\alpha=\pi$, $x\cos\beta+y\sin\beta=\pi$, $x\cos\gamma+y\sin\gamma=\pi$ is (α,β) then $\alpha+\beta$ =

1) 0 2) 1 3) 2 4) 4

34. The orthocentre of the triangle formed by the points $A(a\cos\alpha, a\sin\alpha)$ $B(a\cos\beta, a\sin\beta)$ $C(a\cos\gamma, a\sin\gamma)$ is

1) $(\cos\alpha+\cos\beta+\cos\gamma, \sin\alpha+\sin\beta+\sin\gamma)$

2) $[a(\cos\alpha+\cos\beta+\cos\gamma), a(\sin\alpha+\sin\beta+\sin\gamma)]$

3) $[a(\cos\alpha+\sin\beta+\sin\gamma), a(\sin\alpha+\cos\beta+\cos\gamma)]$

4) $(\cos\alpha\ \cos\beta\ \cos\gamma, \sin\alpha\ \sin\beta\ \sin\gamma)$

35. If $2x+3y+4=0$ & $\lambda x+ky+2=0$ are identical lines then $3\lambda-2k=$ [EAM - 2017]

1) 1 2) 0 3) -1 4) 2

KEY

01) 1	02) 2	03) 2	04) 1	
05) 2	06) 4	07) 4	08) 4	09) 3
10) 3	11) 4	12) 3	13) 2	14) 2
15) 2	16) 1	17) 2	18) 3	
19) 2	20) 3	21) 2	22) 2	23) 1
24) 3		25) 1	26) 4	27) 4
28) 2	29) 3	30) 2	31) 3	32) 3
33) 1	34) 2	35) 2		

SOLUTIONS

1. let the given sides are AB, BC, CD
   $AB \parallel CD \Rightarrow b = 6$
   $AB \perp BC \Rightarrow a = 3$
2. Coefficient of x = 0
3. $m = \tan 150^0 = \tan(180-30) = -\tan 30 = \frac{-1}{\sqrt{3}}$

   $y = m(x-8)$

   $y = \frac{-1}{\sqrt{3}}(x-8)$

   $x + \sqrt{3y} - 8 = 0$
4. Write the equation to a line perpendicular to $7x - y + 8 = 0$ and sub $(-4,5)$
5. Two lines parallel to any given line make intercept of same length k between the axes in opposite quadrants
6. $(x-y+1)+k(y-2x+4)=0$

   $\Rightarrow x(1-2k)+y(k-1)+1+4k-0$

   $1-2k = k-1 \qquad 2 = 3k \Rightarrow k = \frac{2}{3}$
7. Intercepts α, β

   $\therefore \left(\frac{\alpha}{3}, \frac{\beta}{3}\right) = (a,a)$
8. $P = 5, \alpha = 60^0$

   $x\cos\alpha + y\sin\alpha = P$
9. Eq. of the given line is 2x+y=4

   required distance= $\frac{4}{\sqrt{5}}$
10. Given lines $6x+8y+7=0$ _____ (1)

    $3x+4y-14=0$ _______ (2)

    $(2)\times 2 \Rightarrow 6x+8y-28=0$ _______ (3)

    distance between the parallel lines (1) and (3)

    $2R = \frac{7+28}{\sqrt{36+64}} = \frac{35}{10} = \frac{7}{2}$

    Radius = $\frac{7}{4}$
11. : $t^2 + 4\sqrt{2}t - 42 = 0$

    $t = \frac{-4\sqrt{2} \pm \sqrt{32+168}}{2(1)}$

    $t = \frac{-4\sqrt{2} \pm 10\sqrt{2}}{2}$

    $x+7y = 3\sqrt{2}, \quad x+7y = -7\sqrt{2}$

    distance between the lines $= \frac{10\sqrt{2}}{\sqrt{1+49}} = \frac{10\sqrt{2}}{5\sqrt{2}} = 2$
12. Given $L = 3x+4y-8=0$

    $A(x_1, y_1) = (2,-5)$

    $B(x_2, y_2) = (-5,2)$

    $L_{11} = 3x_1 + 4y_1 - 8 = 3(2) + 4(-5) - 8 = 6 - 28 = -22 < 0$

    $L_{22} = 3x_2 + 4y_2 - 8 = 3(-5) + 4(2) - 8 = -15 < 0$

    The points A,B lies on same side of L = 0
13. $-\frac{L_{11}}{L_{22}} = \frac{4}{9} \Rightarrow 3 - 3k = 9$
14. $\begin{vmatrix} k & 4 & 2 \\ 1 & -3 & 5 \\ 2 & 7 & -3 \end{vmatrix} = 0$
15. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$ a+b+c=0
16. $a\left(\frac{4}{6}\right) + b\left(\frac{5}{6}\right) + c = 0$

17. Given lines 2x +3y-1 = 0.........(1)
3x + 4y -6 = 0 ....(2)
solving (1) (2) 6x+9y-3=0
6x+8y-12=0
y + 9 = 0, y = -9substituing (1)
$2x-27-1=0$, $x=14$ point of intersection (14,-9) given line 5x-2-7=0, slope = 582
Perpendicular slope m = -2/5

Equation of line $y+9=-2/5(x-14)$

$5y+45=-2x+28$

$2x+5y+17=0$

18. Intersecting point of $\frac{x}{a}+\frac{y}{b}=1$ and

$\frac{x}{b}+\frac{y}{a}=1$ is $\left(\frac{ab}{a+b},\frac{ab}{a+b}\right)$

19. Given lines $kx+y+9=0$ ______ (1)

$y-3x=4$ ______ (2)

$m_1=-k \quad m_2=3 \quad \theta=45^0$

$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right| \qquad 1=\left|\frac{3+k}{1-3k}\right|$

$|1-3k|=|3+k|$ s.o.b.s

$(1-3k)^2=(3+k)^2$

$1+9k^2-6k=9+k^2+6k$

$8k^2-12k-8=0$

$2k^2-3k-2=0$

$2k^2-4k+k-2=0$

$2k(k-2)+1(k-2)=0$

$k=\frac{-1}{2}(or)k=2$

20. $c^2=ab$

$\Delta=\frac{c^2}{2ab}=\frac{1}{2}$.

21. The line perpendicular to given line is

$4x+3y+k=0 \quad \therefore \frac{k^2}{24}=6$

22. Given line $x+y=2$ point $=(2,-1)$ p=perpendicular distance from (2,-1) to x + y - 2 = 0

$p=\frac{|2-1-2|}{\sqrt{2}}=\frac{1}{\sqrt{2}}$ area $=\frac{p^2}{\sqrt{3}}=\frac{1}{2\sqrt{3}}=\frac{\sqrt{3}}{6}$

23. Adjacent sides are perpendicular and distance between parallel sides are not equal.

24. Mid point becuase $OA=OB$

25. Given point $A\left(1,\frac{1}{3}\right)$ L = 2x+3y-5=0

$B(h,k)$ is a image of $A\left(1,\frac{1}{3}\right)$ w.r.t L =2x + 3y - 5 = 0

$$\frac{h-1}{2}=\frac{k-\frac{1}{3}}{3}=\frac{-2\left(2(1)+3\left(\frac{1}{3}\right)-5\right)}{2^2+3^2}$$

$$\Rightarrow\frac{h-1}{2}=\frac{k-\frac{1}{3}}{3}=\frac{-2(2-4)}{13}$$

$\Rightarrow h-1=\frac{8}{13}$

$h=1+\frac{8}{13}=\frac{21}{13}$

$k-\frac{1}{3}=\frac{12}{13}$

$k=\frac{1}{3}+\frac{12}{13}=\frac{13+36}{39}=\frac{49}{39}$,

$B(h,k)=\left(\frac{21}{13},\frac{49}{39}\right)$

26. Image of (0,0) in line is (1,1)

$\therefore$ image circle is $(x-1)^2+(y-1)^2=1$

27. Line equation y =2 Image with respect to x-axex is y =-2

28. Given ABCD is a spuare. A(-1,1) diagonal BD is 3x + y - 8 =0
C is image of A. w.r.t 3x + y - 8 = 0

$$\frac{h+1}{3}=\frac{k-1}{1}=\frac{-2(3(-1)+1-8)}{3^2+1^2}$$

$$\Rightarrow \frac{h+1}{3}=\frac{k-1}{1}=\frac{-2(-10)}{10}$$

$h+1=6$ $\quad$ $k-1=2$

$h=5$ $\quad$ $k=3$

$c(h,k)=(5,3)$

29. Centroid
30. $A(2,0)B(0,2)C(4,6)$

    mid point of BC is $D(2,4)$

    Equation of AD is $x=2$
31. $AB \perp BC$
32. Hypotenous is diameter
33. (0,0) is equidistance from sides
34. If S= 0 then H = 3G
35. $\frac{2}{\lambda}=\frac{3}{k}=\frac{4}{2}$

EXERCISE - II

1. **The lines $p(p^2+1)x-y+q=0$ and $(p^2+1)^2x+(p^2+1)y+2q=0$ are perpendicular to a common line for [MAINS-2016]**
   1) exactly one value of p
   2) exactly two values of p
   3) more than two values of p
   4) no values of p
2. **The perpendicular bisector of the line segment joining $P(1,4)$ and $Q(K,3)$ has Y intercept -4. then a possible value of K is (AIEEE-2008)**
   1) -4 2) 1 3) 2 4) -2
3. **$P(\alpha,\beta)$ lies on the line y=6x-1 and $Q(\beta,\alpha)$ lies on the line 2x-5y=5. Then the equation of the line $\overleftrightarrow{PQ}$ is**
   1) $2x+y=3$ 2) $3x+2y=5$
   3) $x+y=6$ 4) $3x+y=7$
4. **If t_1, t_2 are roots of the equation $t^2+\lambda t+1=0$ where λ is an arbitary constant, then the line joining the points $(at_1^{\,2}, 2at_1)$ $(at_2^{\,2}, 2at_2)$ always passes through the fixed point**
   1) (-a, 0) 2) (0,a) 3) (a,0) 4) (0,-a)
5. **A line joining A(2,0) and B(3,1) is rotated about A in anticlock wise direction through angle 15o, then the equation of AB in the new position is**
   1) $y=\sqrt{3}\,x-2$ 2) $y=\sqrt{3}\,(x-2)$
   3) $y=\sqrt{3}\,(x+2)$ 4) $x-2=\sqrt{3}\,y$
6. **ABCD is a parallelogram. Equations of AB and AD are 4x + 5y = 0 and 7x + 2y = 0 and the equation of diagonal BD is 11x + 7y + 9=0. The equation of AC is** [EAM -2018]
   1) $x+y=0$ 2) $x-y=0$
   3) $x+y+1=0$ 4) $x+y-1=0$
7. **The line 2x+3y=6, 2x+3y=8 cut the X-axis at A,B respectively. A line L = 0 drawn through the point (2,2) meets the X-axis at C in such a way that abscissa of A,B,C are in arithmetic Progression. then the equation of the line L is**
   1) 2x+3y=10 2) 3x+2y=10
   3) 2x-3y=10 4) 3x-2y=10
8. **The sum of the intercepts cut off by the axes on lines**

   $x+y=a$, $x+y=ar$, $x+y=ar^2$,............

   where $a\neq 0$ and $r=\frac{1}{2}$ [EAM -2016]

   1) $2a$ 2) $a\sqrt{2}$ 3) $2\sqrt{2}a$ 4) a
9. **The equation of the straight line whose intercepts on x-axis and y-axis are respectively twice and thrice of those by the line 3x + 4y = 12, is**
   1) 9x + 8y = 72 2) 9x - 8y = 72
   3) 8x + 9y = 72 4) 8x+9y+72=0
10. **If (4, –3) divides the line segment between the axes in the ratio 4 : 5 then its equation is**
    1) 15x + 16y – 12 = 0 2) 3x – 4y – 24 = 0
    3) 15x – 16y +108 = 0 4) 15x – 16y – 108 = 0
11. **A straight line is such that its distance of 5 units from the origin and its inclination is 135°. The intercepts of the line on the co-ordinate axes are**

1) 5, 5 2) $\sqrt{2},\sqrt{2}$

3) $5\sqrt{2},5\sqrt{2}$ 4) $5/\sqrt{2},5/\sqrt{2}$

12. Angles made with the x - axis by two lines drawn through the point (1, 2) and cutting the line $x+y=4$ at a distance $\sqrt{\frac{2}{3}}$ from the point (1,2) are

1) $\frac{\pi}{6}$ and $\frac{\pi}{3}$ 2) $\frac{\pi}{8}$ and $\frac{3\pi}{8}$

3) $\frac{\pi}{12}$ and $\frac{5\pi}{12}$ 4) $\frac{\pi}{4}$ *and* $\frac{\pi}{2}$

13. Equation of the line through the point of intersection of the lines 3x+2y+4=0 and 2x+5y-1=0 whose distance from (2,-1) is 2.

1) 2x-y+5=0 2) 4x+3y+5=0
3) x+2=0 4) 3x+y+5=0

14. If p,q denote the lengths of the perpendicu lars from the origin on the lines $x\sec\alpha - y\operatorname{cosec}\alpha = a$ and $x\cos\alpha + y\sin\alpha = a\cos 2\alpha$ then (Eam 2017)

1) $4p^2+q^2=a^2$ 2) $p^2+q^2=a^2$

3) $p^2+2q^2=a^2$ 4) $4p^2+q^2=2a^2$

15. The equations of the lines parallel to 4x + 3y + 2 = 0 and at a distance of '4' units from it are

1) $4x + 3y + 22 = 0, 4x + 3y – 20 = 0$
2) $4x + 3y + 22 = 0, 4x + 3y – 18 = 0$
3) $4x + 3y – 18 = 0, 4x + 3y – 20 = 0$
4) $4x – 3y – 18 = 0, 4x + 3y – 20 = 0$

16. The range of α for which the points $(\alpha,\alpha+2)$ and $\left(\frac{3\alpha}{2},\alpha^2\right)$ lie on opposite sides of the line $2x+3y-6=0$

1) $(-\infty,-2)$ 2) $(0,1)$

3) $(-\infty,-2)\cup(0,1)$ 4) $(-\infty,1)\cup(2,\infty)$

17. If $P\left(1+\frac{t}{\sqrt{2}},2+\frac{t}{\sqrt{2}}\right)$ be any point on a line then the range of values of t for which the point P lies between the parallel lines $x+2y=1$ *and* $2x+4y=15$ is

1) $-\frac{4\sqrt{2}}{5}<t<\frac{5\sqrt{2}}{6}$ 2) $-\frac{4\sqrt{2}}{3}<t<\frac{5\sqrt{2}}{6}$

3) $t<\frac{-4\sqrt{2}}{3}$ 4) $t<\frac{5\sqrt{2}}{6}$

18. The distance of the point (3, 5) from the line 2x + 3y – 14 = 0 measured parllel to the line x – 2y = 1 is

1) $\frac{7}{\sqrt{5}}$ 2) $\frac{7}{\sqrt{13}}$ 3) $\sqrt{5}$ 4) $\sqrt{13}$

19. Equation of the straight line passing through (1,1) and at a distance of 3 units from (–2, 3) is

1) $x – 2 = 0$ 2) $5x – 12y + 6 = 0$
3) $5x – 12y + 7 = 0$ 4) $y – 1 = 0$

20. If the point **(*a, a*) falls between the lines |*x*+*y*|=2, then:**

1) $|a|=2$ 2) $|a|=1$ 3) $|a|<1$ 4) $|a|<\frac{1}{2}$

21. A line L cuts the sides AB, BC of ΔABC in the ratio 2 : 5, 7 : 4 respectively. Then the line L cuts CA in the ratio

1) 7 : 10 2) 7 : –10 3) 10 : 7 4) 10 : –7

22. The number of integral values of m for which x-coordinate of point of intersection of the lines 3x+4y=9 and y = m*x* +1 is also an integer is

1) 2 2) 0 3) 4 4) 11

23. The line parallel to the x-axis and passing through the intersection of the lines ax+2by+3b=0 and bx – 2ay – 3a = 0, where (a, b) $\neq$ (0, 0) is

1)Above the x-axis at a distance of 3/2 from it
2)Above the x-axis at a distance of 2/3 from it
3)Below the x-axis at a distance of 3/2 from it
4)Below the x-axis at a distance of 2/3 from it

24. Equation of line which is equally inclined to the axis and passes through a common points of family of lines 4acx + y(ab + bc + ca – abc) + abc = 0 (where a, b, c > 0 are in H.P.) is

1) $y-x=\frac{7}{4}$ 2) $y+x=\frac{7}{4}$

3) $y-x=\frac{1}{4}$ 4) $y+x=\frac{3}{4}$

25. If a,b,c in GP then the line $a^2x+b^2y+ac=0$ always passes through the fixed point
1) (0, 1) 2) (1, 0) 3) (0, -1) 4) (1, -1)

26. The straight lines x+2y-9=0, 3x+5y-5=0 and ax+by-1 are concurrent if the straight line 22x-35y-1=0 passes through the point
1) (a, b) 2) (b,a) 3) (-a,b) 4) (-a, -b)

27. If $a \neq b \neq c$, if $ax+by+c=0$ $bx+cy+a=0$ and $cx+ay+b=0$ are concurrent. Then the value of
$2^{a^2b^{-1}c^{-1}}\,2^{b^2c^{-1}a^{-1}}2^{c^2a^{-1}b^{-1}}$
1) 1 2) 4 3) 8 4) 16

28. If p, q, r are distinct, then (q-r)x + (r-p) y + (p-q)=0 and (q^3-r^3) x+ (r^3-p^3) y + (p^3-q^3) = 0 represents the same line if
1) p+q+r=0 2) p=q=r
3) $p^2+q^2+r^2=0$ 4) $p^3+q^3+r^3=0$

29. If 2(sina + sinb) x – 2sin (a – b) y = 3 and 2(cosa+cos b) x+2cos(a-b)y=5 are perpendicular then sin2a + sin 2b =
1) sin (a-b) – 2sin (a+b)
2) sin 2(a-b) – 2sin (a+b)
3) 2sin (a-b) – sin (a+b)
4) sin2 (a-b) – sin (a+b).

30. The acute angle between the lines lx + my = l+m, l (x-y) + m (x+y) = 2m is
1) $\frac{\pi}{4}$ 2) $\frac{\pi}{6}$ 3) $\frac{\pi}{2}$ 4) $\frac{\pi}{3}$

31. The angle between the lines xcosα +ysinα = p_1 and xcosβ +ysinβ =p_2 where $\alpha > \beta$ is
1) $\alpha+\beta$ 2) $\alpha-\beta$ 3) $\alpha\beta$ 4) $2\alpha-\beta$

32. Two equal sides of an isoceles triangle are given by $7x-y+3=0$ and $x+y-3=0$ and the third side passes through the point (1, 10) then the slope m of the third side is given by
1) $3m^2-1=0$ 2) $m^2+1=0$
3) $3m^2+8m-3=0$ 4) $m^2-3=0$

33. Area of triangle formed by angle bisectors ofcoordinate axes and the line x=6 in sq.units is
1) 36 2) 18 3) 72 4) 9

34. A line passing through (3,4) meets the axes $\overrightarrow{OX}$ and $\overrightarrow{OY}$ at A and B respectively. The minimum area of the triangle OAB in square units is [EAM -2019]
1) 8 2) 16 3) 24 4) 32

35. The equation to the base of an equilateral triangle is $\left(\sqrt{3}+1\right)x+\left(\sqrt{3}-1\right)y+2\sqrt{3}=0$ and opposite vertex is A(1,1) then the Area of the triangle is
1) $3\sqrt{2}$ 2) $3\sqrt{3}$ 3) $2\sqrt{3}$ 4) $4\sqrt{3}$

36. Area of the quadrilateral formed by the lines $4y-3x-a=0$, $3y-4x+a=0$, $4y-3x-3a=0$, $3y-4x+2a=0$ is
1) $\frac{a^2}{5}$ 2) $\frac{a^2}{7}$ 3) $\frac{2a^2}{7}$ 4) $\frac{2a^2}{9}$

37. The equation of perpendicular bisectors of sides AB,BC of $\triangle$ABC are x-y-5=0, x+2y=0 respectively and A(1,-2) then coordinate of C are [EAM -2020]
1) (1,0) 2) (0,1) 3) (5,0)
4) (0,0)

38. If the straight lines 2x+3y-1=0, x +2y-1=0 and ax + by –1 = 0 form a triangle with origin as orthocentre, then (a,b) is giveny by
1) (6,4) 2) (-3,3) 3) (-8,8) 4) (0,7)

39. The vertices A,B of a triangle are (2, 5), (4, -11). If C moves on the line L$\equiv$9x+7y+4=0, then the locus of centroid of triangle ABC is parallel to
1) AB 2) AC 3) BC 4) L

40. The acute angle bisector between the lines 3x-4y-5=0, 5x+12y-26=0 is
1) 7x-56y+32=0 2) 9x-3y+13=0
3) 14x-112y+65=0 4) 7x-13y+9=0

41. Find the equation of the bisector of the angle between the lines x+2y–11=0, 3x–6y–5=0 which contains the point (1,–3).

1) $2x-19=0$ 2) $2x+19=0$

3) $3x-19=0$ 4) $3x+19=0$

42. If 2x+y-4=0 is bisector of the angle between the lines a(x–1)+b(y–2)=0, c(x–1)+d(y–2)=0, then the other bisector is

1) $x-2y+1=0$ 2) $x-2y-3=0$

3) $x-2y+3=0$ 4) $x-2y-5=0$

43. Let P = (-1,0) Q=(0,0) and R=(3, $3\sqrt{3}$) be three points. Then the equation of the bisector of angle PQR is (AIEEE 2007)

1) $\frac{\sqrt{3}}{2}x+y=0$ 2) $x+\sqrt{3}y=0$

3) $\sqrt{3}x+y=0$ 4) $x+\frac{\sqrt{3}}{2}y=0$

44. A ray of light along $x+\sqrt{3}y=\sqrt{3}$ gets reflected upon reaching x-axis, the equation of the reflected ray is [JEE-MAINS 2013]

1) $y=x+\sqrt{3}$ 2) $\sqrt{3}y=x-\sqrt{3}$

3) $y=3x-\sqrt{3}$ 4) $\sqrt{3}y=x-1$

45. Consider the points A(0,1) and B(2,0) and P be a point on the line $4x+3y+9=0$. Coordinates of P such that $|PA-PB|$ is maximum are

1) $\left(\frac{-24}{5},\frac{17}{5}\right)$ 2) $\left(\frac{-84}{5},\frac{13}{5}\right)$

3) $\left(\frac{-6}{5},\frac{17}{5}\right)$ 4) (0, –3)

46. A straight line which make equal intercepts on +ve x and y axes and which is at a distance '1' unit from the origin intersects the straight line $y=2x+3+\sqrt{2}$ at (x_0,y_0) then $2x_0+y_0=$ [EAM 2010]

1) $3+\sqrt{2}$ 2) $\sqrt{2}-1$ 3) 1 4) 0

47. p is the length of the perpendicular drawn from the origin upon a straight line then the locus of mid point of the portion of the line intercepted between the coordinate axes is

1) $\frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{p^2}$ 2) $\frac{1}{x^2}+\frac{1}{y^2}=\frac{2}{p^2}$

3) $\frac{1}{x^2}+\frac{1}{y^2}=\frac{4}{p^2}$ 4) $\frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{p}$

48. The number of circles that touch all the 3 lines 2x + y = 3, 4x - y = 3, x + y = 2 is

1) 0 2) 1 3) 2 4) 4

49. Let P(1,1) and Q(3,2) be given points. The point R on the x-axis such that PR+RQ is minimum is

1) $\left(\frac{5}{3},0\right)$ 2) $(2,0)$ 3) $(3,0)$ 4) $\left(\frac{3}{2},0\right)$

50. Number of circles touching the lines 3x+4y–1=0, 4x–5y+2=0 and 6x+8y+3=0 is

1) 0 2) 2 3) 4 4) infinite

51. A point moves in the xy plane such that the sum of its distance from two mutually perpendicular lines is always equal to 5 units.The area (in square units) enclosed by the locus of the point (EAM- 2012)

1) $\frac{25}{4}$ 2) 25 3) 50 4) 100

KEY

01) 1	02) 1	03) 3	04) 1	05) 2	06) 2
07) 1	08) 3	09) 1	10) 4	11) 3	12) 3
13) 2	14) 1	15) 2	16) 3	17) 2	18) 3
19) 3	20) 3	21) 4	22) 1	23) 3	24) 1
25) 3	26) 2	27) 3	28) 1	29) 2	30) 1
31) 2	32) 3	33) 1	34) 3	35) 3	36. 3
37) 3	38) 1	39) 3	40) 4	41) 3	42) 3
43) 3	44) 3	45) 2	46) 1	47) 2	48) 3
48) 2	49) 1	50) 2	51) 3		

SOLUTIONS

1. Given lines $p\left(p^2+1\right)x-y+\gamma=0$

$\left(p^2+1\right)^2x+\left(p^2+1\right)y+2\gamma=0$ are parallel

$\frac{p\left(p^2+1\right)}{\left(p^2+1\right)^2}=\frac{-1}{p^2+1}$, $p=-1$

2. Given $P(1,4)Q(k,3)$ midpoint of $PQ=\left(\frac{1+k}{2},\frac{7}{2}\right)$ slope of $PQ=\frac{-1}{k-1}$ equation of perpendicular bisector of PQ is

$y-\frac{7}{2}=(k-1)\left(x-\left(\frac{1+k}{2}\right)\right)$ passes through (0,-4)

$$\Rightarrow -4-\frac{7}{2}=(1-k)\left(\frac{1+k}{2}\right)\Rightarrow -\frac{15}{2}=\frac{1-k^2}{2}$$

$k^2=16$

$k=\pm 4$

k = -4

3. By solving $\beta=6\alpha-1$ and $2\beta-5\alpha=5$ we get P(1,5) , Q(5,1)
4. Equation of the line is $y(t_1+t_2)-2at_1(t_1+t_2)$ $=2x-2at_1^2$
5. $A(2,0)B(3,1)$ slope of AB = 1

   $\theta=15^0+45^0=60^0\Rightarrow m=\tan 60^0=\sqrt{3}$

   equation of new position $y-0=m(x-2)$

   $y=\sqrt{3}(x-2)$
6. by solving AB,BD we get B(-5/3, 4/3)
   by solving AD,BD we get D(2/3, -7/3)
   mid point of B.D lies on AC
7. $A=(3,0)B(4,0)$; $c=\left(2-\frac{2}{m},0\right)$
8. Intercepts between the axes made by the given lines are $a\sqrt{2},ar\sqrt{2},ar^2\sqrt{2}$ .... Sum of intercepts $=a\sqrt{2}+a\gamma\sqrt{2}+a\gamma^2\sqrt{2}$ – – – – – – –

$$=a\sqrt{2}\left(1-\gamma+\gamma^2+-----\infty\right)=a\sqrt{2}\frac{1}{1-\gamma}=2a\sqrt{2}$$

9. Given line $3x+4y=12$

   $\frac{x}{4}+\frac{y}{3}=1$ required intercepts a = 8 , b = 9

   $\frac{x}{8}+\frac{y}{9}=1$

   $9x+8y=72 \qquad a=8,b=6$
10. $\frac{nx}{x_1}+\frac{my}{y_1}=m+n$
11. $\alpha=135^0-90^0$, P = 5
12. Given $\gamma=\sqrt{\frac{2}{3}},(x_1,y_1)=(1,2),L=x+y=4$

    $\gamma=\left|\frac{ax_1+by_1+c_1}{a\cos\theta+b\sin\theta}\right|$

    $\sqrt{\frac{2}{3}}=\left|\frac{1+2-4}{\cos\theta+\sin\theta}\right|$ S.O.B.S

    $\frac{2}{3}=\frac{1}{(\cos\theta+\sin\theta)^2}$

    $1+\sin 2\theta=\frac{3}{2}$, $\qquad \sin 2\theta=\frac{1}{2}=\sin 30^0$

    , $\qquad 2\theta=\frac{\pi}{6}$,

    $\theta=\frac{\pi}{\sqrt{2}}(or)\frac{5\pi}{12}$

    $r=\sqrt{2/3},(x_1,y_1)=(1,2)$
13. Point of intersection $(-2,1)$ and verification
14. Given lines $x\sec\alpha-y\operatorname{cosec}\alpha=a$ and $x\cos\alpha+y\sin\alpha=a\cos 2\alpha$

    $\alpha=45^0$

    $\sqrt{2x}-\sqrt{2y}=a$, x + y = 0 given distance from (0,0) to (1) (2) is p,q

    $p=\frac{Q}{2}$ $q=0,2p=a$ now $4p^2+q^2=a^2$
15. $ax+by+c\pm d\sqrt{a^2+b^2}=0$
16. Points lie on opposite sides of the line

    $\Rightarrow L_{11}L_{22}<0$

    $\Rightarrow 5\alpha(3\alpha+3\alpha^2-6)<0\rightarrow\alpha(\alpha+2)(\alpha-1)<0$

    $\Rightarrow\alpha\in(-\infty,-2)\cup(0,1)$
17. Origin, P lies opposite side to the first line and same side to the second line
18. Given point $p(3,5)=(x_1,y_1)L=2x+3y-14=0$

    x - 2y 1, $\tan\theta=m=\frac{1}{2},\sin\theta=\frac{1}{\sqrt{5}},\cos\theta=\frac{2}{\sqrt{5}}$

$$\gamma=\left|\frac{ax_1+by_1+c}{a\cos\theta+b\sin\theta}\right|=\left|\frac{2(3)+3(5)-14}{2\left(\frac{2}{\sqrt5}\right)+3\left(\frac{1}{\sqrt5}\right)}\right|,$$

$$\gamma=\frac{7}{\frac{7}{\sqrt5}}=\sqrt5 \quad \left|\frac{ax_1+by_1+c}{a\cos\theta+b\sin\theta}\right| \quad \text{where}$$

$$\tan\theta=\frac{1}{2}$$

19. Equation of line passing through (1,1) having slope m is $y-1=m(x-1)$

$mx-y+1-m=0$ ____ (1)

Given distance from (-2,3) to (1) is 3

$$3=\frac{|-2m-3+1-m|}{\sqrt{m^2+1}}$$

$3\sqrt{m^2+1}=|2+3m|$ S . O . B . S

$$9(m^2+1)=(2+3m)^2 \Rightarrow 9m^2+9=4+9m^2+12m$$

$12m=5$

$$m=\frac{5}{12}$$

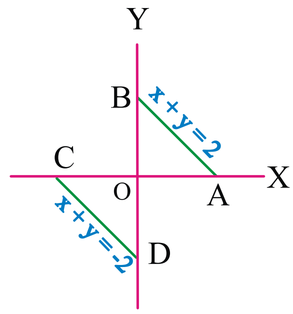
Required line $y-1=\frac{5}{12}(x-1)$

$12y-12=5x-5$

$5x-12y+7=0$

20. From the figure

$-1<a<1$ i.e. $|a|<1$.



21. $\left(\frac{BD}{DC}\right)\left(\frac{CE}{EA}\right)\left(\frac{AF}{FB}\right)=-1$

22. By solving, given equations we get $x=\frac{5}{3+4m}$

x is an ineger of $3+4m=\pm1,\pm5$,

$\therefore$ integral values of m are -1,-2

23. Eq. of required line parallel to x-axis

$\Rightarrow$ slope $=0\Rightarrow\lambda=-a/b$

Equation $=2y+3=0$

24. Lines can be written

$$\frac{4}{b}x+y\left(\frac{3}{b}\right)+1-y=0,\ \frac{1}{b}(4x+3y)+1-y=0$$

$\Rightarrow$ Lines are concurrent at $\left(-\frac{3}{4},1\right)$

$\therefore$ Required line is $y-1=\pm1\left(x+\frac{3}{4}\right)$

25. Given equation is $a^2x+b^2(y+1)=0$

26. $$\begin{vmatrix}1&2&-9\\3&5&-5\\a&b&-1\end{vmatrix}=0$$

27. Given lines $ax+by+c=0, bx+cy+a=0, cx+ay+b=0$ are concurrnet

$$\Rightarrow\begin{vmatrix}a&b&c\\b&c&a\\c&a&b\end{vmatrix}=0\Rightarrow a(bc-a^2)-b(b^2-ac)+c(ab-c$$

$a^3+b^3+c^3=3abc\Rightarrow\frac{a^2}{bc}+\frac{b^2}{ac}+\frac{c^2}{ab}=3$ now

$$2^{\frac{a^2}{bc}}.2^{\frac{b^2}{ca}}.2^{\frac{c^2}{ab}}=2^{\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}}=2^3=8$$

28. $$\frac{q^3-r^3}{q-r}=\frac{r^3-p^3}{r-p}=\frac{p^3-q^3}{p-q}$$

$q^2+r^2+qr=r^2+p^2+pr=p^2+q^2+pq$

$-r(p-q)=(p-q)(p+q)$

$p+q+r=0 \qquad \frac{q^3-r^3}{q-r}=\frac{r^3-p^3}{r-p}=\frac{p^3-q^3}{p-q}$

29. $m_1m_2=-1$

30. $\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$

31. $\cos\theta=\frac{a_1a_2+b_1b_2}{\sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}}$

32. $\frac{m-7}{1+7m}=-\frac{m+1}{1-m}\Rightarrow 3m^2+8m-3=0$

33. Equations of the angular bisectors of the axes are $y=x$ and $y=-x$

34. $(p,q)=(3,4)$ then minimum area = 2pq

35. Area of an equilateral triangle is

$\frac{h^2}{\sqrt{3}}$ where h is the height of the triangle

36. The area of the parallelogram formed by the lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+d_1=0$, $a_1x+b_1y+c_2=0$, $a_2x+b_2y+d_2=0$ is $\left|\frac{(c_1-c_2)(d_1-d_2)}{a_1b_2-a_2b_1}\right|$ Sq.units.

37. Given $A=(1,-2)\, x-y-5=0$

The $B(h,k)$ is iage of $A(1,-2)$ w.r.t $x-y-5=0$

$$\frac{h-1}{1}=\frac{k+2}{-1}=\frac{-2(1+2-5)}{1+1},$$

$$\frac{h-1}{1}=\frac{k+2}{-1}=2$$

h = 3, k = - 4 b(h,k) = (3,-4), c is image of B(3,-4)w.r.t x+2y=0

$$\frac{h-3}{1}=\frac{k+4}{2}=\frac{-2(3-8)}{5}$$

$$\frac{h-3}{1}=2,\ \frac{k+4}{2}=2$$

$h=5$ k = 0

c(5,0)

38. Equation of AO is

$(2x+3y-1)+\lambda(x+2y-11)=0$ passes through (0,0) $\Rightarrow \lambda=-1$

Since $AO\perp BC$ we have a=-b

similarly apply $BO\perp AC$

39. Choose $C=(\alpha,\beta)$

$$G(x_1,y_1)=\left(\frac{\alpha+6}{3},\frac{\beta-6}{3}\right)$$

$$\Rightarrow \alpha=3x_1-6,\beta=3y_1+6$$

Substance (α,β) lies on L=0

40. $a_1a_2+b_1b_2=-32<0,\ c_1c_2=130>0$

Use $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}}=\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$

41. Given lines $x+2y-11=0, 3x-6y-5=0\ (x_1,y_1)=(1,-3)$

$$\frac{|x_1+2y_1-11|}{\sqrt{5}}=\frac{|3x_1-6y_1-5|}{3\sqrt{5}}$$

$$\frac{-(x+2y-11)}{1}=\frac{3x-6y-5}{3}$$

$$\Rightarrow -3x-6y+33=3x-6y-5\Rightarrow 6x=38=0$$

3x = 19 = 0

42. required bisector is perpendicular to given and passes through (1,2)

43. 'T' divides Prise the ratio PQ; QR = 1:6

$$T=\left(\frac{3-6}{7},\frac{3\sqrt{3}+0}{7}\right)=\left(\frac{-3}{7},\frac{3\sqrt{3}}{7}\right)$$

Equation of

$$Q(0,0)T\left(\frac{-3}{7},\frac{3\sqrt{3}}{7}\right)y=0=\frac{3\sqrt{3}-0}{-3-0}(x-0)$$

$$y=-\sqrt{3}x, \sqrt{3x}+y=0$$

44. Slope of reflected ray is $\frac{1}{\sqrt{3}}$ and it passing through

$(\sqrt{3},0)$ is $\frac{y-0}{x-\sqrt{3}}=\frac{1}{\sqrt{3}}\quad \sqrt{3}y=x-\sqrt{3}$

45. Given points $A(0,1), B(2,0)$ line $4x+3y+9=0$

minimum the point 'P' on the line $|PA-PB|$ is

A,B are lies on same side of the given line equation of $A(0,1)B(2,0)$ is

$$y-1=\frac{-1}{2}(x-0) \Rightarrow 2y-2=-x$$

$x+2y-2=0$ ______ (1)

$4x+3y+9=0$ _____ (2)

solving (1) and (2)

$4x+8y-8=0$

$4x+3y+9=0$

5y - 17 = 0 , y = 17/5 substituting in ----(1)

$$x+\frac{34}{5}-2=0, x=\frac{-24}{5}$$

$$p\left(\frac{-24}{5},\frac{17}{5}\right)$$

46. Equation of the straight line having equal intercepts is x+y = k and proceed.

47. Equate the distance from $(0,0)$ to the line

$$\frac{x}{x_1}+\frac{y}{y_1}=2$$

48. Given lines are concurrent.

49. Image of P in x-axis is $P^1=(1,-1)$, R is intersection of x-axis and line QP^1

50. Two lines are parallel

51. From given data $|x|+|y|=5$ hence required

$$\text{area} = \frac{2(5)^2}{|(1)(1)|}=50$$

JEE MAINS QUESTIONS

1.If the perpendicular bisector of the line segment — oiningthe points P (1, 4) and Q (k, 3) has y-intercept equal to – 4,then a value of k is :

(1) – 2 (2) – 4 (3) 14 (4) 15

2.If a DABC has vertices A(–1, 7), B(–7, 1) and C(5, –5), thenits orthocentre has coordinates

(1). $\left(-\frac{3}{5},\frac{3}{5}\right)$ (2). $(-3,3)$

(3). $\left(\frac{3}{5},-\frac{3}{5}\right)$ (4). $(3,-3)$

3.Two vertices of a triangle are (0, 2) and (4, 3). If its orthocentre is at the origin, then its third vertex lies in which quadrant?

(1) third (2) second
(3) first (4) fourth

4..Let C be the centroid of the triangle with vertices (3, –1),(1, 3) and (2, 4). Let P be the point of intersection of thelines x + 3y – 1 = 0 and 3x – y + 1 = 0. Then the line passingthrough the points C and P also passes through the point

(1) (–9, –6) (2) (9, 7) (3) (7, 6) (4) (–9, –7)

5..Slope of a line passing through P(2, 3) and intersectingthe line x + y = 7 at a distance of 4 units from P, is:

1) $\frac{1-\sqrt{5}}{1+\sqrt{5}}$ 2) $\frac{1-\sqrt{7}}{1+\sqrt{7}}$ 3) $\frac{\sqrt{7}-1}{\sqrt{7}+1}$

4) $\frac{\sqrt{5}-1}{\sqrt{5}+1}$

6.A point on the straight line, 3x + 5y = 15 which is equidistant from the coordinate axes will lie only in :

(1) 4th quadrant (2) 1st quadrant
(3) 1st and 2nd quadrants
(4) 1st, 2nd and 4th quadrants

7.Two vertical poles of heights, 20 m and 80 m stand aparton a hori”ontal plane. The height (in meters) of the pointof intersection of the lines joining the top of each pole tothe foot of the other, from this hori”ontal plane is :

(1) 15 (2) 18 (3) 12 (4) 16

8.If a straight line passing through the point P(–3, 4) is suchthat its intercepted portion between the coordinate axes isbisected at P, then its equation is

(1) 3x – 4y + 25 = 0 (2) 4x – 3y + 24 = 0
(3) x – y + 7 = 0 (4) 4x + 3y = 0

9.If in a parallelogram ABDC, the coordinates of A, B and Care respectively (1, 2), (3, 4) and (2, 5), then the equation ofthe diagonal AD is

(1) 5x – 3y + 1 = 0 (2) 5x + 3y – 11 = 0
(3) 3x – 5y + 7 = 0 (4) 3x + 5y – 13 = 0

10.If the line 3x + 4y – 24 = 0 intersects the x-axis at the pointA and the y-axis at the point B, then the incentre of thetriangle OAB, where O is the origin, is

(1) (3, 4) (2) (2, 2) (3) (4, 3) (4) (4, 4)

11.Let L denote the line in the xy-plane with x and y interceptsas 3 and 1 respectively. Then the image of the point (–1, –4)in this line is:

1 . $\left(\frac{11}{5},\frac{28}{5}\right)$ 2. $\left(\frac{29}{5},\frac{8}{5}\right)$ 3. $\left(\frac{8}{5},\frac{29}{5}\right)$

4. $\left(\frac{29}{5},\frac{11}{5}\right)$

12. The locus of the mid-points of the perpendiculars drawnfrom points on the line, x = 2y to the line x = y is

(1) 2x – 3y = 0 (2) 5x – 7y = 0
(3) 3x – 2y = 0 (4) 7x – 5y = 0

13.A rectangle is inscribed in a circle with a diameter lyingalong the line 3y = x + 7. If the two adjacent vertices of therectangle are (–8, 5) and (6, 5), then the area of the rectangle(in sq. units) is:

(1) 84 (2) 98 (3) 72 (4) 56

14.Suppose that the points (h, k), (1, 2) and (– 3, 4) lie on theline L1. If a line L2 passing through the points (h, k) and(4, 3) is perpendicular on L1
, then equals :

1) $\frac{1}{3}$ 2.) 0 3) 3 4) $-\frac{1}{7}$

15.Two sides of a parallelogram are along the lines, x + y = 3and x – y + 3 = 0. If its diagonals intersect at (2, 4), thenone of its vertex is

(1) (3, 5) (2) (2, 1) (3) (2, 6) (4) (3, 6)

16. Let the equations of two sides of a triangle be 3x – 2y + 6 = 0 and 4x + 5y – 20 = 0. If the orthocentreof this triangle is at (1, 1), then the equation of its thirdside is:

(1) 122y – 26x – 1675 = 0
(2) 122y + 26x + 1675 = 0
(3) 26x + 61y + 1675 = 0
(4) 26x – 122y – 1675 = 0

17. The foot of the perpendicular drawn from the origin, onthe line, 3x + y = λ is P. If the line meets x-axis at A andy-axis at B, then the ratio BP : PA is

(1) 9 : 1 (2) 1 : 3 (3) 1 : 9 (4) 3 : 1

18. Let a, b, c and d be non-”ero numbers. If the point ofintersection of the lines 4ax + 2ay + c = 0 and 5bx + 2by + d=0 lies in the fourth quadrant and is equidistant from thetwo axes then

(1) 3bc – 2ad = 0 (2) 3bc + 2ad = 0
(3) 2bc – 3ad = 0 (4) 2bc + 3ad = 0

19.A straight line through a fixed point (2, 3) intersects thecoordinate axes at distinct points P and Q. If O is the originand the rectangle OPRQ is completed, then the locus of Ris

1) $2x+3y=xy$

2) $3x+2y=xy$

3) $3x+2y=6xy$

4) $3x+2y=6$

20.The equation y = sin x sin (x + 2) – sin2 (x + 1) representsa straight line lying in

(1) second and third quadrants only
(2) first, second and fourth quadrant
(3) first, third and fourth quadrants
(4) third and fourth quadrants only

KEY

01)2	02) 2	03)2	04)1	05)2	06)3
07)4	08)2	09)1	10) 2	11) 1	12) 2
13)1	15)4	14)1	16)4	17)4	18)1
19)2	20)4				

SOLUTIONS

1.

Mid point of line segment PQ be $\left(\frac{k+1}{2}, \frac{7}{2}\right)$.

$\therefore$ Slope of perpendicular line passing through

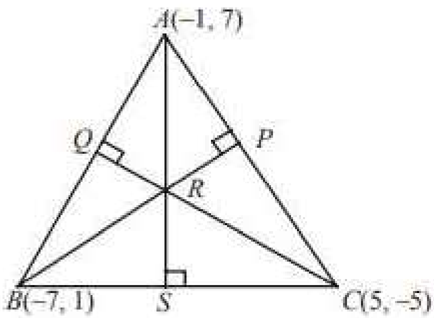
$(0, -4)$ and $\left(\frac{k+1}{2}, \frac{7}{2}\right) = \frac{\frac{7}{2}+4}{\frac{k+1}{2}-0} = \frac{15}{k+1}$

Slope of $PQ = \frac{4-3}{1-k} = \frac{1}{1-k}$

$\therefore \frac{15}{1+k} \times \frac{1}{1-k} = -1$

$1-k^2 = -15 \Rightarrow k = \pm 4.$

2.



$m_{BC} = \frac{6}{-12} = -\frac{1}{2}$

$\therefore$ Equation of AS is $y - 7 = 2(x+1)$

$y = 2x + 9$ ...(i)

$m_{AC} = \frac{12}{-6} = -2$

$\therefore$ Equation of BP is $y - 1 = \frac{1}{2}(x+7)$

$y = \frac{x}{2} + \frac{9}{2}$ ...(ii)

From equs. (i) and (ii),

$2x + 9 = \frac{x+9}{2}$

$\Rightarrow 4x + 18 = x + 9$

$\Rightarrow 3x = 9 \Rightarrow x = -3$

$\therefore y = 3$

3.

Since, $m_{QR} \times m_{PH} = -1$

$\Rightarrow m_{QR} = -\frac{1}{m_{PH}}$

$\Rightarrow m_{QR} = \frac{y-3}{x-4} = 0$

$\Rightarrow y = 3$

$m_{PQ} \times m_{RH} = -1$

$\Rightarrow \frac{1}{4} \times \frac{y}{x} = -1$

$\Rightarrow y = -4x$

$\Rightarrow x = -\frac{3}{4}$

Vertex R is $\left(\frac{-3}{4}, 3\right)$

Hence, vertex R lies in second quadrant.

4.

Coordinates of centroides

$C = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$

$= \left(\frac{3+1+2}{3}, \frac{-1+3+4}{3}\right) = (2, 2)$

The given equation of lines are

$x + 3y - 1 = 0$

$3x - y + 1 = 0$

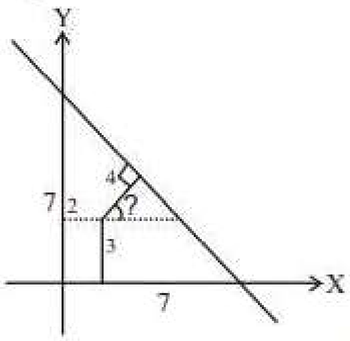
Then, from (i) and (ii)

point of intersection $P\left(-\frac{1}{5}, \frac{2}{5}\right)$

equation of line DP

$8x - 11y + 6 = 0$

5.



Since point at 4 units from P (2, 3) will be A (4 cosθ + 2, 4 sinθ + 3) and this point will satisfy the equation of line x + y = 7

$\Rightarrow \cos\theta + \sin\theta = \frac{1}{2}$

On squaring

$\Rightarrow \sin 2\theta - \frac{3}{4} \Rightarrow \frac{2\tan\theta}{1+\tan^2\theta} = -\frac{3}{4}$

$\Rightarrow 3\tan^2\theta + 8\tan\theta + 3 = 0$

$\Rightarrow \tan\theta = \frac{-8 \pm 2\sqrt{7}}{6}$ (ignoring –ve sign)

$\Rightarrow \tan\theta = \frac{-8 \pm 2\sqrt{7}}{6} = \frac{1-\sqrt{7}}{1+\sqrt{7}}$

6. A point which is equidistant from both the axes li on either $y = x$ and $y = -x$.

Since, point lies on the line $3x + 5y = 15$

Then the required point

$$\begin{array}{r} 3x + 5y = 15 \\ x + y = 0 \\ \hline x = -\frac{15}{2} \end{array}$$

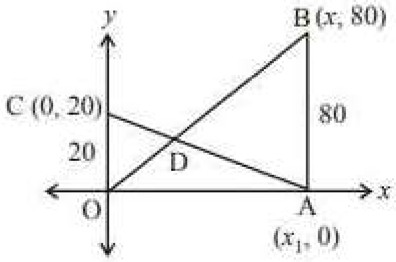
$y = \frac{15}{2} \Rightarrow (x, y) = \left(-\frac{15}{2}, \frac{15}{2}\right)$ {2nd quadrant}

or
$$\begin{array}{r} 3x + 5y = 15 \\ x - y = 0 \\ \hline x = \frac{15}{8} \end{array}$$

$y = \frac{15}{8} \Rightarrow (x, y) = \left(\frac{15}{8}, \frac{15}{8}\right)$ {1st quadrant}

Hence, the required point lies in 1st and 2nd quadrant.

7.



Equations of lines OB and AC are respectively

$y = \frac{80}{x_1}x$ ...(i)

$\frac{x}{x_1} + \frac{y}{20} = 1$ ...(ii)

∵ equations (i) and (ii) intersect each other

∴ substitute the value of x from equation (i) to equation (ii), we get

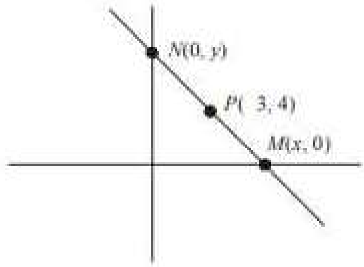
$\frac{y}{80} + \frac{y}{20} = 1$

$\Rightarrow y + 4y = 80 \Rightarrow y = 16$ m

Hence, height of intersection point is 16 m.

8.

Since, P is mid point of MN



Then, $\frac{0+x}{2} = -3$

$\Rightarrow x = -3 \times 2 \Rightarrow x = -6$

and $\frac{y+0}{2} = 4 \Rightarrow y + 0 = 2 \times 4 \Rightarrow y = 8$

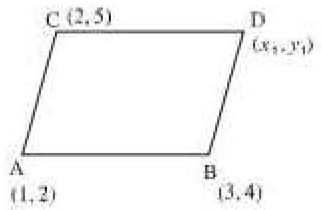
Hence required equation of straight line MN is

$\frac{x}{-6} + \frac{y}{8} = 1 \Rightarrow 4x - 3y + 24 = 0$

9.

Since, in parallelogram mid points of both diagonals coinsides.

$\therefore$ mid-point of AD = mid-point of BC



$$\left(\frac{x_1+1}{2}, \frac{y_1+2}{2}\right) = \left(\frac{3+2}{2}, \frac{4+5}{2}\right)$$

$\therefore \quad (x_1, y_1) = (4, 7)$

Then, equation of AD is,

$$y-7 = \frac{2-7}{1-4}(x-4)$$

$$y-7 = \frac{5}{3}(x-4)$$

$3y-21 = 5x-20$

$5x-3y+1=0$

10.

Equation of the line is:

$3x+4y=24$

or $\frac{x}{8}+\frac{y}{6}=1$

$\therefore$ coordinates of A, B & O are (8, 0), (0, 6) & (0, 0) respectively.

$\Rightarrow \; OA = 8, OB = 6 \;\&\; AB = 10.$

$\therefore$ Incentre of ΔOAB is given as:

$$I \equiv \left(\frac{8\times0+6\times8+10\times0}{8+6+10}, \frac{8\times6+6\times0+10\times0}{8+6+10}\right) \equiv (2,2).$$

11.

The line in xy-plane is,

$$\frac{x}{3}+y=1 \Rightarrow x+3y-3=0$$

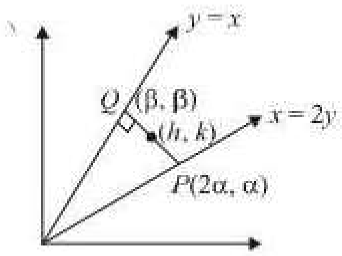
Let image of the point $(-1, -4)$ be (α, β), then

$$\frac{\alpha+1}{1} = \frac{\beta+y}{3} = -\frac{2(-1-12-3)}{10}$$

$$\Rightarrow \alpha+1 = \frac{\beta+4}{3} = \frac{16}{5}$$

$$\Rightarrow \alpha = \frac{11}{5}, \beta = \frac{28}{5}$$

12.



Since, slope of $PQ = \frac{k-\alpha}{h-2\alpha} = -1$

$\Rightarrow \; k-\alpha = -h+2\alpha$

$\Rightarrow \; \alpha = \frac{h+k}{3}$

Also, $2h = 2\alpha+\beta$ and

$2k = \alpha+\beta$

$\Rightarrow \; 2h = \alpha+2k$

$\Rightarrow \; \alpha = 2h-2k$

From (i) and (ii), we have

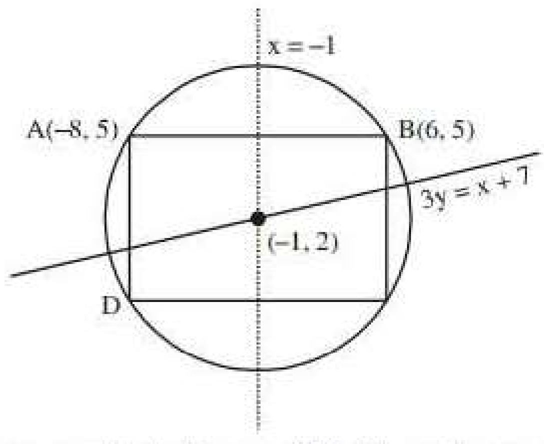
$$\frac{h+k}{3} = 2(h-k)$$

So, locus is $6x-6y = x+y$

$\Rightarrow \; 5x = 7y \;\Rightarrow\; 5x-7y=0$

13.

Given situation



$\because$ perpendicular bisector of AB will pass from centre.

$\therefore$ equation of perpendicular bisector $x = -1$

Hence centre of the circle is $(-1, 2)$

Let co-ordinate of D is (α, β)

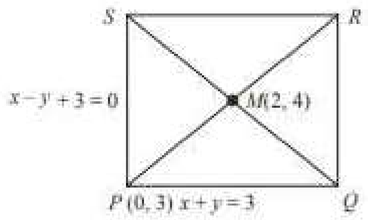
$$\Rightarrow \frac{\alpha+6}{2}=-1 \text{ and } \frac{\beta+5}{2}=2$$

$\Rightarrow \alpha=-8$ and $\beta=-1$, $\therefore D \equiv (-8,-1)$

$|AD|=6$ and $|AB|=14$

Area $=6\times 14=84$

15.



Since, $x-y+3=0$ and $x+y=3$ are perpendicular lines and intersection point of $x-y+3=0$ and $x+y=3$ is $P(0,3)$.

$\Rightarrow$ M is mid-point of $PR \Rightarrow R(4,5)$

Let $S(x_1, x_1+3)$ and $Q(x_2, 3-x_2)$

M is mid-point of SQ

$\Rightarrow x_1+x_2=4, x_1+3+3-x_2=8$

$\Rightarrow x_1=3, x_2=1$

Then, the vertex D is $(3,6)$.

14.

$\because (h,k), (1,2)$ and $(-3,4)$ are collinear

$$\therefore \begin{vmatrix} h & k & 1 \\ 1 & 2 & 1 \\ -3 & 4 & 1 \end{vmatrix}=0 \Rightarrow -2h-4k+10=0$$

$\Rightarrow h+2k=5$

Now, $m_{L_1}=\frac{4-2}{-3-1}=-\frac{1}{2} \Rightarrow m_{L_2}=2$ $\quad [\because L_1 \perp L_2]$

By the given points (h,k) and $(4,3)$,

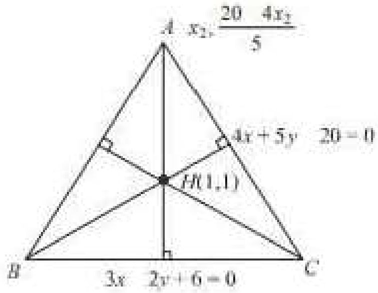
$$m_{L_2}=\frac{k-3}{h-4} \Rightarrow \frac{k-3}{h-4}=2 \Rightarrow k-3=2h-8$$

$2h-k=5$

From (i) and (ii)

$$h=3, k=1 \Rightarrow \frac{k}{h}=\frac{1}{3}$$

16.



$$\left(x_1, \frac{3x_1+6}{2}\right)$$

Since, AH is perpendicular to BC

Hence, $m_{AH} \cdot m_{BC}=-1$

$$\left(\frac{\frac{20-4x_2}{5}-1}{x_2-1}\right)\times\frac{3}{2}=-1$$

$$\frac{15-4x_2}{5(x_2-1)}=-\frac{2}{3}$$

$45-12x_2=-10x_2+10$

$$2x_2=35 \Rightarrow x_2=\frac{35}{2}$$

$$\Rightarrow A\left(\frac{35}{2}, -10\right)$$

Since, BH is perpendicular to CA.

Hence, $m_{BH}\times m_{CA}=-1$

$$\left(\frac{\frac{3x_1}{2}+3-1}{x_1-1}\right)\left(-\frac{4}{5}\right)=-1$$

$$\frac{(3x_1+4)}{2(x_1-1)}\times 4=5$$

$$\Rightarrow 6x_1+8=5x_1-5 \Rightarrow x_1=-13 \Rightarrow \left(-13, \frac{-33}{2}\right)$$

$\Rightarrow$ Equation of line AB is

$$y+10=\left(\frac{-\frac{33}{2}+10}{-13-35}\right)\left(x-\frac{35}{2}\right)$$

-

$\Rightarrow -61y - 610 = -13x + \frac{455}{2}$

$\Rightarrow -122y - 1220 = -26x + 455$

$\Rightarrow 26x - 122y - 1675 = 0$

17.

Equation of the line, which is perpendicular to the line, $3x + y = \lambda (\lambda \neq 0)$ and passing through origin, is given by

$$\frac{x-0}{3} = \frac{y-0}{1} = r$$

For foot of perpendicular

$$r = \frac{-((3\times 0) + (1\times 0) - \lambda)}{3^2 + 1^2} = \frac{\lambda}{10}$$

So, foot of perpendicular $P = \left(\frac{3\lambda}{10}, \frac{\lambda}{10}\right)$

Given the line meets X-axis at $A = \left(\frac{\lambda}{3}, 0\right)$ and meets Y-axis at $B = (0, \lambda)$

So, $BP = \sqrt{\left(\frac{3\lambda}{10}\right)^2 + \left(\frac{\lambda}{10} - \lambda\right)^2} \Rightarrow BP = \sqrt{\frac{9\lambda^2}{100} + \frac{81\lambda^2}{100}}$

$$\Rightarrow BP = \sqrt{\frac{90\lambda^2}{100}}$$

Now, $PA = \sqrt{\left(\frac{\lambda}{3} - \frac{3\lambda}{10}\right)^2 + \left(0 - \frac{\lambda}{10}\right)^2}$

$\Rightarrow PA \sqrt{\frac{\lambda^2}{900} + \frac{\lambda^2}{100}} \Rightarrow PA = \sqrt{\frac{10\lambda^2}{900}}$

Therefore $BP : PA = 3 : 1$

18.

Given lines are

$4ax + 2ay + c = 0$

$5bx + 2by + d = 0$

The point of intersection will be

$$\frac{x}{2ad - 2bc} = \frac{-y}{4ad - 5bc} = \frac{1}{8ab - 10ab}$$

$\Rightarrow x = \frac{2(ad - bc)}{-2ab} = \frac{bc - ad}{ab}$

$\Rightarrow y = \frac{5bc - 4ad}{-2ab} = \frac{4ad - 5bc}{2ab}$

$\because$ Point of intersection is in fourth quadrant so x is positive and y is negative.

Also distance from axes is same

So $x = -y$

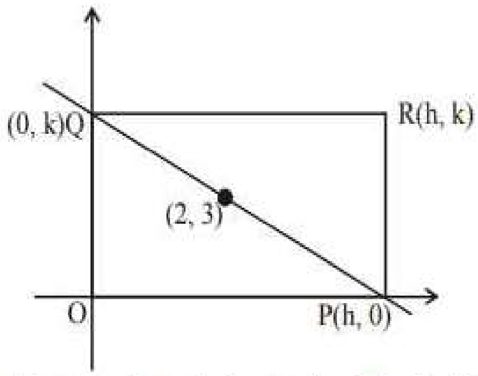
($\because$ distance from x-axis is $-y$ as y is negative)

$\frac{bc - ad}{ab} = \frac{5bc - 4ad}{2ab} \Rightarrow 3bc - 2ad = 0$

19.

Equation of PQ is

$\frac{x}{h} + \frac{y}{k} = 1$ ...(i)



Since, (i) passes through the fixed point (2, 3) Then,

$\frac{2}{h} + \frac{3}{k} = 1$

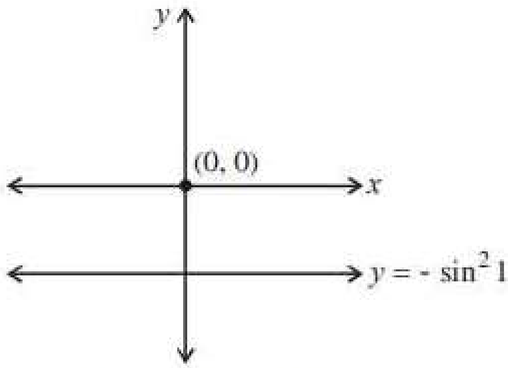
Then, the locus of R is $\frac{2}{x} + \frac{3}{y} = 1$ or $3x + 2y = xy$.

20.

$y = \sin x . \sin(x+2) - \sin^2(x+1)$

$= \frac{1}{2}\cos(-2) - \frac{\cos(2x+2)}{2} - \left[\frac{1-\cos(2x+2)}{2}\right]$

$= \frac{(\cos 2) - 1}{2} = -\sin^2 1$



By the graph y lies in III and IV quadrant.

PAIR OF STRAIGHT LINES

SYNOPSIS

Homogeneous equations :

Combined Equation of a Pair of Straight lines :

- i)If $L_1 = 0, L_2 = 0$ are any two lines, then the combined equation of $L_1 = 0, L_2 = 0$ is $L_1 L_2 = 0$

  ii) Any second degree equation in x and y represents a pair of straight lines if the expression on the left hand side can be expressed as a product of two linear factors in x and y.

Separate equations of pair of lines :

- The equations of the separate lines of $ax^2 + 2hxy + by^2 = 0$ are

  $ax + \left(h + \sqrt{h^2 - ab}\right)y = 0$,

  $ax + \left(h - \sqrt{h^2 - ab}\right)y = 0$

Nature of pair of lines :

- The second degree homogeneous equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines passing through the origin and it represents

  (i) two real and distinct lines if $h^2 > ab$

  (ii) two coincident lines if $h^2 = ab$

  (iii) Imaginary lines if $h^2 < ab$

Slopes of pair of lines :

- i) If $y = m_1 x, y = m_2 x$ are the two lines represented by the pair of lines $ax^2 + 2hxy + by^2 = 0$, $b \neq 0$ with slopes m_1 and m_2 then

  a) The slopes of the lines are the roots of the quadratic equation

  $bm^2 + 2hm + a = 0$

  b) $m_1 + m_2 = \frac{-2h}{b}; m_1 m_2 = \frac{a}{b}; |m_1 - m_2| = \frac{2\sqrt{h^2 - ab}}{|b|}$

  c) The combined equation of pair of lines with slopes m_1, m_2 is

  $(y - m_1 x)(y - m_2 x) = 0$

  $\Rightarrow y^2 - (m_1 + m_2)xy + m_1 m_2 x^2 = 0$

  ii) The slopes of the straight lines represented by $ax^2 + 2hxy + by^2 = 0$ are reciprocal to each other if $a = b$

  iii) If the slopes of two lines represented by $ax^2 + 2hxy + by^2 = 0$ are in the ratio $l : m$ then $(l + m)^2 ab = 4h^2 lm$

  iv) If the slope of one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ is k times the slope of other line then $4kh^2 = (k + 1)^2 ab$

  v) $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines if the slope of one line is the n[th] power of the other then $(ab^n)^{1/n+1} + (a^n b)^{1/n+1} + 2h = 0$

  vi) If the slope of one line of pair of lines $ax^2 + 2hxy + by^2 = 0$ is square of the slope of the other line then $ab(a + b - 6h) + 8h^3 = 0$

Angle between the pair of lines :

- If θ is an acute angle between the pair of lines $ax^2 + 2hxy + by^2 = 0$ then

  $$\cos\theta = \frac{|a + b|}{\sqrt{(a - b)^2 + 4h^2}}$$ or

  $$\sin\theta = \frac{2\sqrt{h^2 - ab}}{\sqrt{(a - b)^2 + 4h^2}}$$ or

  $$\tan\theta = \frac{2\sqrt{h^2 - ab}}{|a + b|}\ ;\ a + b \neq 0$$

i)The lines represented by $ax^2+2hxy+by^2=0$ are perpendicular,
if $a+b=0$.

i.e., coefficient of x^2 + coefficient of $y^2=0$

Pair of parallel & perpendicular lines :

✈ i) The equation to the pair of lines passing through the point (x_1, y_1) and parallel to the pair of straight lines $ax^2+2hxy+by^2=0$ is

$$a(x-x_1)^2+2h(x-x_1)(y-y_1)+b(y-y_1)^2=0$$

ii) The equation to the pair of lines passing through the origin and perpendicular to

$ax^2+2hxy+by^2=0$ is

$$bx^2-2hxy+ay^2=0$$

iii) The equation to the pair of lines passing through the point (x_1, y_1) and perpendicular to the pair of straight lines $ax^2+2hxy+by^2=0$ is

$$b(x-x_1)^2-2h(x-x_1)(y-y_1)+a(y-y_1)^2=0$$

Common line to pair of lines :

✈ i) If the pairs of lines $a_1x^2+2h_1xy+b_1y^2=0$, $a_2x^2+2h_2xy+b_2y^2=0$ have one line in common then

$$\begin{vmatrix} a_1 & 2h_1 \\ a_2 & 2h_2 \end{vmatrix} . \begin{vmatrix} 2h_1 & b_1 \\ 2h_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2$$ **(or)**

$$(a_1b_2-a_2b_1)^2+4(h_1a_2-h_2a_1)(h_1b_2-h_2b_1)=0$$

ii) If one of the lines represented by $a_1x^2+2h_1xy+b_1y^2=0$ is perpendicular to one of the lines represented by $a_2x^2+2h_2xy+b_2y^2=0$ then

$$\begin{vmatrix} a_1 & 2h_1 \\ b_2 & -2h_2 \end{vmatrix} . \begin{vmatrix} 2h_1 & b_1 \\ -2h_2 & a_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ b_2 & a_2 \end{vmatrix}^2$$ **(or)**

$$(a_1a_2-b_1b_2)^2+4(h_1a_2+h_2b_1)(h_1b_2+h_2a_1)=0$$

iii) If the pair of lines $a_1x^2+2h_1xy+b_1y^2=0$ and $a_2x^2+2h_2xy+b_2y^2=0$ are such that they have one line in common and the remaining lines are perpendicular then

$$h_1\left(\frac{1}{a_1}-\frac{1}{b_1}\right)=h_2\left(\frac{1}{a_2}-\frac{1}{b_2}\right)$$

Types of triangles :

✈ i) The equation of the pair of lines passing through the origin and forming an isosceles triangle with the line $ax+by+c=0$ is

$$(ax+by)^2-k(bx-ay)^2=0.$$

(a) If $k=1$ then the triangle is right angled isosceles.

(b) If $k=3$ then the triangle is equilateral.

(c) If $k=\frac{1}{3}$ then the triangle is an isosceles and obtuse angled

ii) The triangle formed by the pair of lines $S\equiv ax^2+2hxy+by^2=0$ and the line $lx+my+n=0$ is

a) equilateral if $ax^2+2hxy+by^2=(lx+my)^2-3(mx-ly)^2$

b) Isosceles if $h(l^2-m^2)=(a-b)lm$

c) Right angled if $a+b=0$ or $S(l,m)=0$

Centres related with triangles :

✈ i) If (α,β) is the centroid of the triangle whose sides are $ax^2+2hxy+by^2=0$ and $lx+my+n=0$, then

$$\frac{\alpha}{bl-hm}=\frac{\beta}{am-hl}=\frac{-2n}{3(bl^2-2hlm+am^2)}$$ **(or)**

$$\frac{\alpha}{\left(\frac{\partial F}{\partial x}\right)_{(l,m)}}=\frac{\beta}{\left(\frac{\partial F}{\partial y}\right)_{(l,m)}}=\frac{-n}{3F_{(l,m)}}$$

where $F=bx^2-2hxy+ay^2$

ii) The pair of lines $S\equiv ax^2+2hxy+by^2=0$ represents two sides of a triangle and (x_1, y_1) is the mid point of the third side then the equation of third side is $S_1=S_{11}$ i.e.,

$$axx_1+h(xy_1+x_1y)+byy_1=ax_1^2+2hx_1y_1+by_1^2$$

iii) If $ax^2+2hxy+by^2=0$ represents two sides of a triangle, $G(x_1,y_1)$ be its centroid then the mid point of the third side of the triangle is $\frac{3}{2}G$ *i.e.*, $\left(\frac{3x_1}{2},\frac{3y_1}{2}\right)$

iv) If (kl,km) is the orthocentre of the triangle formed by the lines $ax^2+2hxy+by^2=0$ and $lx+my+n=0$ then $k=\frac{-n(a+b)}{am^2-2hlm+bl^2}$

v) The distance from the origin to the orthocentre of the triangle formed by the lines $\frac{x}{\alpha}+\frac{y}{\beta}=1$ and $ax^2+2hxy+by^2=0$ is

$$\sqrt{\alpha^2+\beta^2}\left|\frac{(a+b)\alpha\beta}{a\alpha^2-2h\alpha\beta+b\beta^2}\right|$$

vi) If $ax^2+2hxy+by^2=0$ represents two sides of a triangle for which (c,d) is the orthocentre, then the equation of the third side of triangle is $(a+b)(cx+dy)=ad^2-2hcd+bc^2$

Product of perpendiculars :

✈ i) The product of the perpendiculars from (α,β) to the pair of lines

$ax^2+2hxy+by^2=0$ is $\frac{\left|a\alpha^2+2h\alpha\beta+b\beta^2\right|}{\sqrt{(a-b)^2+4h^2}}$

Area of the triangle :

✈ i) The area of the triangle formed by the line $lx+my+n=0$ and the pair of lines

$ax^2+2hxy+by^2=0$ is $\frac{n^2\sqrt{h^2-ab}}{\left|am^2-2hlm+bl^2\right|}$

ii) The equation of the pair of lines through the origin and making an angle 'α' with the line $lx+my+n=0$ is

$(lx+my)^2-\tan^2\alpha(mx-ly)^2=0$ and

the area of the triangle is $\frac{n^2}{\tan\alpha\left(l^2+m^2\right)}$

iii) The area of an equilateral triangle formed by the line $ax+by+c=0$ with the pair of lines

$(ax+by)^2-3(bx-ay)^2=0$ is $\frac{c^2}{\sqrt{3}\left(a^2+b^2\right)}$

$=\frac{p^2}{\sqrt{3}}$ where p is the perpendicular distance from the origin to the line $ax+by+c=0$

Pair of angular bisectors :

✈ i) The equation to the pair of bisectors of the angles between the pair of straight lines

$ax^2+2hxy+by^2=0$ is $h\left(x^2-y^2\right)=(a-b)xy$

ii) The angle between pair of angular bisectors of any pair of lines is $\frac{\pi}{2}$.

iii) The equation to the pair of bisectors of the co-ordinate axes is $x^2-y^2=0$

iv) If one of the line in $ax^2+2hxy+by^2=0$ bisects the angle between the coordinate axes then $(a+b)^2=4h^2$

Equally inclined with a line :

✈ i) A pair of lines $L_1L_2=0$ is said to be equally inclined to a line $L=0$ if the lines $L_1=0, L_2=0$ subtend the same angle with the line $L=0$

ii) Every pair of lines is equally inclined to either of its angular bisectors

iii) A pair of lines is equally inclined to a line $L=0$, if $L=0$ is parallel to one of the angular bisectors.

iv) Given pair of lines through origin is equally inclined to the coordinate axes $\Leftrightarrow$ the pair of angular bisectors of given pair of lines through origin is the coordinate axes

v) If the pair of lines $ax^2+2hxy+by^2=0$ equally inclined to the coordinate axes then $h=0$ and $ab<0$

vi) The pair of lines $L_1L_2=0$ bisects the angle between the pair of lines $L_3L_4=0 \Leftrightarrow$ pair of angular bisectors of $L_3L_4=0$ and pair of lines $L_1L_2=0$ represents the same equation

vii) Two pairs of lines $L_1L_2=0$, $L_3L_4=0$ are such that each bisects the angle between the other pair $\Leftrightarrow$ pair of angular bisector of $L_1L_2=0$, pair of lines $L_3L_4=0$ represents same and vice versa.

viii) Two pairs of lines are equally inclined to each other $\Leftrightarrow$ two pairs of lines have same pair of angular bisectors

Non homogeneous equations :

Condition for pair of lines :

✈ i) If the equation

$S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$

represents a pair of lines then

a) $\Delta\equiv abc+2fgh-af^2-bg^2-ch^2=0$

i.e $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$

b) $h^2\geq ab,\ g^2\geq ac,\ f^2\geq bc$

Angle between the pair of lines :

✈ i) The angle between the pair of lines

$ax^2+2hxy+by^2+2gx+2fy+c=0$ is

same as the angle between the pair of lines

$ax^2+2hxy+by^2=0$

Distance between the pair of lines :

✈ The equation

$ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of parallel lines iff

$\Delta=0, f^2\geq bc, g^2\geq ac, h^2=ab \text{ and } af^2=bg^2$ or

$\frac{a}{h}=\frac{h}{b}=\frac{g}{f}$ and the distance between the parallel lines is $2\sqrt{\frac{g^2-ac}{a(a+b)}}$ or $2\sqrt{\frac{f^2-bc}{b(a+b)}}$

Product of perpendiculars :

✈ i) The product of the perpendiculars drawn from (α,β) to the pair of lines

$ax^2+2hxy+by^2+2gx+2fy+c=0$ is

$$\frac{|a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c|}{\sqrt{(a-b)^2+4h^2}}$$

ii) The product of the perpendiculars from origin to the pair of lines

$ax^2+2hxy+by^2+2gx+2fy+c=0$ is

$$\frac{|c|}{\sqrt{(a-b)^2+4h^2}}$$

iii) If the pair of lines

$ax^2+2hxy+by^2+2gx+2fy+c=0$ are equidistant from the origin then

$f^4-g^4=c(bf^2-ag^2)$

Point of intersection of pair of lines :

✈ i) If $S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of lines and $h^2>ab$ then the point of intersection of the lines is

$\left(\frac{hf-bg}{ab-h^2},\frac{gh-af}{ab-h^2}\right)$ i.e., obtained by solving

$\frac{\partial s}{\partial x}=0$ and $\frac{\partial s}{\partial y}=0$

ii) If the pair of lines

$ax^2+2hxy+by^2+2gx+2fy+c=0$ intersect at (α,β) then (α,β) satisfy the equations

$ax+hy+g=0$, $hx+by+f=0$ and

$gx+fy+c=0$

i.e., $(\alpha,\beta)=\left(\frac{hf-bg}{ab-h^2},\frac{gh-af}{ab-h^2}\right)$

$=\left(\frac{bc-f^2}{hf-bg},\frac{fg-ch}{hf-bg}\right)=\left(\frac{hc-gf}{af-gh},\frac{g^2-ac}{af-gh}\right)$

iii) The coordinates of the point of intersection of the lines represented by $S=0$ is

$\left(\sqrt{\frac{f^2-bc}{h^2-ab}},\sqrt{\frac{g^2-ac}{h^2-ab}}\right)$

iv) If the equation

$ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of intersecting lines, then the square of the distance of their point of intersection from the origin is $\frac{c(a+b)-f^2-g^2}{ab-h^2}$

v) If the equation

$ax^2+2hxy+by^2+2gx+2fy+c=0$

represents a pair of perpendicular lines, then the square of the distance of their point of intersection from the origin is $\frac{f^2+g^2}{h^2-ab}$ (or) $\frac{f^2+g^2}{a^2+h^2}$ (or) $\frac{f^2+g^2}{b^2+h^2}$

Area of the triangle :

✈ The area of the triangle formed by the line $lx+my+n=0$ and the pair of lines $ax^2+2hxy+by^2+2gx+2fy+c=0$ whose point of intersection is (x_1,y_1) is

$\frac{(lx_1+my_1+n)^2\sqrt{h^2-ab}}{|am^2-2hlm+bl^2|}$ sq. units

Intercepts of a pair of lines on coordinate axes :

✈ i) Length of the intercept made by the pair of lines represented by

$ax^2+2hxy+by^2+2gx+2fy+c=0$ on

a) $x-axis\ is\ \frac{2\sqrt{g^2-ac}}{|a|}$

b) $y-axis\ is\ \frac{2\sqrt{f^2-bc}}{|b|}$

ii) If the pair of lines

$ax^2+2hxy+by^2+2gx+2fy+c=0$

intersect on

a) x-axis, then $g^2=ac$ and

$2fgh=af^2+ch^2$

b) y-axis, then $f^2=bc$ and

$2fgh=bg^2+ch^2$

Pair of angular bisectors :

✈ If (α,β) be the point of intersection of the pair of lines

$ax^2+2hxy+by^2+2gx+2fy+c=0$ then the equation to the pair of angular bisectors is

$h\left[(x-\alpha)^2-(y-\beta)^2\right]=(a-b)(x-\alpha)(y-\beta)$

Quadrilateral formed by S = 0 and S¹ = 0 :

✈ i) The pair of lines $S\equiv ax^2+2hxy+by^2=0$ and $S^1\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ forms a

a) rhombus

$\Leftrightarrow a+b\neq 0,(a-b)fg+h(f^2-g^2)=0$

b) square

$\Leftrightarrow a+b=0,(a-b)fg+h(f^2-g^2)=0$

c) rectangle

$\Leftrightarrow a+b=0,(a-b)fg+h(f^2-g^2)\neq 0$

d) parallelogram

$\Leftrightarrow a+b\neq 0,(a-b)fg+h(f^2-g^2)\neq 0$

e) Area of the parallelogram is $\frac{|c|}{2\sqrt{h^2-ab}}$

f) Equation of diagonal not passing through origin is $2gx+2fy+c=0$ i.e., $S^1-S=0$

g) Equation of diagonal passing through origin is $(hf-bg)y=(gh-af)x$

ii) If $ax^2+2hxy+by^2=0$ are two sides of a parallelogram and $lx+my+n=0$ is one of the diagonals of the parallelogram then the equation of other diagonal is

$(bl-hm)y=(am-hl)x$

iii) Given (x_1,y_1) as opposite vertex of a parallelogram with $S\equiv ax^2+2hxy+by^2=0$ as one pair of sides then the equation of the diagonal not passing through the orgin is $2S_1-S_{11}=0$

iv) The pair of lines $xy+ax+by+ab=0$, $xy+cx+dy+cd=0$ form a square

a) If $|a-c|=|b-d|$

b) area is $|a-c||b-d|$

c) point of intersection of diagonals is $\left(\frac{-(b+d)}{2},\frac{-(a+c)}{2}\right)$

d) Equation of diagonals are

$(a-c)x+(b-d)y+ab-cd=0$

$(a-c)x-(b-d)y+ad-bc=0$

Homogenisation :

✈ i) The Combined equation to the pair of lines joining the origin to the points of intersection of the curve

$ax^2+2hxy+by^2 \ +2gx+2fy+c=0$

and the line $lx+my+n=0$ is $ax^2+2hxy+by^2+$

$(2gx+2fy)\left(\frac{lx+my}{-n}\right)+c\left(\frac{lx+my}{-n}\right)^2=0$

ii) The condition that the pair of lines joining the origin to the points of intersection of

$ax^2+2hxy+by^2+2gx+2fy+c=0$ and

$lx+my+n=0$ to be perpendicular is

$n^2(a+b)-2n(lg+mf)+c(l^2+m^2)=0$

Some standard results :

✈ i) The equation to the pair of lines passing through the origin and each is at a distance of d from (α,β) is $(\beta x-\alpha y)^2=d^2(x^2+y^2)$.

ii) If L,M are the feet of the perpendiculars from $(c,0)$ to the lines $ax^2+2hxy+by^2=0$ then the angle made by LM with positive X-axis is $Tan^{-1}\left(\frac{b-a}{2h}\right)$ and the equation of LM is

$(b-a)x-2hy-bc=0$

iii) Point of intersection of diagonals of a rect angle formed by the pairs

$a_1x^2+b_1x+c_1=0$,

$a_2y^2+b_2y+c_2=0$ is $\left(\frac{-b_1}{2a_1},\frac{-b_2}{2a_2}\right)$

iv) The image of pair of lines $f(x,y)=0$ with respect to x-axis is $f(x,-y)=0$ and with re spect y-axis is . $f(-x,y)=0$

EXERCISE - I

1. The range of 'a' so that $a^2x^2+2xy+4y^2=0$ **represents distinct lines**

1) $a>\frac{1}{2}$ or $a<\frac{-1}{2}$ 2) $\frac{-1}{2}\le a\le\frac{1}{2}$

3) $\frac{-1}{2}<a<\frac{1}{2}$ 4) $a\ge\frac{1}{2}$ or $a\le\frac{-1}{2}$

2. The difference of the slopes of the lines represented by

$x^2(\sec^2\theta-\sin^2\theta)-(2\tan\theta)xy+y^2\sin^2\theta=0$

1) 1 2) 2 3) 3 4) 4

3. If the slopes of the lines represented by $ax^2+2hxy+by^2=0$ are in the ratio 3 : 2, then

1) $25ab=24h^2$ 2) $8h^2=9ab$

3) $16h^2=25ab$ 4) $h^2=ab$

4. The combined equation to a pair of straight lines passing through the origin and inclined at an angles 30^0 and 60^0 respectively with X-axis is

1) $\sqrt{3}(x^2+y^2)=4xy$

2) $4(x^2+y^2)=\sqrt{3}xy$

3) $x^2+\sqrt{3}y^2-2xy=0$

4) $x^2+3y^2-2xy=0$

5. If the slope of one line is twice the slope of the other in the pair of straight lines $ax^2+2hxy+by^2=0$ **then $8h^2$ =**

1) 7 ab 2) -7ab 3) 9 ab 4) -9ab

6. The equation of the pair of lines passing through the origin whose sum and product of slopes are respectively the arthemetic mean and geometric mean of 4 and 9 is

(EAM-2016)

1) $12x^2-13xy+2y^2=0$

2) $12x^2+13xy+2y^2=0$

3) $12x^2-15xy+2y^2=0$

4) $12x^2+15xy-2y^2=0$

7. **If the sum of the slopes of the lines given by $x^2+2cxy+y^2=0$ is eight times their product, then c has the value**

1) 1 2) -1 3) -4 4) -2

8. **If the pair of lines given by $\left(x^2+y^2\right)\sin^2\alpha=\left(x\,Cos\,\alpha-y\,Sin\,\alpha\right)^2$ are perpendicular to each other then $\alpha=$**

(EAM-2018) 1) $\pi/2$ 2) 0
3) $\pi/4$ 4) $\pi/3$

9. **If the angle 2θ is acute, then the acute angle between the pair of straight lines $x^2(\cos\theta-\sin\theta)+2xy\cos\theta+y^2(\cos\theta+\sin\theta)=0$ is**

(EAM - 2002)

1) 2θ 2) $\frac{\theta}{2}$ 3) $\frac{\theta}{3}$ 4) θ

10. **The equation to the pair of lines passing through the point $(-2,3)$ and parallel to the pair of lines $x^2+4xy+y^2=0$ is**

1) $x^2-4xy+y^2-8x+2y-11=0$

2) $x^2+4xy+y^2-8x+2y-11=0$

3) $x^2+4xy-y^2-8x+2y-11=0$

4) $x^2-4xy+y^2-8x-2y-11=0$

11. **The equation to the pair of lines passing through the origin and perpendicular to $5x^2+3xy=0$** **[EAM -2017]**

1) $5xy+3y^2=0$ 2) $x^2-2y^2=0$

3) $3xy-5y^2=0$ 4) $3x^2-2xy=0$

12. **If the product of perpendiculars from (k, k) to the pair of lines $x^2+4xy+3y^2=0$ is $4/\sqrt{5}$ then k is**

1) ± 4 2) ± 3 3) ± 2 4) ± 1

13. **If the area of the triangle formed by the lines $3x^2-2xy-8y^2=0$ and the line 2x+y-k=0 is 5sq. units, then k =**

1) 5 2) 6 3) 7 4) 8

14. **If the sides of a triangle are $ax^2+2hxy+by^2=0$ and y = x+c, then its area is**

1) $\frac{c^2\sqrt{h^2-ab}}{|a+b+2h|}$ 2) $\left|\frac{c\sqrt{h^2-ab}}{a+b+2h}\right|$

3) $\frac{\sqrt{h^2-ab}}{|a+b+c|}$ 4) $\frac{\sqrt{h^2-ab}}{|a+b+2h|}$

15. **If the equation of the pair of bisectors of the angle between the pair of lines $3x^2+xy+by^2=0$ is $x^2-14xy-y^2=0$ then b =**

1) 4 2) -4 3) 8 4) -8

16. **If the lines $x^2+(2+k)xy-4y^2=0$ are equally inclined to the coordinate axes, then k =**

1) -1 2) -2 3) -3 4) -4

17. **If the pair of straight lines $x^2-2pxy-y^2=0$ and $x^2-2qxy-y^2=0$ be such that each pair bisects the angle between the other pair, then**

1) $pq=-1$ 2) $p=q$

3) $p=-q$ 4) $pq=1$

18. **If one of the lines in the pair of stright lines given by $4x^2+6xy+ky^2=0$ bisects the angle between the coordinate axes, then k $\in$**

1){-2,-10} 2){-2,10} 3){-10,2} 4){2,10}

19. **If $x^2-y^2=0$, $lx+2y=1$ form an isosceles triangle then $l=$**

1) 1 2) 2 3) 3 4) 0

20. $x^2+k_1y^2+2k_2y=a^2$ **represents a pair of perpendicular lines if**

1) $k_1=1,\ k_2=a$ 2) $k_1=1,\ k_2=-a$

3) $k_1=-1,\ k_2=-a$ 4) $k_1=-1, k_2=a^2$

21. **If $kx^2+10xy+3y^2-15x-21y+18=0$ represents a pair of straight lines then k =**

1) 3 2) 4 3) -3 4) 5

22. **If $x^2+\alpha y^2+2\beta y=a^2$ represents a pair of perpendicular lines, then $\beta=$ (EAM-2014)**
1) 2a 2) 3a 3) 4a 4) a

23. **The equation $x^2-5xy+py^2+3x-8y+2=0$ represents a pair of straight lines. If θ is the angle between them, then $\sin\theta=$ (EAM-2013)**
1) $\frac{1}{\sqrt{50}}$ 2) $\frac{1}{7}$ 3) $\frac{1}{5}$ 4) $\frac{1}{\sqrt{10}}$

24. **If the angle between the lines respresented by $2x^2+5xy+3y^2+6x+7y+4=0$ is $\tan^{-1}(m)$ and $a^2+b^2-ab-a-b+1\le 0$, then $2a+3b=$**
1) $1/m$ 2) m 3) $-m$ 4) m^2

25. **If $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of parallel lines then $\sqrt{\frac{g^2-ac}{f^2-bc}}=$ (EAM-2011)**
1) $\frac{a}{b}$ 2) $\sqrt{\frac{a}{b}}$ 3) $\sqrt{\frac{b}{a}}$ 4) $\frac{b}{a}$

26. **The square of the distance of the point of intersection of the lines $6x^2-5xy-6y^2+x+5y-1=0$ from the origin is**
1) 74/169 2) 85/169 3) 74/185 4) 2/13

27. **If the lines represented by $ax^2+4xy+y^2+8x+2fy+c=0$ intersect on Y-axis, then (f, c) =**
1) (2, 4) 2) (4, 2) 3) (-2, -4) 4) (-4, -2)

28. **If the adjacent sides of a parallelogram are $2x^2-5xy+3y^2=0$ and one diagonal is x+y+2=0 then the other diagonal is**
1) 9x-11y=0 2) 9x+11y=0
3) 11x-9y=0 4) 11x+9y=0

29. **The area of the square formed by the lines $6x^2-5xy-6y^2=0$ and $6x^2-5xy-6y^2+x+5y-1=0$ in sq. units is**
1) $1/\sqrt{3}$ 2) $4/\sqrt{13}$ 3) $\sqrt{13}$ 4) $1/13$

30. **If the lines $x^2+2xy-35y^2-4x+44y-12=0$ and $5x+\lambda y-8=0$ are concurrent, then the value of λ is (EAM-2007)**
1) 0 2) 1 3) -1 4) 2

31. **The length of the side of the square formed by the lines $2x^2+3xy-2y^2=0$ and $2x^2+3xy-2y^2+3x+y+1=0$ is**
1) $\frac{1}{\sqrt{3}}$ 2) $\frac{1}{\sqrt{5}}$ 3) $\frac{1}{\sqrt{7}}$ 4) $\frac{1}{\sqrt{10}}$

32. **The angle between the pair of straight lines formed by joining the points of intersection of $x^2+y^2=4$ and $y=3x+c$ to the origin is a right angle. Then c^2 is equal to (EAM-2007)**
1) 20 2) 13 3) 1/5 4) 5

33. **The angle between the lines joining the origin to the points of intersection of the lines $\sqrt{3}x+y=2$ and the curve $x^2+y^2=4$ is**
1) $\frac{\pi}{6}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{3}$ 4) $\frac{\pi}{2}$

34. **The triangle formed by the pair of lines $3x^2+48xy+23y^2=0$ and the line $3x-2y+4=0$ is**
1) Equilateral 2) Isosceles
3) Right angled 4) Scalane

35. **If the pair of straight lines $Ax^2+2Hxy+By^2=0$ $(H^2>AB)$ forms an equilateral triangle with the line $ax+by+c=0$ then $(A+3B)(3A+B)=$**
1) H^2 2) $-H^2$ 3) $2H^2$ 4) $4H^2$

36. **The equation of the line common to the pair of lines $m^2x^2-(m^2+1)xy+y^2=0$ and $mx^2-(m+1)xy+y^2=0$ is**
1) mx-y=0 2) x+y=0
3) x-y=0 4) x+y=m

37. **If the pair of lines given by**

$ax^2+2hxy+by^2+2gx+2fy+c=0$

intersect on X-axis then

1) b, f, c are in A.P 2) a, f, c are in G.P
3) a, g, c are in G.P 4) a, g, c are in A.P

38. **The rectangle formed by the pair of lines** $2hxy+2gx+2fy+c=0$ **with the coordinate axes has the area equal to**

1) $\frac{|fg|}{h^2}$ 2) $\frac{|gh|}{f^2}$ 3) $\frac{|hf|}{g^2}$ 4) $\frac{|fg|}{h}$

39. **If the pair of lines *2hxy+2gx+2fy+c=0* and the coordinate axes form a rectangle, then the equations of its diagonals are**

1) *2gx+2fy+c=0, gx-fy=0*
2) *2gx+2fy-c=0, gx+fy=0*
3) *gx+fy-c=0, gx–fy=0*
4) *gx+fy=0, gx–fy=0*

40. **The lines $ax^2+2hxy+by^2+2gx+2fy+c=0$ intersect x-axis in A, B and y-axis in C, D respectively. Then the combined equation of** $\overleftrightarrow{AB}$ *and* $\overleftrightarrow{CD}$ **is**

1) $xy=0$
2) $ax^2+2hxy+by^2+2gx+2fy+c=0$
3) $ax^2+2hxy+by^2+2gx+2fy+c+\frac{4gf}{c}xy=0$
4) $ax^2-2hxy+by^2+2gx+2fy+c+\frac{4gf}{c}xy=0$

41. **The locus represented by the equation** $(x-y+c)^2+(x+y-c)^2=0$ **is**

1) A line parallel to x-axis
2) A point
3) Pair of lines
4) Line parallel to y-axis

KEY

01) 3	02) 2	03) 1	04) 1	05) 3	06) 1
07) 3	08) 3	09) 4	10) 2	11) 3	12) 4
13) 1	14) 1	15) 2	16) 2	17) 1	18) 3
19) 4	20) 3	21) 1	22) 4	23) 1	24) 1
25) 2	26) 4	27) 1	28) 1	29) 4	30) 4
31) 2	32) 1	33) 3	34) 1	35) 4	36) 3
37) 3	38) 1	39) 1	40) 1	41) 4	

SOLUTIONS

1. $h^2-ab>0 \Rightarrow 1-4a^2>0 \Rightarrow a^2-\frac{1}{4}<0$

2. Given $x^2\left(\sec^2\theta-\sin^2\theta\right)-2\tan\theta xy+y^2\sin^2\theta=0$ put $\theta=45^0$

$\Rightarrow x^2\left(2-\frac{1}{2}\right)-2xy+\frac{y^2}{2}=0$

$3x^2-4xy+y^2=0$, $m_1+m_2=\frac{4}{1}, m_1m_2=\frac{3}{1}$

$|m_1-m_2|=\frac{2\sqrt{h^2-ab}}{|b|}=\frac{2\sqrt{4-3}}{1}=2$

3. $l:m=3:2$ *and* $\frac{(l+m)^2}{4lm}=\frac{h^2}{ab}$

4. Given $\theta_1=30^0$ $\theta_2=60^0$ $m_1=\tan\theta_1=\frac{1}{\sqrt{3}}$, $m_2=\tan\theta_2=\sqrt{3}$

the combined equation of pair of lines

$y^2-\left(\frac{1}{\sqrt{3}}+\sqrt{3}\right)xy+1\left(x^2\right)=0$

$\sqrt{3}x^2-4xy+\sqrt{3}y^2=0$

5. $4kh^2=(k+1)^2ab$ & $k=2$

6. Given $m_1+m_2=\frac{4+9}{2}=\frac{13}{2}$, $m_1m_2=\sqrt{4\times 9}=6$

combined equation

$y^2-(m_1+m_2)xy+m_1m_2x^2=0$

$12x^2-13xy+2y^2=0$

7. $x^2+2cxy+y^2=0$

Here $m_1+m_2=8m_1m_2$, c = -2

8. Given pair of lines $\left(x^2+y^2\right)\sin^2\alpha=(x\cos\alpha-y\sin\alpha)^2$ are perpendicular

$x^2\sin^2\alpha+y^2\sin^2\alpha=x^2\cos^2\alpha+y^2\sin^2\alpha-2xy\sin\alpha\cos\alpha$

$x^2\left(\sin^2\alpha-\cos^2\alpha\right)+2xy\sin\alpha\cos\alpha=0$,

$\sin^2\alpha-\cos^2\alpha=0$, $\sin^2\alpha=\cos^2\alpha$

$\alpha=\frac{\pi}{4}$

9. Given $x^2(\cos\theta-\sin\theta)+2xy\cos\theta+y^2(\cos\theta+\sin\theta)=0$

$a=\cos\theta-\sin\theta, h=\cos\theta, b=\cos\theta+\sin\theta$

$$\cos=\frac{|a+b|}{\sqrt{(a-b)^2+h^2}}=\frac{2\cos\theta}{\sqrt{4\sin^2\theta+4\cos^2\theta}}=\cos\theta$$

$\alpha=\theta$

10. $a(x-x_1)^2+2h(x-x_1)(y-y_1)+b(y-y_1)^2=0$

11. $bx^2-2hxy+ay^2=0$

12. $\frac{|a\alpha^2+2h\alpha\beta+b\beta^2|}{\sqrt{(a-b)^2+4h^2}}$

13. Given pari of line

Area = $\frac{k^2\sqrt{1+24}}{|3(1)+4-32|}=5 \Rightarrow k^2=25$

k = 5

14. $\frac{n^2\sqrt{h^2-ab}}{|am^2-2hlm+bl^2|}$

15. $h(x^2-y^2)=(a-b)xy$

16. $h=0$

17. Equation of the bisector of the angles between $x^2-2pxy-y^2=0$ is

$px^2+2xy-py^2=0$ this same as

$x^2-2qxy-y^2=0 \therefore \frac{p}{1}=\frac{1}{-q}=\frac{p}{1}\Rightarrow pq=-1$

18. $(a+b)^2=4h^2$

19. $h(l^2-m^2)=(a-b)lm$

20. Given pair of lines $x^2+k_1y^2+2k_2y=a^2$ represents a pair of perpendicular lines

$a+b=0$ and

$\Delta=abc+2fgh-af^2-bg^2-ch^2=0$

$1+k_1=0$ $a^2+0-k_2^2-0-0, k_2=-a$

21. $\Delta=0$

22. $a+b=0$ *and* $\Delta=0$

23. Given equation $x^2-5xy+py^2+3x-8y+2=0$ represents a pair of perpendicular lines

$$\sin\theta=\frac{2\sqrt{\frac{25}{4}-16}}{\sqrt{25+4.\left(\frac{25}{4}\right)}}=\frac{1}{\sqrt{50}}$$

$8p+120-64-9p-50=0$, p = 6

24. Here $\tan\theta=\left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$

$\therefore \theta=\tan^{-1}\left(\frac{1}{5}\right)=\tan^{-1}(m)$

$\therefore m=\frac{1}{5}$ from the given condition we get $a=1, b=1$

25. $2\sqrt{\frac{g^2-ac}{a(a+b)}}=2\sqrt{\frac{f^2-bc}{b(a+b)}}$

26. $\frac{f^2+g^2}{a^2+h^2}$

27. $f^2=bc, hf=bg$

28. $(bl-hm)y=(am-hl)x$

29. $\frac{|c|}{2\sqrt{h^2-ab}}$

30. Given lines $x^2+2xy-35y^2-4x+44y-12=0$ and $5x+\lambda y-8=0$ are concurrent

$\frac{\delta f}{\delta x}=2x+2y-4$,

$x+y-2=0$ ______ (1)

$$\frac{\delta f}{\delta y} = 2x - 70y + 44, \; x - 35y + 22 = 0 \_\_\_\_\_\_ (2)$$

solving (1) and (2)

$36y - 24 = 0$,

$y = \frac{2}{3}$ substituing in (1)

$x + \frac{2}{3} - 2 = 0$, $x = \frac{4}{3}$ lies on

$5x + \lambda y - 8 = 0$

$$\frac{20}{3} + \frac{2\lambda}{3} - 8 = 0$$

$2\lambda = 4$

$\lambda = 2$ P is the point of intersection of $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$ and p lies on the line

31.Given lines $2x^2 + 3xy - 2y^2 = 0$

$$2x^2 + 3xy - 2y^2 + 3x + y + 1 = 0$$

here $a = 2,\ h = \frac{3}{2}\ b = -2,\ g = \frac{3}{2}, f = \frac{1}{2}\ c = 1$

$$a^2 = \frac{1}{2\sqrt{\frac{9}{4} + 4}} \Rightarrow a^2 = \frac{1}{5} \Rightarrow a = \frac{1}{\sqrt{5}}$$

32. $n^2(a+b) - 2n(lg + mf) + c(l^2 + m^2) = 0$

33. Homogenising

$$x^2 + y^2 - \cancel{4}\frac{\left(\sqrt{3}x + y\right)^2}{\cancel{4}} = 0$$

$\Rightarrow 2x^2 + 2\sqrt{3}xy = 0$

34. The pair of lines passing through origin and which forms an equilateral triangle with the given line is

$$(3x - 2y)^2 - 3(2x + 3y)^2 = 0$$

$\Rightarrow 3x^2 + 48xy + 23y^2 = 0$

35. S . O . B . S

$$(A - B)^2 + 4H^2 = 4(A + B)^2 \Rightarrow A^2 + B^2 - 2AB + 4H^2 = 4A^2 + 4B^2 + 8AB$$

$3A^2 + 3B^2 + 10AB = 4H^2$ Now

$(A + 3B)(3A + B) = 4H^2$

36. In both pair of lines sum of the coefficients is zero. Thecommon line is y = x,

37. $g^2 = ac$

38. $\Delta = 0$ and apply $\frac{|c|}{2\sqrt{h^2 - ab}}$ Given pair of tenes $2hxy + 2gx + 2fy + c = 0$

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$0 + 2hfg = ch^2 \Rightarrow 2fg = ch$$

Area $\frac{|c|}{2\sqrt{h^2 - ab}} = \frac{|c|}{2h} = \frac{2fg}{2h^2} = \frac{|fg|}{h^2}$

39. Diagonals are

$$2gx + 2fy + c = 0, (gh - af)x = (hf - bg)y$$

40. Since the given pair intersects x-axis at A, B then the equation of AB is y = 0, Since the pair cuts y-axis at C,D then the equation of CD is x = 0. Then the combined equation of $\overline{AB}$ and $\overline{CD}$ is xy = 0

41. $x - y + c = 0, x + y - c = 0$ solving we get $(0, c)$ which is a point

EXERCISE - II

1. **The triangle formed by the pair of lines $x^2 - 4y^2 = 0$ and the line x-a=0 is always**
   1) Equilateral 2) Isosceles
   3) Right angled 4) Scalene

2. **The triangle formed by x + 3y = 1 and $9x^2 - 12xy + ky^2 = 0$ is right angled triangle and $k \neq -9$. Then k =**
   1) 3 2) 5 3) 7 4) 1

3. **If the pair of lines $2x^2 + 3xy + y^2 = 0$ makes angles θ_1 and θ_2 with X-axis then $\tan(\theta_1 - \theta_2) =$**
   1) 1 2) 1/2 3) 1/3 4) 1/4

4. **If the equation $2x^2 - 5xy + 2y^2 = 0$ represents two sides of an isosceles triangle then the equation of the third side passing through the point (3,3) is**
   1) x+y=3 2) x-y=0
   3) 2x-y=3 4) x + y – 6 = 0

5. **If the equation $x^2+py^2+y=a^2$ represents a pair of perpendicular lines, then the point of intersection of the lines is**
   1) (1, a) 2) (1, -a) 3) (0, a) 4) (0, 2a)

6. **The condition that one of the pair of lines $ax^2 + 2hxy + by^2 = 0$ be coincident with one line of the pair $3x^2 + 12xy + 2y^2 = 0$ and the remaining lines are at right angles, then h (a-b) =** [E AM -2020]
   1) a + b 2) ab 3) 2 ab 4) a / b

7. **If one of the lines represents by $3x^2-4xy+y^2=0$ is perpendicular to one of the line $2x^2-5xy+ky^2=0$ then k =**
   1) -3, 13/9 2) 3, -13/9
   3) -3, -13/94) -7, -33

8. **If the pair of lines $3x^2-5xy+py^2=0$ and $6x^2-xy-5y^2=0$ have one line in common, then p =**
   1) $2, \frac{25}{4}$ 2) $-2, \frac{25}{4}$
   3) $-2, \frac{-25}{4}$ 4) $2, \frac{-25}{4}$

9. **The lines $33y^2 - 136xy + 135x^2 = 0$ are equally inclined to**
   1) x+2y + 7 = 0 2) 2x + y – 7 = 0
   3) x + 2y – 7 = 0 4) x + y = 1

10. **If $ax^2 + 6xy + by^2 - 10x + 10y - 6 = 0$ represents a pair of perpendicular straight lines, then |a| is equal to**
    1) 2 2) 4 3) 1 4) 3

11. **The centroid of the triangle formed by the pair of straight lines $12x^2 - 20xy + 7y^2 = 0$ and the line $2x - 3y + 4 = 0$ is (EAM-2016)**
    1) $\left(-\frac{7}{3}, \frac{7}{3}\right)$ 2) $\left(-\frac{8}{3}, \frac{8}{3}\right)$
    3) $\left(\frac{8}{3}, \frac{8}{3}\right)$ 4) $\left(\frac{4}{3}, \frac{4}{3}\right)$

12. **If $2x^2 - 5xy + 2y^2 = 0$ represents two sides of a triangle whose centroid is (1, 1) then the equation of the third side is**
    1) x+y-3=0 2) x-y-3=0
    3) x+y+3=0 4) x-y+3=0

13. **The orthocentre of the triangle formed by the lines $x^2-3y^2=0$ and the line x=a is**
    1) $\left(\frac{a}{3}, 0\right)$ 2) $\left(\frac{2a}{3}, 0\right)$
    3) (a, 0) 4) $\left(\frac{4a}{3}, 0\right)$

14. **If $x^2+4xy+y^2=0$ represents two sides of ΔOAB and the orthocentre is (-1, -1), then the third side is**
    1) x+y = 2 2) x+y=1
    3) x+y+1=0 4) x+y=3

15. **The circumcentre of the triangle formed by the lines $2x^2 - 3xy-2y^2=0$ and 3x-y=10 is**
    1) (2,1) 2) (1,-2) 3) (3,-6) 4) (3, -1)

16. **The coordinates of the orthocentre of the triangle formed by the lines $2x^2 - 3xy + y^2 = 0$ and x+y=1 are is**
    1) (1, 1) 2) (1/2, 1/2)
    3) (1/3, 1/3) 4) (1/4, 1/4)

17. **If $\frac{x}{a} + \frac{y}{b} = 1$ intersects**

$5x^2+5y^2+5bx+5ay-9ab=0$ **at P and Q, $\angle POQ=\pi/2$ then the relation between a and b is**

1) a = b 2) a = 2b or b = 2a
3) a = 3b or b = 3a 4) a + b = 5

18. If the pair of lines which joins the origin to the point of intersection of

$ax^2+2hxy+by^2+2gx=0$,

$a_1x^2+2h_1xy+b_1y^2+2g_1x=0$ **are at right angles then**

1) $\frac{g}{g_1}=\frac{a_1+b_1}{a+b}$ 2) $\frac{g}{g_1}=\frac{a+b}{a_1+b_1}$

3) $\frac{h}{h_1}=\frac{a+b}{a_1+b_1}$ 4) $\frac{h}{h_1}=\frac{a_1+b_1}{a+b}$

19. The angle between the lines joining the origin to the point of intersection of $lx+my=1$ and $x^2+y^2=a^2$ is [EAM -2018]

1) $\frac{\pi}{2}$ 2) $\frac{\pi}{4}$

3) $\cos^{-1}\left(\frac{1}{a\sqrt{l^2+m^2}}\right)$ 4) $2\cos^{-1}\left(\frac{1}{a\sqrt{l^2+m^2}}\right)$

20. The curve $x^2+y^2+2gx+2fy+c=0$ intercepts on the line $lx+my=1$, a length which subtends a right angle at the origin, then $\frac{lg+mf+1}{l^2+m^2}=$

1) $\frac{c}{2}$ 2) $\frac{-c}{2}$ 3) $\frac{2}{c}$ 4) $\frac{-2}{c}$

21. The lines joining the origin to the points of intersection of $x^2+y^2+2gx+c=0$ and $x^2+y^2+2fy-c=0$ are at right angles is

1) $g^2+f^2=c$ 2) $g^2-f^2=0$

3) $g^2-f^2=2c$ 4) $g^2+f^2=c^2$

22. The line $4y-3x+48=0$ cuts the curve $y^2=64x$ in A and B. If AB subtends an angle θ at the origin, then $\tan\theta=$

1) $\frac{20}{9}$ 2) $\frac{10}{9}$ 3) $\frac{5}{9}$ 4) $\frac{40}{9}$

23. The combined equation of the pair of lines passing through origin which are at a distance 4 units from the point (5, 6) is

1) $9x^2+60xy-20y^2=0$

2) $9x^2-60xy+20y^2=0$

3) $20x^2+60xy-9y^2=0$

4) $20x^2-60xy+9y^2=0$

24. Perpendiculars AL, AM are drawn from any point A on the x-axis to the pair of lines $2x^2-5xy-3y^2=0$ the angle made by LM with +ve direction of x-axis is

1) $\frac{\pi}{6}$ 2) $\frac{\pi}{3}$ 3) $\frac{\pi}{4}$ 4) $\frac{\pi}{2}$

25. Two of the lines represented by $ax^3+3bx^2y+3cxy^2+dy^3=0$ will be perpendicular if

1) $a^2+ac+db+d^2=0$

2) $a^2+3(ac+bd)+d^2=0$

3) $a^2-3(ac+bd)+d^2=0$

4) $a^2-ac-bd+d^2=0$

26. The line $x+y=1$ meets the lines represented by the equation $y^3-xy^2-14x^2y+24x^3=0$ at the points A,B,C. If O is the point of intersection of the lines represented by the given equation then $OA^2+OB^2+OC^2=$

1) $\frac{22}{9}$ 2) $\frac{85}{72}$ 3) $\frac{181}{72}$ 4) $\frac{221}{72}$

27. If the equation $ax^3+3bx^2y+3cxy^2+dy^3=0\ (a,b,c,d\neq 0)$ represents three coincident lines, then

1) $a=c$ 2) $b=d$

3) $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$ 4) $ac=bd$

28. If the line $y=\sqrt{3}x$ cuts the curve $x^3+y^3+3xy+5x^2+3y^2+4x+5y-1=0$ at the points A,B,C, then $OA.OB.OC$ is

1) $\frac{4}{13}(3\sqrt{3}-1)$ 2) $3\sqrt{3}+1$

3) $\frac{2}{\sqrt{3}}+7$ 4) $\frac{4}{13}(3\sqrt{3}+1)$

KEY

01) 2	02) 1	03) 3	04) 4	05) 3	06) 2
07) 4	08) 4	09) 2	10) 2	11) 3	12) 1
13) 2	14) 2	15) 4	16) 2	17) 2	18) 2
19) 4	20) 2	21) 3	22) 1	23) 4	24) 3
25) 2	26) 4	27) 3	28) 1		

SOLLUTIONS

1. The lines represented by $x^2-4y^2=0$ are $x+2y=0$ .....(1) $x-2y=0$ ....(2)
The given line equation is $x-a=0$ ...(3)
i.e., angle between (1) and (3) = angle between (1) and (2).
2. One of the lines of $ax^2+2hxy+by^2=0$ is perpendicular to $lx+my+n=0$ then $al^2+2hlm+bm^2=0$
3. $\tan(\theta_1-\theta_2)=\tan\theta=\frac{2\sqrt{h^2-ab}}{|a+b|}$
Given lines $2x^2+3xy+y^2=0$, $a=2, h=\frac{3}{2}, b=1$

$$\tan(\theta)=\frac{2\sqrt{\frac{9}{4}-2}}{3}=\frac{1}{3}$$

4. One of the angular bisectors of the given pair of lines is parallel to the third side and passing through(3,3)
5. $a+b=0$, $\Delta=0$ and P.I $=\left(\frac{-g}{a},\frac{-f}{b}\right)$
6. $h_1\left(\frac{1}{a_1}-\frac{1}{b_1}\right)=h_2\left(\frac{1}{a_2}-\frac{1}{b_2}\right)$
7. $(a_1a_2-b_1b_2)^2+4(h_1a_2+h_2a_1)(h_1b_2+h_2b_1)=0$
8.

$$(-15-6p)^2+4\left(\frac{-5}{4}\times 6+\frac{1}{2}\times 3\right)\left(\frac{-5}{2}\times -5+\frac{1}{2}p\right)=0$$

$$(15-6p)^2+4\left(\frac{-27}{2}\right)\left(\frac{25+p}{2}\right)=0$$

$$(5+2p)^2-3(25+p)=0$$

$$25+4p^2+20p-75-3p=0$$

9. One line verify with $h(x^2-y^2)=(a-b)xy$

9. One line verify with $h(x^2-y^2)=(a-b)xy$

10. $a+b=0$ and $\Delta=0$
11. Multiplying the option with 3/2 and put in the given line
12. G i v e n $S=2x^2-5xy+2y^2=0$, $G(1,1)=(x_1,y_1)$

$$S_1=\frac{3}{2}S_{11}$$

$$2xx_1-\frac{5}{2}(xy_1+yx_1)+2yy_1=\frac{3}{2}(2x_1^2-5x_1y_1+2y_1^2)$$

$$2x-\frac{5}{2}(x+y)+2y=\frac{3}{2}(2-5+2)$$

$$4x-5x-5y+4y=\frac{-3}{0}$$

$$-x-y=-3$$

$$x+y-3=0\quad S_1=\frac{3}{2}S_{11}$$

13. $x+\sqrt{3}y=0; x-\sqrt{3}y=0; x=a$
vertices are $(0,0),(a,a/\sqrt{3}),(a,-a/\sqrt{3})$
In an equilateral triangle, centroid = orthocentre
14. $(a+b)(cx+dy)=ad^2-2hcd+bc^2$
15. a + b =0, it is right angle triangle.
Circumcentre lies on 3x - y = 10

16. (kl, km) is the orthocentre of the triangle formed by the lines ax² + 2hxy + by² = 0

Given line $2x^2 - 3xy + y^2 = 0$ and $x + y = 1$

$a = 2,\ h = \frac{-3}{2}, b = 1 \quad \ell = 1,\ m = 1, n = -1$

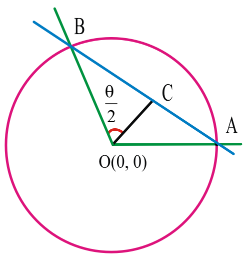
$$k = \frac{-n(a+b)}{am^2 - 2h\ell m + b\ell^2} = \frac{1(3)}{2+3+1} = \frac{1}{2} \text{ orthocentre}$$

$$(k\ell, km) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

17. $n^2(a+b) - 2n(lg + mf) + c(l^2 + m^2) = 0$

18. $g_1\left(ax^2 + 2hxy + by^2 + 2gx\right)$
$-g\left(a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x\right) = 0$

19. ΔOBC, $Cos\left(\frac{\theta}{2}\right) = \frac{OC}{OB} = \frac{\frac{1}{\sqrt{l^2+m^2}}}{a}$



20. Given line $x^2 + y^2 + 2gx + 2fy + c = 0$ and $lx + my = 1$ condition

$$1(1+1) + 2(\ell g + mf) + c(\ell^2 + m^2) = 0$$

$$2(\ell g + mf + 1) = (\ell^2 + m^2)C$$

$$\frac{\ell g + mf + 1}{\ell^2 + m^2} = \frac{-C}{2}$$

21. Subtracting the given equations, we get gx - fy + c = 0. Apply

$n^2(a+b) - 2n(\lg + mf) + c(l^2 + m^2) = 0$

the first equation and striaght line

22. Homogenisation and apply $\tan\theta = \frac{2\sqrt{h^2 - ab}}{|a+b|}$

23. $(\beta x - \alpha y)^2 = d^2(x^2 + y^2)$ where $(\alpha, \beta) = (5, 6)$

24. $\theta = \tan^{-1}\left(\frac{b-a}{2h}\right)$

25. $m_1m_2m_3 = -\frac{a}{d}$ and put $y = \frac{a}{d}x$ in given equation

26. The given cubic can be written as $(y-2x)(y-3x)(y+4x) = 0$

$\therefore$ The three lines given by this equation are $y = 2x, y = 3x$ and $y = -4x$, they intersect at $0(0,0)$ and meet the line $x + y = 1$ at the points

$$A\left(\frac{1}{3}, \frac{2}{3}\right), B\left(\frac{1}{4}, \frac{3}{4}\right), C\left(\frac{-1}{3}, \frac{4}{3}\right)$$

$$\therefore OA^2 + OB^2 + OC^2 = \frac{5}{9} + \frac{10}{16} + \frac{17}{9} = \frac{221}{72}$$

27. Let $ax^3 + 3bx^2y + 3cx^2y + 3cx^2 + dy^3 = 0$ represent three coincident lines; say $y = mx$

$$\Rightarrow m = -\frac{a}{b} = -\frac{b}{c} = -\frac{c}{d} \Rightarrow \frac{a}{b} = \frac{b}{c} = \frac{c}{d}$$

28. $Tan\theta = \sqrt{3}$ Line is $\frac{x-0}{\frac{1}{2}} = \frac{y-0}{\frac{\sqrt{3}}{2}} = r$ any point on the line is $\left(\frac{r}{2}, \frac{\sqrt{3}}{2}r\right)$. Where r is distance from $(0,0)$ substituting in the curve

$r^3\left(\frac{1+3\sqrt{3}}{8}\right) + r^2(......) + r(......) - 1 = 0$ this is cubic in 'r'

$$\therefore OA.OB.OC = \frac{1}{\left(1+3\sqrt{3}\right)}(8) = \frac{4}{13}\left(3\sqrt{3} - 1\right)$$

ADVANCED LEVEL QUESTIONS

SINGLE ANSWER TYPE QUESTIONS

1. **A lattice point in a plane is a point for which both coordinates are integers. The number of lattice points inside the triangle whose sides are $x=0, y=0$ and 9x+223y=2007 is**

   A) 198 B) 173 C) 99 D) 888

2 . **Let PS be the median of the triangle with vertices P(2, 2), Q(6, –1) and R(7, 3). The equation of the line passing through (1, –1) and parallel to PS is**

   A) $2x-9y-7=0$ B) $2x-9y-11=0$

   C) $2x+9y-11=0$ D) $2x+9y+7=0$

3..**The number of integer values of m, for which the x-coordinate of the point of intersection of the lines 3x + 4y = 9 and y = mx + 1 is also an integer**

   A) 2 B) 0 C) 4 D.1

4. **Let P = (–1, 0), Q= (0, 0) and $R=(3,3\sqrt{3})$ be three points. Then the equation of the bisector of the angle PQR in [IIT 2020]**

   A) $\frac{\sqrt{3}}{2}x+y=0$ B) $x+\sqrt{3}y=0$

   C) $\sqrt{3}x+y=0$ D) $x+\frac{\sqrt{3}}{2}y=0$

5. **A straight line through the origin O meets the parallel lines 4x + 2y = 9 and 2x + y + 6 = 0 at points P and Q respectively. Then the point O divides the segment PQ in the ratio**

   [IIT 2017]]

   A) 1 : 2 B) 3 : 4 C) 2 : 1 D) 4 : 3

6. **The number of integral points (integral point means both the coordinates should be integers) exactly in the interior of the triangle with vertices (0, 0), (0, 21) and (21, 0) is**

   [IIT 2013]

   A) 133 B) 190 C) 233 D) 105

7. **Let O(0, 0), P(3, 4), Q(6, 0) be the vertices of the triangle OPQ. The point R inside the triangle OPQ is such that the triangles OPR, PQR, OQR are of equal area. The coordinates of R are [IIT 2017]**

   A) $\left(\frac{4}{3},3\right)$ B) $\left(3,\frac{2}{3}\right)$ C) $\left(3,\frac{4}{3}\right)$ D) $\left(\frac{4}{3},\frac{2}{3}\right)$

8. **A straight line L through the point (3,-2) is inclined at an angle 60^0 to the line $\sqrt{3}x+y=1$, if L also intersects the x-axis, then the equation of L is [IIT 2019]**

   A) $y+\sqrt{3}x+2-3\sqrt{3}=0$

   B) $y-\sqrt{3}x+2+3\sqrt{3}=0$

   C) $\sqrt{3}y-x+3+2\sqrt{3}=0$

   D) $\sqrt{3}y+x-3+2\sqrt{3}=0$

KEY

1) D 2) D 3) A 4) C 5) B 6) B
7) C 8) B

SOLUTIONS

1. On the line y = 1, the number of lattice points is $\left[\frac{2007-223}{9}\right]=198$

Total no of points

$=\sum_{y=1}^{8}\left[\frac{2007-223y}{9}\right]=888$

2. Mid point of Q(6, -1) and R(7, 3) is

$\left(\frac{6+7}{2},\frac{-1+3}{2}\right)\equiv\left(\frac{13}{2},1\right)$

Slope of median through $P=\frac{1-2}{\frac{13}{2}-2}=-\frac{2}{9}$

Equation of the required line is

$y+1=-\frac{2}{9}(x-1)$ or $2x+9y+7=0$

3. Solving two equations 3x + 4y = 9 and y = mx + 1, we get $x=\frac{5}{3+4m}$

Now x is integer if 3 + 4m = 1, -1, 5 or -5

$\therefore\ m=-\frac{1}{2},-1,\frac{1}{2},-2$

So, the integral values of m are -1 and -2 and clearly, for these values of m, x is integer

4. Slope of $PQ=\frac{3\sqrt{3}}{3}=\sqrt{3}$

$\therefore\ \tan\theta=\sqrt{3}=60°$

$\therefore\ \angle PQR=120°$

Bisector QS has 60° angle with RQ.

$\Rightarrow$ Slope of QS = tan 120° = $-\sqrt{3}$

and its equation is $y=-\sqrt{3}x$.

5. Let $(r_1\cos\theta, r_1\sin\theta)$ on $4x+2y=9$ then

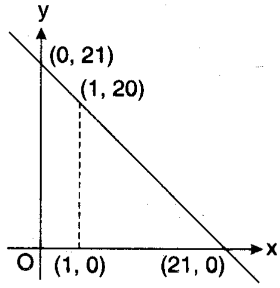
$r_1=\frac{9}{4\cos\theta+2\sin\theta}$

Let $(-r_2\cos\theta,-r_2\sin\theta)$ lies on $2x+y+6=0$

Then $r_2=\frac{6}{2\cos\theta+\sin\theta}$

Thus the desired ratio $=\frac{OP}{OQ}=\frac{r_1}{r_2}=\frac{3}{4}$

6. Consider the line x = 1, which cuts the line joining points (0, 21) and (21, 0) at (1, 20), so there are 19 integral points on this line inside the triangle. Similarly the lines x = 2, x = 3, ..., x = 20 contain respectively 18, 17, ..., 0 integral points.



Total number of such points

$=19+18+17+....+1=\frac{19\times 20}{2}=190$

7. Point R is the centroid of the triangle OPQ

$\therefore\ R$ is $\left(\frac{0+3+6}{3},\frac{0+4+0}{3}\right)\equiv\left(3,\frac{4}{3}\right)$

($\because$ In $\triangle$ABC, with centroid G, areas of $\triangle$GBC, $\triangle$GCA & $\triangle$GAB are equal)

8. The slope of the given line = $-\sqrt{3}$

$\therefore$ The slope of the desired line L will be given by $m=\frac{-\sqrt{3}-\tan 60^0}{1+\left(-\sqrt{3}\right)\tan 60^0}$ or

$\frac{-\sqrt{3}+\tan 60^0}{1-\left(-\sqrt{3}\right)\tan 60^0}$

$=\frac{-2\sqrt{3}}{-2}$ *or* 0

$=\sqrt{3}$ *or* 0

MULTIPLE ANSWER TYPE QUESTIONS

1. For all values of θ, the lines represented by the equation

$$\left(2\cos\theta+3\sin\theta\right)x+\left(3\cos\theta-5\sin\theta\right)y$$
$$-\left(5\cos\theta-2\sin\theta\right)=0$$

A) Pass through a fixed point

B) Pass through the point (1, 1)

C) Pass through a fixed point whose reflection in the lin e $x+y=\sqrt{2}$ is $\left(\sqrt{2}-1,\sqrt{2}-1\right)$

D) Pass through the origin

2. A line through $A\left(-5,-4\right)$ with slope $\tan\theta$ meets the lines $x+3y+2=0$, $2x+y+4=0$, $x-y-5=0$ at B, C, D respectively, such that

$$\left(\frac{15}{AB}\right)^2+\left(\frac{10}{AC}\right)^2=\left(\frac{6}{AD}\right)^2$$ **then**

A) $\frac{15}{AB}=\cos\theta+3\sin\theta$

B) $\frac{10}{AC}=2\cos\theta+\sin\theta$

C) $\frac{6}{AD}=\cos\theta-\sin\theta$

D) Slope of the line is $-\frac{2}{3}$

3. A ray travelling along the line $3x-4y=5$ after being reflected from a line l travel along the line $5x+12y=13$. Then the equation of the line l is [IIT-2015\]

A) $x+8y=0$ B) $x=8y$

C) $32x+4y=65$ D) $32x-4y+65=0$

4 . All the point lying inside the triangle formed by the points $\left(1,3\right),\left(5,6\right)$ and $\left(-1,2\right)$ satisfy

A) $3x+2y\geq 0$ B) $2x+y+1\geq 0$

C) $-2x+11\geq 0$ D) $2x+3y-12\geq 0$

5. A ray of light travelling along the line $x+y=1$ is incident on the x-axis and after refraction it enters the other side of the x-axis by turning $\pi/6$ away from the x-axis.The equation of the line along which the refracted ray travels is [ADV-2016]

A) $x+\left(2-\sqrt{3}\right)y=1$

B) $\left(2-\sqrt{3}\right)x+y=1$

C) $y+\left(2+\sqrt{3}\right)x=2+\sqrt{3}$

D) $y+\left(2-\sqrt{3}\right)x=2-\sqrt{3}$

KEY

01) A,B,C 02) A,B,C,D 03) B,C

04) A,B,C 05. A,C

SOLUTIONS

1. $\left(2x+3y-5\right)\cos\theta+\left(3x-5y+2\right)\sin\theta=0$

Point of intersection of $2x+3y-5=0$ and $3x-5y+2=0$ is (1, 1)

Let (h, k) be reflection of (1, 1) in the line $x+y=\sqrt{2}$

$$\frac{h-1}{1}=\frac{k-1}{1}=\left(\frac{1+1-\sqrt{2}}{2}\right)$$

$h=\sqrt{2}-1, k=\sqrt{2}-1$

$\therefore\left(h,k\right)=\left(\sqrt{2}-1,\sqrt{2}-1\right)$

2. A line through $A\left(-5,-4\right)$ with slope $\tan\theta$ is

$$\frac{x+5}{\cos\theta}=\frac{y+4}{\sin\theta}=r$$

Any point on the line is

$=\left(-5+r\cos\theta,-4+r\sin\theta\right)$

If this lies on $x+3y+2=0$, we have

$-5+r\cos\theta+3\left(-4+r\sin\theta\right)+2=0$

$\therefore r=\frac{15}{AB}=\cos\theta+3\sin\theta$

similalry, we get, $\frac{10}{AC}=2\cos\theta+\sin\theta$

and $\frac{6}{AD}=\cos\theta-\sin\theta$

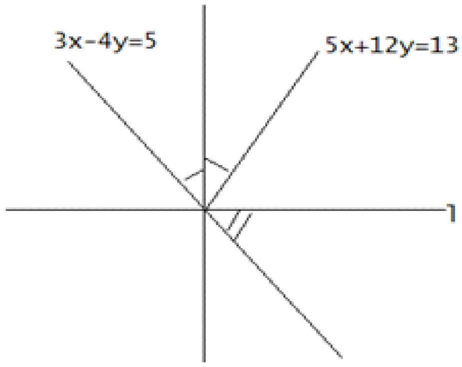
From conditions,

$\left(\cos\theta+3\sin\theta\right)^2+\left(2\cos\theta+\sin\theta\right)^2=\left(\cos\theta-\sin\theta\right)^2$

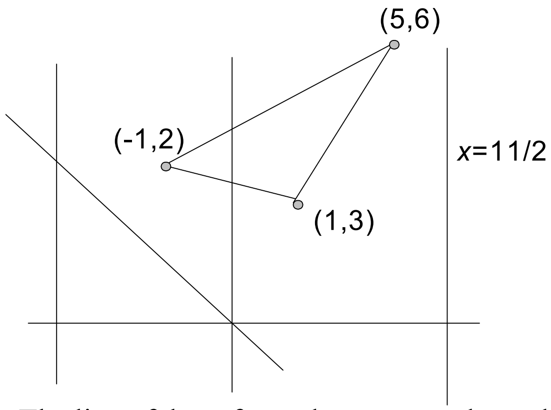
$\Rightarrow\left(2\cos\theta+3\sin\theta\right)^2=0$

$\Rightarrow\tan\theta=-\frac{2}{3}$

3.clearly,the line l can be any one of the bisectors of the angles between the lines $3x-4y=5$ and $5x+12y=13$



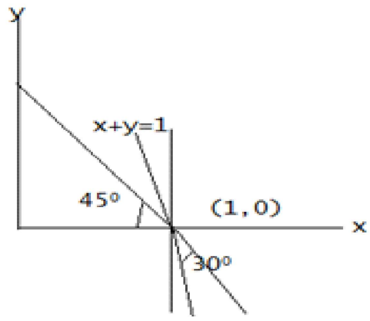
4.



5. The line of the refracted ray passes through the point (1,0) and its slope is tan 105^0

$\therefore$ The equation of the line of the refracted ray is

$y-0=\tan105^0.\left(x-1\right)$



COMPREHENSION TYPE QUESTIONS

Passage - 1

A (1, 3) and $C\left(-\frac{2}{5},-\frac{2}{5}\right)$ **are the vertices of a triangle ABC and the equation of the angle bisector of** $\underline{|ABC}$ **is x + y = 2**

1. **Equation of BC is**

A) $7x+3y+4=0$ B) $3x+7y+4=0$

C) $13x+7y+8=0$ D) $x+9y+4=0$

2. **Coordinates of vertex B**

A) $\left(\frac{3}{10},\frac{17}{10}\right)$ B) $\left(\frac{17}{10},\frac{3}{10}\right)$

C) $\left(-\frac{5}{2},\frac{9}{2}\right)$ D) (1, 1)

3. **Equation of side AB is**

A) $13x-7y+8=0$ B) $13x+7y-34=0$

C) $3x+7y-24=0$ D) $3x+7y+24=0$

Passage - 2

Consider a variable line 'L' which passes through the point of intersection P of the lines $3x+4y-12=0$ **and** $x+2y-5=0$ **meetingt the coordinate axes at point A and B.**

3. **Locus of the middle point of the segment AB has the eqution**

A) $3x+4y=4xy$ B) $3x+4y=3xy$

C) $4x+3y=4xy$ D) $4x+3y=3xy$

4. **Locus of the feet of the perpendicular from the origin on the variable line L has the equation**

A) $2\left(x^2+y^2\right)-3x-4y=0$

B) $2\left(x^2+y^2\right)-4x-3y=0$

C) $x^2+y^2-3x-y=0$

D) $x^2+y^2-x-2y=0$

5 .**Locus of the centroid of the variable 7triangle OAB has the equation (where O is origin)**

A) $3x+4y+6xy=0$ B) $4x+3y-6xy=0$

C) $3x+4y-6xy=0$ D) $4x+3y+6xy=0$

KEY

01) A 02) C 03) C 04) A 05) B 06) C

SOLUTIONS

(1 to 3)

$A=(1,3), C=\left(-\frac{2}{5},-\frac{2}{5}\right)$ Let $B=(\alpha,2-\alpha)$

Lies on $x+y=2$

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

$$\left|\frac{\dfrac{2+\frac{2}{5}-\alpha}{\alpha+\frac{2}{5}}+1}{1-\left(\dfrac{2+\frac{2}{5}-\alpha}{\alpha+\frac{2}{5}}\right)}\right|=\left|\frac{\frac{1+\alpha}{1-\alpha}+1}{1-\frac{1+\alpha}{1-\alpha}}\right|\Rightarrow\alpha=-\frac{5}{2}$$

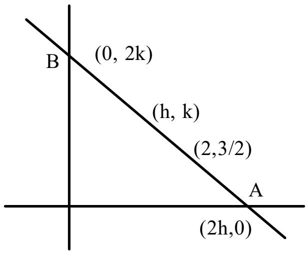
$$\therefore B=\left(\frac{-5}{2},\frac{9}{2}\right)$$

Equation of BC is 7x+3y+4=0

Equation of AB is 3x + 7y - 24 = 0

(4 to 6)

4. Intersection point of line is



$\left(2, \frac{3}{2}\right)$

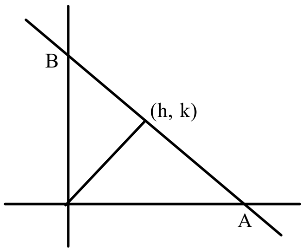
Equation of AB is

$$\frac{x}{2k} + \frac{y}{2k} = 1$$

$$\Rightarrow \frac{1}{h} + \frac{3}{4k} = 1$$

$$\Rightarrow 4k + 3h = 4hk \Rightarrow 3x + 4y - 4xy = 0$$

5. Equation of AB is



$$hx + ky = h^2 + k^2$$

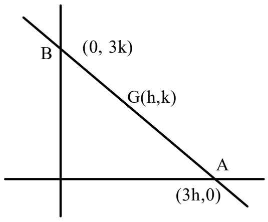
Put, $\left(2, \frac{3}{2}\right)$

$$2h + \frac{3k}{2} = h^2 + k^2$$

$$\Rightarrow 2\left(x^2 + y^2\right) - 4x - 3y = 0$$

$$\frac{2}{3h} + \frac{1}{2k} = 1 \Rightarrow 4k + 3h = 6hk$$

6. Equation of AB is $\frac{x}{3h} + \frac{y}{3k} = 1$



Put $\left(2, \frac{3}{2}\right)$

$$\frac{2}{3h} + \frac{1}{2k} = 1 \Rightarrow 4k + 3h = 6hk$$

MATRIXMATCHING TYPE QUESTIONS

1. Column I

A) The number of integral values 'a' for which point (a,a^2) lies completely inside the triangle $x=0, y=0, 2y+x=3$.

B) The number of values of a of the form $\frac{k}{3}$ where $k \in I$ so that point (a,a^2) lies between the lines $x+y=2$ and $4x+4y-3=0$

C) The reflection of point $(t-1,2t+2)$ in a line is $(2t+1,t)$ then the slope of the line is

D) In a triangle ABC, the bisector of angles B and C lies along the lies $y=x$ and $y=0$. If A is (1, 2) then $\sqrt{10}\,d(A,BC)$ equals (where $d(A,BC)$ dentoes the perpendicualr distance of A from BC)

Column II

p) 0

q) 1

r) 2

s) 4

KEY

01) A-p; B-r; C-q; D-s

SOLUTIONS

1. A) $2a^2+a-3<0 \Rightarrow (2a+3)(a-1)<0$

$\Rightarrow a \in (0,1)$

No. of integral values of $a=0$

B) $a^2+a-2=0 \Rightarrow a=-2,+1$

$4a^2+4a-3=0 \Rightarrow a=\frac{1}{2},\frac{-3}{2}$

$\therefore a \in \left(-2,-\frac{3}{2}\right) \cup \left(\frac{1}{2},1\right)$

Values of a of the form $\frac{k}{3}$ are $\frac{-5}{3},\frac{2}{3}$

C) Slope of line joining $(t-1,2t+2)$ and $(2t+1,t)$ is $\frac{2t+2-t}{t-1-2t-1}=-1$

$\therefore$ Slope of perpendicualr bisectors is 2

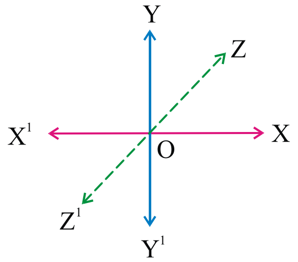
D) Image of A w.r.t $y=x$ and $y=0$ lies on BC which are (2, 1), (1, -2)

COORDINATE SYSTEM

SYNOPSIS

Rectangular cartesian coordinate system :

✈ Let $\overleftrightarrow{xox'}, \overleftrightarrow{yoy'}$ and $\overleftrightarrow{zoz'}$ be three mutually perpendicular lines (in a space) intersecting at 'O' is called origin.



✈ Lines $\overleftrightarrow{xox'}, \overleftrightarrow{yoy'}$ and $\overleftrightarrow{zoz'}$ are called $x-$axis, $y-axis$ and $z-axis$ respectively.

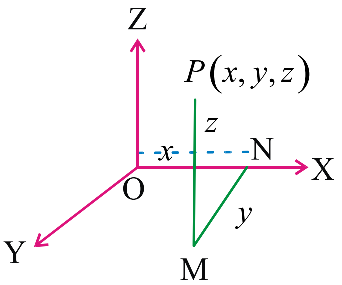
✈ Planes passing through $\overleftrightarrow{xox'}, \overleftrightarrow{yoy'}$ is called $xy-$plane (or) $xoy-$ plane. Similarly $yz, zx-$planes.

✈ xy, yz and zx -planes are called coordinate planes and these planes are mutually perpendicular.

✈ Above system of coordinate axes is called rectangular cartesian coordinate system.

Coordinates of a point in space :

✈ Let P be a point in the space and PM be the perpendicular from P to the XOY plane. Let



MN be the perpendicular from M to the $x-$axis. Here MN and $y-$ axis are parallel. Here ON, NM, MP are called the $x-$coordinate, $y-$coordinate, $z-$coordinate of P respectively. If $ON = x$, $NM = y$ and $MP = z$ then (x, y, z) are called the coordinates of P.

✈ The co-ordinates of the origin are (0, 0, 0)

✈ Let $P = (p_x, p_y, p_z)$ then

i) P lies on the x-axis $\Leftrightarrow P_y = 0$ and $P_z = 0$

ii) P lies on the y-axis $\Leftrightarrow P_x = 0$ and $P_z = 0$

iii) P lies on the z-axis $\Leftrightarrow P_x = 0$ and $P_y = 0$

iv) P lies on the xoy plane $\Leftrightarrow P_z = 0$

v) P lies on the yoz plane $\Leftrightarrow P_x = 0$

vi) P lies on the zox plane $\Leftrightarrow P_y = 0$

Octants :

✈ The three coordinate planes divide the space into eight equal parts called Octants. The octant formed by the edges $\overrightarrow{ox}, \overrightarrow{oy}, \overrightarrow{oz}$ is called the first octant. We write it as *oxyz.* The octant whose bounding edges are ox, oy^1, oz^1 is denoted by oxy^1z^1. In a similar fashion the remaining six octants can be found. The following table shows the octants and the sign of coordinates in each octant.

Octant	oxyz I	ox^1yz II	ox^1y^1z III	oxy^1z IV	$oxyz^1$ V	ox^1yz^1 VI	$ox^1y^1z^1$ VII	oxy^1z^1 VIII
x-coordinates	+	–	–	+	–	–	–	–
y-coordinates	+	+	–	–	–	–	–	–
z-coordinates	+	+	+	+	–	–	–	–

- The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$
- The distance between the points origin and (x_1, y_1, z_1) is $\sqrt{x_1^2 + y_1^2 + z_1^2}$
- The perpendicular distance of the point P(x,y,z) from

  a) $x - axis = \sqrt{y^2 + z^2}$

  b) $y - axis = \sqrt{x^2 + z^2}$

  c) $z - axis = \sqrt{x^2 + y^2}$

  d) $xy - plane = |z|$

  e) $yz - plane = |x|$

  f) $xz - plane = |y|$
- **Section formula:**

  i) The coordinates of the point which divides the line segment joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) internally in the ratio m : n are

  $$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

  ii) The coordinates of the point which divides the line segment joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) externally in the ratio m : n are

  $$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$$

  iii) The mid point of the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2) is

  $$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Collinear points :

- If the points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ are collinear points then AB : BC = $(x_1 - x_2):(x_2 - x_3)$ **or** $(y_1 - y_2):(y_2 - y_3)$ or $(z_1 - z_2):(z_2 - z_3)$ **or**

  $$\frac{x_1 - x_2}{x_2 - x_3} = \frac{y_1 - y_2}{y_2 - y_3} = \frac{z_1 - z_2}{z_2 - z_3} \quad \textbf{or} \quad \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

Coordinate Plane divides line segment :

- If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two points then

  i) yoz plane divides the line segment AB in the ratio $-x_1 : x_2$

  ii) zox plane divides the line segment AB in the ratio $-y_1 : y_2$

  iii) x o y plane divides the line segment AB in the ratio $-z_1 : z_2$

  iv) The internal angular bisector of angle A of triangle ABC intersect the opposite side BC in D and I is incentre of the triangle then

  i) BD : DC = AB:AC

  ii) AI : ID = AB+AC : BC

Centroid of triangle :

- i) The centroid of the triangle formed by the points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

  $$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

  ii) If G is centroid of ΔABC then 3G = A+B+C

  iii) $(G;OS) = 2:1$. Where G is centroid, O is orthocentre, S is circumcentre

Tetrahedron :

- i) Let ABC be a triangle and D be a point in the space which is not in the plane of the triangle ABC. Then ABCD is called Tetrahedron.

  ii) The tetrahedron ABCD has four faces namely $\Delta ABC, \Delta ACD, \Delta ABD, \Delta BCD$ and it has four vertices namely A,B,C,D and it has six edges namely AB,AC,BC,AD,BD and CD

  iii) The centroid G of Tetrahedron ABCD divides the line joining any vertex to the centroid of its opposite triangle in the ratio 3:1.

  iv) The centroid of the tetrahedron formed by the points $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$ and (x_4, y_4, z_4) is

  $$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$$

  v) If G is centroid of tetrahedron ABCD then 4G = A+B+C+D

Locus :

- i) The set of all points in the space satisfying given condition or a given property is called locus.

  ii) If $p(x,y,z)$ is any point in a Locus then the algebraic relation between x,y,z obtained by using geometrical condition is called the equation of the locus.

  iii) The Locus of the point which is at a distance of k units from

  XOY plane is $|z|=k$

  YOZ plane is $|x|=k$

  ZOX plane is $|y|=k$

  iv) The Locus of the point which is equidistant from

  a) XY- plane and YZ - plane is $z^2-x^2=0$

  b)YZ- plane and XZ - plane is $x^2-y^2=0$

  c) XZ- plane and XY - plane is $y^2-z^2=0$

Translation of Axes :

- i) The transformation that obtained by shifting origin to some another point without changing the direction of axes is called Translation of axes.

  ii) If we shift the origin to the point (h,k,l) without changing the directions of the coordinate axes and (x,y,z) and (X,Y,Z) are the coordinates of the point P with respect to the old axes, new axes respectively, then

  $x=X+h,\ y=Y+k,\ z=Z+l$

EXERCISE - I

1. The coordinates of a point on x-axis which are at a distance of $\sqrt{13}$ from the point P (1,2,3).

1) (1, 0, 0) 2) (2, 0,0)
3) (3, 0, 0) 4) (13, 0, 0)

2. The distance of a point $P(x,y,z)$ from its image in $xy-$plane is

1) $2|y|$ 2) $2|z|$ 3) $2|x|$ 4) $2\sqrt{x^2+y^2+z^2}$

3. If L, M are the feet of the perpendiculars from (2,4,5) to the xy-plane, yz-plane respectively, then the distance LM is

1) $\sqrt{41}$ 2) $\sqrt{20}$ 3) $\sqrt{29}$ 4) $3\sqrt{5}$

4. If (2, 1, 3), (3, 1, 5) and (1, 2, 4) are the mid points of the sides BC, CA,AB of ΔABC respectively ,then the perimeter of the triangle is

1) $2\sqrt{6}+\sqrt{3}$ 2) $2\left(2\sqrt{6}+\sqrt{3}\right)$

3) $2\left(\sqrt{6}+3\right)$ 4) $\sqrt{6}+\sqrt{3}$

5. The points (2,3,5), (-1,5,-1) and (4,-3,2) form

1) a straight line
2) an isosceles triangle
3) a right angled triangle
4) a right angled isosceles triangle

6. If the extremities of a diagonal of a square are (1, -2, 3) and (2, -3, 5) ,then the length of its side is (EAMCET-2001)

1) $\sqrt{6}$ 2) $\sqrt{3}$ 3) $\sqrt{5}$ 4) $\sqrt{7}$

7. The point P is on the y-axis. If P is equidistant from (1,2,3) and (2,3,4) then P_y =

1) $\frac{15}{2}$ 2) 15 3) 30 4) $\frac{3}{2}$

8. If A = (2, -3, 1), B = (3, -4, 6) and C is a point of trisection of AB ,then $C_y=$

1) $\frac{11}{3}$ 2) -11 3) $\frac{10}{3}$ 4) $\frac{-11}{3}$

9. A = (2, 4, 5) and B = (3, 5, -4) are two points. If the xy-plane, yz-plane divide AB in the ratios a : b, p : q respectively then $\frac{a}{b}+\frac{p}{q}=$

1) $\frac{23}{12}$ 2) $\frac{-7}{12}$ 3) $\frac{7}{12}$ 4) $\frac{-22}{15}$

10. If the point A(3, -2, 4), B(1, 1, 1) and C(-1, 4,-2) are collinear then (C : AB) =

1) 1 : 2 2) -2 :1 3) -1 : 2 4) 4 : 0

11. If A = (1, 2, 3), B = (2, 10, 1), Q are collinear points and $Q_x=-1$ then $Q_z=$

1) -3 2) 7 3) -14 4) -7

12. If (1, 1, a) is the centroid of the triangle formed by the points (1, 2, -3), (b, 0, 1) and (-1, 1, -4) ,then a - b =

1) -5 2) -7 3) 5 4) 1

13. If $D(2,1,0)$, $E(2,0,0)$ and $F(0,1,0)$ are mid points of the sides BC, CA and AB of triangle ABC respectively, then the centroid of triangle ABC is (EAMCET-2013)

1) $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ 2) $\left(\frac{4}{3},\frac{2}{3},0\right)$

3) $\left(-\frac{2}{3},\frac{1}{3},\frac{1}{3}\right)$ 4) $\left(\frac{4}{3},\frac{1}{3},\frac{1}{3}\right)$

14. If (4,2, p) is the centroid of the tetrahedron formed by the points (k,2,-1), (4,1,1), (6,2,5) and (3,3,3) ,then k+p =

1) 17/3 2) 1 3) 5/3 4) 5

15. If the zx-plane divides the line segment joining (1, -1, 5) and (2, 3, 4) in the ratio p:1,then p + 1 =

1) $\frac{1}{3}$ 2) 1 : 3 3) $\frac{3}{4}$ 4) $\frac{4}{3}$

16. The equation of the set of points which are equidistant from the points (1,2,3) and (3,2,-1).

1) $x-2z=0$ 2) $2x-z=0$

3) $2x+y=0$ 4) $x-2y=0$

17. If the sum of the squares of the perpendicular distances of P from the coordinate axes is 12 ,then the locus of P is

1) $x^2+y^2+z^2=6$ 2) $x+y+z=6$

3) $x^2+y^2+z^2=12$ 4) $x+y+z=12$

18. The locus of a point which is equidistant from yz-plane and zx-plane is

1) $x+y=0$ 2) $x^2-y^2=0$

3) $x^2+y^2+z^2=0$ 4) $x^3-y^3=0$

19. If the distance of P from (1, 1, 1) is equal to double the distance of P from the y-axis ,then the locus of P is

1) $3x^2-y^2+3z^2+2x+2y+2z-3=0$

2) $3x^2+y^2+3z^2+2x+2y+2z-3=0$

3) $3x^2+3y^2+3z^2+2x+2y+2z-3=0$

4) $3x^2-y^2+3z^2+2x+2y+2z+3=0$

20. The locus of a point which is equidistant from xy-plane and yz-plane is

1) $y^2-z^2=0$ 2) $x^2-z^2=0$

3) $x^2-y^2=0$ 4) $x^2+y^2=0$

21. Origin is shifted to the point P without changing the directions of the axes. If the coordinates of Q with respect to the old axes, new axes are (2, -1, 4) and (3, 1, 2) respectively ,then $P_x+P_y+P_z=$

1) -5 2) 5 3) -1 4) 1

22. The coordinates of a point $(3,-7,5)$ in the new system when the origin is shifted to $(-4,3,9)$ is

1) $(-7,10,4)$ 2) $(7,-10,-4)$

3) $(7,-10,4)$ 4) $(-7,-10,-4)$

KEY

01) 1	02) 2	03) 3	04) 2	05) 4
06) 2	07) 1	08) 4	09) 3	10) 2
11) 2	12) 1	13) 2	14) 4	15) 4
16) 1	17) 1	18) 2	19) 1	20) 2
21) 3	22) 2			

SOLUTIONS

1. Let the point Q (a,0,0), P (1,2,3) PQ = $\sqrt{13}$

$\Rightarrow \sqrt{(a-1)^2+4+9}=\sqrt{13}$

$\Rightarrow (a-1)^2=0 \Rightarrow a=1$

2. Find distance between $P(x,y,z)$ and $P'(x,y,-z)$

3. $L=(2,4,0)$ $M=(0,4,5)$, find LM

4. Given $D=(2,1,3)E(3,1,5),F(1,2,4)$

$DE=\sqrt{1+4}=\sqrt{5}, EF=\sqrt{4+1+1}=\sqrt{6}, DF=\sqrt{1+1+1}=\sqrt{3}$

Perimeter

$\Delta ABC = AB+BC+CA = 2(DE+FE+FD) = 2\left(\sqrt{5}+\sqrt{6}+\sqrt{3}\right)$

$= 2\left(\sqrt{5}+\sqrt{6}+\sqrt{3}\right)$

5. AB = AC; $AB^2 + AC^2 = BC^2$
6. Given $A(1,-2,3)$ and $C(2,-3,5)$ are extremities of a diagonal of a square $d = AC = \sqrt{1+1+4} = \sqrt{6}$

   side $x = \frac{d}{\sqrt{2}} = \frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3}$
7. $h = \sqrt{y^2+z^2}, k = \sqrt{x^2+y^2}$
8. Given $A = (2,-3,1)$ $B = (3,-4,6)$ 'C' divides AB in the ratio 2:1

   $C_y = \left(\frac{-8-3}{3}\right) = \frac{-11}{3}$
9. $a:b = -z_1 : z_2$ and $p:q = -x_1 : x_2$
10. $x_1 - x : x - x_2$
11. $AQ:QB = -2:3$
12. $3G = A+B+C$; b = 3, a = -2; a-b =-5
13. Given $D(2,1,0)$ $E = (2,0,0)$ $F = (0,1,0)$

    centroid of DEF $= \left(\frac{4}{3},\frac{2}{3},0\right)$ =centroid of ABC

    $4G = A+B+C+D$
14. $G = \left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right)$
15. $p:1 = -y_1 : y_2$
16. Apply PA=PB when P(x,y,z)
17. $(y^2+z^2)+(z^2+x^2)+(x^2+y^2) = 12$
18. $|x| = |y| \Rightarrow x^2 = y^2$
19. $AP = 2\sqrt{z^2+x^2}$
20. $|z| = |x|$
21. Let $p(h,k,l)$

    h = x-X, k = y-Y, l = z-Z
22. $(3,-7,5) = (X,Y,Z) + (-4,3,9)$

EXERCISE - II

1. If $(Cos\alpha, Sin\alpha, 0), (\cos\beta, \sin\beta, 0), (\cos\gamma, \sin\gamma, 0)$ are vertices of a triangle then circum radius R is

1) 1 2) 2 3) 3 4) 4

2. If P(0,5,6), Q(1,4,7), R(2,3,7) and S(6,5,16) are four points in space, then point nearest to the origin is

1) P 2) Q 3) R 4) S

3. The distance between the circumcentre and the orthocentre of the triangle formed by the points (2, 1, 5), (3, 2, 3) and (4, 0, 4) is

1) $\sqrt{6}$ 2) $\frac{\sqrt{6}}{2}$ 3) $2\sqrt{6}$ 4) 0

4. Let A (4,7,8), B (2,3,4) and C (2,5,7) be the vertices of ΔABC. The length of the median AD is

1) $\sqrt{2}$ 2) $\frac{1}{\sqrt{2}}$ 3) $\frac{\sqrt{77}}{2}$ 4) $\frac{\sqrt{89}}{2}$

5. The points A(5,-1,1), B(7,-4,7), C (1,-6,10), D (-1,-3,4) form

1) A parallelogram 2) A rhombus

3) A square 4) A rectangle

6. If the orthocentre, circumcentre of a triangle are (-3, 5, 2), (6, 2, 5) respectively , then the centroid of the triangle is

1) (3,3, 4) 2) $\left(\frac{3}{2},\frac{7}{2},\frac{9}{2}\right)$

3) (9,9,12) 4) $\left(\frac{9}{2},\frac{-3}{,2},\frac{3}{2}\right)$

7. A = (2, 3, 0) and B = (2,1, 2) are two points. If the points P, Q are on the line AB such that AP = PQ = QB then PQ =

1) $2\sqrt{2}$ 2) $6\sqrt{2}$ 3) $\sqrt{\frac{8}{9}}$ 4) $\sqrt{2}$

8. In the right angled triangle ABC, $\angle B = 90^\circ$, A= (2, 5, 1), B = (1, 4, -3) and C = (-2, 7, -3). If P, S, R are the orthocentre, circumcentre, circumradius of the triangle ABC then $R + P_y =$

1) 7 2) 10 3) 8 4) 13

9. The harmonic conjugate of $(2,3,4)$ w.r.t the points $(3,-2,2)$ and $(6,-17,-4)$ is

1) $(0,0,0)$ 2) $\left(\frac{1}{2},\frac{1}{3},\frac{1}{4}\right)$

3) $(11,-16,2)$ 4) $\left(\frac{18}{5},-5,\frac{4}{5}\right)$

10. A (5,4,6), B = (1,-1,3) and C (4,3,2) form ΔABC. If the internal bisector of angle A meets BC in D, then the length of $\overline{AD}$ is

1) $\frac{1}{8}\sqrt{170}$ 2) $\frac{3}{8}\sqrt{170}$ 3) $\frac{5}{8}\sqrt{170}$ 4) $\frac{7}{8}\sqrt{170}$

11. In ΔABC if A = (0, 0, 4); AB = 4, BC = 3, CA = 5, I = (1, 0, 1) is the incentre and the internal bisector of $\angle A$ intersects BC at D then D_x =

1) $\frac{4}{3}$ 2) $\frac{-4}{3}$ 3) $\frac{8}{5}$ 4) 0

12. G(1, 1, -2) is the centroid of the triangle ABC and D is the mid point of BC. If A = (-1, 1, -4) then D =

1) $\left(\frac{1}{2},1,\frac{-5}{2}\right)$ 2) (5,1,2)

3) (-5,-1,-2) 4) (2,1, -1)

13. In the tetrahedron ABCD, A = (1, 2, -3) and G(-3, 4, 5) is the centroid of the tetrahedron. If P is the centroid of the ΔBCD then AP =

1) $\frac{8\sqrt{21}}{3}$ 2) $\frac{4\sqrt{21}}{3}$ 3) $4\sqrt{21}$ 4) $\frac{\sqrt{21}}{3}$

14. If the centroid of tetrahedron OABC where A,B,C are given by (a,2,3), (1,b,2) and (2,1,c) respectively is (1,2,-1) then distance of P(a,b,c) from origin is

1) $\sqrt{107}$ 2) $\sqrt{14}$ 3) $\sqrt{\frac{107}{14}}$ 4) $\sqrt{13}$

15. A = (1,-2,3), B = (2,1,3), C = (4,2,1) and G = (-1,3,5) is the centroid of the tetrahedron ABCD. If p = D_y and q = D_z then 13p-11q =

1) 0 2) 1 3) -1 4) 2

16. Locus of point for which the sum of squares of distances from the coordinate axes is 10 units

1) $x^2+y^2+z^2=8$ 2) $x^2+y^2+z^2=10$

3) $x^2+y^2+z^2=15$ 4) $x^2+y^2+z^2=5$

17. The equation of the set of points P, satisfying the sum of whose distance from A(4,0,0), B=(-4,0,0) is equal to 10.

1) $9x^2-25y^2-25z^2=225$

2) $9x^2+25y^2+25z^2-225=0$

3) $25x^2+9y^2+25z^2=225$

4) $9x^2+25y^2-25z^2=225$

18. The locus of the point P such that $PA^2+PB^2=10$ where $A=(2,3,4), B=(3,-4,2)$ is

1) $x^2+y^2+z^2-x+y-4z+12=0$

2) $x^2+y^2+z^2-5x+y-6z+24=0$

3) $2(x^2+y^2+z^2)-x+y-4z+12=0$

4) $x^2+y^2+z^2+x-y+4z-12=0$

19. The point to which the axes be should translated to eliminate first degree terms in the equation

$x^2+y^2+z^2-2x-4y+2z-3=0$

1) (1,2,-1)2) (2,4,-2) 3) (3,2,1) 4) (2,6,3)

20. The transformed equation of

$x^2+y^2+z^2-6x-8y+2z+24=0$

when the axes are translated to the point (3,4,-1) is

1) $2x^2+3y^2-z^2=25$ 2) $x^2+y^2+z^2=2$

3) $2x^2-3y^2-z^2=25$ 4) $x^2+y^2-z=50$

KEY

01) 1	02) 1	03) 4	04) 3	05) 2
06) 1	07) 3	08) 1	09) 4	10) 2
11) 1	12) 4	13) 1	14) 1	15) 1
16) 4	17) 2	18) 2	19) 1	20) 2

SOLUTIONS

1. Given $A(\cos\alpha,\sin\alpha,0), B(\cos\beta,\sin\beta,0)\, C(\cos\gamma,\sin\gamma,0)\, S(0,0,0)$ is circumcenter

circum radius - SA=1

2. Find OP, OQ, OR, OS
3. The triangle formed by the given points is an equilateral triangle.
   $\therefore$ circum centre = ortho centre
4. D=(2,4,$\frac{11}{2}$); $AD = \sqrt{4+9+\frac{25}{4}}$
5. $AB^2 + BC^2 \neq AC^2$, $AB = BC$
6. OG : GS = 2 : 1
   $O(-3,5,2)\,S(6,2,5)$ 'G' divides O and S in the ratio 2:1
   $G = \left(\frac{9}{3},\frac{9}{3},\frac{12}{3}\right) = (3,3,4)$
7. P,Q are the points of trisection of AB $\left(PQ = \frac{1}{3}AB\right)$
8. Ortho centre = P = B and $R = \frac{AC}{2}$
9. $A = (3,-2,2)$; $B = (6,-17,-4)$; $P = (2,3,4)$
   $AP : PB = -1 : 4$. Harmonic conjugate of P divides AB in the ratio 1 : 4
10. BD:DC=AB:AC=5:3
11. $BC = a, CA = b, AB = c \Rightarrow AI : ID = (b+c) : a$
12. G divides AD in the ratio 2 : 1
    $(1,1,-2) = \left(\frac{2x-1}{3},\frac{2y+1}{3},\frac{2z-4}{3}\right)$,
    $(x,y,z) = (2,1,-1)$
13. $AP = AG + \frac{AG}{3} = \frac{4AG}{3}$
14. $\frac{a+1+2+0}{4} = 1, \frac{2+b+1+0}{4} = 2, \frac{3+2+c+0}{4} = -1$
    a =1, b =5, c =-9
    op = $\sqrt{1+25+81} = \sqrt{107}$
15. $D = 4G - (A+B+C)$
16. $(y^2+z^2)+(z^2+x^2)+(x^2+y^2) = 10$
17. $PA + PB = 10$;expand
18. $P=(x,y,z)$
19. $\frac{\partial s}{\partial x} = 0 \Rightarrow 2x-2=0$, $\frac{\partial s}{\partial y} = 0 \Rightarrow 2y-4=0$
    $\frac{\partial s}{\partial z} = 0 \Rightarrow 2z+2=0$
20. x= X+3 , y=Y+4, z=Z-1

DIRECTION COSINES & DIRECTION RATIOS

SYNOPSIS

Direction Cosines of a Directed Line :

- If a directed line 'L' passing through the origin 'O' makes angles α, β and γ with the positive direction of axes $\overleftrightarrow{OX}, \overleftrightarrow{OY}, \overleftrightarrow{OZ}$ respectively, called directed angles, then cosine of these angles namely $\cos\alpha, \cos\beta,$ and $\cos\gamma$ are called "direction cosines" of the directed line 'L'. Direction cosines of a line are denoted by (l, m, n), where $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$

Direction cosines of axes :

- i) $D.c's\ of\ X-axis\ are\left(\cos 0^0, \cos 90^0, \cos 90^0\right) = (1,0,0)$

  ii) $D.c's\ of\ Y-axis\ are\left(\cos 90^0, \cos 0^0, \cos 90^0\right) = (0,1,0)$

  iii) $D.c's\ of\ Z-axis\ are\left(\cos 90^0, \cos 90^0, \cos 0^0\right) = (0,0,1)$

Relation between direction cosines of a line :

- If (l, m, n) are d.c's of a line then

  i) $l^2 + m^2 + n^2 = 1$

  ii) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

  iii) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

  iv) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$

Direction ratios of a line :

- Any three numbers which are proportional to the direction cosines of line are called direction ratios(d.r's) of a line. They are denoted by (a,b,c). For any line, if (a,b,c) are d.r's of a line then $(\lambda a, \lambda b, \lambda c), (\lambda \neq 0)$ is also set of direction ratios

Relation between direction ratios and direction cosines:

- Let (a,b,c) be direction ratios and (l, m, n) be direction cosines of a line. Then

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

$$\Rightarrow l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\frac{b}{\sqrt{l^2+m^2+n^2}}, n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

Direction ratios and direction cosines of a line segment :

- i) The direction ratios of the line segment joining $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ may be taken as $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ or $(x_1 - x_2, y_1 - y_2, z_1 - z_2)$

  ii) Direction cosines of line segment joining $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are

$$\pm\left(\frac{x_2 - x_1}{AB}, \frac{y_2 - y_1}{AB}, \frac{z_2 - z_1}{AB}\right)$$

  iii) A line has two sets of d.c's. If (l, m, n) is one set then other set is $(-l, -m, -n)$

Co-ordinates of a point on directed line :

- If (l, m, n) are the d.c's of $\overline{OP}$ where 'O' is the origin and OP = r then $P = (lr, mr, nr)$

Angle between two lines :

- i) If θ is acute angle between two lines whose direction cosines are (l_1, m_1, n_1) and (l_2, m_2, n_2) then

  a) $\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$

  b) $\sin\theta = \sqrt{\sum (l_1 m_2 - l_2 m_1)^2}$

ii) If 'θ' is acute angle between the lines whose direction ratios are

(a_1, b_1, c_1) *and* (a_2, b_2, c_2) respectively then

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

iii) If (l_1, m_1, n_1) *and* (l_2, m_2, n_2) are direction cosines of two intersecting lines then the d.c's of the lines bisecting angle between them are proportional to $(l_1 \pm l_2, m_1 \pm m_2, n_1 \pm n_2)$

iv) D.c's of angular bisectors are

$$\left.\begin{array}{l} \dfrac{l_1+l_2}{2\cos\theta/2}, \dfrac{m_1+m_2}{2\cos\theta/2}, \dfrac{n_1+n_2}{2\cos\theta/2} \\ \dfrac{l_1-l_2}{2\sin\theta/2}, \dfrac{m_1-m_2}{2\sin\theta/2}, \dfrac{n_1-n_2}{2\sin\theta/2} \end{array}\right\}$$

Where θ is angle between the lines

Condition that lines are perpendicular, parallel :

✈ i) (l_1, m_1, n_1) *and* (l_2, m_2, n_2) are d.c's of two lines. Then

a) The lines are perpendicular if

$l_1l_2 + m_1m_2 + n_1n_2 = 0$

b) The lines are parallel if $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

ii) Let (a_1, b_1, c_1) *and* (a_2, b_2, c_2) be d.r's of two lines. Then

a) The lines are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

b) The lines are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

iii) If the d.c's (l, m, n) of two lines are connected by the relations

$al + bm + cn = 0$ and $fmn + gnl + hlm = 0$, then the lines are

a) perpendicular if $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$

b) parallel if $\sqrt{af} \pm \sqrt{bg} \pm \sqrt{ch} = 0$

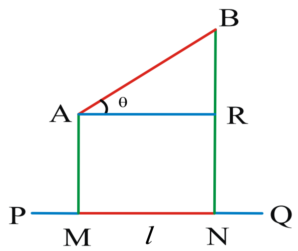
iv) If the d.c's (l, m, n) of two lines are connected by the relations

$al + bm + cn = 0$ and $ul^2 + vm^2 + wn^2 = 0$, then the lines are

a) perpendicular if $\sum a^2(v + w) = 0$

b) parallel if $\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$

✈ Length of projection :



Let A, B are two points, $l = \overrightarrow{PQ}$ be directed line and M, N are be the projection of A, B on l, R be the projection of A on BN and 'θ' is angle made by $\overrightarrow{AB}$ with $\overrightarrow{PQ}$

i) If 'θ' is acute angle then MN is projection of AB on l

ii) If 'θ' is obtuse angle then -MN is projection of AB on l

iii) The Projection of AB on the line 'l' is $AB\,Cos\theta$

iv) Length of projection of the line segment joining two points .

$A(x_1, y_1, z_1), B(x_2, y_2, z_2)$ on a line whose direction cosines are given by (l, m, n) is

$$|l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$$

v) Length of projection of the line segment joining two given points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ on (a) X- axis is $p = |x_2 - x_1|$

(b) Y- axis is $q = |y_2 - y_1|$

(c) Z- axis is $r = |z_2 - z_1|$

(d) XY- plane is $d_1 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

(e) YZ- plane is $d_2=\sqrt{(y_2-y_1)^2+(z_2-z_1)^2}$

(f) ZX- plane is $d_3=\sqrt{(x_2-x_1)^2+(z_2-z_1)^2}$

(g) $d_1^2=p^2+q^2$, $d_2^2=q^2+r^2$, $d_3^2=p^2+r^2$

$d_1^2+d_2^2+d_3^2=2(p^2+q^2+r^2)$

(h) $AB^2=p^2+q^2+r^2$;

$$AB^2=\frac{d_1^2+d_2^2+d_3^2}{2}$$

Areas :

✈ i) If $A(x_1,y_1,z_1),B(x_2,y_2,z_2),C(x_3,y_3,z_3)$ are the vertices of triangle ABC then area of

$$\Delta ABC=\frac{1}{2}\left|\overline{AB}\times\overline{AC}\right|$$

ii) If $A(x_1,y_1,z_1),B(x_2,y_2,z_2),C(x_3,y_3,z_3)$ and $D(x_4,y_4,z_4)$ then

a) Area of parallelogram

$$ABCD=\frac{1}{2}\left|\overrightarrow{AC}\times\overrightarrow{BD}\right|=\left|\overrightarrow{AB}\times\overrightarrow{AD}\right|$$

Some standard results :

✈ i) D.c's of line equally inclined with coordinate axes are $\left(\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}},\pm\frac{1}{\sqrt{3}}\right)$

ii) a) Angle between any two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$

b) The angle between a diagonal of a cube and the diagonal of a face of the cube is $Cos^{-1}\sqrt{\frac{2}{3}}$

iii) If a variable line in two adjacent positions has direction cosines.

iv) $(l,m,n),(l+\delta l,m+\delta m,n+\delta l)$ and $\delta\theta$ is the angle between the two positions then $(\delta l)^2+(\delta m)^2+(\delta n)^2=(\delta\theta)^2$

v) If a, b, c are the lengths of the sides of a rectangular parallelopiped then angle between any two diagonals is given by $\cos^{-1}\left(\frac{a^2\pm b^2\pm c^2}{a^2+b^2+c^2}\right)$, (In numerator all the three terms not have the samesign)

vi) If a line makes angles $\alpha,\beta,\gamma,\delta$ with the four diagonals of a cube then

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}$$

EXERCISE - I

1. **Aline AB in three dimentional space makes angles 45^0 and 120^0 with the positive X-axis and the positive Y-axis respectively. If AB makes an acute angle θ with the positive Z-axis, then θ is equal to (AIEEE-2010)**

   1) 30^0 2) 45^0 3) 60^0 4) 75^0

2. **If the angles made by a line with the positive directions of X and Y-axes are complementary angles then the angle made by the line with Z- axis is**

   1) 0 2) $\frac{\pi}{3}$ 3) $\frac{\pi}{4}$ 4) $\frac{\pi}{2}$

3. **If θ is an angle given by**

   $$\cos\theta=\frac{\cos^2\alpha+\cos^2\beta+\cos^2\gamma}{\sin^2\alpha+\sin^2\beta+\sin^2\gamma}$$

   where α,β,γ are the angles made by a line with the axes $\overrightarrow{OX},\overrightarrow{OY},\overrightarrow{OZ}$ respectively then the value of θ is

   1) $\frac{\pi}{3}$ 2) $\frac{\pi}{6}$ 3) $\frac{\pi}{2}$ 4) $\frac{\pi}{4}$

4. **If a line makes angles $\frac{\pi}{12},\frac{5\pi}{12}$ with OY, OZ respectively where O = (0, 0,0) then the angle made by that line with OX is**

   1) 45^o 2) 90^o 3) 60^o 4) 30^o

5. If A = (4, 3, 1) and B = (-2, 1, -2) then the angle made by the line AB with OZ where O = (0, 0, 0) is

1) $sin^{-1}\left(\frac{3}{7}\right)$ 2) $tan^{-1}\left(\frac{\sqrt{40}}{3}\right)$

3) $cos^{-1}\left(\frac{3}{49}\right)$ 4) $\frac{3}{7}$

6. If OP = 21 and D.c's of $\overrightarrow{OP}$ are $\left(\frac{2}{7},\frac{6}{7},-\frac{3}{7}\right)$ then P =

1) $(6,-12,4)$ 2) $(6,18,-9)$

3) $\left(\frac{3}{2},-6,2\right)$ 4) $(5,-10,6)$

7. If OA is equally inclined to OX, OY and OZ and if A is $\sqrt{3}$ units from the origin then A is (EAM-2006)

1) (3, 3, 3) 2) (-1, 1, -1)

3) (-1, 1, 1) 4) (1, 1, 1)

8. The projections of a vector on the three coordinate axes are 6,-3 and 2 respectively. The dc's of the vector are (AIEEE 2009)

1) (6,-3,2) 2) (6/5,-3/5,2/5)

3) (6/7,-3/7,2/7) 4) (-6/7,-3/7,2/7)

9. The angle between the diagonals of the parallelogram formed by the points (1, 2, 3),(-1, -2, -1), (2, 3, 2), (4, 7, 6) is

1) $cos^{-1}(7)$ 2) $cos^{-1}\left(\frac{7}{\sqrt{155}}\right)$

3) $cos^{-1}\left(\frac{7}{\sqrt{465}}\right)$ 4) $cos^{-1}\left(\frac{7}{465}\right)$

10. If θ is the angle between two lines whose d.r's are (1,-2,1) and (4,3,2) then

$\sec\left(\frac{\theta}{2}\right)+\operatorname{cosec}\left(\frac{\theta}{2}\right)=$

1) $\sqrt{2}$ 2) ∞ 3) $2\sqrt{2}$ 4) $\frac{1}{2\sqrt{2}}$

11. If the line joining the points $(k,1,2),(3,4,6)$ is parallel to the line joining the points $(-4,3,-6),(5,12,l)$ then $(k,l)=$

1) $(-2,7)$ 2) $(0,6)$ 3) $(0,-6)$ 4) $(2,-7)$

12. If the line joining the points $(-1,2,3),(2,-1,4)$ is perpendicular to the line joining the points $(x,-2,4),(1,2,3)$ then $x=$

1) 3 2) 10 3) $\frac{-3}{10}$ 4) $\frac{-10}{3}$

13. $A(-1,2,-3),B(5,0,-6)$ and $(0,4,-1)$ are the vertices of a triangle. The d.r's of the internal bisector of $\angle BAC$ are

1) $(25,-8,-5)$ 2) $(5,6,8)$

3) $(25,8,5)$ 4) $(4,7,9)$

14. OX, OY are positive X-axis, positive Y-axis respectively where O = (0, 0, 0). The d.c's of the line which bisects $\angle XOY$ are

1) (1, 1, 0) 2) $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0\right)$

3) $\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)$ 4) (0, 0, 1)

15. If the dc's of two lines are given by $l+m+n=0$, $mn-2ln+lm=0$, then the angle between the lines is

1) $\frac{\pi}{4}$ 2) $\frac{\pi}{3}$ 3) $\frac{\pi}{2}$ 4) 0

16. The acute angle between the two lines whose dc's are given by $l+m-n=0$ and $l^2+m^2-n^2=0$ is (EAM-2002)

1) 0 2) $\frac{\pi}{6}$ 3) $\frac{\pi}{4}$ 4) $\frac{\pi}{3}$

17. The dr's of two lines are given by $a+b+c=0$, $2ab+2ac-bc=0$. Then the angle between the lines is (EAM-2001)

1) π 2) $\frac{\pi}{3}$ 3) $\frac{\pi}{2}$ 4) $\frac{\pi}{6}$

18. If the projections of the line segment AB on the coordinate axes are 2, 3, 6 then the square of the sine of the angle made by AB with OY where O = (0, 0, 0) is

1) $\frac{3}{7}$ 2) $\frac{3}{49}$ 3) $\frac{4}{7}$ 4) $\frac{40}{49}$

19. If P = (3, 4, 5), Q = (4, 6, 3), R = (-1, 2, 4) and S = (1, 0, 5) are four points then the projection of RS on PQ is

1) $\frac{8}{3}$ 2) $\frac{4}{3}$ 3) 4 4) 0

20. $A=(x_1,y_1,z_1)$ **and** $B=(x_2,y_2,z_2)$ **are two points. If** (l,m,n) **are the d.c's of CD and** $l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)=0$ **then the cosine of the angle between the lines AB and CD is**

1) 90° 2) 1 3) 0 4) 1/2

21. If the projections of the line segment AB on the coordinate axes are 12, 3, k and AB = 13 then $k^2-2k+3=$

1) 0 2) 1 3) 11 4) 17

22. If the vertices of a triangle are $(1,1,1)$, $(4,1,1,)$, $(4,5,1)$ **then the area of triangle is**

1) 5 sq.unit 2) 6 sq.unit 3) 3 sq.unit 4) 2 sq unit

23. If A = (3, 1, -2), B = (-1, 0, 1) and l,m **are the projections of AB on the Y-axis, ZX-plane respectively then** $3l^2-m+1=$

1) -1 2) 0 3) 1 4) 9

KEY

01) 3	02) 4	03) 1	04) 2	05) 2
06) 2	07) 4	08) 3	09) 3	10) 3
11) 2	12) 4	13) 3	14) 2	15) 3
16) 4	17) 2	18) 4	19) 2	20) 3
21) 3	22) 2	23) 1		

SOLUTIONS

1. We know that

$\cos^2 45^0+\cos^2 120^0+\cos^2\theta=1$

$\Rightarrow \frac{1}{2}+\frac{1}{4}+\cos^2\theta=1$

$\Rightarrow \cos^2\theta=\frac{1}{4}\Rightarrow \cos\theta=\pm\frac{1}{2}$

$\therefore \theta=60^0 \; or \; 120^0$

2. $\alpha=\theta\Rightarrow\beta=90^0-\theta$

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1\Rightarrow\gamma=90^0$

3. $\cos\theta=\frac{1}{2}$

4. Use $cos^2\alpha+cos^2\beta+cos^2\gamma=1$

5. d.r's of AB = (a, b, c)=(6,2,3)

$Cos\theta=\frac{|c|}{\sqrt{a^2+b^2+c^2}}=\frac{3}{7}$

$Tan\theta=\frac{\sqrt{40}}{3}$

6. $P=\left(\frac{2}{7}(21),\frac{6}{7}(21),\frac{-3}{7}(21)\right)$=(6,18,-9)

7. If $A=(1,1,1)$ then $OA=\sqrt{3}$ and

$\angle AOX=\angle AOY=\angle AOZ$

8. (a,b,c) = $(x_2-x_1,y_2-y_1,z_2-z_1)$ = (6,-3,2)

$(l,m,n)=\left(\frac{6}{7},\frac{-3}{7},\frac{2}{7}\right)$

9. D.r's of AC = (1,1,-1), D.r's of BD = (5,9,7)

$\cos\theta=\frac{|5+9-7|}{\sqrt{3}.\sqrt{155}}\Rightarrow\theta=\cos^{-1}\left(\frac{7}{\sqrt{465}}\right)$

10. $a_1a_2+b_1b_2+c_1c_2=0\Rightarrow\theta=90^o$

$\therefore \sec\frac{\theta}{2}+\cos ec\frac{\theta}{2}=\sqrt{2}+\sqrt{2}=2\sqrt{2}$

11. D.r's of a line joining $(k,1,2),(3,4,6)$ are $(3-k,3,4)$

D.r's of a line joining $(-4,3,-6),(5,12,l)$ are $(9,9,l+6)$

Since these two lines are parallel

$\frac{3-k}{9}=\frac{3}{9}=\frac{4}{l+6}, \quad \Rightarrow k=0, l=6$

12. D.r's of the line joining $(-1,2,3),(2,-1,4)$ are $(3,-3,1)$

D.r's of the line joining $(x,-2,4),(1,2,3)$ are $(1-x,4,-1)$

The two lines are perpendicular

$\Rightarrow 3(1-x)-12-1=0$, $\Rightarrow x=-\frac{10}{3}$

13. D.r's of BA $=(5+1,0-2,6+3)=(6,-2,-3)$

$\Rightarrow$ D.c's of BA are $\left(\frac{6}{7},-\frac{2}{7},-\frac{3}{7}\right)$

D.r's of CA are $(0+1,4-2,-1+3)\equiv(1,2,2)$

$\Rightarrow$ D.c's of CA are $\left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$

$\therefore$ D.r's of the internal bisector of $\angle BAC$ are

$\left(\frac{6}{7}+\frac{1}{3},-\frac{2}{7}+\frac{2}{3},-\frac{3}{7}+\frac{2}{3}\right)=\left(\frac{25}{21},\frac{8}{21},\frac{5}{21}\right)=(25,8,5)$

14. $\cos 45^o, \cos 45^o, \cos 90^o$

15. $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}=\frac{1}{1}-\frac{2}{1}+\frac{1}{1}=0$

$\Rightarrow$ Angle between the lines $=\frac{\pi}{2}$

16. $l+m-n=0 \Rightarrow n=l+m$

$l^2+m^2-n^2=0 \qquad \Rightarrow l^2+m^2-(1+m)^2=0$

$\Rightarrow -2lm=0 \qquad \Rightarrow l=0$ or $m=0$

If $l=0$ then

$n=m \Longrightarrow l:m:n=0:m:n=0:1:1$.

If $m=0$ then

$n=l \Rightarrow l:m:n=l:0:l=1:0:1$

$\cos\theta=\frac{0.1+1.0+1.1}{\sqrt{2}.\sqrt{2}}=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}$

17. $a+b+c=0 \rightarrow (1)$

$2ab+2ac-bc=0 \rightarrow (2)$

$\Rightarrow 2a(b+c)-bc=0, \qquad [a=-(b+c)]$

$\Rightarrow (b+2c)(2b+c)=0$

$\Rightarrow c=-2b$ or $-\frac{b}{2}$

If $c=-2b \Rightarrow a+b-2b=0 \Rightarrow a=b$

$a:b:c=$ 1: 1 : -2

If $c=-\frac{b}{2}$ then $a+b-\frac{b}{2}=0$, $\Rightarrow a=-\frac{b}{2}$

$\Rightarrow a:b:c=-\frac{b}{2}:b:-\frac{b}{2}=-1:2:-1=1:-2:1$

If θ is an angle between the lines then

$\cos\theta=\frac{(1)(1)+(1)(-2)+(-2)(1)}{\sqrt{1+1+4}\sqrt{1+4+1}}$

$=-\frac{1}{2} \Rightarrow \theta=\frac{2\pi}{3}$ or $\frac{\pi}{3}$

18. D.r's of AB = (2, 3, 6) = (a, b, c)

Use $\cos\beta=\frac{b}{\sqrt{a^2+b^2+c^2}}$

19. Use $\left|l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)\right|$

Where *(l,m,n)* are d.c's of PQ

20. $\theta=90^o \Rightarrow \cos\theta=0$

21. p, q, r = 12, 3, k

Use $AB^2=p^2+q^2+r^2$

22. Let $A(1,1,1), B(4,1,1), C(4,5,1)$

$AB^2+BC^2=CA^2$, $\triangle ABC$ is a right angled triangle.

Area of the triangle

$=\frac{1}{2}AB.BC=\frac{1}{2}(3)(4)=6$ sq.unit

23. $l=|y_1-y_2|$;

$m=\sqrt{(x_1-x_2)^2+(z_1-z_2)^2}$

EXERCISE - II

1. **A line makes the same angle θ with each of the X- axis and Z- axis. It makes β angle with Y-axis such that $\sin^2\beta = 3\sin^2\theta$ then $\cos^2\theta =$**

1) $\frac{2}{5}$ 2) $\frac{1}{5}$ 3) $\frac{3}{5}$ 4) $\frac{4}{5}$

2. **The d.r's of the line AB are (6, -2, 9). If the line AB makes angles α, β with OY, OZ respectively where $O = (0,0,0)$ then $sin^2\alpha - sin^2\beta =$ (AIEEE 2004)**

1) $\frac{77}{121}$ 2) $\frac{-32}{121}$ 3) 77 4) $\frac{85}{121}$

3. **A line makes angles α, β, γ with the coordinate axes. If $\alpha + \beta = \frac{\pi}{2}$ then $(\cos\alpha + \cos\beta + \cos\gamma)^2$ is equal to**

1) $1+\sin 2\alpha$ 2) $1+\cos 2\alpha$

3) $1-\sin 2\alpha$ 4) 1

4. **A line OP where O = (0, 0, 0) makes equal angles with OX, OY, OZ. The point on OP, which is at a distance of 6 units from 'O' is**

1) $\left(\frac{12}{\sqrt{3}}, \frac{12}{\sqrt{3}}, \frac{12}{\sqrt{3}}\right)$ 2) $\left(2\sqrt{3}, -2\sqrt{3}, 2\sqrt{3}\right)$

3) $\left(2\sqrt{3}, 2\sqrt{3}, 2\sqrt{3}\right)$ 4) $\left(6\sqrt{3}, 6\sqrt{3}, 6\sqrt{3}\right)$

5. **If O = (0, 0, 0), OP = 5 and the d.r's of OP are (1, 2, 2) then $P_x + P_y + P_z =$**

1) 25 2) $\frac{25}{9}$ 3) $\frac{25}{3}$ 4) $\left(\frac{5}{3}, \frac{10}{3}, \frac{10}{3}\right)$

6. **If $(x,3,5)$ and $(2,-1,2)$ are d.r's of two lines and angle between the lines is 45^0 then the values of x are**

1) $-4,-52$ 2) $3,42$ 3) $4,52$ 4) $-3,32$

7. **The d.r's of the line x = ay + b, z = cy + d are**

1) 1, a, c 2) a, 1, c 3) b, 1, c 4) c, a, 1

8. **If the d.c's (l,m,n) of two lines are connected by the relations $2l - m + 2n = 0$ and $mn + nl + lm = 0$ then the angle between the lines is**

1) $\frac{\pi}{4}$ 2) $\frac{\pi}{3}$ 3) $\frac{\pi}{6}$ 4) $\frac{\pi}{2}$

9. **If the dr's of two lines are given by $3lm - 4l\text{n} + mn = 0$ and $l + 2m + 3n = 0$ then the angle between the lines is**

1) $\frac{\pi}{2}$ 2) $\frac{\pi}{3}$ 3) $\frac{\pi}{4}$ 4) $\frac{\pi}{6}$

10. **If the d.c's (l,m,n) of two lines are connected by the relations $l + 5m + 3n = 0$, $7l^2 + 5m^2 - 3n^2 = 0$ then the d.c's of the two lines are**

1) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

2) $\left(\frac{1}{14}, \frac{2}{14}, \frac{3}{14}\right), \left(\frac{1}{26}, \frac{3}{26}, \frac{4}{26}\right)$

3) $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right), \left(\frac{1}{\sqrt{26}}, \frac{3}{\sqrt{26}}, \frac{4}{\sqrt{26}}\right)$

4) $\left(\frac{-1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right), \left(\frac{-1}{\sqrt{26}}, \frac{3}{\sqrt{26}}, \frac{4}{\sqrt{26}}\right)$

11. **The triangle formed by the points $(4,2,4)(10,2,-2),(2,0,-4)$ is**

1) Equilateral triangle
2) Right angled triangle
3) Isosceles triangle
4) Right angled isosceles triangle

12. **The vertices of a triangle are $(2,3,5)$, $(-1,3,2)$, $(3,5,-2)$, then the angles are**

1) $30^0, 30^0, 120^0$

2) $Cos^{-1}\left(\frac{1}{\sqrt{5}}\right), 90^0, Cos^{-1}\left(\frac{\sqrt{5}}{\sqrt{3}}\right)$

3) $30^0, 60^0, 90^0$

4) $Cos^{-1}\left(\frac{1}{\sqrt{3}}\right), 90^0, Cos^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right)$

13. If the d.c's (l,m,n) of two lines are connected by the relations $l+m+n=0$, $2lm-mn+2nl=0$ then the d.c's of the two lines are

1) $\left(\frac{1}{\sqrt{16}},\frac{1}{\sqrt{6}},\frac{-2}{\sqrt{6}}\right),\left(\frac{1}{\sqrt{6}},\frac{-2}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$

2) $\left(\frac{1}{14},\frac{2}{14},\frac{3}{14}\right),\left(\frac{1}{26},\frac{3}{26},\frac{4}{26}\right)$

3) $\left(\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}}\right),\left(\frac{1}{\sqrt{26}},\frac{3}{\sqrt{26}},\frac{4}{\sqrt{26}}\right)$

4) $\left(\frac{1}{\sqrt{16}},\frac{-1}{\sqrt{6}},\frac{-2}{\sqrt{6}}\right),\left(\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$

14. $A=(-1,2,-3),\ B=(5,0,-6),\quad C=(0,4,-1)$ **are the vertices of a triangle. The d.c's of the internal bisector of $\angle BAC$ are**

1) $\left(\frac{25}{\sqrt{714}},\frac{-8}{\sqrt{714}},\frac{-5}{\sqrt{714}}\right)$ 2) $\left(\frac{5}{\sqrt{74}},\frac{6}{\sqrt{74}},\frac{8}{\sqrt{74}}\right)$

3) $\left(\frac{25}{\sqrt{714}},\frac{8}{\sqrt{714}},\frac{5}{\sqrt{714}}\right)$ 4) $\left(\frac{-5}{\sqrt{74}},\frac{6}{\sqrt{74}},\frac{-8}{\sqrt{74}}\right)$

15. The foot of the perpendicular from $(1,2,3)$ to the line joining the points $(6,7,7)$ & $(9,9,5)$ is

1) $(5,3,9)$ 2) $(3,5,9)$

3) $(3,9,5)$ 4) $(3,9,9)$

16. If a line in the space makes angles α,β and γ with the coordinate axes, then

$\cos 2\alpha+\cos 2\beta+\cos 2\gamma+\sin^2\alpha+\sin^2\beta+\sin^2\gamma=$

1) -1 2) 0 3) 1 4) 2

17. If a line makes angles α,β,γ with positive axes, then the range of $\sin\alpha\sin\beta+\sin\beta\sin\gamma+\sin\gamma\sin\alpha$ is

1) $\left(\frac{-1}{2},1\right)$ 2) $\left(\frac{1}{2},2\right)$

3) $(-1,2)$ 4) $(-1,2]$

KEY

01) 3	02) 1	03) 1	04) 3	05) 3
06) 3	07) 2	08) 4	09) 1	10) 1
11) 1	12) 4	13) 1	14) 3	15) 2
16) 3	17) 4			

SOLUTIONS

1. $\cos^2\theta+\cos^2\beta+\cos^2\theta=1$

$\Rightarrow 2\cos^2\theta=\sin^2\beta \qquad \Rightarrow 2\cos^2\theta=3\sin^2\theta$

$\Rightarrow c\,os^2\theta=\frac{3}{5}$

2. (a, b, c) = (6, -2, 9)

Use $cos\alpha=\frac{b}{\sqrt{a^2+b^2+c^2}},cos\beta=\frac{c}{\sqrt{a^2+b^2+c^2}}$

3. $\beta=\frac{\pi}{2}-\alpha$

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1\Rightarrow\gamma=90^0$

$(\cos\alpha+\cos\beta+\cos\gamma)^2=(\cos\alpha+\sin\alpha)^2$

4. $OP=6$; $d.c's of\ OP=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)=(l,m,n)$

P = (lr, mr, nr)

5. OP = r; OP d.c's = (l, m, n) ; P = (lr, mr,nr)

6. $\frac{1}{\sqrt{2}}=\frac{2x+7}{3\sqrt{34+x^2}}$

$\Rightarrow x^2-56x+208=0 \qquad \Rightarrow x=4,52$

7. $Use\ \cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}.\sqrt{a_2^2+b_2^2+c_2^2}}$

8. $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}=\frac{1}{2}-\frac{1}{1}+\frac{1}{2}=0$

$\Rightarrow$ Angle = 90^0

9. $l+2m+3n=0\ \Rightarrow l=-2m-3n$

$3lm-4\ln+nm=0$

$\Rightarrow 3m(-2m-3n)-4n(-2m-3n)+mn=0$

$\Rightarrow -6m^2-9mn+8mn+12n^2+mn=0$

$\Rightarrow 12n^2=6m^2 \qquad \Rightarrow m=\pm\sqrt{2}n, l=\left(+2\sqrt{2}\right)n$

D.r's of the lines are

$\left(-2\sqrt{2}-3,\sqrt{2},1\right)\left(2\sqrt{2}-3,-\sqrt{2},1\right)$

$\therefore a_1a_2+b_1b_2+c_1c_2=\left(-2\sqrt{2}-3\right)\left(2\sqrt{2}-3\right)-2+1$

$=9-8-2+1=0$

$\Rightarrow$ Required angle $=\dfrac{\pi}{2}$

10. $l+5m+3n=0 \Rightarrow l=-5m-3n$

$7l^2+5m^2-3n=0$

$\Rightarrow 7(1-5m-3n)^2+5m^2-3n^2=0$

$\Rightarrow 3m+2n=0, 2m+n=0$

$\Rightarrow 6m^2+7nm+2n^2=0$

$\Rightarrow (3m+2n)(2m+n)=0$

$\Rightarrow 3m+2n=0, 2m+n=0$

If $2m+n=0$ then $m=k, n=-2k, l=k$

D.r's of one line are $(k,k,-2k)=(1,1,-2)$

If $3m+2n=0$ then $l=\dfrac{p}{3},\ m=\dfrac{-2p}{3},\ n=p$

D.r's of second line are $(p,-2p,3p)=(1,-2,3)$

D.c's of two lines are

$\left(\dfrac{1}{\sqrt{6}},\dfrac{1}{\sqrt{6}},\dfrac{-2}{\sqrt{6}}\right),\left(\dfrac{1}{\sqrt{14}},\dfrac{-2}{\sqrt{14}},\dfrac{3}{\sqrt{14}}\right)$

11. Let $A(4,2,4), B(10,2,-2), C(2,0,-4)$

D.r's of $\overrightarrow{AB}$ are $=(1,0,-1)$

D.r's of $\overrightarrow{BC}$ are $=(4,1,1)$

D.r's of $\overrightarrow{CA}$ are $=(1,1,4)$

If α is an angle between $\overleftrightarrow{AB},\overleftrightarrow{AC}$ then

$$\cos\alpha=\frac{|1\times1+0\times1+(-1)\times4|}{\sqrt{1+0+1}\sqrt{1+1+16}}=\frac{1}{2}$$

$\Rightarrow \alpha=60^0$

If β is an angle between $\overleftrightarrow{BC},\overleftrightarrow{AC}$ then

$$\cos\beta=\frac{|1\times4+1\times1+(4)\times1|}{\sqrt{1+0+1}\sqrt{1+1+1}}=\frac{1}{2} \quad \Rightarrow \beta=60^0$$

If γ is an angle between $\overleftrightarrow{AB},\overleftrightarrow{CA}$ then

$$\cos\gamma=\frac{|1\times1+0\times1+(-1)\times4|}{\sqrt{1+1+16}\sqrt{1+0+1}}=\frac{1}{2} \quad \Rightarrow \gamma=60^0$$

Angles of the triangle are 60^0, 60^0, 60^0

$\therefore$ ABC is an equilateral triangle

12. Let $A(2,3,5), B(-1,3,2), C(3,5,-2)$

D.r's of $\overleftrightarrow{AB}$ are $(-1-2, 3-3, 2-5)$

$=(-3,0,-3)=(1,0,1)$,

D.r's of $\overleftrightarrow{BC}$ are

$(3+1,5-3,-2-2)=(4,2,-4)=(2,1,-2)$

D.r's of $\overleftrightarrow{CA}$ are $(3-2,5-3,-2-5)=(1,2,-7)$

If α is an angle between $\overleftrightarrow{AB},\overleftrightarrow{AC}$ then

$$Cos\,\alpha=\frac{1}{\sqrt{3}} \Rightarrow \alpha=Cos^{-1}\frac{1}{\sqrt{3}}$$

If β is an angle between $\overleftrightarrow{BC},\overleftrightarrow{AB}$ then

$Cos\beta=0 \Rightarrow \beta=90^0$

If γ is an angle between $\overleftrightarrow{BC},\overleftrightarrow{CA}$ then

$$\gamma=Cos^{-1}\sqrt{\frac{2}{3}}$$

Angles of the triangle are

$$Cos^{-1}\frac{1}{\sqrt{3}},90^0,Cos^{-1}\sqrt{\frac{2}{3}}$$

13. $l+m+n=0 \Rightarrow l=-m-n$

$2lm-mn+2nl=0$

$\Rightarrow (-m-n)-nm+2n(-m-n)=0$

$\Rightarrow -2m^2-2nm-mn-2mn-2n^2=0$

$\Rightarrow 2m^2+5mn+2n^2=0$

$\Rightarrow (2m+n)(m+2n)=0$

$\Rightarrow 2m+n=0$ or $m+2n=0$

If $2m+n=0$ then $m=k, n=-2k, l=k$

D.r's of one line are $(k,-k,-2k)=(1,1,-2)$

If $m+2n=p$ then $n=p, m=-2p, l=p$

D.r's of second line are $(p,-2p,p)=(1,-2,1)$

14. Bisector of $\angle A$ meets BC at D

$BD : DC = AB : AC = 7:3$

$\Rightarrow D = \left(\frac{15}{10}, \frac{28}{10}, \frac{-25}{10}\right)$

d.r's of $AD = \left(\frac{25}{10}, \frac{8}{10}, \frac{5}{10}\right)$ = (25, 8, 5)

d.c's of AD = $\left(\frac{25}{\sqrt{714}}, \frac{8}{\sqrt{714}}, \frac{5}{\sqrt{714}}\right)$

15. Any point on the line joining the given points can be taken as $(6+3t, 7+2t, 7-2t)$

If it is the required foot of the perpendicular of $(1,2,3)$

we get $3(5+3t)+2(5+2t)-2(4-2t)=0$

$\Rightarrow t=-1$

$\therefore$ Foot of the perpendicular

$=(6-3, 7-2, 7+2)=(3,5,9)$

16. $\cos 2\alpha + \cos 2\beta + \cos 2\gamma + \sin^2\alpha + \sin^2\beta + \sin^2\gamma$

$= 2\cos^2\alpha - 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1 +$

$1 - \cos^2\alpha + 1 - \cos^2\beta + 1 - \cos^2\gamma$

$= \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

17. $(\sin\alpha + \sin\beta + \sin\gamma)^2 > 0$

and $\sin^2\alpha + \sin^2\beta + \sin^2\gamma - \sin\alpha\sin\beta$

$-\sin\beta\sin\gamma - \sin\gamma\sin\alpha \geq 0$

But $\sin\alpha, \sin\beta, \sin\gamma > 0$

EXERCISE - III

1. If O is the origin and the line OP of length r makes an angle α with X-axis and lies in the XY-plane then the coordinates of P are

1) $(r\cos\alpha, 0, r\sin\alpha)$ 2) $(r\cos\alpha, r\sin\alpha, 0)$

3) $(0, 0, r\cos\alpha)$ 4) $(r\sin\alpha, r\cos\alpha, 0)$

2. The three lines with d.r's $(1,1,2)$ $(\sqrt{3}-1, -\sqrt{3}-1, 4)$, $(-\sqrt{3}-1, \sqrt{3}-1, 4)$ **forms**

1) An equilateral triangle
2) A right angled triangle
3) An isosceles triangle
4) A right angled isosceles triangle

3. Let a line makes an angle 'θ' with X and Z-axes and β with Y-axis. If $\sin(\beta) = \sqrt{3}\sin\theta$, then $\cos^2\theta =$

1) $\frac{3}{5}$ 2) $\frac{5}{3}$ 3) $\frac{2}{5}$ 4) $\frac{1}{5}$

4. A line makes acute angles α, β, γ with the coordinate axes such that $\cos\alpha\cos\beta = \cos\beta\cos\gamma = \frac{2}{9}$ and $\cos\gamma\cos\alpha = \frac{4}{9}$ then $\cos\alpha + \cos\beta + \cos\gamma$ value is

1) $\frac{25}{9}$ 2) $\frac{5}{9}$ 3) $\frac{5}{3}$ 4) $\frac{2}{3}$

5. If the dr's of a line are $(1+\lambda, 1-\lambda, 2)$ and it makes an angle 60^0 with the Y- axis then λ is

1) $1\pm\sqrt{3}$ 2) $4\pm\sqrt{5}$ 3) $2\pm2\sqrt{5}$ 4) $2\pm\sqrt{5}$

6. If the angle between line with d.c's $\left(-\frac{2}{\sqrt{21}}, \frac{a}{\sqrt{21}}, \frac{b}{\sqrt{21}}\right)$ and other line with d.c's $\left(\frac{3}{\sqrt{54}}, \frac{3}{\sqrt{54}}, -\frac{6}{\sqrt{54}}\right)$ is 90^0 then a pair of possible values of 'a' and 'b' respectively are

1) -1, 4 2) 4, 2 3) 4,1 4) -4,-2

7. If three consecutive vertices of a parallelogram are $A(4,3,5)$, $B(0,6,0)$, $C(-8,1,4)$ and D is the fourth vertex then the angle between $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ is

1) $Cos^{-1}\left(\frac{55}{\sqrt{149}\sqrt{161}}\right)$ 2) $Cos^{-1}\left(\frac{65}{\sqrt{149}\sqrt{161}}\right)$

3) $Cos^{-1}\left(\frac{15}{\sqrt{149}\sqrt{161}}\right)$ 4) $Cos^{-1}\left(\frac{3}{\sqrt{149}\sqrt{161}}\right)$

8. If A = (2, 1,9), B = (-4, 1, -3), C = (0, 7, 6) and in the ΔABC the equation of the median through C is $\frac{x}{a} = \frac{y-7}{b} = \frac{z-6}{c}$ then a + b + c =

1) 9 2) 7 3) 10 4) 4

9. $P(1,2,-2,)$, $Q(8,10,11)$, $R(1,2,3)$, $S(3,5,7)$ **if λ denotes the length of projection of PQ on RS then $29\lambda^2+29$ is equal to**
1) 8100 2) 8029 3) 8129 4) 90

10. **If the lengths of the sides of a rectangular parallelopiped are 3,2,1 then the angle between two diagonals out of four diagonals is**
1) $cos^{-1}\left(\frac{6}{7}\right)$ 2) $cos^{-1}\left(\frac{2}{3}\right)$
3) $cos^{-1}\left(\frac{13}{14}\right)$ 4) $cos^{-1}\left(\frac{9}{14}\right)$

11. **I) If $P=(0,1,2), Q=(4,-2,1)$ Then $\angle POQ=\pi/2$ where 'O' is origin.**
II) If the d.r's of two lines are $(1,-1,0)$ and $(1,-2,1)$ then the angle between them is $\frac{\pi}{6}$.
Which of the above statements are correct
1) only I 2) only II
3) Both I & II 4) Neither I nor II.

12. **Observe the following statements**
Statement I : The dr's of a straight line L_1 are (a_1, b_1, c_1) and dr's of another straight line L_2 are (a_2, b_2, c_2). The straight lines L_1, L_2 are perpendicular if $a_1a_2+b_1b_2+c_1c_2=0$
Statement II : The dr's of L_1 are (2, 5, 7) and dr's of L_2 are $\left(\frac{4}{\sqrt{19}},\frac{10}{\sqrt{19}},\frac{14}{\sqrt{19}}\right)$. The lines L_1, L_2 are parallel
Which of the following is correct?
1)I is true, II is true & II is correct explanation of I
2) I is true, II is true & II is not correct explanation of I
3) I is true, II is false
4) I is false, II is true

13. **I) If the d.c's of two non-parallel lines satisfy $l+m+n=0$ and $l^2+m^2-n^2=0$ then the angle between the lines is $\frac{\pi}{3}$**
II) If the d.r's of two non-parallel lines are $(0,\lambda,-\lambda)$ and $(\mu,0,-\mu)$ then angle between the lines is $\frac{\pi}{3}$ $(\lambda>0,\mu>0)$
1) Both I and II are true and II is the correct explanation of I
2) Both I and II are true and II is not the correct explanation of I
3) I is true but II is false
4) I is false but II is true

KEY

01) 2	02) 1	03) 1	04) 3	05) 4
06) 3	07) 1	08) 3	09) 3	10) 1
11) 3	12) 2	13) 2		

SOLUTIONS

1. OP lies in XY- plane and makes α angle with X-axis $\Rightarrow$ it makes $\frac{\pi}{2}-\alpha$ with Y- axis and $\frac{\pi}{2}$ with Z- axis.
d.c's of OP are
$$(l,m,n)=\left(\cos\alpha,\cos\left(\frac{\pi}{2}-\alpha\right),\cos\frac{\pi}{2}\right)$$
$=(\cos\alpha,\sin\alpha,0)$, $P=(lr,mr,nr)$

2. If α is the angle between (1), (2) then $\cos\alpha=\frac{1}{2}\Rightarrow\alpha=60^o$ and β is the angle between (1),(3) then $\cos\beta=\frac{1}{2}\Rightarrow\beta=60^0$

3. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$
$\therefore\cos^2\theta+\cos^2\theta+\cos^2\beta=1$
$2\cos^2\theta=\sin^2\beta$ ....(1)
$\sin(\beta)=\sqrt{3}\sin\theta(\text{given})$
$\sin^2\beta=3\sin^2\theta$ ....(2)
$\therefore\ 2\cos^2\theta=3\sin^2\theta$
$=3(1-\cos^2\theta), 5\cos^2\theta=3$, $\cos^2\theta=\frac{3}{5}$

4. $(\cos\alpha+\cos\beta+\cos\gamma)^2=1+\frac{4}{9}+\frac{4}{9}+\frac{8}{9}=\frac{25}{9}$

$\Rightarrow \cos\alpha+\cos\beta+\cos\gamma=\frac{5}{3}$

5. $\cos\beta=\frac{b}{\sqrt{a^2+b^2+c^2}}$

$\Rightarrow \lambda^2-4\lambda-1=0 \Rightarrow \lambda=2\pm\sqrt{5}$

6. $\left(\frac{-2}{\sqrt{21}}\right)^2+\left(\frac{a}{\sqrt{21}}\right)^2+\left(\frac{b}{\sqrt{21}}\right)^2=1 \quad (\because l^2+m^2+n^2=1)$

$4+a^2+b^2=21 \quad \Rightarrow a^2+b^2=17 \; ....(1)$

Angle between the given lines is 90^0

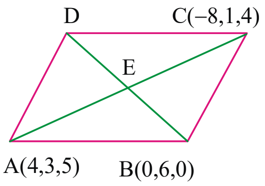
$\Rightarrow\left(\frac{-2}{\sqrt{21}}\right)\left(\frac{3}{\sqrt{54}}\right)+\left(\frac{a}{\sqrt{21}}\right)\left(\frac{3}{\sqrt{54}}\right)$

$+\left(\frac{b}{\sqrt{21}}\right)\left(\frac{-6}{\sqrt{54}}\right)=0$

$-6+3a-6b=0 \Rightarrow 3a-6b=6 \Rightarrow a-2b=2 .....(2)$

Solving (1) & (2), we get a possible solution given by $a=4:b=1$

7.



In the figure E is mid point of AC and BD

$\therefore E=\left(-2,2,\frac{9}{2}\right)$

Since it is also midpoint BD,

we have $D=(-4,-2,9)$

D.r's of AC are (12, 2, 1)

D.c's of AC are $\left(\frac{12}{\sqrt{149}},\frac{2}{\sqrt{149}},\frac{1}{\sqrt{149}}\right)$

D.r's of BD are (-4, -8, 9)

D.c's of BD are $\left(\frac{-4}{\sqrt{161}},\frac{-8}{\sqrt{161}},\frac{9}{\sqrt{161}}\right)$

Angle between diagonals $\theta=Cos^{-1}\left(\frac{55}{\sqrt{149}\sqrt{161}}\right)$

8. F = mid point of AB , d.r's of CF= (a, b, c)

9. D.c's of $RS=\left(\frac{2}{\sqrt{29}},\frac{3}{\sqrt{29}},\frac{4}{\sqrt{29}}\right)$

$\lambda=\left|l(x_1-x_2)+m(y_1-y_2)+n(z_1-z_2)\right|$

10. (a, b, c) = (3, 2, 1); Use $\cos\theta=\frac{a^2+b^2-c^2}{a^2+b^2+c^2}$

11. Use $\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

12. I is true, because the line L_1 with d.r's (a_1,b_1,c_1) and the line L_2 with d.r's (a_2,b_2,c_2) are perpendicular if , $a_1a_2+b_1b_2+c_1c_2=0$

II is true because , $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

$\frac{2}{4/\sqrt{19}}=\frac{5}{10/\sqrt{19}}=\frac{7}{14/\sqrt{19}},\ \frac{1}{2}=\frac{1}{2}=\frac{1}{2}$

II is not correct explantion of I

13. I) Solve the given equations for the d.r's of the lines and use "$\cos\theta$" formula

II) Use "$\cos\theta$" formula

JEE MAINS QUESTIONS

1. An angle between the lines whose direction cosines are given by the equations, l + 3m + 5n = 0 and 5lm – 2mn + 6nl = 0, is [Online April 15, 2018]

1) $\cos^{-1}\frac{1}{8}$ 2) $\cos^{-1}\frac{1}{6}$ 3) $\cos^{-1}\frac{1}{3}$ 4) $\cos^{-1}\frac{1}{4}$

2. The angle between the lines whose direction cosines satisfy the equations l +m +n= 0 and $l^2 + m^2 + n^2$ is [2014]

1) $\frac{\pi}{6}$ 2) $\frac{\pi}{2}$ 3) $\frac{\pi}{3}$ 4) $\frac{\pi}{4}$

KEY

1)2 2)3

SOLUTIONS

1)Given

l + 3m + 5n = 0 .. (1)

and 5lm – 2mn + 6nl = 0 ..(2)

From eq. (1) we have

l = – 3m – 5n

Put the value of l in eq. (2), we get;

5 (– 3m – 5n) m – 2mn + 6n (– 3m – 5n) = 0

$$15m^2 + 45mn + 30n^2 = 0$$

$$m^2 + 3mn + 2n^2 = 0$$

$$m^2 + 2mn + mn + 2n^2 = 0$$

(m + n) (m + 2n) = 0

Therfore, m = – n or m = – 2n

For m = – n, l = – 2n

And for m = – 2n, l = n

(l, m, n) = (– 2n, – n, n) Or (l, m, n) = (n, – 2n, n)

(l, m, n) = (– 2, – 1, 1) Or (l, m, n) = (1, – 2, 1)

Therefore, angle between the lines is given as:

$$COS(\theta) = \frac{1}{6} \qquad \theta = \cos^{-1}\frac{1}{6}$$

2. Given, l + m + n = 0 and $l^2 = m^2 + n^2$

$$(-m-n)^2 = m^2 + n^2$$

mn =0 and m =0 or n=0

If m = 0 then l = –n

we know

$l^2 + m^2 + n^2 = 1$ and $n = \pm\frac{1}{\sqrt{2}}$

i.e $(l_1, m_1, n_1) = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$

If n = 0 then l = –m

$l^2 + m^2 + n^2 = 1 \quad \rightarrow 2m^2 = 1$

$$m = \pm\frac{1}{\sqrt{2}}$$

let , $m = \frac{1}{\sqrt{2}}$

$l = -\frac{1}{\sqrt{2}}$ and n=0

$$(l_2, m_2, n_2) = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$$

$$\cos\theta = \frac{1}{2} \quad \rightarrow \theta = \frac{\pi}{3}$$

3D-PLANES

SYNOPSIS

Equation of a Plane :

- Every first degree equation in x,y,z always represents a plane.
- Plane surface is a surface in which line joining every two points P and Q on it lies entirely in the surface.
- The general form of equation of plane is $ax+by+cz+d=0$, a,b,c are not all zero i.e., $a^2+b^2+c^2\neq 0$

Equation of Planes with Different Conditions :

- i) The equation of the plane passing through the point (x_1,y_1,z_1) and having d.r's of normal as (a,b,c) is
$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ or
$ax+by+cz=ax_1+by_1+cz_1$

ii) The equation of the plane passing through a point (x_1,y_1,z_1) and parallel to the plane $ax+by+cz+d=0$ is
$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$
=> $ax+by+cz=ax_1+by_1+cz_1$

Equation of plane which is Parallel to lines :

- i) The equation of the plane passing through the point (x_1,y_1,z_1) and parallel to lines whose d.r's are (a_1,b_1,c_1) and (a_2,b_2,c_2) is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$

ii) The equation of the plane passing through the points (x_1,y_1,z_1), (x_2,y_2,z_2) and parallel to the line whose d.r's are (a,b,c) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a & b & c \end{vmatrix}=0$$

iii) The equation of the plane passing through three non collinear points $(x_1,y_1,z_1),(x_2,y_2,z_2),(x_3,y_3,z_3)$ is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

iv) If $(x_1,y_1,z_1),(x_2,y_2,z_2)(x_3,y_3,z_3)$ and (x_4,y_4,z_4) are coplanar, then

$$\begin{vmatrix} x_4-x_1 & y_4-y_1 & z_4-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

General equation of a plane with different conditions :

- i) The equation of a plane with **d.r's** of normal as (a , b ,c) is $ax+by+cz+d=0$.

ii) If a) a=0, $b\neq 0, c\neq 0$ Then equation $by+cz+d=0$ represents a plane which is parallel to x-axis and $\perp^{er}$ to YZ - plane.

b) $b=0, a\neq 0, c\neq 0$ then equation $ax+cz+d=0$ represents a plane which is parallel to y-axis and $\perp^{er}$ to xz -plane.

c) $a\neq 0, b\neq 0, c=0$ then equation $ax+by+d=0$ represents a plane which is parallel to z-axis and $\perp^{er}$ to XY -plane.

iii) The equation of the plane passing through (x_1, y_1, z_1) and parallel to

a) yz- plane and $\perp^{er}$ to X-axis is $x = x_1$

b) xy-plane and $\perp^{er}$ to Z-axis is $z = z_1$

c) zx-plane and $\perp^{er}$ to Y-axis is $y = y_1$

iv) Equation of plane parallel to the plane $ax + by + cz + d_1 = 0$ is of the form $ax + by + cz + d_2 = 0$

v) Distance between the above two parallel planes is $\dfrac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$

vi) Equation of plane parallel to $\vec{r}.\vec{n} = d_1$ is $\vec{r}.\vec{n} = d_2$ (vector form)

vii) The equation of the plane, mid way between the parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is

$$ax + by + cz + \left(\frac{d_1 + d_2}{2}\right) = 0$$

viii) The equation of the plane which bisects the line joining $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ and perpendicular to AB is

$$(x_1 - x_2)x + (y_1 - y_2)y + (z_1 - z_2)z = \frac{(x_1^2 + y_1^2 + z_1^2) - (x_2^2 + y_2^2 + z_2^2)}{2}$$

ix) The reflection of $a^1x + b^1y + c^1z + d^1 = 0$ in the plane $ax + by + cz + d = 0$ is given by

$$2(aa^1 + bb^1 + cc^1)(ax + by + cz + d)$$
$$= (a^2 + b^2 + c^2)(a^1x + b^1y + c^1z + d^1)$$

Intercept form of a plane :

✈ i) If a plane cuts X-axis at $A(a,0,0)$, Y-axis at $B(0,b,0)$ and Z-axis at $C(0,0,c)$ then a,b,c are called X-intercept,Y-intercept, Z-intercept of the plane.

ii) The equation of the plane in intercept form is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

iii) If $ax + by + cz + d = 0$ is a plane if $a \neq 0, b \neq 0, c \neq 0$ then X-intercept $= -\dfrac{d}{a}$

Y-intercept $= -\dfrac{d}{b}$, Z-intercept $= -\dfrac{d}{c}$

iv) The equation of the plane whose intercepts are K times the intercepts made by the plane $ax + by + cz + d = 0$ on corresponding axes is $ax + by + cz + kd = 0$.

Foot and image :

✈ i) The foot of the perpendicular of the point $P(x_1, y_1, z_1)$ on the plane $ax + by + cz + d = 0$ is $Q(h,k,l)$ then

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{l - z_1}{c} = \frac{-(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

ii) If Q (h, k, l) is the image of the point $p(x_1, y_1, z_1)$ w.r.to the plane $ax + by + cz + d = 0$ then

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = \frac{l - z_1}{c} = \frac{-2(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

iii) If 'd ' is the distance from the origin and (l,m,n) are the dc's of the normal to the plane through the origin, then the foot of the perpenducular is (ld, md, nd)

Ratio formula :

✈ i) The ratio in which the plane ax + by + cz + d = 0 divides the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$-(ax_1 + by_1 + cz_1 + d) : (ax_2 + by_2 + cz_2 + d)$$

ii) Position of the points w.r.to the plane

a) If $\dfrac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d} > 0$ then the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ lie on same side of the plane $ax + by + cz + d = 0$

b) If $\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}<0$ then the points $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ lie on opposite sides of the plane $ax+by+cz+d=0$

Normal form of a plane :

✈ i) If (l,m,n) are the direction cosines of normal to plane π and p is the $\perp^{er}$ distance from origin to the plane then the equation of plane is $lx+my+nz=p$

ii) The normal form of the plane representing by the equation $ax+by+cz+d=0$ is

a) If d < 0

$$\frac{a}{\sqrt{a^2+b^2+c^2}}x+\frac{b}{\sqrt{a^2+b^2+c^2}}y+\frac{c}{\sqrt{a^2+b^2+c^2}}z=\frac{-d}{\sqrt{a^2+b^2+c^2}}$$

b) If d > 0

$$\frac{-a}{\sqrt{a^2+b^2+c^2}}x+\frac{-b}{\sqrt{a^2+b^2+c^2}}y+\frac{-c}{\sqrt{a^2+b^2+c^2}}z=\frac{d}{\sqrt{a^2+b^2+c^2}}$$

Perpendicular distance from point to the plane :

✈ i) The perpendicular distance from (x_1,y_1,z_1) to the plane ax + by + cz + d = 0 is $\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

ii) The perpendicular distance of the plane ax+by+cz+d=0 from the origin is $\frac{|d|}{\sqrt{a^2+b^2+c^2}}$.

Areas :

✈ i) Area of the triangle formed by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ with

a) X – axis , Y –axis is $\frac{1}{2}|ab|$ Sq. units

b) Y– axis, Z– axis is $\frac{1}{2}|bc|$ Sq. units

c) Z– axis, X– axis is $\frac{1}{2}|ca|$ Sq. units

ii) If the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the co-ordinate axes in the points A,B,C. then the area of the triangle ABC is $\frac{1}{2}\sqrt{(ab)^2+(bc)^2+(ca)^2}$.

Angle between Two Planes :

✈ i) The angle between two planes is equal to the angle between the perpendiculars from the origin to the planes.

ii) If 'θ ' is the angle between the planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ then

$$\cos\theta=\pm\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

iii) If the above two planes are parallel then $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

iv) If the above two planes are perpendicular then $a_1a_2+b_1b_2+c_1c_2=0$

v) Angle between the line with d.c's(l_1,m_1,n_1) and the plane whose normal with d.c's (l_2,m_2,n_2) is θ then $\cos(90-\theta)=|l_1l_2+m_1m_2+n_1n_2|$

vi) If θ is angle between a line L and a plane π then the angle between L and normal to the plane π is $90\pm\theta$.

Equations of planes bisecting the angles between given planes :

✈ i) Equations of two planes bisecting the angles between the planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

ii) If $d_1,d_2>0$

Condition	Acute	Obtuse
$a_1a_2+b_1b_2+c_1c_2>0$	–	+
$a_1a_2+b_1b_2+c_1c_2<0$	+	–

iii) a) The Bisector planes are perpendicular to each other

b) Positive sign bisector is the bisector containing the origin.

The projection of line segment on a line (plane) :

✈ Let P, Q be two points and L be a line (Π plane). If M, N are feet of perpendiculars from P,Q to the line L (to the plane Π) respectively then MN is called projection of PQ on the line L(the plane Π). The length of projection of PQ is always non-negative.

EXERCISE - I

1. The equation of the plane passing through the point (3, -6, 9) and perpendicular to the x-axis is

1) $x+2=0$ 2) $y-3=0$ 3) $z-7=0$ 4) $x-3=0$

2. The product of the d.r's of a line perpendicular to the plane passing through the points (4,0,0), (0,2,0) and (1,0,1) is

1) 6 2) 2 3) 0 4) 1

3. Equation of the plane through the mid-point of the join of A(4,5,-10) and B(-1,2,1) and perpendicular to AB is

1) $5x+3y-11z+\frac{135}{2}=0$

2) $5x+3y-11z=\frac{135}{2}$

3) $5x+3y+11z=135$

4) $5x+3y-11z+\frac{185}{2}=0$

4. A plane which passes through the point (3, 2, 0) and the line $\frac{x-4}{1}=\frac{y-7}{5}=\frac{z-4}{4}$ is

1) $x-y+z=1$ 2) $x+y+z=5$

3) $x-2y-z=1$ 4) $2x-y+z=5$

5. The equation of the plane parallel to the plane $2x+3y+4z+5=0$ and passing through the point (1,1,1) is

1) $2x+3y+4z-9=0$ 2) $2x+3y+4z+9=0$

3) $2x+3y+4z+7=0$ 4) $2x+3y+4z-7=0$

6. Distance between two parallel planes $7x+4y-4z+3=0$ and $14x+8y-8z-12=0$ is

1) $\frac{15}{9}$ 2) 1 3) $\frac{9}{15}$ 4) $\frac{1}{2}$

7. In the space the equation $by+cz+d=0$ represents a plane perpendicular to the plane

1)YOZ 2) ZOX 3)XOY 4) Z = k

8. If the foot of perpendicular from (0, 0, 0) to a plane is (1, 2, 2) then the equation of the plane is

1) $-x+2y+8z-9=0$ 2) $x+2y+2z-9=0$

3) $x+y+z-5=0$ 4) $x+2y=3z+1=0$

9. The foot of the perpendicular from (7, 14, 5) to $2x+4y-z=2$ is

1) $(-1,1,0)$ 2) $(1,2,8)$ 3) $(2,-1,-2)$ 4) $(1,2,3)$

10. The ratio in which the plane $\bar{r}.(\bar{i}-2\bar{j}+3\bar{k})=17$ divides the line joining the points $-2\hat{i}+4\hat{j}+7\hat{k}$ and $3\hat{i}-5\hat{j}+8\hat{k}$ is

1) 1 : 52) 1 : 10 3) 3 : 5 4) 3 : 10

11. For the plane $\Pi \equiv 2x+3y+5z+10=0$, the point $(2,3,-5)$ lie in the

1) Opposite to the origin side

2) Origin side 3) Plane 4) can not say

12. The normal form of $2x-2y+z=5$ is

1) $12x-4y+3z=39$ 2) $-\frac{6}{7}x+\frac{2}{7}y+\frac{3}{7}z=1$

3) $\frac{12}{13}x-\frac{4}{13}y+\frac{3}{13}z=3$ 4) $\frac{2}{3}x-\frac{2}{3}y+\frac{1}{3}z=\frac{5}{3}$

13. The d.c's of the normal to the plane $2x-y+2z+5=0$ are

1) (3, -2, 6) 2) $\left(\frac{2}{7},\frac{3}{7},\frac{-6}{7}\right)$

3) $\left(\frac{3}{7},\frac{-2}{7},\frac{6}{7}\right)$ 4) $\left(\frac{2}{3},\frac{-1}{3},\frac{2}{3}\right)$

14. A plane passes through $(2,3-1)$ and is perpendicular to the line having dr's $(3,-4,7)$. The perpendicular distance from the origin to this plane is (EAM-2017)

1) $\frac{3}{\sqrt{74}}$ 2) $\frac{5}{\sqrt{74}}$ 3) $\frac{6}{\sqrt{74}}$ 4) $\frac{13}{\sqrt{74}}$

15. 5, 7 are the intercepts of a plane on the Y-axis, Z-axis respectively, if the plane is parallel to the X- axis then the equation of that plane is [EAM - 2018]

1) $5y+7z=35$ 2) $7y+5z=1$

3) $\frac{y}{5}+\frac{z}{7}=35$ 4) $7y+5z=35$

16. If the plane $7x+11y+13z=3003$ meets the coordinate axes in A, B, C then the centroid of the ΔABC is

1) $(143,91,77)$ 2) $(143,77,91)$

3) $(91,143,77)$ 4) $(143,66,91)$

17. If the areas of triangles formed by a plane with the positive $X,Y:Y,Z:Z,X$ axes respectively are 12, 9, 6 sq. unit respectively then the equation of the plane is

1) $\frac{x}{4}+\frac{y}{6}+\frac{z}{3}=1$ 2) $\frac{x}{6}+\frac{y}{3}+\frac{z}{4}=1$

3) $\frac{x}{3}+\frac{y}{4}+\frac{z}{6}=1$ 4) $\frac{x}{3}+\frac{y}{6}+\frac{z}{4}=1$

18. The area of the triangle formed by $\frac{x}{4}+\frac{y}{3}-\frac{z}{2}=1$ with X-axis and Y-axis is

1) 2 2) 3 3) 6 4) 12

19. The angle between the planes $2x+y+z=3,\ x-y+2z=5$ is [EAM - 2019]

1) $\frac{\pi}{2}$ 2) $\frac{\pi}{6}$ 3) $\frac{3\pi}{4}$ 4) $\frac{\pi}{3}$

20. If the planes $2x+3y-z+5=0$, $x+2y-kz+7=0$ are perpendicular then $k=$

1) 4 2) 6 3) 8 4) -8

21. If $\lambda x+4y+5z=7$, $4x+4\lambda y+10z-14=0$ represent the same plane then the value of $\lambda=$

1) 1 2) 2 3) 0 4) 3

22. If the planes $x+2y+kz=0$ and 2x+y-2z+3=0 are at right angles, then the values of k is [EAM -2020]

1) $-\frac{1}{2}$ 2) $\frac{1}{2}$ 3) -2 4) 2

23. If the points $(1,1,p)$ and $(-3,0,1)$ be equidistant from the plane $\bar{r}.(3\bar{i}+4\bar{j}+12\bar{k})+13=0$ then the value of P =

1) $-\frac{1}{3}$ 2) 6 3) 3 4) $\frac{1}{3}$

KEY

01) 4	02) 1	03) 2	04) 1	05) 1
06) 2	07) 1	08) 2	09) 2	10) 4
11) 1	12) 4	13) 4	14) 4	15) 4
16) 1	17) 1	18) 3	19) 4	20) 4
21) 2	22) 4	23) 1		

SOLUTIONS

1. Equation of the plane passing through (3, -6, 9) and perpendicular to x-axis is $x=x_1$

$\Rightarrow x-3=0$

2. Equation of the plane passing through the given points is $x+2y+3z-4=0$,

D. r's of normal = (1, 2, 3)

3. Mid point of AB = $\left(\frac{3}{2},\frac{7}{2},\frac{-9}{2}\right)$

D.r's of AB are (5,3,-11)

Equation of plane is

$$\left[\bar{r}-\left(\frac{3}{2}\bar{i}+\frac{7}{2}\bar{j}-\frac{9}{2}\bar{k}\right)\right].(5\bar{i}+3\bar{j}-11\bar{k})=0$$

$$\bar{r}.(5\bar{i}+3\bar{j}-11\bar{k})=\frac{135}{2}$$

4. Verify options

5. $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

6. Distance $= \frac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}} = 1$

7. Verification method

8. D.r's of the perpendicular to the plane are (1,2,2)

9. D.r's of the normal to $2x+4y-z=2$ is $(2,4,-1)$

The point $(1,2,8)$ lies on a plane and D.r's of a line joining $(1,2,8)$ and $(7,14,5)$ are $(6,12,-3)=(2,4,-1)$.,

$\therefore$ Required point $=(1,2,8)$

10. Plane is $\bar{r}.(\bar{i}-2\bar{j}+3\bar{k})=17$ --(1)

A point P dividing the join of $-2\bar{i}+4\bar{j}+7\bar{k}$ and $3\bar{i}-5\bar{j}+8\bar{k}$ in the ratio $\lambda:1$ is $\left(\frac{3\lambda-2}{\lambda+1}\right)\bar{i}+\left(\frac{-5\lambda+4}{\lambda+1}\right)\bar{j}+\left(\frac{8\lambda+7}{\lambda+1}\right)\bar{k}$

It lies on (1) then we get $,\lambda=\frac{3}{10} \Rightarrow \lambda:1=3:10$

11. $\Pi_{111}=4+9-25+10=-2$, $\Pi_{222}=10 \Rightarrow$ The point lie on the opposite to the origin side.

12. $\sqrt{a^2+b^2+c^2}=\sqrt{4+4+1}=3\cdot$

Normal form is $\frac{2}{3}x-\frac{2}{3}y+\frac{1}{3}z=\frac{5}{3}$

13. $D.c's=\pm\left(\frac{a}{\sqrt{a^2+b^2+c^2}},\frac{b}{\sqrt{a^2+b^2+c^2}},\frac{c}{\sqrt{a^2+b^2+c^2}}\right)$

$=\pm\left(\frac{2}{3},\frac{-1}{3},\frac{2}{3}\right)$

14. Equation of the plane is

$3(x-2)-4(y-3)+7(z+1)=0$

$\Rightarrow 3x-4y+7z+13=0$

Perpendicular distance from origin

$=\frac{13}{\sqrt{9+16+49}}=\frac{13}{\sqrt{74}}$

15. Plane equation is $\frac{y}{5}+\frac{z}{7}=1$

16. The plane $7x+11y+13z=3003$ meets the coordinate axes in $A(429,0,0)$, $B(0,273,0)$, $C(0,0,231)$.

Centroid of ΔABC is $\left(\frac{429}{3},\frac{273}{3},\frac{231}{3}\right)$

$=(143,91,77)$

17. If a,b,c are the intercepts of the required plane then $\frac{1}{2}ab=12,\frac{1}{2}bc=9,\frac{1}{2}ca=6$

$\Rightarrow ab=24, bc=18, ca=12$

$a^2=\frac{abac}{bc}=\frac{24\times 12}{18}=16 \Rightarrow a=4$

$ab=24, ac=12 \Rightarrow b=6, c=3$

$\therefore$ The equation of the plane is $\frac{x}{4}+\frac{y}{6}+\frac{z}{3}=1$

18. Area $=\frac{1}{2}|4\times 3|=6$ sq. units

19. $\cos\theta=\frac{|(2)(1)+(1)(-1)+(1)(2)|}{\sqrt{4+1+1}\sqrt{1+1+4}}=\frac{3}{6}=\frac{1}{2}$

$\Rightarrow \theta=\frac{\pi}{3}$

20. Given planes are perpendicular $a_1a_2+b_1b_2+c_1c_2=0$

$\Rightarrow (2)(1)+(3)(2)+(-1)(-k)=0$

$\Rightarrow 2+6+k=0, \Rightarrow k=-8$

21. Given equations represent the same plane

$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}=\frac{d_1}{d_2} \Rightarrow \frac{\lambda}{4}=\frac{5}{10} \Rightarrow \lambda=2$

22. Use $a_1a_2+b_1b_2+c_1c_2=0$

23. Distance from point $A(\bar{a})$ to plane $\bar{r}.\bar{n}+k=0$

is $\frac{|\bar{a}.\bar{n}+k|}{|\bar{n}|}$

EXERCISE - II

1. **The vertices of a tetrahedron are A(3,4,2) B(1,2,1) C(4,1,3) D(-1,-1,3). The height of A above the base BCD.**

   1) $\frac{27}{\sqrt{237}}$ 2) $\frac{23}{\sqrt{237}}$ 3) $\frac{20}{\sqrt{237}}$ 4) $\frac{27}{\sqrt{247}}$

2. **If the equation of the plane passing through the points (1,2,3), (-1,2,0) and perpendicular to the ZX - plane is** $ax+by+cz+d=0$ $(a>0)$ **then** **[EAM -2015]**

   1) $a=0$ *and* $c=0$ 2) $a+d=0$

   3) $c+d-5=0$ 4) $a+c+d-4=0$

3. **The dr's of a normal to the plane passing through** $(0,0,1),(0,1,2)$ **and** $(1,2,3)$ **are**

   1) $(0,1,-1)$ 2) $(1,0,-1)$

   3) $(0,0,-1)$ 4) $(1,0,0)$

4. **The equation of the plane which passes through the line of intersection of the planes** $2x-y=0$ **and** $3z-y=0$ **and is perpendicular to the plane** $4x+5y-3z=8$ is

   1) $28x-17y+9z=0$ 2) $28x+17y+9z=0$

   3) $2x+17y-9z=0$ 4) $2x-y-z=0$

5. **The equation of the plane through the line of intersection of planes** $ax+by+cz+d=0$, $a'x+b'y+c'z+d'=0$ **and parallel to the lines y = 0 = z is**

   1) $(ab'-a'b)x+(bc'-b'c)y+(ad'-a'd)=0$

   2) $(ab'-a'b)y+(ac'-a'c)z+(ad'-a'd)=0$

   3) $(ab'-a'b)x+(bc'-b'c)z+(ad'-a'd)=0$

   4) $(ab'-a'b)x-(bc'-b'c)y+(ad'+a'd)=0$

6. **The equation to the plane through the line of intersection of** $2x+y+3z-2=0$, $x-y+z+4=0$ **such that each plane is at a distance of 2 unit from the origin is**

   1) $x+y+2z+13=0, x+y+z-3=0$

   2) $2x+y-2z+3=0, x-2y-2z-3=0$

   3) $15x-12y+16z+50=0, x+2y+2z-6=0$

   4) x-y+2z-13=0, x+y-z-3=0

7. **A plane** π **passes through the point (1,1,1). If b, c, a are the dr's of a normal to the plane, where a, b, c** $(a<b<c)$ **are the prime factors of 2001, then the equation of the plane** π **is**

   1) $29x+31y+3z=63$ 2) $23x+29y-29z=23$

   3) $23x+29y+3z=55$ 4) $31x+27y+3z=71$

8. **The dr's of a normal to the plane through (1,0,0), (0,1,0) which makes an angle of** $\frac{\pi}{4}$ **with the plane** $x+y=3$ **are**

   1) $1,\sqrt{2},1$ 2) $1,1,\sqrt{2}$ 3) $1,1,2$ 4) $\sqrt{2},1,1$

9. **Let** $A(1,1,1), B(2,3,5)$ *and* $C(-1,0,2)$ **be three points, then equation of a plane parallel to the plane ABC which is at a distance 2 units is**

   1) $2x-3y+z+2\sqrt{14}=0$ 2) $2x-3y+z-\sqrt{14}=0$

   3) $2x-3y+z+2=0$ 4) $2x-3y+z-2=0$

10. **A variable plane is at a constant distance 3p from the origin and meets the axes in A, B and C. The locus of the centroid of the triangle ABC is** **[EAM -2016]**

    1) $x^{-2}+y^{-2}+z^{-2}=p^{-2}$

    2) $x^{-2}+y^{-2}+z^{-2}=4p^{-2}$

    3) $x^{-2}+y^{-2}+z^{-2}=16p^{-2}$

    4) $x^{-2}+y^{-2}+z^{-2}=9p^{-2}$

11. **A variable plane intersects the coordinate axes at A,B,C and is at a constant distance 'p' from 0(0,0,0). Then the locus of the centroid of the tetrahedron OABC is**

1) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$ 2) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{4}{p^2}$

3) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{p^2}$ 4) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=16p^2$

12. The equation of the plane which is parallel to X-axis and making intercepts 3 and 8 on Y and Z-axes respectively is

1) $3y+8z=24$ 2) $3y-8z=24$

3) $8y-3z=24$ 4) $8y+3z=24$

13. The sum of the intercepts of the plane which bisects the line segment joining (0,1,2) and (2,3,0) perpendicularly is

1) 2 2) 4 3) 6 4)12

14. A plane meets the coordinate axes at A, B, C so that the centroid of the triangle ABC is (1, 2, 4). Then the equation of the plane is (EAM-2020)

1) $x+2y+4z=12$ 2) $4x+2y+z=12$

3) $x+2y+4z=3$ 4) $4x+2y+z=3$

15. The reflection of the plane $2x-3y+4z-3=0$ in the plane $x-y+z-3=0$ is the plane

1) $4x-3y+2z-15=0$ 2) $x-3y+2z-15=0$

3) $4x+3y-2z+15=0$ 4) $4x+3y+2z+15=0$

16. The equations of bisectors of angles between YZ-plane and XZ-plane is

1) $x-z=0, x+2z=0$ 2) $x-z+2=0$

3) $x+z=0, x-z=0$ 4) $x+y=0, x-y=0$

KEY

01) 2	02) 4	03) 1	04) 1	05) 2
06) 3	07) 3	08) 2	09) 1	10) 1
11) 3	12) 4	13) 1	14) 2	15) 1
16) 4				

SOLUTIONS

1. The equation of the plane containing BCD

$$\begin{vmatrix} x-1 & y-2 & z-1 \\ 3 & -1 & 2 \\ -2 & -3 & 2 \end{vmatrix}=0$$

$\Rightarrow 4x-10y-11z+27=0$

distance from $A=\frac{23}{\sqrt{237}}$.

2. The plane equation is $\frac{x}{l}+\frac{z}{m}=1$.

$\frac{1}{l}+\frac{3}{m}=1; \frac{-1}{l}=1 \Rightarrow l=-1; m=3/2$

3. Let $A(0,0,1), B(0,1,2), C(1,2,3)$

$\overrightarrow{AB}=(0,1,1), \overrightarrow{AC}=(1,2,2)$,

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix}=$$

$\vec{i}(2-2)-\vec{j}(0-1)+\vec{k}(0-1)=\vec{j}-\vec{k}$

$\therefore$ D.r's of normal to the plane are (0, 1, -1)

4. Any plane through the line is

$2x-y+\lambda(3z-y)=0$ (1)

given $4x+5y-3z-8=0$ (2)

(1) and (2) are perpendicular we have

$2(4)+5(-(1+\lambda))+3\lambda(-3)=0 \Rightarrow \lambda=\frac{3}{14}$

$\Rightarrow 14(2x-y)+3(3z-y)=0$.

$\Rightarrow 28x-17y+9z=0$

5. Equation of the plane through the intersection of the planes $ax+by+cz+d=0$ and

$a^1x+b^1y+c^1z+d^1=0$

$(a^1x+b^1y+c^1z+d^1)+\lambda(ax+by+cz+d)=0$

which is parallel to Y = 0 = Z

parallel to X - axis

$\Rightarrow (a^1+a\lambda)^1=0 \quad \Rightarrow a\lambda=-a^1=\lambda=\frac{-a^1}{a}$

the equation of the plane is

$(a^1b-ab^1)y+(a^1c-ac^1)z+a^1d-ad^1=0$

6. Equation of the plane is

$(2x+y+3z-2)+k(x-y+z+4)=0$

$$p=\frac{|-2+4k|}{\sqrt{(2+k)^2+(1-k)^2+(3+k)^2}}=2$$

$\Rightarrow k = 13, -1$

$\therefore$ Planes are $(2x + y + 3z - 2) + 13$

$(x - y + z + 4) = 0, (2x + y + 3z - 2) - 1$

$(x - y + z + 4) = 0$

$\Rightarrow 15x - 12y + 16z + 50 = 0, x + 2y + 2z - 6 = 0$

7. $2001 = 3 \times 23 \times 29$ *and*

$(3 + 23 + 29) = 55 \Rightarrow a = 3, b = 23, c = 29$

8. Any plane through (1,0,0) is

$A(x-1) + B(y-0) + C(z-0) = 0$ ....(1)

It contains (0,1,0) if $-A + B = 0$......(2)

Also (1) makes an angle of $\frac{\pi}{4}$ with the plane

$x + y = 3$,

$\therefore \cos\frac{\pi}{4} = \frac{|A+B|}{\sqrt{A^2+B^2+C^2}\sqrt{1^2+1^2}}$

$\Rightarrow 2AB = C^2$ ......(3)

From (2) and (3),

$C^2 = 2A^2 \qquad \Rightarrow C = \pm\sqrt{2}A$

Hence A:B:C =A:A: $\pm\sqrt{2}.A$

$\therefore$ D.r's are $\quad (1, 1, \pm\sqrt{2})$

9. $A(1,1,1), B(2,3,5), C(-1,0,2)$ d.r's of AB are (1, 2, 4)

D.r's of AC are $(-2,-1,1)$.

D.r's of normal to plane ABC are (2,-3,1) As a result, equation of the plane ABC is $2x - 3y + z = 0$ Let the equation of the required plane is $2x - 3y + z = k$, then

$\left|\frac{k}{\sqrt{4+9+1}}\right| = 2k = \pm 2\sqrt{14}$

Hence, equation of the required plane is $2x - 3y + z + 2\sqrt{14} = 0$

10. If $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

distance from origin = 3p

$\Rightarrow \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{9p^2}$

$\Rightarrow$ Then $a = 3x_1, b = 3y_1, c = 3z_1$

$\therefore$ Locus is $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$.

11. Intercepts of the plane = 4x, 4y, 4z where (x,y, z) is the centroid of the tetrahedron OABC.

Here, $\frac{1}{16x_1^2} + \frac{1}{16y_1^2} + \frac{1}{16z_1^2} = \frac{1}{p^2}$

12. Required plane equation is

$\frac{y}{3} + \frac{z}{8} = 1 \Rightarrow 8y + 3z = 24$

13. The plane equation is

$2x(x_2 - x_1) + 2y(y_2 - y_1) + 2z(z_2 - z_1)$

$+ x_1^2 + y_1^2 + z_1^2 - x_2^2 - y_2^2 - z_2^2 = 0$

where $(x_1, y_1, z_1) = (0, 1, 2)$ and

$(x_2, y_2, z_2) = (2, 3, 0)$

14. Let the equation of the plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$\therefore A = (a,0,0), B(0,b,0), C(0,0,c)$

Centroid of $\Delta ABC = (1,2,4)$

$\Rightarrow \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right) = (1,2,4) \quad \Rightarrow a = 3, b = 6, c = 12$

The equation of the plane is $\frac{x}{3} + \frac{y}{6} + \frac{z}{12} = 1$

$\Rightarrow 4x + 2y + z = 12$

15. Equation of the required plane be obtained using the reflection of $a^1x + b^1y + c^1z + d^1 = 0$ in the plane $\quad ax + by + cz + d = 0$ is given by

$2(aa^1 + bb^1 + cc^1)(ax + by + cz + d)$

$= (a^2 + b^2 + c^2)(a^1x + b^1y + c^1z + d^1)$

16. Equation of yz-plane is $x = 0$, Equation of xz-

plane is $y=0$

$\therefore$ Equation of the bisectors of the angles between the planes are

$$\frac{x}{1}=\pm\frac{y}{1}\Rightarrow x+y=0, x-y=0$$

EXERCISE - III

1. If the points (1, 1, -3) and (1, 0, -3) lie on opposite sides of the plane $x+y+3z+d=0$ then

1) $d<7$ 2) $d>8$

3) $7<d<8$ 4) $d<7\ or\ d>8$.

2. P is a point such that the sum of the squares of its distances from the planes $x+y+z=0$, $x+y-2z=0, x-y=0$ is 5 then the locus of P is

1) $x^2+y^2+z^2=10$ 2) $x^2+y^2+z^2=25$

3) $x^2+y^2+z^2=5$ 4) $x^2+y^2+z^2=50$

3. The plane $ax+by+cz+(-3)=0$ meet the co-ordinate axes in A,B,C. Then centroid of the triangle is

1) (3a,3b,3c) 2) $\left(\frac{3}{a},\frac{3}{b},\frac{3}{c}\right)$

3) $\left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$ 4) $\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$

4. The areas of triangles formed by a plane with the positive $X,Y;Y,Z;Z,X$ axes respectively are 12, 9, 6 square units then the equation of the plane is

1) $\frac{x}{4}+\frac{y}{6}+\frac{z}{3}=1$ 2) $\frac{x}{6}+\frac{y}{3}+\frac{z}{4}=1$

3) $\frac{x}{4}+\frac{y}{4}+\frac{z}{6}=1$ 4) $\frac{x}{3}+\frac{y}{6}+\frac{z}{4}=1$

5. Equation of the plane passing through the point (-1,3,2) and perpendicular to each of the planes $x+2y+3z=5$ and $3x+3y+z=0$ is

1) $7x+8y-3z=0$ 2) $7x-8y-3z=-37$

3) $7x-8y+3z+25=0$ 4) $7x+8y+3z=23$

6. If P= (0,1,0) and Q = (0,0,1) then the projection of PQ on the plane $x+y+z=3$ is

1) 2 2) $\sqrt{2}$ 3) 3 4) $\sqrt{3}$

7. A parallelopiped is formed by the planes drawn through the points (2, 3, 5) and (5, 9, 7) parallel to the coordinate planes. The length of diagonal of the paralleopiped is

1) 7 2) $\sqrt{38}$ 3) $\sqrt{155}$ 4) $\sqrt{7}$

8. If the angles made by the normal of the plane $2x+3y-4z-16=0$ with the coordinates axes X, Y, Z are $Cos^{-1}k_1, Cos^{-1}k_2, Cos^{-1}k_3$, then k_1,k_2,k_3 respecitvely are

1) $\frac{2}{\sqrt{29}},\frac{3}{\sqrt{29}},\frac{4}{\sqrt{29}}$ 2) $\frac{2}{\sqrt{29}},-\frac{3}{\sqrt{29}},\frac{4}{\sqrt{29}}$

3) $\frac{2}{\sqrt{29}},\frac{3}{\sqrt{29}},-\frac{4}{\sqrt{29}}$ 4) $\frac{1}{2},\frac{1}{3},-\frac{1}{4}$

9. If the plane $4(x-1)+k(y-2)+8(z-5)=0$ contains the line $\frac{x-1}{2}=\frac{y-2}{4}=\frac{z-5}{3}$, then k is

1) 2 2) 4 3) -8 4) 8

10. If the plane $3(x-2)+(y-2)+6(z+3)=0$ contains the line $\frac{x-2}{a}=\frac{y-2}{b}=\frac{z+3}{1}$ whose inclination with X-axis is 60^0, then it satisfies the equation

1) $6a^2+36a+37=0$ 2) $36a^2+37=0$

3) $36a^2+37a+36=0$ 4) $a+3=0$

11. If the equation of the plane passing through the line of intersection of the planes $ax+by+cz+d=0, a_1x+b_1y+c_1z+d_1=0$ and perpendicular to the XY-plane is $px+qy+rz+s=0$ then S=

1) dc_1-d_1c 2) dc_1+d_1c

3) dd_1+cc_1 4) $aa_1+bb_1+cc_1$.

12. The two planes represented by $12x^2-2y^2-6z^2-7yz+6zx-2xy=0$ are

1) $2x+y+2z=0, 6x-2y+3z=0$

2) $2x-y+2z=0, 6x+2y-3z=0$

3) $2x-y+2z+4=0, 6x+2y-3z=0$

4) $2x-y+2z=0, 6x+2y-3z+1=0$

13. The angle between the planes represented by

$2x^2-6y^2-12z^2+18yz+2zx+xy=0$ **is**

1) $Cos^{-1}\left(\frac{16}{21}\right)$ 2) $Cos^{-1}\left(\frac{17}{21}\right)$

3) $Cos^{-1}\left(\frac{19}{21}\right)$ 4) $\frac{\pi}{2}$

14. The equation of the plane through the line of intersection of the planes $x-2y+3z-1=0$, $2x+y+z-2=0$ and the point $(1,2,3)$ is

1) $7x-9y+8z=0$ 2) $7x+y+8z=0$

3) $x+3y-2z-1=0$ 4) x-3y-2z+1=0

15. The equation of the plane which is parallel to Y-axis and making intercepts of lengths 3 and 4 on X-axis and Z-axis is

1) $2x+2z=20$ 2) $4x+3z=12$

3) $4x-3z=12$ 4) $6x+13z=15$

KEY

01) 3 02) 3 03) 4 04) 1 05) 3 06) 2
07) 1 08) 3 09) 3 10) 1 11) 1 12) 2
13) 1 14) 3 15) 2

SOLUTIONS

1. d-7 and d-8 must have opposite signs
$\Rightarrow$ 7<d < 8.

2. $\left(\frac{x_1+y_1+z_1}{\sqrt{3}}\right)^2+\left(\frac{x_1+y_1-2z_1}{\sqrt{6}}\right)^2+\left(\frac{x_1-y_1}{\sqrt{2}}\right)^2=5$
$\Rightarrow x^2+y^2+z^2=5$

3. A plane meet co-ordinate axes at
$A\left(\frac{3}{a},0,0\right), B\left(0,\frac{3}{b},0\right), C\left(0,0,\frac{3}{c}\right)$
$\therefore$ *Centroid* $G=\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$

4. $\frac{1}{2}|ab|=12$, $\frac{1}{2}|bc|=9$, $\frac{1}{2}|ca|=6$

5. Equation of the plane is
$a(x+1)+b(y-3)+c(z+2)=0$
$a+2b+3c=0$; $3a+3b+c=0$

6. If L, M are the feet of the perpendiculars from P, Q to the plane then projection of PQ is LM.

7. The lengths of edges are $a=5-2=3$, $b=9-3=6$, $c=7-5=2$
$\therefore$ Length of the diagonal $=\sqrt{a^2+b^2+c^2}=7$

8. The d.r's of normal of the plane $2x+3y-4z-16=0$ are $(2,3,-4)$ and the d.c's are $\left(\frac{2}{\sqrt{29}},\frac{3}{\sqrt{29}},\frac{-4}{\sqrt{29}}\right)$
$\therefore \cos\alpha=\frac{2}{\sqrt{29}}, \cos\beta=\frac{3}{\sqrt{29}}, \cos\gamma=\frac{-4}{\sqrt{29}}$
Where α,β,γ are the angle made by the normal with X, Y, Z axes respectiely.
$\therefore \alpha=Cos^{-1}\left(\frac{2}{\sqrt{29}}\right), \beta=Cos^{-1}\left(\frac{3}{\sqrt{29}}\right)$
$\gamma=Cos^{-1}\left(\frac{-4}{\sqrt{29}}\right)$
$\therefore k_1,k_2,k_3$ values are $\frac{2}{\sqrt{29}},\frac{3}{\sqrt{29}},\frac{-4}{\sqrt{29}}$ respectively

9. If a plane contains the line, then its normal and line are perpendicular
$\therefore 4(2)+k(4)+8(3)=0$
$(i.e., al+bm+cn=0)$
$8+4k+24=0 \Rightarrow k=-8$

10. The d.r's of line are $(a,b,1)$
$\therefore$ The d.c.'s of line are
$\left(\frac{a}{\sqrt{a^2+b^2+1}},\frac{b}{\sqrt{a^2+b^2+1}},\frac{1}{\sqrt{a^2+b^2+1}}\right)$
$\therefore \cos 60^0=\frac{a}{\sqrt{a^2+b^2+1}} \Rightarrow \frac{1}{2}=\frac{a}{\sqrt{a^2+b^2+1}}$
i.e., $2a=\sqrt{a^2+b^2+1}$
i.e., $3a^2=b^2+1$ ....(1)
As the plane contains the given line, we have
$3(a)+1(b)+6(1)=0$ ....(2)
Eliminating b from (1) & (2),

we get $3a^2 = \{-(6+3a)\}^2 + 1$

i.e., $6a^2 + 36a + 37 = 0$

11. The plane equation is (ax + by + cz + d) - $\left(\frac{c}{c_1}\right)$ $(a_1x + b_1y + c_1z + d_1) = 0$.

12. The product of the equations of planes for the option (2) is $(2x - y + 2z)(6x + 2y - 3z) = 0$

$12x^2 - 2y^2 - 6z^2 + 7yz + 6xz - 2xy = 0$

$\therefore$ correct answer is (2)

13. The equation $2x^2 - 6y^2 - 12z^2 + 18yz + 2zx + xy = 0$ represents two planes and they are

$2x - 3y + 6z = 0$ ....(1)

$x + 2y - 2z = 0$ ....(2)

If θ is the angle between the planes (1) and (2)

then $\cos\theta = \frac{|(2)(1)+(-3)(2)+6(-2)|}{\sqrt{2^2+(-3)^2+6^2}\sqrt{1^2+2^2+2^2}} = \frac{16}{21}$

$\therefore \theta = Cos^{-1}\frac{16}{21}$

14. Equation of the plane is

$(x - 2y + 3z - 1) + k(2x + y + z - 2) = 0$

It passes through (1, 2, 3) then

$5 + 5k = 0 \Rightarrow k = -1$

$\therefore$ Plane is

$(x - 2y + 3z - 1) - 1(2x + y + z - 2) = 0$

$\Rightarrow x + 3y - 2z - 1 = 0$

15. Equation of plane parallel to Y-axis is of the

both $\frac{x}{a} + \frac{z}{c} = 1 \Rightarrow \frac{x}{3} + \frac{z}{4} = 1 \Rightarrow 4x + 3z = 12$

JEE MAINS QUESTIONS

1. *The shortest distance between the lines*

$\frac{x-1}{0} = \frac{y+1}{-1} = \frac{z}{1}$ and [2020]

$x + y + z + 1 = 0,\ 2x - y + z + 3 = 0$ *is* :

1) 1 2) $\frac{1}{\sqrt{3}}$ 3) $\frac{1}{\sqrt{2}}$ 4) $\frac{1}{2}$

2. If for some a $\in$ R, the lines

$L_1 : \frac{x+1}{2} = \frac{y-2}{-1} = \frac{z-1}{1}, L_2 : \frac{x+2}{\alpha} = \frac{y+1}{5-\alpha} = \frac{z+1}{1}$

are coplanar, then the line L2passes through the point : [2020]

1) (10,2,2) 2) (2,-10,-2)

3) (10,-2,-2) 4) (-2,10,-2)

3.If the equation of a plane P, passing through the intersection of the planes, x + 4y – z + 7 = 0 and 3x + y + 5z= 8 is ax + by + 6z = 15 for some a b, , ÎR then thedistance of the point (3, 2, –1) from the plane P is

___________. [2020]

4.The distance of the point (1, –2, 3) from the plane

x-y+z= 5 measured parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$ is [Jan. , 2020]

1) $\frac{7}{5}$ 2) 1 3) $\frac{1}{7}$ 4) 7

5.The foot of the perpendicular drawn from the point (4, 2, 3)to the line joining the points (1, –2, 3) and (1,1, 0) lies onthe plane : [Jan., 2020]

1) 2x + y – z = 1 2) x – y – 2z = 1

3)x – 2y + z = 1 4) x + 2y – z = 1

6.The plane which bisects the line —oining the points (4, – 2, 3)and (2, 4, – 1) at right angles also passes through thepoint: [2020]

(1) (4, 0, 1) (2) (0, –1, 1)
(3) (4, 0, –1) (4) (0, 1, –1)

7.The plane passing through the points (1, 2, 1), (2, 1, 2) andparallel to the line, 2x = 3y, z = 1 also through the point : [2020]

(1) (0, 6, –2) (2) (–2, 0, 1)
(3) (0, –6, 2) (4) (2, 0, –1)

8.A plane passing through the point (3, 1, 1) contains twolines whose direction ratios are 1, –2, 2 and 2, 3, –1respectively. If this plane also passes through the point(a, 3, 5), then a is equal to : [2019]

(1) 5 (2) –10
(3) 10 (4) –5

9.Let P be a plane passing through the points (2, 1, 0),(4, 1, 1) and (5, 0, 1) and R be any point (2, 1, 6). Then theimage of R in the plane P is: [2019]

(1) (6, 5, 2) (2) (6, 5, –2)
(3) (4, 3, 2) (4) (3, 4, –2)

10.A plane which bisects the angle between the two givenplanes 2x – y + 2z – 4 = 0 and x + 2y + 2z – 2 = 0, passesthrough the point : [2019]

(1) (1, –4, 1) (2) (1, 4, –1)
(3) (2, 4, 1) (4) (2, –4, 1)

11.If Q (0, –1, –3) is the image of the point P in the plane 3x –y + 4z = 2 and R is the point (3, –1, –2), then the area (in sq.units) of trianglePQR is :

[2019]

a) $2\sqrt{13}$ b) $\frac{\sqrt{91}}{4}$ c) $\frac{\sqrt{91}}{2}$ d) $\frac{\sqrt{65}}{2}$

12. Let P be the plane, which contains the line of intersectionof the planes, x + y + z– 6 = 0 and 2x + 3y + z+ 5 = 0 and it is perpendicular to the xy-plane. Then the distance of thepoint (0, 0, 256) from P is equal to:

[2019]

a) $\frac{17}{\sqrt{5}}$ b) $\frac{63}{\sqrt{5}}$ c) $\frac{205}{\sqrt{5}}$ d) $\frac{11}{\sqrt{5}}$

13.The equation of a plane containing the line of intersectionof the planes 2x – y – 4 = 0 and y + 2z – 4 = 0 and passingthrough the point (1, 1, 0) is :

[2019]

(1) x – 3y – 2z = –2 (2) 2x – z = 2
(3) x – y – z = 0 (4) x + 3y + z = 4

14.The sum of the intercepts on the coordinate axes of theplane passing through the point (– 2, – 2, 2) and containingthe line joining the points (1, – 1, 2) and (1, 1, 1) is [2018]

(1) 12 (2) – 8
(3) – 4 (4) 4

15.A plane bisects the line segment joining the points (1, 2, 3)and (– 3, 4, 5) at right angles. Then this plane also passesthrough the point. [2018]

(1) (– 3, 2, 1) (2) (3, 2, 1)
(3) (1, 2, – 3) (4) (– 1, 2, 3)

KEY

01)2	02)2	03)3.00	04)2	05)1
06) 3	07)2	08)1	09)2	10)4
11)3	12)4	13)3	14)3	15)1

SOLUTIONS

1.For line of intersection of planes x + y + z + 1 = 0 and 2x – y + z + 3 = 0 :

Put y = 0, we get x = –2 and z = 1

$L_2 : \bar{r} = (-2\hat{i} + \hat{k}) + \lambda(2\hat{i} + \hat{j} - 3\hat{k})$

$L_1 : \bar{r} = (\hat{i} - \hat{j}) + \mu(-\hat{j} + \hat{k})$

now $\bar{b_1} X \bar{b_2} = -2[\hat{i} + \hat{j} + \hat{k}]$ and a2-a1= $-3\hat{i} + \hat{j} + \hat{k}$

so shortest distnace is $\frac{1}{\sqrt{3}}$

2. Since, lince are coplanar

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & 1 \\ \alpha & 5-\alpha & 1 \end{pmatrix} = 0$$

$1(-1-5+\alpha)-3(2-\alpha)+2(10-2\alpha+\alpha)$

therfore, $\alpha = -4$

Equation of L2 is $\frac{x+2}{-4} = \frac{y+1}{9} = \frac{z+1}{1}$

Point (2, –10, –2) lies on line L2

.Equation of plane P is

(x+4y-z+7)+ λ (3x+y-5z-8)=0

x(1+3 λ)+y(4+ λ)+z(-1+5 λ)+(7-8 λ) = 0

$$\frac{1+3\lambda}{a} = \frac{4+\lambda}{b} = \frac{5\lambda-1}{6} = \frac{7-8\lambda}{-15}$$

From last two ratios, λ =-1

$$\frac{-2}{a} = \frac{3}{b} = -1$$

a=2, b= -3

Equation of plane is, 2x-3y +6z -15= 0

$$\text{distance} = \frac{21}{7} = 3$$

4.Equation of line through point P (1, – 2, 3) and parallelto the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$ is

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6}$$

So, any point on line = Q(2 λ +1, 3 λ -2, -6 λ +3)

Since, this point lies on plane x -y +2= 5

$$2\lambda + 1 - 3\lambda + 2 - 6\lambda + 3 = 5$$

so, $\lambda = \frac{1}{7}$

Point of intersection line and plane, Q $(\frac{9}{7}, \frac{11}{7}, \frac{15}{7})$

Required distance PQ = 1

5.Equation of line through points (1, – 2, 3) and (1, 1, 0)is $\frac{x-1}{0} = \frac{y-1}{-3} = \frac{z}{3}$

$M(1, 1-\lambda, \lambda)$

Direction ratios of P $M[-3, -1-\lambda, \lambda-3]$

$PM \perp AB$, SO $\lambda = 1$

\ Foot of perpendicular = (1, 0, 1)

This point satisfy the plane 2x + y – z = 1

6.Direction ratios of normal to the plane are <1, –3, 2>.Plane passes through (3, 1, 1)

Equation of plane is,

$1[x-3]-3[y-1]+2[z-1]=0$

x-3y+2z=0

7.Let plane passes through (2, 1, 2) be

a(x-2)+b(y-1)+z-2=0

It also passes through (1, 2, 1)

-a+b-c=0 , a-b+c=0

The given line is

$\frac{x}{3} = \frac{y}{2} = \frac{z-1}{0}$ is parallel to plane

3a+2b=0

$$= \quad \frac{a}{0-2} = \frac{b}{3} = \frac{c}{2+3}$$

$$= \quad \frac{a}{2} = \frac{b}{-3} = \frac{c}{-5}$$

$$2x - 3y - 4 + 3 - 5z + 10 = 0$$

$$2x - 3y - 5z + 9 = 0$$

8. Plane contains two lines

$$\bar{n} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 2 & 3 & -1 \end{pmatrix} = -4\hat{i} + 5\hat{j} + 7\hat{k}$$

-4(x-3)+5(y-1)+7(z-1)=0

-4x+5y+7z=0

This also passes through (a, –3, 5)

-4a-15+35=0

$-4a=-20 \qquad a=5$

9.

Equation of plane is $x + y - 2z = 3$

$$\frac{x-2}{1}=\frac{y-1}{1}=\frac{z-6}{-2}=\frac{-2(2+1-12-3)}{6}$$

$(x, y, z) = (6, 5, -2)$

10.The equations of angle bisectors are,

$$\frac{x+2y+2z-2}{3}=\pm\frac{2x-y+2z-4}{3}$$

$x - 3y - 2 = 0$ or $3x + y + 4z - 6 = 0$

$(2, -4, 1)$ lies on the second plane.

11.

(c) Image of Q $(0, -1, -3)$ in plane is,

$$\frac{(x-0)}{3}=\frac{(y+1)}{-1}=\frac{z+3}{+4}=\frac{-2(1-12-2)}{9+1+16}=1$$

$\Rightarrow x=3, y=-2, z=1$

$\Rightarrow P(3,-2,1), Q(0,-1,-3), R(3,-1,-2)$

$\therefore$ Area of ΔPQR is

$$\frac{1}{2}|\vec{QP}\times\vec{QR}|=\frac{1}{2}\left\|\begin{matrix}\hat{i} & \hat{j} & k\\ 3 & -1 & 4\\ 3 & 0 & 1\end{matrix}\right\|$$

$$=\frac{1}{2}|\{\hat{i}(-1)-\hat{j}(3-12)+\hat{k}(3)\}|$$

$$=\frac{1}{2}\sqrt{(1+81+9)}=\frac{\sqrt{91}}{2}$$

12.

(d) Let the plane be

$P\equiv(2x+3y+z+5)+\lambda(x+y+z-6)=0$

$\because$ above plane is perpendicular to xy plane.

$\therefore\left((2+\lambda)\hat{i}+(3+\lambda)\hat{j}+(1+\lambda)\hat{k}\right).\hat{k}=0\Rightarrow\lambda=-1$

Hence, the equation of the plane is,

$P\equiv x+2y+11=0$

Distance of the plane P from $(0, 0, 256)$

$$\left|\frac{0+0+11}{\sqrt{5}}\right|=\frac{11}{\sqrt{5}}$$

13.Let the equation of required plane be;

$(2x - y - 4) + \lambda\ (y + 2z - 4) = 0$

This plane passes through the point $(1, 1, 0)$ then $(2 - 1 - 4) + \lambda(1 + 0 - 4) = 0$

$\lambda = -1$

Then, equation of required plane is

$(2x - y - 4) - (y + 2z - 4) = 0$

$2x - 2y - 2z = 0$ Þ $x - y - z = 0$

14.

Equation of plane passing through three given points is:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x+2 & y+2 & z-2 \\ 1+2 & -1+2 & 2-2 \\ 1+2 & 1+2 & 1-2 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x+2 & y+2 & z-2 \\ 3 & 1 & 0 \\ 3 & 3 & -1 \end{vmatrix}=0$$

$$\Rightarrow -x+3y+6z-8=0$$

$$\Rightarrow \frac{x}{8}-\frac{3y}{8}-\frac{6z}{8}+\frac{8}{8}=0$$

$$\Rightarrow \frac{x}{8}-\frac{y}{\frac{8}{3}}-\frac{z}{\frac{8}{6}}=-1$$

$$\Rightarrow \frac{x}{-8}+\frac{y}{\frac{8}{3}}+\frac{z}{\frac{8}{6}}=1$$

15.Since the plane bisects the line —oining the points (1, 2, 3) and (–3, 4, 5) then the plane passes through the midpoint of the line which is :

$$\left(\frac{1-3}{2},\frac{2+4}{2},\frac{5+3}{2}\right)\equiv\left(\frac{-2}{2},\frac{6}{2},\frac{8}{2}\right)\equiv(-1,3,4).$$

As plane cuts the line segment at right angle, so the direction cosines of the normal of the plane are (– 3 – 1, 4 – 2, 5 – 3) = (– 4, 2, 2)

So the equation of the plane is : $-4x+2y+2z=\lambda$

As plane passes through (– 1, 3, 4)

so

$-4(-1)+2(3)+2(4)=\lambda$

$\lambda=18$

Therefore, equation of plane is : $-4x+2y+2z=18$

Now, only (– 3, 2, 1) satisfies the given plane as $-4(-3)+2(2)+2(1)=18$

****

3D–LINES

SYNOPSIS

Equation of a line :

✈ **General Form (Unsymmetrical form) of a line :**

The intersection of two plane s is a line.The equations

$a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2$

represents a line.

i) Equation to the X-axis is y = 0 , z = 0

ii) Equation to the Y- axis is x = 0 , z = 0

iii) Equation to the Z- axis is x = 0 , y = 0

iv) Equation of the line parallel to x-axis is y=p, z=q, p,q,$\in$ R

v) Equation of the line parallel to y-axis is x=h, z=q, $h,q\in$ R

vi) Equation of the line parallel to z-axis is x=h, y=p, $h,p\in$ R

Symmetrical form of a line :

✈ i) The equation of the line passing through the point (x_1,y_1,z_1) and having d.c's (l,m,n) is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

ii) The equation of a line passing through the point (x_1,y_1,z_1) and having d.r's (a, b, c) is

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

iii) The equation of the line passing through two points (x_1,y_1,z_1) and (x_2,y_2,z_2) is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Vector form of a line :

✈ Cartesian equation of a line passing through the point (x_1,y_1,z_1) and having d.r's (a,b,c) is $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$

Let $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=\lambda$

$x-x_1=a\lambda\Rightarrow x=x_1+a\lambda$

$y-y_1=b\lambda\Rightarrow y=y_1+b\lambda$

$z-z_1=c\lambda\Rightarrow z=z_1+c\lambda$, Now,

$x\bar{i}+y\bar{j}+z\bar{k}=x_1\bar{i}+\lambda a\bar{i}+y_1\bar{j}+\lambda b\bar{j}+z_1\bar{k}+\lambda c\bar{k}$

$\bar{r}=(x_1\bar{i}+y_1\bar{j}+z_1\bar{k})+\lambda(a\bar{i}+b\bar{j}+c\bar{k})$

Which is the vector equation of the line passing through (x_1,y_1,z_1) and having d.r's (a,b,c) (or) vector equation of the line passing through (x_1, y_1, z_1) and parallel to the vector $a\bar{i}+b\bar{j}+c\bar{k}$ where $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$.

Conversion of non-symmetrical form to symmetrical form :

✈ Let the equation of the line in non-symmetrical form be

$a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z +d_2$

To find the equation of the line in symmetrical form, we must know (i) its d.r's (ii) coordinates of any point on it.

i) To find the d.rs of the line :

Let l, m, n be the d.r's of the line.

Since the line lies in both the planes, it must be perpendicular to normals of both planes.

So, $a_1l + b_1m + c_1n = 0$

$a_2l + b_2m + c_2n = 0$

From these equations proportional values of l, m,n found by cross multiplication method

$$\begin{matrix} & l & & m & & n & \\ b_1 & & c_1 & & a_1 & & b_1 \\ b_2 & & c_2 & & a_2 & & b_2 \end{matrix}$$

$$\Rightarrow \frac{l}{b_1c_2 - b_2c_1} = \frac{m}{c_1a_2 - c_2a_1} = \frac{n}{a_1b_2 - a_2b_1}$$

ii) To find a point on the line :

At least one of the d.r's must be non-zero.

Let $a_1b_2 - a_2b_1 \neq 0$

The line cannot be parallel to xy-plane.

Let it intersect the xy-plane in $(x_1, y_1, 0)$

then $a_1x_1 + b_1y_1 + d_1 = 0$

and $a_2x_1 + b_2y_1 + d_2 = 0$

By solving these equations we get the point $(x_1,y_1,0)$ on the line.

Hence the equation of the line in symmetric form is $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-0}{n}$

Note: If $l \neq 0$, take a point on yz-plane as $(0,y_1,z_1)$ and if $m \neq 0$ take a point on xz-plane as $(x_1,0,z_1)$.

Parametric form :

✈ The parametric equations of the line passing through the point $P(x_1, y_1, z_1)$ and having d.c's (l,m,n) are $x = x_1 + lr$, $y = y_1 + mr$, $z = z_1 + nr$ Where r = OP

Remark: The coordinates of a point on the line whose d.c's are (l, m, n) which is at a distance of 'r' units from the point (x_1,y_1,z_1) are $(x_1 \pm lr, y_1 \pm mr, z_1 \pm nr)$

Angle between two lines :

✈ If θ is the angle between the lines given by

$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and

$\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ then

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{\sum a_1^2}\sqrt{\sum a_2^2}}\right|$$

i) a)If the lines are parellel then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

b) If the lines are $\perp^r$ then $a_1a_2 + b_1b_2 + c_1c_2 = 0$

ii) If 'θ' is the acute angle between the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and the plane $ax + by + cz + d = 0$ then

$$\sin\theta = \frac{|al + bm + cn|}{\sqrt{a^2+b^2+c^2}.\sqrt{l^2+m^2+n^2}}$$

iii) If the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ is parallel to the plane $ax + by + cz + d = 0$ then $al + bm + cn = 0$ (Normal to the plane is perpendicular to the line)

iv) If the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ perpendicular to the plane $ax + by + cz + d = 0$ then $\frac{a}{l} = \frac{b}{m} = \frac{c}{n}$.

v) d.c's of the line which makes equal angles with coordinate axes are $\pm\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and the d.r's of the line are (1, 1, 1).

Coplanar lines :

✈ Two lines are said to be coplanar if they are either parallel or intersect.

Non–Coplanar Lines :

✈ Two lines are said to be non coplanar or skew lines if they are neither parallel nor intersecting.

Condition for two lines to be coplanar :

✈ The line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ lies in the plane $ax + by + cz + d = 0$ if

$ax_1 + by_1 + cz_1 + d = 0$, $al + bm + cn = 0$.

Note: The lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$,

$\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ are coplanar

$$\Leftrightarrow \begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Equation of a plane containing lines :

✈ The equation of the plane containing the lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$, $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$

is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$ (or)

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

✈ If the lines $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$, $a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2$ are coplanar then

$$\frac{a_1x_1+b_1y_1+c_1z_1+d_1}{a_1l+b_1m+c_1n}=\frac{a_2x_1+b_2y_1+c_2z_1+d_2}{a_2l+b_2m+c_2n}$$

Skew lines :

✈ Two straight lines are said to be skew lines if they are neither parallel nor intersecting. i.e. the lines which do not lie in a plane.

Shortest distance :

✈ If L_1 and L_2 are skew lines then there is one and only one line perpendicular to both of the lines L_1 and L_2 which is called the line of shortest distance. If PQ is the line of shortest distance then the distance between P and Q is called distance between the given skew lines.

i) The shortest distance between the skew lines $\bar{r}=\bar{a}_1+\lambda\bar{b}_1, \bar{r}=\bar{a}_2+\mu\bar{b}_2$ is

$$\frac{|(\bar{a}_1-\bar{a}_2)\cdot(\bar{b}_1\times\bar{b}_2)|}{|\bar{b}_1\times\bar{b}_2|} \ (or) \ \frac{|[\bar{a}_1-\bar{a}_2 \ \ \bar{b}_1 \ \ \bar{b}_2]|}{|\bar{b}_1\times\bar{b}_2|}$$

ii) If the above two lines are coplanar or intersecting then $[\bar{a}_1-\bar{a}_2 \ \ \bar{b}_1 \ \ \bar{b}_2]=0$

iii) Shortest distance between the lines $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$

and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is

$$\left|\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}\right| \Big/ \sqrt{\sum(b_1c_2-b_2c_1)^2}$$

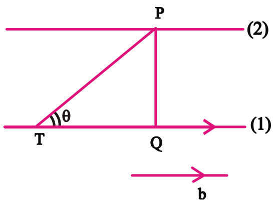
Distance between parallel lines :

✈ The distance between the parallel lines $\bar{r}=\bar{a}_1+\lambda\bar{b}, \bar{r}=\bar{a}_2+\mu\bar{b}$ is $\frac{|\bar{b}\times(\bar{a}_1-\bar{a}_2)|}{|\bar{b}|}$

Proof: Given parallel lines are:

$\bar{r}=\bar{a}_1+\lambda\bar{b}$ --- (1)

$\bar{r}=\bar{a}_2+\lambda\bar{b}$ --- (2)



Let PQ be the distance between (1) and (2)

let T be a point on (1) with $OT=\bar{a}_1$

Let $\overline{OP}=\bar{a}_2$

Let Q be the projection of P on (1)

Let θ be the angle between PT and $\bar{b}$

$\bar{b}\times\overline{TP}=\left(|\bar{b}||\overline{TP}|\sin\theta\right).\hat{n}$ --- (3)

Where $\hat{n}$ is the unit vector perpendicular to the plane of the lines (1) and (2)

$\overline{TP}=\overline{OP}-\overline{OT}=\bar{a}_2-\bar{a}_1$

$$\left[\begin{array}{l} In \ \Delta PTQ \\ \sin\theta=\frac{PQ}{PT}\Rightarrow PQ=PT.\sin\theta \end{array}\right]$$

From (3) $\bar{b}\times\overline{TP}=\left(|\bar{b}|\,|\overline{TP}|\sin\theta\right)\hat{n}$

$\Rightarrow \bar{b}\times(\bar{a}_2-\bar{a}_1)=|\bar{b}|(PQ)\hat{n}$

$\Rightarrow |\bar{b}\times(\bar{a}_2-\bar{a}_1)|=|\bar{b}|\cdot|\overline{PQ}| \quad (|\hat{n}|=1)$

$\Rightarrow |\overline{PQ}|=\frac{|\bar{b}\times(\bar{a}_2-\bar{a}_1)|}{|\bar{b}|}\Rightarrow PQ=\frac{|\bar{b}\times(\bar{a}_2-\bar{a}_1)|}{|\bar{b}|}$

$\Rightarrow PQ=\frac{|\bar{b}\times(\bar{a}_1-\bar{a}_2)|}{|\bar{b}|}$

EXERCISE - I

1. The equation of the line joining (-2,1,3) and (1,1,4) is

1) $\frac{x+2}{3}=\frac{y-1}{0}=\frac{z-3}{1}$ 2) $\frac{x-2}{3}=\frac{y+1}{0}=\frac{z+3}{1}$

3) $\frac{x+2}{4}=\frac{y+1}{3}=\frac{z-3}{2}$ 4) $\frac{x-3}{1}=\frac{y-1}{1}=\frac{z-2}{1}$

2. The equation of the line through (3,1,2) and equally inclined to the axes is

1) $\frac{x-3}{1}=\frac{y-1}{0}=\frac{z-2}{0}$ 2) $\frac{x-3}{0}=\frac{y-1}{1}=\frac{z-2}{0}$

3) $\frac{x-3}{0}=\frac{y-1}{0}=\frac{z-2}{1}$ 4) $\frac{x-3}{1}=\frac{y-1}{1}=\frac{z-2}{1}$

3. The equation of the line passing through (-1, 2, -3) and perpdendicular to the plane $2x+3y+z+5=0$ is [EAM 2017]

1) $\frac{x-1}{-1}=\frac{y+2}{1}=\frac{z-3}{-1}$ 2) $\frac{x+1}{-1}=\frac{y-2}{1}=\frac{z+3}{1}$

3) $\frac{x+1}{2}=\frac{y-2}{3}=\frac{z+3}{1}$ 4) $\frac{x+1}{1}=\frac{y-2}{1}=\frac{z+3}{1}$

4. The equation to the plane which passes through the z-axis and is perpendicular to the line $\frac{x-1}{\cos\alpha}=\frac{y+2}{\sin\alpha}=\frac{z-3}{0}$ is

1) $x\sin\alpha+y\cos\alpha=0$ 2) $x\sin\alpha-y\cos\alpha=0$

3) $x\cos\alpha+y\sin\alpha=0$ 4) $x\cos\alpha-y\sin\alpha=0$

5. The cartesian equation of line is $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$ its vector form is

1) $\bar{r}=\left(5\bar{i}+4\bar{j}+6\bar{k}\right)+\lambda\left(3\bar{i}+2\bar{j}+2\bar{k}\right)$

2) $\bar{r}=\left(5\bar{i}-4\bar{j}+6\bar{k}\right)+\lambda\left(3\bar{i}+7\bar{j}+2\bar{k}\right)$

3) $\bar{r}=\left(5\bar{i}+4\bar{j}+6\bar{k}\right)+\lambda\left(3\bar{i}+7\bar{j}+2\bar{k}\right)$

4) $\bar{r}=\left(-5\bar{i}+4\bar{j}+6\bar{k}\right)+\lambda\left(3\bar{i}+7\bar{j}+2\bar{k}\right)$

6. Parametric form of the equation of the line $3x-6y-2z-15=0=2x+y-2z-5$ is

1) $\frac{x-5}{14}=\frac{y}{2}=\frac{z}{15}$ 2) $\frac{x-1}{14}=\frac{y-5}{2}=\frac{z-1}{15}$

3) $\frac{x-3}{14}=\frac{y+1}{2}=\frac{z}{15}$ 4) $\frac{x+5}{14}=\frac{y}{2}=\frac{z}{15}$

7. The value of p so that the lines $\frac{1-x}{3}=\frac{7y-14}{2p}=\frac{z-3}{2}$ and $\frac{7-7x}{3p}=\frac{y-5}{1}=\frac{6-z}{5}$ are at right angles are

1) 70/11 2) 7/11 3) 10/7 4) 17/11

8. The angle between the lines $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z+3}{2}$ and $\frac{x}{1}=\frac{y}{1}=\frac{z}{0}$ is

1) 0^0 2) 30^0 3) 45^0 4) 90^0

9. The sine of the angle between the straight line $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ and the plane $2x-2y+z=5$ is

1) $\frac{\sqrt{5}}{3}$ 2) $\frac{2\sqrt{2}}{5}$ 3) $\frac{1}{5\sqrt{2}}$ 4) $\frac{2\sqrt{3}}{5}$

10. The angle between the lines 2x = 3y = –z and 6x = –y = – 4z is [EAM 2019]

1) 0º 2) 30º 3) 45º 4) 90º

11. The angle between the pair of lines $\bar{r}=\left(3\bar{i}+2\bar{j}-4\bar{k}\right)+\lambda\left(\bar{i}+2\bar{j}+2\bar{k}\right)$ and $\bar{r}=\left(5\bar{i}-2\bar{j}\right)+\mu\left(3\bar{i}+2\bar{j}+6\bar{k}\right)$ is

1) $\tan^{-1}\left(19/21\right)$ 2) $\cos^{-1}\left(19/21\right)$

3) $\sin^{-1}\left(19/21\right)$ 4) $\cos^{-1}\left(19/20\right)$

12. The angle between the lines x = 1, y = 2 and y = –1, z = 0 is [EAM 2020]

1) $\frac{\pi}{2}$ 2) $\frac{\pi}{6}$ 3) $\frac{\pi}{3}$ 4) 0º

13. The lines

$\frac{x-1}{a}=\frac{y-2}{3}=\frac{z-3}{4}$; $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$

are coplanar. Then a =

1) 1 2) 2 3) -1 4) -2

14. If the lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar then k can have (MAIN-2013)

1) any value 2) exactly one value
3) exactly two values 4) exactly three values

15. The equation of the plane containing the line $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z-3}{4}$ and perpendicular to the plane $x+2y+z=12$ is

1) $9x-2y+5z+4=0$ 2) $9x-2y-5z+4=0$
3) $9x-2y-5z-4=0$ 4) $9x+2y-5z-4=0$

16. A plane which passes through the point (3, 2, 0) and the line $\frac{x-4}{1}=\frac{y-7}{5}=\frac{z-4}{4}$ is (AIEEE - 2002)

1) x – y + z = 1 2) x + y + z = 5
3) x + 2y – 2 = 1 4) 2x – y + z = 5

17. The value of *m* for which staight line $3x-2y+z+3=0=4x-3y+4z+1$ is parallel to the plane $2x-y+mz-2=0$ is

1) –2 2) 8 3) –18 4) 11

18. If the line $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect then k= (JEE MAINS -2012)

1) -1 2) 2/9 3) 9/2 4) 0

19. Let L be the line of intersection of the planes 2x + 3y + z = 1 and x + 3y + 2z = 2. If L makes an angle α with the positive X-axis then $\cos\alpha =$ (AIEEE - 2007)

1) $\frac{1}{\sqrt{3}}$ 2) $\frac{1}{2}$ 3) 1 4) $\frac{1}{\sqrt{2}}$

20. The d.r's of the line given by the planes x – y + z – 5 = 0, x – 3y – 6 = 0 are

1) (3, 1,–2) 2) (2, –4, 1)
3) (1, –1, 1) 4) (0, 2, 1)

21. The equation of the plane containing the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_2}{n}$ is a $(x-x_1)+b(y-y_1)+c(z-z_1)=0$ where

1) $ax_1+by_1+cz_1=0$ 2) $al+bm+cn=0$
3) $\frac{a}{l}=\frac{b}{m}=\frac{c}{n}$ 4) $lx_1+my_1+nz_1=0$

KEY

01) 1	02) 4	03) 3	04) 3	05) 2	06) 3
07) 1	08) 3	09) 3	10) 4	11) 2	12) 1
13) 2	14) 3	15) 2	16) 1	17) 1	18) 3
19) 1	20) 1	21) 2			

SOLUTIONS

1. D.r's of the line = $(x_2-x_1, y_2-y_1, z_2-z_1)$
Let (a, b, c) = (3, 0, 1)
Use the formula $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$
2. Use the formula $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$, where $l=m=n=\frac{1}{\sqrt{3}}$
3. By verify the options
4. By verification D.r's of normal are $(\cos\alpha, \sin\alpha, 0)$
5. $\frac{x-5}{3}-\frac{y+4}{7}=\frac{z-6}{2}=\lambda$
6. $\pi_1=3x-6y-2z-15=0$
$\pi_2=2x+y-2z-5=0$
now the dr's of the common line two planes are $(14,2,15)$ and Put $z=0\, in\, \pi_1\, and\, \pi_2$ and solve them we get a point
7. $\frac{x-1}{-3}=\frac{y-2}{\frac{2p}{7}}=\frac{z-3}{2}$ --- (1)
$\frac{x-1}{\frac{-3p}{7}}=\frac{y-5}{1}=\frac{z-6}{-5}$ --- (2)
(1) and (2) are perpendicular.
$a_1a_2+b_1b_2+c_1c_2=0$
8. Use formula,
$$\cos\theta=\frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

9. Take $(a_1,b_1,c_1)=(3,4,5)$ and $(a_2,b_2,c_2)=(2,-2,1)$.
Use
$$\cos\left(\frac{\pi}{2}-\theta\right)=\frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

10. Given lines are
$$\frac{x}{3}=\frac{y}{2}=\frac{z}{-6} \quad \text{--- (1)}$$
$$\frac{x}{2}=\frac{y}{-12}=\frac{z}{-3} \quad \text{--- (2)}$$
If θ is the required angle
$$a_1a_2+b_1b_2+c_1c_2=0;\ \theta=\frac{\pi}{2}$$

11. D.r's of given lines
$$\vec{b}_1=(1,2,2),\vec{b}_2=(3,2,6);\ \cos\theta=\frac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}$$

12. The line x = 1, y = 2 is parallel to z-axis.
The line y = 1, z = 0 is parallel to x-axis.
Angle between the lines is $\frac{\pi}{2}$

13. $$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

14. $$\begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix}=0 \Rightarrow k^2+3k=0,\ k=0,-3$$

15. Take $$\begin{vmatrix} x-1 & y+1 & z-3 \\ 2 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix}=0$$

16. Verification
Required plane has to pass through the points (3, 2, 0) and (4, 7, 4)

17. $\pi_1=3x-2y+z+3=0$
$\pi_2=4x-3y+4z+1=0$
$$\begin{matrix} a & b & c & \\ -2 & 1 & 3 & -2 \\ -3 & 4 & 4 & -3 \end{matrix} \Rightarrow \frac{a}{-8+3}=\frac{b}{4-12}=\frac{c}{-9+8}$$
D.r's of the common line of the two planes are $(-5,-8,-1)$
D.r's of the normal to the plane are (2, –1, m)
Now apply the perpendicular condition with the given plane we get $m=-2$

18. $$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$$

19. D.r's of the line of intersection $\begin{matrix} 3 & 1 & 2 & 3 \\ 3 & 2 & 1 & 3 \end{matrix}$
$(a_1, b_1\ c_1)=(3,-3,3)$
D.r's of x-axis $(a_2,\ b_2\ c_2)=(1,\ 0,\ 0)$
$$\cos\alpha=\frac{3}{\sqrt{9+9+9}}=\frac{1}{\sqrt{3}}$$

20. By cross multiplication method
$$\begin{matrix} & a & b & c & \\ -1 & 1 & 1 & -1 \\ -3 & 0 & 1 & -3 \end{matrix} \Rightarrow \frac{a}{3}=\frac{b}{1}=\frac{c}{-2}$$

21. Plane contains the given line normal to the plane must be perpendicular to the line so, $al+bm+cn=0$.

EXERCISE - II

1. A line passes through two points A(2,-3,-1) and B(8,-1,2). The coordinates of a point on this line at a distance of 14 units from A are
1) (14, 1, 5) 2) (-10, -7, 7)
3) (86, 25, 41) 4) $(0,0,0)$

2. The distance of the point (1,0,–3) from the plane $x-y-z=9$ measured parallel to the line $\frac{x-2}{2}=\frac{y+2}{3}=\frac{z-6}{-6}$ is
1) 6 2) 7 3) 8 4) 9

3. The distance of the point (1, -5, 9) from the plane $x-y+z=5$ measured along a straight line $x=y=z$ is (AIEEE-2011)
1) $3\sqrt{5}$ 2) $10\sqrt{3}$ 3) $5\sqrt{3}$ 4) $3\sqrt{10}$

4. **The line** $\frac{x-2}{3}=\frac{y+1}{2}=\frac{z-1}{-1}$ **intersects the curve** $xy=c^2, z=0$ **if** ***c*****=** ***[EAM -2018]***

1) ± 1 2) $\pm\sqrt{3}$ 3) $\pm\sqrt{5}$ 4) $\pm\sqrt{7}$

5. **The point of intersection of the line** $\frac{x-3}{3}=\frac{2-y}{4}=\frac{z+1}{1}$ **and planes** $2x+4y+3z+3=0$ **,** $x+2y+3z=0$ **is**

1) (9,6,1) 2) (-9,6,1)
3) (9,-6,1) 4) (-9,-6,-1)

6. **The shortest distance between the lines** $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-1}{2}$; $\frac{x-4}{4}=\frac{y-5}{5}=\frac{z-2}{3}$ **is**

1) $\frac{1}{\sqrt{3}}$ 2) $\frac{1}{\sqrt{6}}$ 3) $\frac{1}{\sqrt{2}}$ 4) $\frac{5}{\sqrt{6}}$

7. **The shortest distance between the lines** $\bar{r}=(1-t)\bar{i}+(t-2)\bar{j}+(3-2t)\bar{k}$ **and** $\bar{r}=(s+1)\bar{i}+(2s-1)\bar{j}-(2s+1)\bar{k}$ **is**

1) $8/\sqrt{17}$ 2) $8/\sqrt{493}$ 3) $8/\sqrt{29}$ 4) $16\sqrt{29}$

8. **The distance between the parallel lines** $\bar{r}=2\bar{i}+3\bar{j}-\bar{k}+\lambda(\bar{i}-\bar{j}+2\bar{k})$ **and** $\bar{r}=-3\bar{i}+4\bar{j}+\bar{k}+\mu(\bar{i}-\bar{j}+2\bar{k})$ **is**

1) $2\sqrt{\frac{22}{3}}$ 2) $\sqrt{\frac{175}{6}}$ 3) $\frac{14}{\sqrt{6}}$ 4) 7

9. **The reflection of the point A (1, 0, 0) in the line** $\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$ **is** **[EAM -2019]**

1) (3, –4, –2) 2) (5, –8, –4)
3) (1, –1, –10) 4) (2, –3, 8)

10. **The foot of the perpendicular from (a,b,c) on the line** $x=y=z$ **is the point** (r,r,r) **where**

1) $r=a+b+c$ 2) $r=3(a+b+c)$
3) $3r=a+b+c$ 4) $r=4(a+b+c)$

11. **The length of the perpendicular from the point (1, 2, 3) to the line** $\frac{x-6}{3}=\frac{y-7}{2}=\frac{z-7}{-2}$ **is**

1) 7 2) $\sqrt{48}$ 3) 8 4) 9

12. **The length of the perpendicular from the point (–1, 3, 9) to the line** $\frac{x-13}{5}=\frac{y+8}{-8}=\frac{z-31}{1}$ **is**

1) 21 2) 22 3) 20 4) $\sqrt{439}$

KEY

01) 1 02) 2 03) 2 04) 3 05) 3 06) 2
07) 3 08) 1 09) 2 10) 3 11) 1 12) 1

SOLUTIONS

1. $\frac{x-2}{6}=\frac{y+3}{2}=\frac{z+1}{3}\Rightarrow\frac{x-2}{\frac{6}{7}}=\frac{y+3}{\frac{2}{7}}=\frac{z+1}{3/7}$ points on the line

$=\left(2\pm\frac{6}{7}r,-3\pm\frac{2r}{7},-1\pm\frac{3r}{7}\right)$ where *r*=14.

2. Equation of the line passing through (1, 0, –3) with d.r's (2, 3, –6) is

$\frac{x-1}{2}=\frac{y-0}{3}=\frac{z+3}{-6}=t$ (say)

x = 1 + 2t, y = 3t, z = –3 – 6t
Let P be a point in the plane x–y–z=9 such that AP is parallel to given line
P = (1 + 2t, 3t, –3 – 6t)
Substitute P in the given plane, t = 1
P = (3, 3, –9), AP = 7.

3. Let P = (1, -5, 9)
Let Q be a point on the given plane such that PQ is parallel to given line
The equation of the line PQ is

$\frac{x-1}{1}=\frac{y+5}{1}=\frac{z-9}{1}$

Let Q = $(1+t,t-5,t+9)$

Sub Q in the given plane, $t=-10$

$\therefore Q=(-9,-15,-1)$

PQ = $\sqrt{300}=10\sqrt{3}$

4. we have $z=0$ for the point, where the line intersects the curve. therefore,

$$\frac{x-2}{3}=\frac{y+1}{2}=\frac{0-1}{-1}$$

$$\Rightarrow \frac{x-2}{3}=1 \text{ and } \frac{y+1}{2}=1$$

$$\Rightarrow x=5 \text{ and } y=1$$

Putting these values in $xy=c^2$, we get

$5=c^2 \Rightarrow c=\pm\sqrt{5}$

5. By verification method

6. $\bar{a}_1=(2,3,1)$, $\bar{a}_2=(4,5,2)$, $\bar{b}_1=(3,4,2)$

$\bar{b}_2=(4,5,3)$

Shortest distance = $\dfrac{\left|\left[\bar{a}_1-\bar{a}_2\ \ \bar{b}_1\ \ \bar{b}_2\right]\right|}{\left|\bar{b}_1\times\bar{b}_2\right|}$

7. Given lines

$\bar{r}=(\bar{i}-2\bar{j}+3\bar{k})+t(-\bar{i}+\bar{j}-2\bar{k})$ and

$\bar{r}=(\bar{i}-\bar{j}-\bar{k})+s(\bar{i}+2\bar{j}-2\bar{k})$

$\vec{a}_1=(1,-2,3),\vec{b}_1=(-1,1,-2)$

$\vec{a}_2=(1,-1,-1),\vec{b}_2=(1,2,-2)$

Find $\dfrac{\left|\left[\bar{a}_1-\bar{a}_2\ \ \bar{b}_1\ \ \bar{b}_2\right]\right|}{\left|\bar{b}_1\times\bar{b}_2\right|}$

8. $\bar{a}_1=2\bar{i}+3\bar{j}-\bar{k}$, $\bar{a}_2=-3\bar{i}+4\bar{j}+\bar{k}$,

$\bar{b}=\bar{i}-\bar{j}+2\bar{k}$, $\bar{a}_1-\bar{a}_2=5\bar{i}-\bar{j}-2\bar{k}$

$$\bar{b}\times(\bar{a}_1-\bar{a}_2)=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 1 & -1 & 2\\ 5 & -1 & -2\end{vmatrix}=4\bar{i}+12\bar{j}+4\bar{k}$$

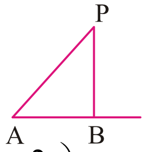
$\left|\bar{b}\times(\bar{a}_1-\bar{a}_2)\right|=\sqrt{176}$, $\left|\bar{b}\right|=\sqrt{6}$

Distance = $\dfrac{\left|\bar{b}\times(\bar{a}_1-\bar{a}_2)\right|}{|\bar{b}|}=\dfrac{\sqrt{176}}{\sqrt{6}}=2\sqrt{\dfrac{22}{3}}$

9. Any point P on the line is (2r+1, –3r–1, 8r–10)
D.r's of AP are (2r, –3r–1, 8r–10)
AP is perpendicular to the given line
2(2r) – 3(–3r–1) + 8(8r–10) = 0 $\Rightarrow$ r = 1
P (3, –4, –2)
Let B be the image of A
B = 2P – A

10. apply the formula for perpendicular condition.

11. Let P = (1, 2, 3)
Let A = (6, 7, 7)
Let B be the foot of the perpendicular of P on the given line
D.r's of given line = (3, 2, –2)



D.c's of given line = $\left(\dfrac{3}{\sqrt{17}},\dfrac{2}{\sqrt{17}},\dfrac{-2}{\sqrt{17}}\right)$

AB = projection of AP on the given line

$=\left|l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)\right|$

AB = $\sqrt{17}$

From ΔABP, $PB^2=AP^2-AB^2$ = 66 – 17

PB = $\sqrt{49}$ = 7

12. Let A = (–1, 3, 9)

Any point P on the line is

(13 + 5t, –8–8t, 31 + t)

Let P be the foot of the perpendicular of A

D.r's of AP = (14+5t, –11–8t, 22 + t)

AP is perpendicular to given line.

$a_1a_2+b_1b_2+c_1c_2=0$

$\Rightarrow$ 5(14 + 5t) + 8(11 + 8t) + 22 + t = 0

$\Rightarrow$ t = –2 , P = (3, 8, 29), AP = 21

EXERCISE - III

1. Equation of the perpendicular line from (3, –1, 11) to the line $\dfrac{x}{2}=\dfrac{y-2}{3}=\dfrac{z-3}{4}$ is

1) $\dfrac{x-3}{1}=\dfrac{y+1}{-6}=\dfrac{z-11}{4}$ 2) $\dfrac{x-3}{2}=\dfrac{y+1}{5}=\dfrac{z-11}{7}$

3) $\dfrac{x-3}{1}=\dfrac{y+1}{11}=\dfrac{z-11}{3}$ 4) $\dfrac{x-3}{1}=\dfrac{y+1}{6}=\dfrac{z-11}{4}$

2. The equation of line of shortest distance between the lines $\dfrac{x-3}{1}=\dfrac{y-5}{-2}=\dfrac{z-7}{1}$;

$\dfrac{x+1}{7}=\dfrac{y+1}{-6}=\dfrac{z+1}{1}$ is

1) $\frac{x-3}{2}=\frac{y-5}{3}=\frac{z-7}{4}$

2) $\frac{x+3}{2}=\frac{y+5}{3}=\frac{z-7}{4}$

3) $\frac{x+3}{2}=\frac{y-5}{3}=\frac{z-7}{4}$

4) $\frac{x-3}{2}=\frac{y-5}{3}=\frac{z+7}{4}$

3. A plane mirror is placed at the origin so that the direction ratios of its normal are (1, -1, 1). A ray of light, coming along the positive direction of the *x*-axis strikes the mirror. Then the direction ratios of the reflected ray are

1) $\frac{1}{3},\frac{2}{3},\frac{2}{3}$ 2) $\frac{-1}{3},\frac{2}{3},\frac{2}{3}$

3) $\frac{-1}{3},\frac{-2}{3},\frac{-2}{3}$ 4) $\frac{-1}{3},\frac{-2}{3},\frac{2}{3}$

4. If the angle between the line $x=\frac{y-1}{2}=\frac{z-3}{\lambda}$ and the plane $x+2y+3z=4$ is $\cos^{-1}\left(\sqrt{5/14}\right)$ then $\lambda=$ (AIEEE-2011)

1) $\frac{3}{2}$ 2) $\frac{5}{3}$ 3) $\frac{2}{3}$ 4) $\frac{2}{5}$

5. If lines $x=y=z$ and $x=y/2=z/3$ and third line passing through $(1,1,1)$ form a triangle of area $\sqrt{6}$ units, then point of intersection of third line with second line will be

1) $(1,2,3)$ 2) $(2,4,6)$

3) $\left(\frac{4}{3},\frac{8}{3},\frac{12}{3}\right)$ 4) $(2,1,3)$

6. Let $P(3,2,6)$ be a point in space and Q be a point on the line $\bar{r}=\left(\bar{i}-\bar{j}+2\bar{k}\right)+\mu(-3\bar{i}+\bar{j}+5\bar{k})$ then the value of μ for which the vector $\overline{PQ}$ is parallel to the plane $x-4y+3z=1$ is (IIT-2009)

1) $\frac{1}{4}$ 2) $\frac{-1}{4}$ 3) $\frac{1}{8}$ 4) $\frac{-1}{8}$

7. The equation of the plane which passes through the z-axis and is perpendcular to the line $\frac{x-a}{\cos\theta}=\frac{y+2}{\sin\theta}=\frac{z-3}{0}$ is

1) $x+y\tan\theta=0$ 2) $y+x\tan\theta=0$

3) $x\cos\theta-y\sin\theta=0$ 4) $x\sin\theta-y\cos\theta=0$

8. The image of the line $\frac{x-1}{3}=\frac{y-3}{1}=\frac{z-4}{-5}$ in the plane 2x – y + z + 3 = 0 is the line (MAINS-2014)

1) $\frac{x-3}{3}=\frac{y+5}{1}=\frac{z-2}{-5}$ 2) $\frac{x-3}{-3}=\frac{y+5}{-1}=\frac{z-2}{5}$

3) $\frac{x+3}{3}=\frac{y-5}{1}=\frac{z-2}{-5}$ 4) $\frac{x+3}{-3}=\frac{y-5}{-1}=\frac{z+2}{5}$

9. For the line $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$, which one of the following is incorrect ?

1) it lies in the plane $x-2y+z=0$

2) it is same as line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$

3) it passes through (2,3,5)

4) it is parallel to the plane $x-2y+z=6$

10. The projection of the line $\frac{x+1}{-1}=\frac{y}{2}=\frac{z-1}{3}$ in the plane x-2y+z=6 is the line of intersection of this plane with the plane is

1) 2x+y+2=0 2) 3x+y-z=2

3) 2x-3y+8z=3 4) x+y-z=1

11. If the straight lines x = 1 + s, y = –3 – λ s, z = 1 + λ s and $x=\frac{t}{2}, y=1+t, z=2-t$ with parameters 's' and 't' respectively are coplanar then λ =

1) –2 2) 0 3) $-\frac{1}{2}$ 4) –1

12. The equation of a plane which passes through the point of intersection of lines $\frac{x-1}{3}=\frac{y-2}{1}=\frac{z-3}{2}$ and $\frac{x-3}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ and at greatest distance from point (0,0,0) is

1) $4x+3y+5z=25$ 2) $4x+3y+5z=50$

3) $4x+3y+5z=49$ 4) $x+7y-5z=2$

KEY

01) 1	02) 1	03) 4	04) 3	05) 2	06) 1
07) 1	08) 3	09) 3	10) 1	11) 1	12) 2

SOLUTIONS

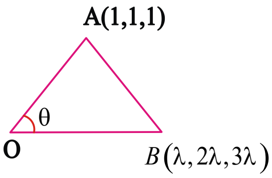
1. Let P be the foot of the perpendicular from A (3, –1, 11) to the given line then
P = (2r , 3r + 2, 4r + 3)
D.r's of A.P are (2r – 3, 3r + 3, 4r – 8)
$\overrightarrow{AP}$ is perpendiciular to the given line
$\Rightarrow$ 2(2r – 3) + 3(3r + 3) + 4(4r – 8)=0 $\Rightarrow$ r = 1
P = (2, 5, 7)
D.r's of $\overrightarrow{AP}$ are (1, –6, 4)

2. Let $P(\alpha+3,-2\alpha+5,\alpha+7)$ and $Q(7\beta-1,-6\beta-1,\beta-1)$ be the points on the given lines so that PQ is the line of shortest distance between the given lines
D.r's of $PQ=(\alpha-7\beta+4,-2\alpha+6\beta+6,\alpha-\beta+8)$
Since PQ is perpendicular to the given lines
$1(\alpha-7\beta+4)-2(-2\alpha+6\beta+6)+1(\alpha-\beta+8)=0$
$\Rightarrow 6\alpha-20\beta=0\Rightarrow 3\alpha-10\beta=0$ --- (1)
and,
$\Rightarrow 7(\alpha-7\beta+4)-6(-2\alpha+6\beta+6)+1(\alpha-\beta+8)=0$
$\Rightarrow 7\alpha-49\beta+28+12\alpha-36\beta-36+\alpha-\beta+8=0$
$\Rightarrow 20\alpha-86\beta=0\Rightarrow 10\alpha=43\beta$ --- (2)
From (1) and (2) $\alpha=0,\beta=0$
$P=(3,5,7),\ Q=(-1,-1,-1)$
D.r's of PQ = (–4, –6, –8) = (2, 3, 4)
Equation of PQ is $\frac{x-3}{2}=\frac{y-5}{3}=\frac{z-7}{4}$

3. Let (*l, m, n*) be the d.c's of reflected ray
We have (1, 0, 0) are the d.c's of incident ray (x-axis).
D.c's of normal are $\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$
Let $(l_1,m_1,n_1)=(l,m,n),\quad (l_2,m_2,n_2)=(1,0,0)$
If θ is the angle between the normal to the plane and incident ray, then $\cos\theta=\frac{1}{\sqrt{3}}$
$$\left(\frac{l_1+l_2}{2\cos\theta},\frac{m_1+m_2}{2\cos\theta},\frac{n_1+n_2}{2\cos\theta}\right)=(l,m,n)$$
$$\frac{l+1}{2\cos\theta}=\frac{1}{\sqrt{3}}\quad\Rightarrow\quad l=\frac{-1}{3}$$
$$\frac{m-0}{2\cos\theta}=\frac{-1}{\sqrt{3}}\Rightarrow m=\frac{-2}{3}$$
$$\frac{n+0}{2\cos\theta}=\frac{1}{\sqrt{3}}\Rightarrow n=\frac{2}{3}$$

4. Let $\cos^{-1}\left(\sqrt{\frac{5}{14}}\right)=\theta\Rightarrow\cos\theta=\frac{\sqrt{5}}{\sqrt{14}}$
$\Rightarrow\sin\theta=\frac{3}{\sqrt{14}}$
D.r's of given line (a_1, b_1, c_1) = (1, 2, λ)
D.r's of normal to the given plane
(a_2, b_2, c_2) = (1, 2, 3)
$$\sin\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$
$$\frac{3}{\sqrt{14}}=\frac{1+4+3\lambda}{\sqrt{1+4+\lambda^2}\sqrt{1+4+9}};\quad \lambda=\frac{2}{3}$$

5. x = y = z--- (1), $x=\frac{y}{2}=\frac{z}{3}$ --- (2)
Clearly point of intersection of (1) and (2) is (0,0,0)
D.r's of (1) are (1, 1, 1)
D.r's of (2) are (1, 2, 3)
Let θ be the angle between (1) and (2)
$$\cos\theta=\frac{6}{\sqrt{42}},\sin\theta=\frac{\sqrt{6}}{\sqrt{42}}$$
Let any point on second line be $(\lambda,2\lambda,3\lambda)$
Third line passing through (1, 1, 1)
(1, 1, 1) lies on (1)
A = (1, 1, 1)
$$Area\ of\ \Delta OAB=\frac{1}{2}(OA)OB\sin\theta$$
$$=\frac{1}{2}\sqrt{3}\lambda\sqrt{14}\times\frac{\sqrt{6}}{\sqrt{42}}=\sqrt{6}\quad\Rightarrow\lambda=2$$
So B is $(2,4,6)$



6. $P = (3, 2, 6)$
$Q = (1 - 3\mu, \mu - 1, 2 + 5\mu)$
D.r's of PQ = $(-3\mu - 2, \mu - 3, 5\mu - 4)$
Equation of the plane $x - 4y + 3z = 1$
D.r's of the normal to the plane (1, –4, 3)
PQ is perpendicular to the normal to the plane
$a_1a_2 + b_1b_2 + c_1c_2 = 0$
$\Rightarrow -3\mu - 2 - 4\mu + 12 + 15\mu - 12 = 0$
$\Rightarrow 8\mu - 2 = 0 \Rightarrow \mu = \frac{1}{4}$

7. The d.r's of the normal of the plane are $(\cos\theta, \sin\theta, 0)$.
Now, the required plane passes through the z-axis. hence the point (0,0,0) lies on the plane.
The required plane is
$x\cos\theta + y\sin\theta = 0, \quad \Rightarrow x + y\tan\theta = 0$

8. 3(2) + 1(–1) + (–5) (1) = 0
Given line and given plane are parallel
$\therefore$ Image line is also parallel to the given line
Image of A (1, 3, 4) w.r.to given plane lies on the image line.
Equation of the normal to the plane is
$\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1}$
Any point on the line B = (2r + 1, –r+3, r+4)
If B is the image of A (1, 3, 4) then mid point of AB lies on the plane.
Mid point = $\left(\frac{2r+2}{2}, \frac{-r+6}{2}, \frac{r+8}{2}\right)$
Mid point lies in the given plane
$\Rightarrow 2\left(\frac{2r+2}{2}\right) - \left(\frac{-r+6}{2}\right) + \frac{r+8}{2} + 3 = 0$
$\Rightarrow r = -2$, B = (–3, 5, 2)
Image line is $\frac{x+3}{3} = \frac{y-5}{1} = \frac{z-2}{-5}$

9. (1,2,3) satisfies the plane $x - 2y + z = 0$ and also $(\hat{i} + 2j + 3k).(i - 2j + k) = 0$
Since the lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ both satisfy (0,0,0) and (1,2,3) both are same. given line is obviously parallel to the plane $x - 2y + z = 6$

10. Equation of a plane through (-1,0,1) is
$a(x+1) + b(y-0) + c(z-1) = 0$
Which is paraallel to the given line and perpendicular to the given plane.
$-a + 2b + 3c = 0$ and $a - 2b + c = 0$
by solving the above, we get , c=0, a=2b

11. Given lines are $\frac{x-1}{1} = \frac{y+3}{-\lambda} = \frac{z-1}{\lambda}$,
$\frac{x}{\frac{1}{2}} = \frac{y-1}{1} = \frac{z-2}{-1}$
Given lines are coplanar
$\Rightarrow \begin{vmatrix} 1 & -4 & -1 \\ 1 & -\lambda & \lambda \\ \frac{1}{2} & 1 & -1 \end{vmatrix} = 0, \quad \Rightarrow \lambda = -2$

12. Let a point $(3\lambda + 1, \lambda + 2, 2\lambda + 3)$ of the first line also lies on the second line Then
$\frac{3\lambda + 1 - 3}{1} = \frac{\lambda + 2 - 1}{2} = \frac{2\lambda + 3 - 2}{3} \Rightarrow \lambda = 1$
Hence, the point of intersection P of the two lines is (4,3,5)

JEE MAINS QUESTIONS

1.A plane P meets the coordinate axes at A, B and C respec tively. The centroid of triangle ABC is given to be (1,1,2). Thenthe equation of the line through this centroid and perpendicular to the plane P is:

[2020]

1) $\frac{x-1}{2}=\frac{y-1}{1}=\frac{z-2}{1}$ 2) $\frac{x+1}{2}=\frac{y-1}{1}=\frac{z-2}{1}$

3) $\frac{x-1}{2}=\frac{y-1}{2}=\frac{z-2}{1}$ 4) none of these

2.If (a, b, c) is the image of the point (1, 2, –3) in the line, $\frac{x+1}{2}=\frac{y-3}{-2}=\frac{z}{-1}$, then a + b + c is equals to:

[2020]

(1) 2 (2) – 1
(3) 3 (4) 1

3.The shortest distance between the lines

$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$

1) 3 2) 5 [2020]

3) $3\sqrt{30}$ 4) none

4.If the foot of the perpendicular drawn from the point (1, 0, 3) on a line passing through (a, 7, 1) is , then a isequal to _________. [2020]

5.Two lines

$\frac{x-3}{1}=\frac{y+1}{3}=\frac{z-6}{-1}$ and $\frac{x+5}{7}=\frac{y-2}{-6}=\frac{z-3}{4}$

intersect at the point R. The reflection of R in the xy-planehas coordinates : [2020]

(1) (2, –4, –7) (2) (2, 4, 7)
(3) (2, –4, 7) (4) (–2, 4, 7)

6. If the lines x = ay + b, z = cy + d and x = a' z + b', y = c' z + d' are perpendicular, then: [2019]

(1) ab' + bc' + 1 = 0
(2) cc' + a + a' = 0
(3) bb' + cc' + 1 = 0
(4) aa' + c + c' = 0

7.The number of distinct real values of l for which the lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z+3}{\lambda^2}$ and $\frac{x-3}{1}=\frac{y-2}{\lambda^2}=\frac{z-2}{1}$ arecoplanar is : [2019]

(1) 2 (2) 4
(3) 3 (4) 1

8.If the length of the perpendicular from the point (b, 0, b) to the line $\frac{x}{1}=\frac{y-1}{0}=\frac{z+1}{-1}$ is $\sqrt{\frac{3}{2}}$, then b is equal to: [2019]

(1) 1 (2) 2
(3) –1 (4) –2

9.If a point R(4, y, z) lies on the line segment —oining thepoints P(2, –3, 4) and Q(8, 0, 10), then distance of R fromthe origin is [2018]

1) $2\sqrt{14}$ 2) $2\sqrt{21}$
3) 6 4) $\sqrt{53}$

10.The length of the projection of the line segment joiningthe points (5, –1, 4) and (4, –1, 3) on the plane, x+y+z=7is: [2018]

1) $\frac{2}{3}$ 2) $\frac{1}{3}$ 3) $\sqrt{\frac{2}{3}}$ 4) $\frac{2}{\sqrt{3}}$

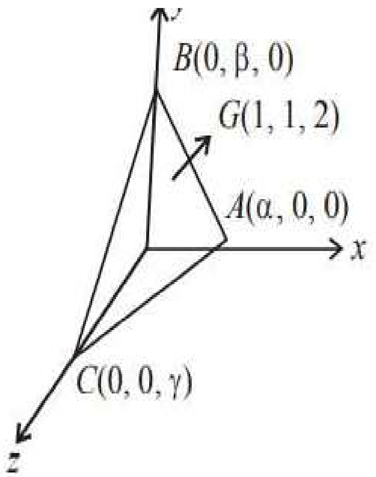
KEY

1) 3 2) 1 3) 3 4) 4 5)1

6) 4 7) 3 8)3 9) 1 10)1

SOLUTIONS

1. C



$\therefore \alpha = 3, \beta = 3$ and $\gamma = 6$ as G is centroid.

$\therefore$ The equation of plane is

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$$

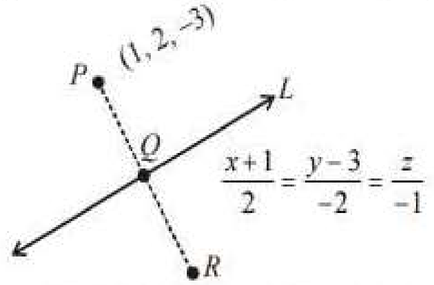
$$\Rightarrow \frac{x}{3}+\frac{y}{3}+\frac{z}{6}=1 \Rightarrow 2x+2y+z=6$$

The required line is $,\frac{x-1}{2}=\frac{y-1}{2}=\frac{z-2}{1}$

2. a

$$\frac{x+1}{2}=\frac{y-3}{-2}=\frac{z}{-1}=\lambda$$

Any point on line $= Q(2\lambda-1, -2\lambda+3, -\lambda)$



$\therefore$ D.r. of $PQ = [2\lambda-2, -2\lambda+1, -\lambda+3]$

D.r. of given line $= [2, -2, -1]$

$\because$ PQ is perpendicular to line L

$\therefore 2(2\lambda-2)-2(-2\lambda+1)-1(-\lambda+3)=0$

$\Rightarrow 4\lambda-4+4\lambda-2+\lambda-3=0$

$\Rightarrow 9\lambda-9=0 \Rightarrow \lambda=1$

$\because$ Q is mid point of $PR = Q = (1, 1, -1)$

$\therefore$ Coordinate of image $R = (1, 0, 1) = (a, b, c)$

$\therefore a+b+c=2.$

3.

$$\overline{AB}=6\hat{i}+15\hat{j}+3\hat{k}$$

$$\vec{p}=\hat{i}+4\hat{j}+22\hat{k}$$

$$\vec{q}=\hat{i}+\hat{j}+7\hat{k}$$

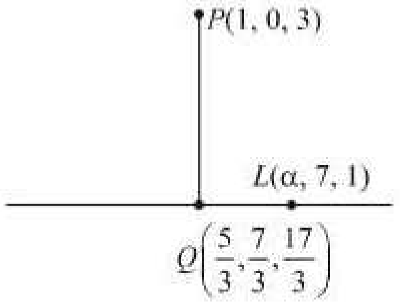
$$\vec{p}\times\vec{q}=\begin{vmatrix} i & j & k \\ 1 & 4 & 22 \\ 1 & 1 & 7 \end{vmatrix}=6\hat{i}+15\hat{j}-3\hat{k}$$

Shortest distance between the lines is

$$=\frac{|\overline{AB}.(\vec{p}\times\vec{q})|}{|\vec{p}\times\vec{q}|}=\frac{|36+225+9|}{\sqrt{36+225+9}}=3\sqrt{30}$$

4.

Since, PQ is perpendicular to L



$$\left(1-\frac{5}{3}\right)\left(\alpha-\frac{5}{3}\right)+\left(\frac{-7}{3}\right)\left(7-\frac{7}{3}\right)$$

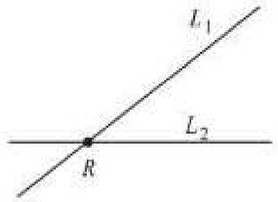
$$+\left(3-\frac{17}{3}\right)\left(1-\frac{17}{3}\right)=0$$

$$\frac{-2\alpha}{3}+\frac{10}{9}-\frac{98}{9}+\frac{112}{9}=0$$

$$\frac{2\alpha}{3}=\frac{24}{9} \Rightarrow \alpha=4$$

5.

Let the coordinate of P with respect to line



$$\frac{x-3}{1}=\frac{y+1}{3}=\frac{z-6}{-1}=\lambda$$

$$\frac{x+5}{7}=\frac{y-2}{-6}=\frac{z-3}{4}=\mu$$

$L_1=(\lambda+3, 3\lambda-1, -\lambda+6)$

and coordinate of P w.r.t.

line $L_2=(7\mu-5, -6\mu+2, 4\mu+3)$

$\therefore\quad \lambda-7\mu=-8, 3\lambda+6\mu=3, \lambda+4\mu=3$

From above equation : $\lambda=-1, \mu=1$

$\therefore\quad$ Coordinate of point of intersection $R=(2,-4,7)$.

Image of R w.r.t. xy plane $=(2,-4,-7)$.

6.

First line is: $x=ay+b, z=cy+d$

$$\frac{x-b}{a}=\frac{y}{1}=\frac{z-d}{c}$$

and another line is: $x=a'z+b', y=c'z+d'$

$$\Rightarrow\quad \frac{x-b'}{a'}=\frac{y-d'}{c'}=\frac{z}{1}$$

$\because$ Both lines are perpendicular to each other

$\therefore\quad aa'+c'+c=0$

7.

Lines are coplanar

$$\begin{vmatrix} 3-1 & 2-2 & 1-(-3) \\ 1 & 2 & \lambda^2 \\ 1 & \lambda^2 & 2 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} 2 & 0 & 4 \\ 1 & 2 & \lambda^2 \\ 1 & \lambda^2 & 2 \end{vmatrix}=0$$

$$\Rightarrow\quad 2(4-\lambda^4)+4(\lambda^2-2)=0$$

$$\Rightarrow\quad 4-\lambda^4+2\lambda^2-4=0 \Rightarrow \lambda^2(\lambda^2-2)=0$$

$$\Rightarrow\quad \lambda=0, \sqrt{2}, -\sqrt{2}$$

8.

Given, $\frac{x}{1}=\frac{y-1}{0}=\frac{z+1}{-1}=p$ (let) and point P $(\beta, 0, \beta)$

Any point on line A $=(p, 1, -p-1)$

Now, DR of AP a" $<p-\beta, 1-0, -p-1-\beta>$

Which is perpendicular to line.

$\therefore (p-\beta)1+0.1-1(-p-1-\beta)=0$

$\Rightarrow p-\beta+p+1+\beta=0 \Rightarrow p=\frac{-1}{2}$

$\therefore$ Point $A\left(\frac{-1}{2}, 1-\frac{1}{2}\right)$

Given that distance AP $=\sqrt{\frac{3}{2}} \Rightarrow AP^2=\frac{3}{2}$

$\Rightarrow\left(\beta+\frac{1}{2}\right)^2+1+\left(\beta+\frac{1}{2}\right)^2=\frac{3}{2}$ or $2\left(\beta+\frac{1}{2}\right)^2=\frac{1}{2}$

$\Rightarrow\left(\beta+\frac{1}{2}\right)^2=\frac{1}{4} \Rightarrow \beta=0,-1,(\beta \neq 0)$

$\therefore \beta=-1$

9.

Here, P, Q, R are collinear

$\therefore \overrightarrow{PR}=\lambda\overrightarrow{PQ}$

$2\hat{i}+(y+3)\hat{j}+(z-4)\hat{k}=\lambda[6\hat{i}+3\hat{j}+6\hat{k}]$

$\Rightarrow 6\lambda=2, y+3=3\lambda, z-4=6\lambda$

$\Rightarrow \lambda=\frac{1}{3}, y=-2, z=6$

$\therefore$ Point $R(4,-2,6)$

Now, OR $=\sqrt{(4)^2+(-2)^2+(6)^2}=\sqrt{56}=2\sqrt{14}$

10.

Let $\vec{v}_1$ and $\vec{v}_2$ be the vectors perpendicular to the plane OPQ and PQR respectively.

$$\vec{v}_1=\overrightarrow{PQ}\times\overrightarrow{OQ}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 1\\ 2 & 1 & 3\end{vmatrix}$$

$=5\hat{i}-\hat{j}-3\hat{k}$

$$\vec{v}_2=\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 2\\ -2 & -1 & 1\end{vmatrix}$$

$=\hat{i}-5\hat{j}-3\hat{k}$

$\because \quad \cos\theta=\frac{\vec{v}_1\cdot\vec{v}_2}{|\vec{v}_1||\vec{v}_2|}=\frac{5+5+9}{25+1+9}=\frac{19}{35}$

$\therefore \quad \theta=\cos^{-1}\left(\frac{19}{35}\right)$

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ADVANCED LEVEL QUESTIONS

SINGLE ANSWER TYPE QUESTIONS

1. The point in which the YZ plane divides the line joining the points (3, 5, –7) and (–2, 1, 8) is (x, y, z). Then the value of x + 5y + z is

(A)10 (B)15 (C)12 (D)20

2. P(1, 1, 1) and Q(l, l, l) are two points in the space such that $PQ = \sqrt{27}$, the value of l can be [JEE 2006]

(A)–4 (B)–2 (C)2 (D)0

3. The direction ratios of the bisector of the angle between the lines whose $l_1, m_1, n_1;\ l_2, m_2, n_2$ are [JEE - 2003]

(A) $l_1 \pm l_2, m_1 \pm m_2, n_1 \pm n_2$

(B) $l_1^2 + l_2^2, m_1^2 + m_2^2, n_1^2 + n_2^2$

(C) $l_1m_2 - l_2m_1,\ m_1n_2 - m_2n_1,\ n_1l_2 - n_2l_1$

(D) $l_1m_2 + l_2m_1,\ m_1n_2 + m_2n_1,\ n_1l_2 + n_2l_1$

4. The plane which contains the line 3x + y = 1, z = 4 and parallel to x + y + z + 1 = 0, y + 2z = 1, cuts the x-axis at

(A)(–2, 0, 0) (B)(–3, 0, 0)
(C)(–4, 0, 0) (D)(–1, 0, 0)

5. The line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and the plane 2x –4y + 2 z = 3 meet in **[JEE 2099]**

(A)at one point (B)no point
(C)infinitely many points
(D)at two points

6. If $\left(4k_1, k_1^2, 1\right)$ and $\left(4k_2, k_2^2, 1\right)$ are two points lying on the plane in which (2, 3, 2) and (1, 2, 1) are mirror image to each other, then k_1k_2 is equal to

(A) $-\frac{3}{2}$ (B) $-\frac{5}{2}$ (C) $-\frac{7}{2}$ (D) $-\frac{9}{2}$

7. The plane $x - y - z = 2$ is rotated through an angle 90^0 about its line of intersection with the plane $x + 2y + z = 2$. Then equation of this plane in new position is

(A) $5x + 4y + z - 10 = 0$

(B) $4x + 5y - 3z = 0$

(C) $2x + y + 2z = 9$

(D) $3x + 4y - 5z = 9$

KEY

01) B 02) B 03) A 04) D 05) B 06) D 07) A

SOLUTIONS

1. Let the yz–plane divide the line joining the given points in the ratio $m_1 : m_2$. Then the coordinates of the point of division are

$$\left(\frac{-2m_1 + 3m_2}{m_1 + m_2}, \frac{m_1 + 5m_2}{m_1 + m_2}, \frac{8m_1 - 7m_2}{m_1 + m_2}\right).$$

Since this point lies on the yz–plane, its x-coordinates is zero. Therefore

$-2m_1 + 3m_2 = 0$, i.e. $m_1 : m_2 = 3 : 2$

The other coordinates of the point of division are now

$$y = \frac{m_1 + 5m_2}{m_1 + m_2} = \frac{3 + 2.5}{3 + 5} = \frac{13}{5}, \text{ and}$$

$$z = \frac{8m_1 - 7m_2}{m_1 + m_2} = \frac{3.8 - 2.7}{3 + 2} = 2$$

$\Rightarrow \quad x + 5y + z = 15$

2. $(PQ)^2 = (\lambda - 1)^2 + (\lambda - 1)^2 + (\lambda - 1)^2$

$= 3(\lambda - 1)^2 = 27 \Rightarrow \quad (\lambda - 1)^2 = 9$

$\Rightarrow \quad \lambda = -2 \text{ or } 4$

3. Let OP = OQ = r

M is the mid point of PQ, then coordinates of

$$M\left[\left(\frac{l_1 + l_2}{2}\right)r,\ \left(\frac{m_1 + m_2}{2}\right)r,\ \left(\frac{n_1 + n_2}{2}\right)r\right]$$

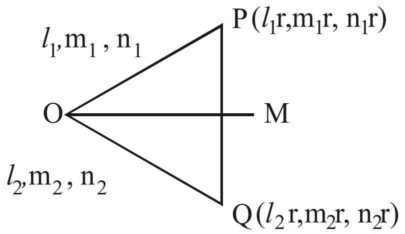
$$\left\{\left(\frac{l_1 + l_2}{2}\right)r, \left(\frac{m_1 + m_2}{2}\right)r, \left(\frac{n_1 + n_2}{2}\right)r\right\}$$

DR's of the bisector are

$l_1 + l_2, m_1, +m_2, n_1 + n_2$

DR's of other bisector are

$l_1 - l_2, m_1 - m_2, n_1 - n_2$



4. Let the plane be $3x + y - 1 + \lambda(z-4) = 0$

   It is parallel to line $\frac{x+2}{1} = \frac{y-1}{-2} = \frac{z}{1}$

   $\Rightarrow \lambda = -1 \qquad \Rightarrow 3x + y - z + 3 = 0$

   Hence point is (–1, 0, 0)

5. Any point on the given line is $(t, 2t, 3t)$.

   It lies in the given plane if $2(t) - 4(2t) + 2(3t) = 3$

   $\Rightarrow 0 = 3$. Which is not true for any $t \in R$. Hence, the given line and given plane does not meet in any point.

6. Required plane is 2x + 2y + 2z – 11 = 0

   $\Rightarrow 2k^2 + 8k - 9 = 0 \qquad \Rightarrow k_1 k_2 = -\frac{9}{2}$

7. $(x - y - z - 2) + \lambda(x + 2y + z - 2) = 0$

   $(\lambda+1)x + (2\lambda-1)y + (\lambda-1)z - 2\lambda - 2 = 0$

   $(\lambda+1).1 + (2\lambda-1)(-1) + (\lambda-1)(-1) = 0$

   $\Rightarrow$ equation of plane is $5x + 4y + z - 10 = 0$

MULTIPLE ANSWER TYPE QUESTIONS

1. If the direction cosines l, m, n of a line are related by the equations l+m+n=0, 2mn+2ml- nl=0 then the ordered triplet (l, m, n) is

(A) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$

(B) $\left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$

(C) $\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

(D) $\left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$

2. The lines $\frac{x-1}{3} = \frac{y-1}{-1} = \frac{z+1}{0}$ and $\frac{x-4}{2} = \frac{y+0}{0} = \frac{z+1}{3}$

(A) do not intersect

(B) intersect

(C) intersect at (4, 0, –1)

(D)intersect at (1, 1, –1)

3. The plane $x - 2y + 7z + 21 = 0$

(A) contains the line $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$

(B) contains the point (0, 7, –1)

(C) is perpendicular to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{7}$

(D) is parallel to the plane x – 2y + 7z = 0

4. The equation of a line $4x - 4y - z + 11 = 0 = x + 2y - z - 1$ can be put as

(A) $\frac{x}{2} = \frac{y-2}{1} = \frac{z-3}{4}$

(B) $\frac{x-4}{2} = \frac{y-4}{1} = \frac{z-11}{4}$

(C) $\frac{x-2}{2} = \frac{y}{1} = \frac{z-3}{4}$

(D) $\frac{x-2}{2} = \frac{y-2}{1} = \frac{z}{4}$

5. If P(2, 3, 1) is a point and $L \equiv x - y - z - 2 = 0$ is a plane then

(A) origin and P lie on the same side of the plane

(B)distance of P from the plane is $\frac{4}{\sqrt{3}}$

(C) foot of perpendicular is $\left(\frac{10}{3}, \frac{5}{3}, -\frac{1}{3}\right)$

(D) image of point P by the plane $\left(\frac{10}{3}, \frac{5}{3}, -\frac{1}{3}\right)$

6. Consider the plane through (2, 3, –1) and at right angles to the vector $3\hat{i} - 4\hat{j} + 7\hat{k}$ from the origin is

(A)The equation of the plane through the given point is $3x-4y+7z+13=0$

(B)perpendicular distance of plane from origin $\frac{1}{\sqrt{74}}$

(C)perpendicular distance of plane from origin $\frac{13}{\sqrt{74}}$

(D)perpendicular distance of plane from origin $\frac{21}{\sqrt{74}}$

7. A line L passing through the point P(1, 4, 3), is perpendicular to both the lines $\frac{x-1}{2}=\frac{y+3}{1}=\frac{z-2}{4}$, and $\frac{x+2}{3}=\frac{y-4}{2}=\frac{z+1}{-2}$. If the position vector of point Q on L is (a_1, a_2, a_3) such that $(PQ)^2 = 357$, then $(a_1 + a_2 + a_3)$ can be

(A) 16 (B) 15 (C) 2 (D) 1

.

KEY

01) A,B,C,D	02) B,C	03) A,B,C,D
04) A,B)	05) A,B,C	06) A,C
07) B,D		

SOLUTIONS

1. l+m+n =0 and 2 mn+2ml-nl=0

l= - (m+n) $\Rightarrow$ 2mn-(2m-n) (m+n) =0

$\Rightarrow 2m^2-n^2-mn=0 \Rightarrow \frac{m}{n}=1 \, or \, \frac{-1}{2}$

when m=n, l = -2n ; when $m=\frac{-n}{2}, l=\frac{-n}{2}$

$hence = \frac{l}{-2}=\frac{m}{1}=\frac{n}{1} \, or \, \frac{l}{-1}=\frac{m}{-1}=\frac{n}{2}$

Since $l^2+m^2+n^2=1$

$(l,m,n)=\left(\frac{-2}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right) or \left(\frac{-1}{\sqrt{6}},\frac{-1}{\sqrt{6}},\frac{2}{\sqrt{6}}\right)$

$or\left(\frac{2}{\sqrt{6}},\frac{-1}{\sqrt{6}},\frac{-1}{\sqrt{6}}\right) or \left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{-2}{\sqrt{6}}\right)$

2. For the given lines

$$\begin{vmatrix} 4-1 & 0-1 & -1-(-1) \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = 0$$

So, the given lines intersect.

Any point on the first line is $(3r_1+1,-r_1+1,-1)$ and any point on the second line is $(2r_2+4,0,3r_2-1)$.

Since, the lines intersect, at the point of intersection.

$3r_1+1=2r_2+4, -r_1+1=0, -1=3r_2-1$

$r_1=1, r_2=0$

Hence, the point of intersection is (4, 0, –1)

3. (a) We know that the plane ax + by + cz + d = 0 contains the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$

if $a\alpha+b\beta+c\gamma+d=0$ and $al + bm + cn = 0$.

Now, since (–1) – 2 (3) + 7 (–2) + 21 = 0

And (–3) (1) + 2 (–2) + 1 (7) = 0

The line given in (a) lies on the given plane.

(b) Since, $0-2(7)+7(-1)+21=0$

The point (0, 7, –1) lies on the plane.

(c) Direction ratio of the normal to the given plane are (1, -2, 7) which are same as those of the given in (c). So, the plane is perpendicular to the lines.

(d) direction ratios of normal to plane are equal hence two planes are parallel.

4. The given equation are

$4x-4y-z+11=0$ …(i)

$x+2y-z-1=0$ ….(ii)

The D.r's of normals to the planes (i) and (ii) are 4, –4, –1 and 1, 2, –1 respectively.

Let Dr's of line of intersection of plane be l, m, n As the line of intersection of the planes isperpendicular to the normals of the both planes, we get

$4l-4m-n=0$

and $l+2m-n=0$

By cross multiplication

$\frac{l}{6}=\frac{m}{3}=\frac{n}{12}$ or $\frac{l}{2}=\frac{m}{1}=\frac{n}{4}$

If x = 0, Eqs. (i) and (ii) becomes

$-4y-z+11=0$

$2y-z-1=0$

Solving, we get $y=2, z=3$

Equation of line is $\frac{x}{2}=\frac{y-2}{1}=\frac{z-3}{4}$

Also x = 4, y = 4, z = 11 satisfies Eqs. (i) and (ii) Hence, (b) is also the correct option.

5. At $(0,0,0), x-y-z-2=-2=(-ve)$
at (2, 3, 1), $x-y-z-2=2-3-1-2=-4$
Since, both have same sign (0, 0, 0) and (2, 3, 1) lie on the same side of the plane.

Distance $=\left|\frac{2-3-1-2}{\sqrt{1^2+1^2+1^2}}\right|=\frac{4}{\sqrt{3}}$

Equation of a line perpendicular to the plane $x-y-z-2=$ and passing through the point (2, 3, 1) is

$\frac{x-2}{1}=\frac{y-3}{-1}=\frac{z-1}{-1}=\lambda$

A point on the line is $(\lambda+2, 3-\lambda, 1-\lambda)$ and it lies and the plane $x-y-z-2=0$

if $\lambda+2-3+\lambda-1=\lambda-2=0 \Rightarrow \quad \lambda=\frac{4}{3}$

Foot of perpendicular on the plane is

$\left(\frac{4}{3}+2, 3-\frac{4}{3}, 1-\frac{4}{3}\right)=\left(\frac{10}{3}, \frac{5}{3}, -\frac{1}{3}\right)$

6. The equation of the plane through $(2, 3, -1)$ and perpendicular to the vector $3\hat{i}-4\hat{j}+7\hat{k}$ is

$3(x-2)+(-4)(y-3)+7(z-(-1))=0$

or $3x-4y+7z+13=0$

Distance of this plane from the origin

$=\frac{|3\times 0-4\times 0+7\times 0+13|}{\sqrt{3^2+(-4)^2+7^2}}=\frac{13}{\sqrt{74}}.$

7. Equation of the line passing through P(1, 4, 3) is

$\frac{x-1}{a}=\frac{y-4}{b}=\frac{z-3}{c}$ ....(1)

Since (1) is perpendicular to

$\frac{x-1}{2}=\frac{y+3}{1}=\frac{z-2}{4}$ and

$\frac{x+2}{3}=\frac{y-4}{2}=\frac{z+1}{-2}$

Hence 2a + b + 4c = 0
and 3a + 2b – 2c = 0

$\frac{a}{-2-8}=\frac{b}{12+4}=\frac{c}{4-3}$

$\Rightarrow \frac{a}{-10}=\frac{b}{16}=\frac{c}{1}$

Hence the equation of the lines is

$\frac{x-1}{-10}=\frac{y-4}{16}=\frac{z-3}{1}$ **....(2)**

Ans. Now any point Q on (2) can be taken as (1 – 10l , 16l + 4 , l + 3)
Distance of Q from P (1, 4, 3)
$=(10l)^2+(16l)^2+l^2=357$
$\Rightarrow (100+256+1)l^2=357 \Rightarrow l=1$ or -1
Q is (–9, 20, 4) or (11, – 12, 2)
Hence $a_1+a_2+a_3=15$ or 1

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COMPREHENSION TYPE QUESTIONS

Passage - 1

Eq. of line is $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$

Equation of plane through the intersection of two planes is

$(a_1x+b_1y+c_1z+d_1)+\lambda(a_2x+b_2y+c_2z+d_2)=0$

1. **The distance of point (1, - 2, 3) from the plane** $x-y+z=5$ **measured parallel to the line**

(A) $\frac{\sqrt{21}}{5}$ (B) $\frac{\sqrt{29}}{5}$ (C) $\frac{\sqrt{13}}{5}$ (D) $\frac{2}{\sqrt{5}}$

2. **The equation of the plane through (0,2,4) and containing the line** $\frac{x+3}{3}=\frac{y-1}{4}=\frac{z-2}{-2}$ **is**

(A) $x-2y+4z-12=0$

(B) $5x+y+9z-38=0$

(C) $10x-12y-9z+60=0$

(D) $7x+5y-3z+2=0$

Passage - 2

$\vec{a}=6\hat{i}+7\hat{j}+7\hat{k},\quad \vec{b}=3\hat{i}+2\hat{j}-2\hat{k},\quad P(1,2,3)$

3. **The position vector of L, the foot of the perpendicular from P on the line** $\vec{r}=\vec{a}+\lambda\vec{b}$ **is**

(A) $6\hat{i}+7\hat{j}+7\hat{k}$ (B) $3\hat{i}+2\hat{j}-2\hat{k}$

(C) $3\hat{i}+5\hat{j}+9\hat{k}$ (D) $9\hat{i}+9\hat{j}+5\hat{k}$

4. **The image of the point P in the line** $\vec{r}=\vec{a}+\lambda\vec{b}$ **is**

(A) (11, 12, 11) (B)(5, 2, -7)

(C) (5, 8, 15) (D)(17, 16, 7)

5. **If A is the point with position vector** $\vec{a}$ **then area of the triangle** PLA **in sq. units is equal to**

(A) $3\sqrt{6}$ (B) $\frac{7\sqrt{17}}{2}$ (C) $\sqrt{17}$ (D) $\frac{7}{2}$

Passage - 3

Given points A (1, -4, 5) and B(0,6,1) and a plane $3x-y+2z=7$

6. **The ratio in which the line segment AB is divided by the plane, is**

(A)2/3 (B)1/11 (C)10/11 (D)12/11

7. **If** $P(\lambda^2+1,\lambda,\lambda-1)$ **is a point on the same side of the plane as the point A, then the set of values of** λ**, is**

(A) $\left(\frac{-1-\sqrt{73}}{6},\frac{-1+\sqrt{73}}{6}\right)$

(B) $\left(-\infty,\frac{-1-\sqrt{73}}{6}\right)\cup\left(\frac{-1+\sqrt{73}}{6},\infty\right)$

(C) $(-\infty,\infty)$ (D) $(0,\infty)$

Passage - 4

Read the following passage and answer the questions Consider the lines

$L_1:\frac{x+1}{3}=\frac{y+2}{1}=\frac{z+1}{2},$

$L_2:\frac{x-2}{1}=\frac{y+2}{2}=\frac{z-3}{3}$

8. **The unit vector perpendicular to both** L_1 **and** L_2 **is**

(A) $\frac{-\hat{i}+7\hat{j}+7\hat{k}}{\sqrt{99}}$ (B) $\frac{-\hat{i}-7\hat{j}+5\hat{k}}{5\sqrt{3}}$

(C) $\frac{-\hat{i}+7\hat{j}+5\hat{k}}{5\sqrt{3}}$ (D) $\frac{7\hat{i}-7\hat{j}-\hat{k}}{\sqrt{99}}$

9. **The shortest distance between** L_1 **and** L_2 **is**

(A) 0 unit (B) $\frac{17}{\sqrt{3}}$ unit

(C) $\frac{41}{5\sqrt{3}}$ unit (D) $\frac{17}{5\sqrt{3}}$ unit

10. The distance of the point $(1,1,1)$ from the plane passing through the point $(-1,-2,-1)$ and whose normal is perpendicular to both the lines L_1 and L_2 is [IIT-JEE 2008]

(A) $\frac{2}{\sqrt{75}}$ unit (B) $\frac{7}{\sqrt{75}}$ unit

(C) $\frac{13}{\sqrt{75}}$ unit (D) $\frac{23}{\sqrt{75}}$ unit

Passage - 5

Consider the line L : $\frac{x-1}{2}=\frac{y}{1}=\frac{z+1}{-2}$ and a point A(1, 1, 1). Let P be the foot of the perpendicular from A on L and Q be the image of the point A in the line L, 'O' being the origin.

11. The distance of the origin from the plane passing through the point A and containing the line L is

(A) $\frac{1}{3}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{2}{3}$ (D) $\frac{1}{2}$

12. The distance of the point A from the line L is

(A) 1 (B) 2 (C) $\sqrt{3}$ (D) $\frac{4}{3}$

13. The distance of the origin from the point Q is

(A) $\sqrt{3}$ (B) $\sqrt{\frac{17}{6}}$ (C) $\sqrt{\frac{17}{3}}$ (D) $\frac{1}{\sqrt{3}}$

KEY

01) B 02) C 03) C 04) C 05) B 06) C
07) B 08) B 09) D 10) C 11) A 12) B
13) C

SOLUTIONS

1. Consider line passing through $P(1,-2,3)$

$$\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{-4}=\lambda$$

Q on line $Q(2\lambda+1,3\lambda-2,-4\lambda+3)$ is also lying on plane.

$$(2\lambda+1)-(3\lambda-2)+(-4\lambda+3)=5\Rightarrow -5\lambda=-1\Rightarrow\lambda=\frac{1}{5}$$

$$PQ=\sqrt{(2\lambda)^2+(3\lambda)^2+(4\lambda)^2}=\frac{\sqrt{29}}{5}$$

2. Normal to plane.

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0-(-3) & z-1 & 4-2\\ 3 & 4 & -2\end{vmatrix}=-10\hat{i}+12\hat{j}+9\hat{k}$$

Equation of plane is

$$-10(x-0)+12(y-2)+9(z-4)=0$$

$$-10x+12y-24+9z-36=0$$

$$10x-12y-9z+60=0$$

3. Let the position vector of L be

$\vec{a}+\lambda\vec{b}=(6+3\lambda)\hat{i}+(7+2\lambda)\hat{j}+(7-2\lambda)\hat{k}$

So $\overrightarrow{PL}=(6+3\lambda)\hat{i}+(7+2\lambda)\hat{j}+(7-2\lambda)\hat{k}-(\hat{i}+2\hat{j}+3\hat{k})$

$=(5+3\lambda)\hat{i}+(5+2\lambda)\hat{j}+(4-2\lambda)\hat{k}$

Since $\overrightarrow{PL}$ is perpendicular to the given line which is parallel to $\vec{b}=3\hat{i}+2\hat{j}-2\hat{k}$

$\Rightarrow 3(5+3\lambda)+2(5+2\lambda)-2(4-2\lambda)=0$

$\Rightarrow \lambda=-1$ and thus the position vector of L is $3\hat{i}+5\hat{j}+9\hat{k}$

4. Let the position vector of Q, the image of P in the given line be $x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$, then L is the mid-point of PQ.

$$\Rightarrow 3\hat{i}+5\hat{j}+9\hat{k}=\frac{\hat{i}+2\hat{j}+3\hat{k}+x_1\hat{i}+y_1\hat{j}+z_1\hat{k}}{2}$$

$$\Rightarrow\frac{x_1+1}{2}=3,\ \frac{y_1+2}{2}=5,\ \frac{z_1+3}{2}=9$$

$\Rightarrow x_1=5,\ y_1=8, z_1=15$

$\Rightarrow$ image of P in the line is (5, 8, 15)

5. Area of the

$$\Delta PLA=\frac{1}{2}\left|\overrightarrow{PL}\right|\left|\overrightarrow{AL}\right|=\frac{1}{2}\left|2\hat{i}+3\hat{j}+6\hat{k}\right|\left|-3\hat{i}-2\hat{j}+2\hat{k}\right|$$

$= \frac{1}{2}\sqrt{4+9+36}\ \sqrt{9+4+4} = \frac{7\sqrt{17}}{2}$ sq. uints.

6. The equation of plane is $3x - y + 2z - 7 = 0$. Let the line segment AB cuts the plane in the ratio $\mu : 1$ $\Rightarrow C\left(\frac{1}{\mu+1}, \frac{6\mu-4}{\mu+1}, \frac{\mu+5}{\mu+1}\right)$

   on $3x - y + 2z = 7$

   $\therefore 3\left(\frac{1}{\mu+1}\right) - \left(\frac{6\mu-4}{\mu+1}\right) + 2\left(\frac{\mu+5}{\mu+1}\right) = 7$

   $\Rightarrow \mu = \frac{10}{11}$

7. As A and P are on same side of the plane , the value of $3x - y + 2z - 7$ has same sign at A and P. $\Rightarrow 3(\lambda^2+1) - \lambda + 2(\lambda - 1) - 7 > 0$

   or $3\lambda^2 + \lambda - 6 > 0$

   $\therefore \left(\lambda - \frac{-1-\sqrt{73}}{6}\right)\left(\lambda - \frac{\sqrt{73}-1}{6}\right) > 0$

   $\Rightarrow \lambda \in \left(-\infty, \frac{-1-\sqrt{73}}{6}\right) \cup \left(\frac{-1+\sqrt{73}}{6}, \infty\right)$

8. The equations of given lines in vector forms may be written as

   $L_1 : \vec{r} = (-\hat{i} - 2\hat{j} - \hat{k}) + \lambda(3\hat{i} + \hat{j} + 2\hat{k})$

   and $L_2 : \vec{r} = (2\hat{i} - 2\hat{j} + 3\hat{k}) + \mu(\hat{i} + 2\hat{j} + 3\hat{k})$

   Since, the vector perpendicular to both L_1 and L_2

   $\therefore \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -\hat{i} - 7\hat{j} + 5\hat{k}$

   $\therefore$ required unit vector

   $= \frac{(-\hat{i} - 7\hat{j} + 5\hat{k})}{\sqrt{(-1)^2 + (-7)^2 + (-5)^2}} = \frac{1}{5\sqrt{3}}(-\hat{i} - 7\hat{j} + 5\hat{k})$

9. The shortest distance between L_1 and L_2 is

   $\left|\frac{\{(2-(-1))\hat{i} + (2-2)\hat{j} + (3-(-1))\hat{k}\}.(-\hat{i} - 7\hat{j} + 5\hat{k})}{5\sqrt{3}}\right|$

   $= \left|\frac{(3\hat{i} + 4\hat{k}).(-\hat{i} - 7\hat{j} + 5\hat{k})}{5\sqrt{3}}\right| = \frac{17}{5\sqrt{3}}$ unit

10. The equation of the plane passing through the point $(-1, -2, -1)$ and whose normal is prependiuclr to the both the given lines L_1 and L_2 may be written as

    $(x+1) + 7(y+2) - 5(z+1) = 0$

    $\Rightarrow x + 7y - 5z + 10 = 0$

    The distance of the point (1,1,1) from the plane

    $= \left|\frac{1+7-5+10}{\sqrt{1+49+25}}\right| = \frac{13}{\sqrt{75}}$ unit.

11, 12, 13

We have $\frac{x-1}{2} = \frac{y}{1} = \frac{z+1}{-2} = t$ (say)

Now $\overline{AP} = 2t\hat{i} + (t-1)\hat{j} - 2(t+1)\hat{k}$

As $\overline{AP} \cdot \vec{V} = 0 \Rightarrow t = \frac{-1}{3}$

Again $a_1 + 1 = \frac{2}{3} \Rightarrow a_1 = \frac{-1}{3}$

$a_2 + 1 = \frac{-2}{3} \Rightarrow a_2 = \frac{-5}{3}$

$a_3 + 1 = \frac{-2}{3} \Rightarrow a_3 = \frac{-5}{3}$

Hence Q is $\left(\frac{-1}{3}, \frac{-5}{3}, \frac{-5}{3}\right)$

Hence OQ = $\sqrt{\frac{1}{9} + \frac{25}{9} + \frac{25}{9}} = \sqrt{\frac{17}{3}}$

Ans.(iii)

Equation of the plane containing the point A and L is given by $[\overrightarrow{PA}, \overrightarrow{RA}\ \vec{V}] = 0$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z-1 \\ 0 & 1 & 2 \\ 2 & 1 & -2 \end{vmatrix} = 0$$

$\Rightarrow (x-1)(x-2) + 2\left(2(y-1)-(z-1)\right) = 0$

$\Rightarrow -4(x-1) + 4(y-1) - 2(z-1) = 0$

$\Rightarrow 2(x-1) - 2(y-1) + (z-1) = 0$

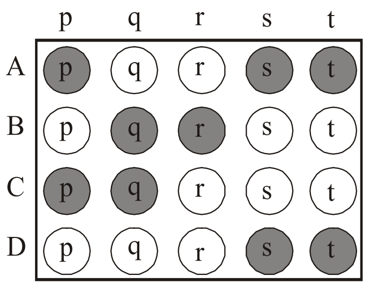
$\Rightarrow 2x - 2y + z = 1$ ......(1)

Distance of origin from (1) is $\frac{1}{\sqrt{9}} = \frac{1}{3}$ **Ans.(i)**

Finally AP $= \sqrt{\frac{4}{9}+\frac{16}{9}+\frac{16}{9}} = \sqrt{4} = 2$ **Ans.(ii)**

MATRIXMATCHING TYPE QUESTIONS

This section contains 1 questions. Each questions contain statements given in two columns, which have to be matched. The statements in **Column I** are labeled A, B, C and D while the statements in **Column II** are labelled p, q, r, s and t. Any given statement in **Column I** can have correct matching with **ONE OR MORE** statement(s) in **Column II.** The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example. If the correct matches are A-p, s and t, B-q and r, C-p and q, and D-s and t, then the correct darkening of bubbles will look like the following



1. Match the statements/expressions given in Column I with the values given in ColumnII

Column I

(A)The area of the triangle whose vertices are (0,0,0), (3,4,7) and (5,2,6) is

(B)Distance of plane ax + by + cz + d = 0 from origin may be (a,b,c,d $\in$ I) is

(C) The value(s) of λ for which the triangle with vertices A(6,10,10) B(1,0,-5) and C(6,-10, λ) will be a right angled triangle (right angled at A) is /are

(D)d is the perpendicular distance from (1, 3, 4) to $\frac{x-1}{-1} = \frac{y-1}{1} = \frac{z}{1}$, then value of $\frac{d}{2\sqrt{3}}$

Column II

(P) 0

(Q) 70/3

(R) $\sqrt{\frac{2}{3}}$

(S) $\frac{3}{2}\sqrt{65}$

2. Match the statements/expressions given in Column I with the values given in ColumnII

Column I

Consider a cube

(A) Angle between any two solid diagonal

(B)Angle betwen a solid diagonal and a plane

(C) Angle between plane diagonals of adjacent faces

(D)If a line makes angle $\frac{\pi}{4}$ and $\frac{\pi}{3}$ with positive X and Y axis then the angle which it makes with positive Z-axis

Column II

(P) $\cos^{-1}\frac{2}{\sqrt{6}}$

(Q) $\cos^{-1}\left(+\frac{1}{2}\right)$

(R) $\cos^{-1}\frac{1}{3}$

(S) $\frac{1}{2}$

3. Match the statements/expressions given in Column I with the values given in ColumnII

Column I

(A) If Acute and obtuse angle bisectors

2x – y + 2z + 3 = 0 and 3x – 2y + 6z + 8 = 0 are represented by A and O, then

(B) If acute and obtuse angle bisectors of the planes x – 2y + 2z – 3 = 0 and

2x – 3y + 6z + 8 = 0 are represented by A and O, then

(C) The acute and obtuse angle bisectors of the planes 2x + y – 2z + 3 = 0 and

6x + 2y – 3z – 8 = 0 are represented by A and O, then

Column II

(P) A : 32x + 13y – 23z – 3 = 0

(Q) O : x – 5y – 4z – 45 = 0

(R) A : 23x – 13y + 32z + 45 = 0

(S) O : 4x – y + 5z – 45 = 0

(T) A : 13x – 23y + 32z + 3 = 0

KEY

01).$(A)\to(S);(B)\to(P,Q,R,S);(C)\to(Q);(D)\to(R)$

02) $(A)\to(R);(B)\to(P);(C)\to(q);(D)\to(Q)$

03) $(A)\to(R);(B)\to(Q,T);(C)\to(P,S)$

SOLUTIONS

1. Let O(0,0,0), A(3,4,7) and B(5,2,6) be the given point

Area of $\Delta OAB = \frac{1}{2} OA.OB\ \sin(\angle AOB)$

Now, $OA = \sqrt{3^2+4^2+7^2} = \sqrt{74}$

$OB = \sqrt{5^2+2^2+6^2} = \sqrt{65}$

Also D.c's of the line OA and OB are

$= \frac{3}{\sqrt{74}}, \frac{4}{\sqrt{74}}, \frac{7}{\sqrt{74}}$ and $\frac{5}{\sqrt{65}}, \frac{2}{\sqrt{65}}, \frac{6}{\sqrt{65}}$

$\therefore$ Required area

$\frac{1}{2}\times\sqrt{74}\times\sqrt{65}\times\frac{3}{\sqrt{74}} = \frac{3}{2}\sqrt{65}$

(B) Distance $= \frac{|d|}{\sqrt{a^2+b^2+c^2}}$

(C) Let the given points be A, B and C respectively. Then find AB, AC, BC and then apply $AB^2 + AC^2 = BC^2$ then solve for the λ.

(D) Any point on the line is $(1-r, r+1, r)$

The direction ratio of the line joining (1, 3, 4) & (1 –r, r + 1, r) is –r, r–2, r–4

$\therefore (-1)(-r) + 1.(r-2) + (r-4) = 0$

$r + r - 2 + r - 4 = 0$, $3r = 6 \Rightarrow r = 2$

$\therefore$ Foot of the perpendicular is (–1, 3, +2)

$\therefore$ distance $\sqrt{(2)^2+0+4} = 2\sqrt{2}$ $\quad \therefore d = 2\sqrt{2}$

$\frac{d}{2\sqrt{3}} = \frac{2\sqrt{2}}{2\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}$

2. The solid diagonals may be taken as the lines joing (0, 0, 0), (a, a, a) and (a, a, 0) and (0, 0, a). The direction ratios wil be a, a, a; a, a, –a.

$\Rightarrow \cos\theta = \frac{a^2+a^2-a^2}{\sqrt{3a^2}\times\sqrt{3a^2}} \frac{1}{3} \Rightarrow \theta = \cos^{-1}\frac{1}{3}$

Let us take the solid diagonal as the one joining (0, 0, 0), (a, a, a) and plane diagonal as joining (0, 0, 0) and (a, a, 0). We easily get the angle as

$\cos^{-1}\frac{2}{\sqrt{6}}$.

The third part is easily found as $\cos^{-1}\left(\frac{1}{2}\right)$

(D) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{3} + \cos^2\gamma = 1$, $\cos\gamma = \pm\frac{1}{2}$

$\gamma = \cos^{-1}\left(\frac{1}{2}\right)$

3. (A) $\because (2)(3) + (-1)(-2) + (2)(6) = 20 > 0$

Bisectors are

$$\frac{(2x-y+2z+3)}{\sqrt{(2)^2+(-1)^2+(2)^2}} = \pm\frac{(3x-2y+6z+8)}{\sqrt{(3)^2+(-2)^2+(6)^2}}$$

or 7 (2x–y+2z+3) = ± 3(3x–2y+6z+8)

Acute angle bisector is

$7(2x-y+2z+3) = -3(3x-2y+6z+8)$

$\Rightarrow 23x - 13y + 32z + 45 = 0$

and obtuse angle bisector is

$7(2x - y + 2z + 3) = 3(3x - 2y + 6z + 8)$

$\Rightarrow \; 5x - y - 4z - 3 = 0$

A : 23x –13y + 32z + 45 = 0

and O : 5x–y–4z–3 = 0

(B) The given planes can be written as

$-x + 2y - 2z + 3 = 0$ and $2x - 3y + 6z + 8 = 0$

$\because (-1)(2) + (2)(-3) + (-2)(6)$

$= -2 - 6 - 12 = -20 < 0$

Bisectors are,

$$\frac{(-x+2y-2z+3)}{\sqrt{(-1)^2+(2)^2+(-2)^2}} = \pm\frac{(2x-3y+6z+8)}{\sqrt{(2)^2+(-3)^2+(6)^2}}$$

$\Rightarrow 7(-x + 2y - 2z + 3) = \pm 3(2x - 3y + 6z + 8)$

Acute angle bisector is

$7(-x + 2y - 2z + 3) = 3(2x - 3y + 6z + 8)$

$\Rightarrow 13x - 23y + 32z + 3 = 0$

and obtuse bisector is

$7(-x + 2y - 2z + 3) = -3(2x - 3y + 6z + 8)$

$\Rightarrow \; x - 5y - 4z - 45 = 0$

$\Rightarrow \; A : 13x - 23y + 32z + 3 = 0$

and $O : x - 5y - 4z - 45 = 0$

(C) The given planes can be written as

$2x + y - 2z + 3 = 0$ and $-6x - 2y + 3z + 8 = 0$

$\because (2)(-6) + (1)(-2) + (-2)(3) = -20 < 0$

Bisectors are

$$\frac{(2x+y-2z+3)}{\sqrt{\{(2)^2+(1)^2+(-2)^2\}}} = \pm\frac{(-6x-2y+3z+8)}{\sqrt{\{(-6)^2+(-2)^2+(3)^2\}}}$$

$\Rightarrow \; 7(2x + y - 2z + 3) = \pm 3(-6x - 2y + 3z + 8)$

Acute angle bisector is

$7(2x + y - 2z + 3) = 3(-6x - 2y + 3z + 8)$

$\Rightarrow \; 32x + 13y - 23z - 3 = 0$

and obtuse bisector is

$7(2x + y - 2z + 3) = -3(-6x - 2y + 3z + 8)$

$\Rightarrow 4x - y + 5z - 45 = 0$

A:32x+13y–23z–3 = 0

and O:4x–y+5z–45=0

INTEGER TYPE QUESTIONS

1. If the area of the triangle whose vertices are A(1, 2, 3), B(2, –1, 1) and C(1, 2, –4) is λ sq unit then $\frac{2\lambda}{\sqrt{10}}$ must be

2. The equation of a plane which bisects the line joining (1,5,7) and (-3,1,-1) is x + y + 2z = λ then λ must be

3. The distance of the point $(3,0,5)$ from the line $x - 2y + 2z - 4 = 0 = x + 3z - 11$ is

4. The $\frac{x+4}{3} = \frac{y+6}{5} = \frac{z-1}{-2}$ and $3x - 2y + z + 5 = 0 = 2x + 3y + 4z - k$ are coplanar for k is equal to

5. If the distance of the point P(4, 3, 5) from the axis of y is λ unit, then the value of $\frac{5\lambda^2}{41}$ must be

6. Let L be the distance between the lines $x = 0, \frac{y}{b} + \frac{z}{c} = 1$ and $y = 0, \frac{x}{a} - \frac{z}{c} = 1$. Then $L^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$ is

KEY

01) 7 02) 8 03) 3 04) 4 05) 5 06) 4

SOLUTIONS

1. The coordinates of the projections of A, B, C on the yz-plane are (0, 2, 3), (0, -1, 1) and (0, 2, -4) respectively

$\therefore \Delta_x$ = area of projection of Δ ABC on yz-plane

$$= \frac{1}{2}\left|\begin{vmatrix} 2 & 3 & 1 \\ -1 & 1 & 1 \\ 2 & -4 & 1 \end{vmatrix}\right| = \frac{21}{2} \text{sq. unit}$$

Similarly, the projection of A, B and C on zx and xy-planes are (1, 0, 3), (2, 0, 1), (1, 0, -4) and (1, 2, 0), (2, -1, 0), (1, 2, 0) respectively

Also, Let Δ_y and Δ_z be teh areas of the projection of the Δ ABC on zx and xy-planes respectively.

Then, $\Delta_y = \frac{1}{2}\left|\begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 1 & -4 & 1 \end{vmatrix}\right| = \frac{7}{2}$

and $\Delta_z = \frac{1}{2}\left|\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix}\right| = 0$

$\therefore$ The required area $= \sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2}$

$= \sqrt{\left(\frac{21}{2}\right)^2 + \left(\frac{7}{2}\right)^2 + (0)^2} = \frac{7}{2}\sqrt{10}$ sq unit

$\lambda = \frac{7}{2}\sqrt{10}$ sq unit $\Rightarrow \quad \frac{2\lambda}{\sqrt{10}} = 7$

2. Plane must pass through

$\left(\frac{1-3}{2}, \frac{5+1}{2}, \frac{7-1}{2}\right)$ or $(-1, 3, 3)$

$\Rightarrow -1 + 3 + 2\times 3 = \lambda \Rightarrow \lambda = 8.$

3. The d.r's of the line are given by $l - 2m + 2n = 0, l + 3n = 0 \Rightarrow \frac{\lambda}{6} = \frac{m}{1} = \frac{n}{-2}$

Taking $y = 0$, we get

$x + 2z = 4, x + 3z = 11 \Rightarrow x = -10, z = 7$

The line is $\frac{x+10}{6} = \frac{y}{1} = \frac{z-7}{-2}$

$b = 6i + j - 2k$

Distance of C from the line is $\frac{|\overline{AC} \times \overline{b}|}{|\overline{b}|}$

$= \frac{|(13i - 2k)\times(6i + j - 2k)|}{\sqrt{41}} = \frac{|2i + 14j + 13k|}{\sqrt{41}} = \frac{\sqrt{369}}{\sqrt{41}} = 3$

4. Any point on the first line in symmetrical form is (3r – 4, 5r – 6, –2r + 1). If the lines are coplanar, this point must lie on both the planes which determine the second line.

$\Rightarrow \; 3(3r - 4) - 2(5r - 6) - 2r + 1 + 5 = 0 \qquad \ldots(i)$

and $2(3r - 4) + 3(5r - 6) + 4(-2r + 1) - k = 0 \; ..(ii)$

From Eq. (i) we get , $\quad r = 2$

Now substituting r = 2 in Eq. (ii), then k = 4

5. The equations of y-axis are $\frac{x}{0} = \frac{y}{1} = \frac{z}{0}$,

Any point N on y-axis is (0, r, 0) $\qquad \ldots(i)$

The direction cosines of the line PN are 0–4, r–3, 0 – 5 ie, –4, r–3, –5 $\ldots(ii)$

Let N be the foot of the perpendicular from P to y-axis, then PN is ^ to the y-axis whose direction cosines are 0, 1, 0 and so from Eq. (ii), we have

$-0.(-4) + 1.(r - 3) + 0.(-5) = 0 \qquad \Rightarrow r = 3$

From Eq. (i) the coordinates of N are (0, 3, 0)

Required distance = PN

$= \sqrt{(4-0)^2 + (3-3)^2 + (5-0)^2} = \sqrt{41}$ unit

$\lambda = \sqrt{41} \Rightarrow \frac{5\lambda^2}{41} = 5$

6. The lines are $\frac{x}{0} = \frac{y-b}{b} = \frac{z}{-c}$ and $\frac{x-a}{a} = \frac{y}{0} = \frac{z}{c}$

$(bj - ck)\times(ai + ck) = bci - caj - abk$

$\overline{n} = \frac{bci - caj - abk}{\sqrt{b^2c^2 + c^2a^2 + a^2b^2}}$

The points on the lines are $\overline{a} = bj, c = ai$

$\Rightarrow c - a = ai - bj \Rightarrow L = \vec{n}.(\vec{c} - \vec{a}) = \frac{2abc}{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}$

$\therefore L^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = 4$

LIMITS

SYNOPSIS

Limit of a function :

- Let f be a function defined over a deleted neighbourhood of the real number 'a' and 'l' be a real number. If to every positive number $\in$ (however small) there exists a positive number 'δ' such that $|f(x)-l|<\in$ for all x such that $0<|x-a|<\delta$, we say that $f(x)$ tends to 'l' as x tends to a and we write it as $\underset{x\to a}{Lt}\ f(x)=l$.

Right handed limit :

- Let $f(x)$ be a function defined in the interval $(a,a+h)$ and 'l' be a real number. If to every positive number $\in$ (however small), there corresponds a positive number δ such that $|f(x)-l|<\in, \forall\ x\in(a,a+\delta)$ then we say that $f(x)$ tends to 'l' as x approaches a through values higher than a and we denote

$\underset{x\to a^+}{Lt}\ f(x)=l\ \ (or)\ \ \underset{h\to 0}{Lt}\ f(a+h)=l$.

Left handed limit :

- Let $f(x)$ be a function defined in the interval $(a-h,a)$ and 'l' be a real number. If to every positive number $\in$ (however small), there corresponds a positive number δ such that $|f(x)-l|<\in$ for all x such that $x\in(a-\delta,a)$ we say that $f(x)$ tends to 'l' as x approaches a through values less than a and we denote it as

$\lim\limits_{x\to a^-} f(x)=l\ (or)\ \lim\limits_{h\to 0} f(a-h)=l$

- Let $a, l\in R$ and f be a function defined on a deleted neighbourhood of 'a' then

$\lim\limits_{x\to a} f(x)=l \Leftrightarrow \lim\limits_{x\to a^+} f(x)=l=\lim\limits_{x\to a^-} f(x)$

- If $\lim\limits_{x\to a} f(x)$ exists, then

$\lim\limits_{x\to a} f(x)=\lim\limits_{x\to 0} f(a+x)=\lim\limits_{x\to 0} f(a-x)$

Infinite limits :

- Let f be a function defined over a deleted neighbourhood of a. If for every positive number k (however large) there corresponds a positive number δ, such that $f(x)>k$, $\forall x$ such that $0<|x-a|<\delta$, we say that $f(x)\to+\infty$ as $x\to a$. We write is as $\underset{x\to a}{Lt} f(x)=+\infty$

Similarly we define (i) $\underset{x\to a}{Lt} f(x)=-\infty$

(ii) $\underset{x\to+\infty}{Lt} f(x)=+\infty,\ \ \underset{x\to+\infty}{Lt} f(x)=-\infty$

(iii) $\underset{x\to-\infty}{Lt} f(x)=+\infty,\ \ \underset{x\to-\infty}{Lt} f(x)=-\infty$

Limit of a function *f* as $x\to+\infty$ or $-\infty$:

- Let *f* be a function and *l* be a real number. If for every positive number $\in$ there corresponds a positive number k (however large) such that $|f(x)-l|<\in\ \forall x>k$, then we say that $f(x)$ tends to l as x tends to ∞. We write it as $\underset{x\to+\infty}{Lt} f(x)=l.$

Similarly wedefine $\underset{x\to-\infty}{Lt} f(x)=l$.

Fundamental Theorem on Limits :

- If $\underset{x\to a}{Lt}\ f(x)=l\ and\ \underset{x\to a}{Lt}\ g(x)=m$ then,

i) $\underset{x\to a}{Lt}\ \{f(x)+g(x)\}=l+m$

ii) $\underset{x\to a}{Lt}\ \{f(x)-g(x)\}=l-m$

iii) $\underset{x\to a}{Lt}\{f(x).g(x)\} = l.m$

iv) $\underset{x\to a}{Lt}\{k.f(x)\} = k.\underset{x\to a}{Lt} f(x)$

v) $\underset{x\to a}{Lt}\frac{f(x)}{g(x)} = \frac{l}{m}$, if $m \neq 0$

vi) $\underset{x\to a}{Lt}(f(x))^k = l^k$, if $k \in Q$ and $l^k \in R$

vii) $\underset{x\to a}{Lt} f(g(x)) = f\left(\underset{x\to a}{Lt}(g(x))\right) = f(m)$

viii) $\underset{x\to a}{Lt}\left(f(x)^{g(x)}\right) = l^m$, if $l^m \in R$

ix) If $f(x) \le g(x)$ on a deleted nbd of 'a'

then $\underset{x\to a}{Lt} f(x) \le \underset{x\to a}{Lt} g(x)$

x) $\underset{x\to a}{Lt} f(x) = l \Rightarrow \underset{x\to a}{Lt}|f(x)| = |l|$.
However the converse need not be true

Ex. $\underset{x\to 0}{Lt}\frac{1}{|x|} = \infty$ where as $\underset{x\to 0}{Lt}\frac{1}{x}$ does not exist.

Indeterminate forms :

- $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0\times\infty, 0^0, \infty^0$ and 1^∞ are called indeterminate forms

Standard Limits :

- For all real values of n, $\underset{x\to a}{Lim}\frac{x^n - a^n}{x-a} = n.a^{n-1}$

  (Provided $n\, a^{n-1}$ is defined)

- $\underset{x\to a}{Lt}\frac{x^m - a^m}{x^n - a^n} = \frac{m}{n}a^{m-n}, (m > n)$

- If $0 < |x| < \frac{\pi}{2}$ and x is measured in radians.

  $\underset{x\to 0}{Lim}\frac{\sin x}{x} = 1$ and $\underset{x\to 0}{Lim}\frac{\tan x}{x} = 1$,

  $\underset{x\to 0}{Lt}\frac{\sin ax}{x} = a$ and $\underset{x\to 0}{Lt}\frac{\tan ax}{x} = a$

  $\underset{x\to 0}{Lim}\frac{\sin x^0}{x} = \frac{\pi}{180}$ and $\underset{x\to 0}{Lim}\frac{\tan x^0}{x} = \frac{\pi}{180}$

- $\underset{x\to\infty}{Lim}\frac{\sin x}{x} = 0$ and $\underset{x\to\infty}{Lim}\frac{\cos x}{x} = 0$

- $\underset{x\to 0}{Lim}\frac{\sin^{-1} x}{x} = 1$ and $\underset{x\to 0}{Lim}\frac{\tan^{-1} x}{x} = 1$

- $\underset{x\to 0}{Lim}\left(\frac{e^x - 1}{x}\right) = 1$

- $\underset{x\to 0}{Lim}\left(\frac{a^x - 1}{x}\right) = \log_e a, (a > 0)$

- $\underset{x\to 0}{Lim}\frac{a^x - b^x}{x} = \log_e\left(\frac{a}{b}\right)$

- $\underset{x\to 0}{Lim}\frac{a^x - 1}{b^x - 1} = \log_b a$

- $\underset{x\to a}{\lim}\frac{|x-a|}{x-a}$ does not exist

- $\underset{x\to 0}{Lim}(1+x)^{\frac{1}{x}} = e$, $\underset{x\to 0}{Lt}(1+ax)^{\frac{1}{x}} = e^a$

- $\underset{x\to\infty}{Lim}\left(1+\frac{1}{x}\right)^x = e$, $\underset{x\to\infty}{Lt}\left(1+\frac{a}{x}\right)^x = e^a$

- $f(x), g(x)$ are two polynomials such that degree of $f(x)$ is m and degree of $g(x)$ is n then

  i) $\lim_{x\to\infty}\frac{f(x)}{g(x)} = 0$ for $m < n$

  ii) $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \infty$ for $m > n$ and coef of $x^m > 0$

  iii) $\lim_{x\to\infty}\frac{f(x)}{g(x)} = -\infty$ for $m > n$ and coef of $x^m < 0$

  iv) $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \frac{coef\ of\ x^m\ in\ Nr}{coef\ of\ x^n\ in\ Dr}$ for $m = n$

- $\lim_{x\to 0^+} e^{\frac{1}{x}} = \infty, \lim_{x\to 0^-} e^{\frac{1}{x}} = 0$

- $\lim_{x\to 0^+} e^{-\frac{1}{x}} = 0, \lim_{x\to 0^-} e^{-\frac{1}{x}} = \infty$

- $\lim_{n\to\infty} x^n = 0$ if $|x|<1$

- $\lim_{n\to\infty} x^n = \infty$ if $|x|>1$

- $\lim_{x\to 0}\sin\left(\frac{1}{x}\right) = \lim_{x\to 0}\cos\left(\frac{1}{x}\right) =$ Does not exist

- $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = \lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0$

L'Hospital's Rule :

- If $\operatorname{Lim}_{x\to a}\frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$\operatorname{Lim}_{x\to a}\frac{f(x)}{g(x)} = \operatorname{Lim}_{x\to a}\frac{f^1(x)}{g^1(x)}$.

If $\operatorname{Lim}_{x\to a}\frac{f^1(x)}{g^1(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then

$\operatorname{Lim}_{x\to a}\frac{f(x)}{g(x)} = \operatorname{Lim}_{x\to a}\frac{f^{11}(x)}{g^{11}(x)}$

If $\operatorname{Lim}_{x\to a}\frac{f^{11}(x)}{g^{11}(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then

This can be continued till we finally arrive at a determinate result.

Some useful results :

- Let $S = \{x, \sin x, \tan x, \sinh x, \tanh x, \sin^{-1} x, \tan^{-1} x, \sinh^{-1} x, \tanh^{-1} x\}$

If $f(x), g(x) \in S$ then $\underset{x\to 0}{Lt}\frac{f(mx)}{g(nx)} = \frac{m}{n}$, .

If $f_1(x), f_2(x), g_1(x), g_2(x) \in S$ then

$$\underset{x\to 0}{Lt}\frac{f_1(mx)\pm f_2(nx)}{g_1(px)\pm g_2(qx)} = \frac{m\pm n}{p\pm q}$$

- $\underset{x\to o}{Lt}\frac{\sin ax}{\tan bx} = \frac{a}{b}$, $\underset{x\to 0}{Lt}\frac{1-\cos ax}{x^2} = \frac{a^2}{2}$

Some frequently used expansions :

- i) $(1+x)^p =$

$1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots\infty$, if $|x|<1$

ii) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\infty$

iii) $a^x = 1 + \frac{x}{1!}.\log_e a + \frac{x^2}{2!}(\log_e a)^2 + \ldots\infty$

iv) $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\infty$

v) $\log_e(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\infty\right)$

vi) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots.\infty$

vii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots..\infty$

viii) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15}\ldots..\infty$

ix) $\sin^{-1} x = x + \frac{1}{2}.\frac{x^3}{3} + \frac{1}{2}.\frac{3}{4}.\frac{x^5}{5} + \ldots..\infty$

x) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots..\infty$

Sandwich theorem or Squeezeprinciple :

- If f, g, h are functions such that $f(x) \le g(x) \le h(x)$

then $\underset{x\to a}{Lt} f(x) \le \underset{x\to a}{Lt} g(x) \le \underset{x\to a}{Lt} h(x)$ and

$\underset{x\to a}{Lt} f(x) = \underset{x\to a}{Lt} h(x) = l$ then $\underset{x\to a}{Lt} g(x) = l$

EXAMPLES

1. $\lim_{n\to\infty}\left(4^n + 5^n\right)^{1/n}$ **is equal to**

Sol: Given limit $=$

$$\lim_{n\to\infty}\left(4^n+5^n\right)^{1/n} = \lim_{n\to\infty} 5\left(1+\left(\frac{4}{5}\right)^n\right)^{1/n} = 5$$

$$\left(\because \left(\frac{4}{5}\right)^n \to 0 \text{ as } n\to\infty\right)$$

2:

$$\lim_{n\to\infty}\frac{5^{n+1}+3^n-2^{2n}}{5^n+2^n+3^{2n+3}}$$ **is equal to**

Sol: $$\lim_{n\to\infty}\frac{5^{n+1}+3^n-2^{2n}}{5^n+2^n+3^{2n+3}}=\lim_{n\to\infty}\frac{5.5^n+3^n-4^n}{5^n+2^n+27.9^n}$$

$$=\lim_{n\to\infty}\frac{5.\frac{5^n}{9^n}+\frac{3^n}{9^n}-\frac{4^n}{9^n}}{\frac{5^n}{9^n}+\frac{2^n}{9^n}+27}=\frac{0+0-0}{0+0+27}=0$$

. 3:

Let $f(x)$ be a twice differentiable function and $f''(0)$=5, then

$$\lim_{x\to 0}\frac{3f(x)-4f(3x)+f(9x)}{x^2}$$ ***is equal to***

Sol: $$\lim_{x\to 0}\frac{3f(x)-4f(3x)+f(9x)}{x^2}\left(\frac{0}{0}\,form\right)$$

$$=\lim_{x\to 0}\frac{3f'(x)-12f'(3x)+9f'(9x)}{2x}\left(\frac{0}{0}\,form\right)$$

$$=\lim_{x\to 0}\frac{3f''(x)-36f''(3x)+81f''(9x)}{2}$$

$$=\frac{3f''(0)-36f''(0)+81f''(0)}{2}$$

$$=24f''(0)=24(5)=120$$

. 4:

$$\underset{x\to 0}{Lt}\frac{\sin 7x+\sin 5x}{\tan 5x-\tan 2x}=\frac{7+5}{5-2}=4$$

✈ If $f_1(x),f_2(x),g_1(x),g_2(x)\in S$ and m + n = p + q then

$$\underset{x\to 0}{Lt}\frac{f_1^{\,m}(ax)f_2^{\,n}(bx)}{g_1^{\,p}(cx)g_2^{\,q}(dx)}=\frac{a^mb^n}{c^pd^q}$$

5:

$$\underset{x\to 0}{Lt}\frac{\sin^3 2x\tan^2 3x}{x\sin^4 4x}=\frac{2^3\times 3^2}{4^4}=\frac{9}{32}$$

✈ If $g_1(x),g_2(x)\in S$ then

$$\underset{x\to 0}{Lt}\frac{1-\cos ax}{g_1(cx)g_2(dx)}=\frac{a^2}{2cd}$$

6:

$$\underset{x\to 0}{Lt}\frac{1-\cos x}{x\sin 3x}=\frac{1}{6}$$

✈ If $g_1(x),g_2(x)\in S$ then

$$\underset{x\to 0}{Lt}\frac{1-\cos^n(ax)}{g_1(cx)g_2(dx)}=\frac{na^2}{2cd}$$

. 7:

$$\underset{x\to 0}{Lt}\frac{1-\cos^3 2x}{\sin 5x\tan 7x}=\frac{3\times 2^2}{2\times 5\times 7}=\frac{6}{35}$$

✈ If $g_1(x),g_2(x),.............,g_{2n}(x)\in S$ then

$$\underset{x\to 0}{Lt}\frac{1-\cos(ax^n)}{g_1(c_1x).g_2(c_2x)........g_{2n}(c_{2n}x)}=\frac{a^2}{2c_1c_2........c_{2n}}$$

8:

$$\underset{x\to 0}{Lt}\frac{1-\cos(2x^3)}{x\sin^2(2x)\tan^3(3x)}=\frac{4}{2\times 2^2\times 3^3}=\frac{1}{54}$$

✈ If $g_1(x),g_2(x)\in S$ then

$$\underset{x\to 0}{Lt}\frac{\cos ax-\cos bx}{f(cx)g(dx)}=\frac{b^2-a^2}{2cd}$$

9:

$$\underset{x\to 0}{Lt}\frac{\cos 3x-\cos 5x}{x^2}=\frac{25-9}{2}=8$$

✈ If $g_1(x),g_2(x)\in S$ then

$$\underset{x\to 0}{Lt}\frac{\cos^n ax-\cos^n bx}{g_1(cx).g_2(dx)}=\frac{n(b^2-a^2)}{2cd}$$

10:

$$\underset{x\to 0}{Lt}\frac{\cos^3 3x-\cos^3 5x}{x^2}=\frac{3(25-9)}{2}=24$$

✈ If $g_1(x),g_2(x),.............,g_{2n}(x)\in S$ then

$$\underset{x\to 0}{Lt}\frac{\cos(ax^n)-\cos(bx^n)}{g_1(c_1x).g_2(c_2x).......g_{2n}(c_{2n}x)}=\frac{b^2-a^2}{2c_1c_2........c_{2n}}$$

11:

$$\underset{x\to 0}{Lt}\frac{\cos(2x^3)-\cos(5x^3)}{x\sin^2(2x)\tan^3(3x)}=\frac{25-4}{2\times 2^2\times 3^3}=\frac{7}{18\times 4}=\frac{7}{72}$$

✈ If $g_1(x)\in S$ then

$$\underset{x\to 0}{Lt}\frac{\tan^n(ax)-\sin^n(ax)}{[g(x)]^{n+2}}=\frac{na^{n+2}}{2}$$

12:

$$\underset{x\to 0}{Lt}\frac{\tan x-\sin x}{x^3}=\frac{1}{2}$$

✈ $$\underset{x\to 0}{Lt}\frac{\sqrt{1+x^n}-\sqrt{1-x^n}}{x^n}=1$$

13:

$$\underset{x\to 0}{Lt}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x^2}=1$$

✈ $$\underset{x\to 0}{Lt}\frac{\sqrt[n]{a+x^m}-\sqrt[n]{a-x^m}}{x^m}=\frac{2}{n}a^{\frac{1}{n}-1}$$

14:

$$\underset{x\to 0}{Lt}\frac{\sqrt{a+x}-\sqrt{a-x}}{x}=a^{\frac{1}{2}-1}=\frac{1}{\sqrt{a}}$$

✈ If $g(x)\in S$ then $\underset{x\to 0}{Lt}\frac{\sqrt{a+x}-\sqrt{a-x}}{g(x)}=\frac{1}{\sqrt{a}}$

. 15:

$$\underset{x\to 0}{Lt}\frac{\sqrt{3+x}-\sqrt{3-x}}{\sin x}=\frac{1}{\sqrt{3}}$$

✈ If $g(x)\in S$ then $\underset{x\to 0}{Lt}\frac{\sqrt{a+x}-\sqrt{a}}{g(x)}=\frac{1}{2\sqrt{a}}$

. 16:

$$\underset{x\to 0}{Lt}\frac{\sqrt{2+x}-\sqrt{2}}{x}=\frac{1}{2\sqrt{2}}$$

✈ $$\underset{x\to 0}{Lt}\frac{x.a^{\alpha x}-x}{1-\cos(mx)}=\frac{2\alpha}{m^2}\log a$$

17:

$$\underset{x\to 0}{Lt}\frac{x.2^{3x}-x}{1-\cos(3x)}=\frac{2\times 3}{3^2}\log 2=\frac{2}{3}\log 2$$

✈ $\lim_{x\to a}f(x)=1$ *and* $\lim_{x\to a}g(x)=\infty$ then

$$\lim_{x\to a}\left[f(x)\right]^{g(x)}=e^{\lim_{x\to a}g(x)[f(x)-1]}$$

18:

If $\lim_{x\to 0}\left(1+ax+bx^2\right)^{2/x}=e^3$, then the values of a and b are

Sol: Let $\lim_{x\to 0}\left(1+ax+bx^2\right)^{2/x}$ is of the form 1^∞

$$e^{\lim_{x\to 0}\left(1+ax+bx^2-1\right).\frac{2}{x}}=e^{\lim_{x\to 0}(2a+2bx)}$$

$$=e^{2a}=e^3(given)\qquad \therefore a=3/2\ and\ b\in R$$

✈ $\lim_{x\to a}f(x)=0$ and $\lim_{x\to a}g(x)=0$, then

$$\lim_{x\to a}[f(x)]^{g(x)}=e^{\lim_{x\to a}g(x)\log f(x)}\qquad (f(x)>0)$$

19:

$$\underset{x\to 1}{Lt}\left(1-x^2\right)^{\frac{1}{\log(1-x)}}=$$

Sol : $$e^{\lim_{x\to 1}\frac{\log\left(1-x^2\right)}{\log(1-x)}}=e^{\lim_{x\to 1}\left[\frac{\log(1-x)+\log(1+x)}{\log(1-x)}\right]}$$

$$=e^{\lim_{x\to 1}\left[1+\frac{\log(1+x)}{\log(1-x)}\right]}=e^{1+0}=e$$

✈ $$\lim_{x\to 0}\left[\frac{a_1^x+a_2^x+a_3^x+...+a_n^x}{n}\right]^{\frac{1}{x}}=\left(a_1.a_2...a_n\right)^{\frac{1}{n}}$$

✈ $$\lim_{x\to\infty}\left[\frac{a_1^{\frac{1}{x}}+a_2^{\frac{1}{x}}+a_3^{\frac{1}{x}}+...+a_n^{\frac{1}{x}}}{n}\right]^{x}=\left(a_1.a_2...a_n\right)^{\frac{1}{n}}$$

20:

Evaluate $\lim_{x\to 0}\left(\frac{2^x+2^{2x}+2^{3x}}{3}\right)^{1/x}$

Sol: $$\lim_{x\to 0}\left(\frac{2^x+2^{2x}+2^{3x}}{3}\right)^{1/x}=(2.4.8)^{1/3}=4$$

. 21:

$$\lim_{x\to 0}[\cos x+m\sin ax]^{\frac{n}{x}}=e^{amn}$$

✈ $$\lim_{x\to\infty}a^x=\begin{cases}0,\ 0\le a<1\\ 1,\ a=1\\ \infty,\ a>1\\ does\ not\ exists,\ a<0\end{cases}$$

. 22:

Evaluate $\underset{x\to 0}{Lim}\left(\dfrac{(1+x)^{\frac{1}{x}}-e+\dfrac{ex}{2}}{\sin^2 x}\right)$

Sol : $\underset{x\to 0}{Lt}\dfrac{(1+x)^{1/x}-e+\dfrac{ex}{2}}{\sin^2 x}$

$= \underset{x\to 0}{Lt}\dfrac{(1+x)^{1/x}-e+\dfrac{ex}{2}}{x^2}.\left(\dfrac{x^2}{\sin^2 x}\right)$

$= \underset{x\to 0}{Lt}\dfrac{(1+x)^{1/x}-e+\dfrac{ex}{2}}{\sin^2 x}.1$

$= \underset{x\to 0}{Lt}\dfrac{e\left[1-\dfrac{x}{2}+\dfrac{11}{24}x^2....\right]-e+\dfrac{ex}{2}}{x^2}=\dfrac{11e}{24}\cdot$

23:

Find $\underset{x\to 0}{Lt}\dfrac{\left(\sin x-x+\dfrac{x^3}{6}\right)}{x^5}$**.**

Sol : $\underset{x\to 0}{Lt}\dfrac{\left(\sin x-x+\dfrac{x^3}{6}\right)}{x^5}$

$= \underset{x\to 0}{Lt}\dfrac{\left(x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+.....\right)-x+\dfrac{x^3}{6}}{x^5}$

$= \underset{x\to 0}{Lt}\dfrac{\left(\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+.....\right)}{x^5}$

$= \underset{x\to 0}{Lt}\left(\dfrac{1}{5!}-\dfrac{x^2}{7!}+\right.$ terms containing positive integral powers of x)

$= \dfrac{1}{5!}=\dfrac{1}{120}.$

24:

If $3-\dfrac{x^2}{12}\le f(x)\le 3+\dfrac{x^3}{9}$ **for all** $x\neq 0$**,**

then the value of $\lim\limits_{x\to 0} f(x)$ **is**

Sol: According to the equation,

$$\lim_{x\to 0}\left(3-\frac{x^2}{12}\right)\le\lim_{x\to 0} f(x)\le\lim_{x\to 0}\left(3+\frac{x^3}{9}\right)$$

$\Rightarrow (3-0)\le\lim\limits_{x\to 0} f(x)\le(3+0)$

Hence, $\lim\limits_{x\to 0} f(x)=3$

✈ $\underset{n\to\infty}{Lt}\dfrac{[1^k x]+[2^k x]+.....[n^k x]}{n^{k+1}}=\dfrac{x}{k+1}(k\in N)$

25:

Show that $\underset{n\to\infty}{Lt}\dfrac{[x]+[2x]+....[nx]}{n^2}=\dfrac{x}{2}$.

Sol : For r=1,2,3.....n, $r.x-1<[rx]\le rx$

$$\Rightarrow\sum_{r=1}^{n}(rx-1)<\sum_{r=1}^{n}[rx]\le\sum_{r=1}^{n}(rx)$$

$$\Rightarrow\frac{\dfrac{n(n+1)x}{2}-n}{n^2}<\frac{\sum\limits_{r=1}^{n}[rx]}{n^2}\le\frac{n(n+1)x}{2n^2}$$

$$\Rightarrow\underset{n\to\infty}{Lt}\left(\left(1+\frac{1}{n}\right)\frac{x}{2}-\frac{1}{n}\right)\le\underset{n\to\infty}{Lt}\frac{\sum\limits_{r=1}^{n}[rx]}{n^2}\le\underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)\frac{x}{2}$$

(Note that x is a constant and n is a variable)

$\Rightarrow\dfrac{x}{2}\le\underset{n\to\infty}{Lt}\dfrac{[1.x]+[2.3]+.....+[n.x]}{n^2}\le\dfrac{x}{2}$

$\therefore$ By sandwich theorem,

$\underset{n\to\infty}{Lt}\dfrac{[1x]+[2x]+....[nx]}{n^2}=\dfrac{x}{2}$.

EXERCISE - I

1. $\underset{x\to 2}{Lim}\dfrac{7x^2-11x-6}{3x^2-x-10}=$

1) $\dfrac{17}{11}$ 2) $\dfrac{11}{17}$ 3) $\dfrac{17}{14}$ 4) $-\dfrac{17}{11}$

2. $\underset{x\to 0}{Lim}\dfrac{\sqrt[K]{1+x}-1}{x}$ **(K is a positive integer)**

1) K 2) $-K$ 3) $\dfrac{1}{K}$ 4) $-\dfrac{1}{K}$

3. $\underset{x\to 1}{Lim}\dfrac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}=$ **[EAM -2019]**

1) $\dfrac{1}{10}$ 2) $-\dfrac{1}{10}$ 3) $\dfrac{2}{5}$ 4) $-\dfrac{2}{5}$

4. $\underset{x\to 0}{Lim}\dfrac{\sqrt[3]{1+\sin x}-\sqrt[3]{1-\sin x}}{x}=$

1) 0 2) 1 3) $\dfrac{2}{3}$ 4) $\dfrac{3}{2}$

5. $\underset{x\to 0}{Lim}\dfrac{\sqrt{4+x}-\sqrt[3]{8+3x}}{x}=$

1) $-\dfrac{1}{2}$ 2) $\dfrac{1}{2}$ 3) -3 4) 0

6. If $\underset{x\to 5}{Lim}\dfrac{x^k-5^k}{x-5}=500$**, then the positive integral value of k is**

1) 3 2) 4 3) 5 4) 6

7. If a > 0 and $\underset{x\to a}{Lim}\dfrac{a^x-x^a}{x^x-a^a}=-1$ **then a =**

1) 0 2) 1 3) e 4) 2e

8. $\underset{x\to -1}{Lim}\dfrac{x+1}{\sqrt{x^2+3}-2}=$

1) -2 2) 1/2 3) 2 4) 0

9. $\underset{x\to 0}{Lim}\dfrac{x\left(1-\sqrt{1-x^2}\right)}{\sqrt{1-x^2}\left(\sin^{-1}(x)^3\right)}=$

1) 1 2) $\dfrac{1}{2}$ 3) $-\dfrac{1}{2}$ 4) -1

10. If $\lim_{x\to 0}\left(\dfrac{\cos 4x+a\cos 2x+b}{x^4}\right)$ **is finite then the value of** a,b **respectively**

1) 5 2) -5,-4 3) -4,3 4) 4,5

11. $\lim_{x\to 1}\left(\sec\left(\dfrac{\pi x}{2}\right)\log x\right)$ **is**

1) $-\dfrac{2}{\pi}$ 2) $-\dfrac{\pi}{2}$ 3) $\dfrac{2}{\pi}$ 4) $\dfrac{\pi}{2}$

12. $\underset{x\to 0}{Lim}\dfrac{\sin x}{\sqrt{x^2}}=$

1) 1 2) -1 3)0 4) doesn't exist

13. $\lim_{x\to 2^+}\left(\dfrac{[x]^3}{3}-\left[\dfrac{x}{3}\right]^3\right)$ **is (where [ ] is g.i.f)**

1) 0 2) $\dfrac{64}{27}$ 3) $\dfrac{8}{3}$ 4) $\dfrac{10}{3}$

14. $\lim_{x\to 0}\dfrac{\sin[\cos x]}{1+[\cos x]}$ **is (where [ ] is g.i.f)**

1) 1 2) 0 3) does not exist 4) 2

15. $\underset{x\to \frac{\pi}{4}}{Lim}\dfrac{\cos x-\sin x}{\left(\dfrac{\pi}{4}-x\right)(\cos x+\sin x)}=$

1) 2 2) 1 3) 0 4) 3

16. $\underset{x\to 0}{Lim}\dfrac{\sin x\sin\left(\dfrac{\pi}{3}+x\right)\sin\left(\dfrac{\pi}{3}-x\right)}{x}=$

1) $\dfrac{3}{4}$ 2) $\dfrac{1}{4}$ 3) $\dfrac{4}{3}$ 4) 0

17. $\underset{x\to 5}{Lim}\dfrac{\sin^2(x-5)\tan(x-5)}{(x^2-25)(x-5)}=$

1) 1 2) 1/10 3) 0 4)-6

18. $\underset{x\to 0}{Lim}\frac{1-\cos^3 x}{x\sin 2x}=$

1) 1/2 2) 3/2 3)3/4 4)1/4

19. $\underset{x\to 1}{Lim}(1-x)Tan\left(\frac{\pi x}{2}\right)=$

1) π 2) 2π 3) $\frac{\pi}{2}$ 4) $\frac{2}{\pi}$

20. $\underset{x\to\frac{\pi}{6}}{Lim}\frac{3\sin x-\sqrt{3}\cos x}{6x-\pi}=$

1) $\sqrt{3}$ 2) $\frac{1}{\sqrt{3}}$ 3) $-\sqrt{3}$ 4) $-\frac{1}{\sqrt{3}}$

21. $\underset{x\to\frac{\pi}{2}}{Lim}\frac{\cot x-\cos x}{\left(\frac{\pi}{2}-x\right)^3}=$

1) $\frac{-1}{2}$ 2) $\frac{1}{2}$ 3) 2 4) -2

22. $\underset{x\to 0}{Lim}\frac{\sec 4x-\sec 2x}{\sec 3x-\sec x}=$

1) $\frac{3}{2}$ 2) $\frac{2}{3}$ 3) $\frac{1}{3}$ 4) $\frac{3}{4}$

23. $\underset{x\to 0}{Lim}\frac{3\sin\left(x^g\right)-\sin\left(3x^g\right)}{x^3}=$

1) $\left(\frac{\pi}{200}\right)^3$ 2) $4\left(\frac{\pi}{200}\right)^3$ 3) $\frac{\pi}{200}$ 4) $\frac{\pi}{100}$

24. $\underset{x\to 0}{Lim}\frac{\log_e(1+x)}{3^x-1}=$ [EAM -2018]

1) $\log_e 3$ 2) 0 3) 1 4) $\log_3 e$

25. $\underset{x\to 0}{Lim}\frac{e^{\alpha x}-e^{\beta x}}{\sin\alpha x-\sin\beta x}=$

1) 0 2) 1/2 3) 1/3 4) 1

26. $\underset{x\to 0}{Lim}\log\left|\frac{\log(1+x)}{x}\right|=$

1) 0 2) 1 3) e 4) 1/e

27. **The value of** $\lim_{x\to 2}\frac{2^x+2^{3-x}-6}{\sqrt{2^{-x}-2^{1-x}}}$ **is**

1) 16 2) 8 3) 4 4) 2

28. $\underset{x\to 0}{Lim}\frac{e^x+\sin x-1}{\log(1+x)}=$ **[EAM -2020]**

1) 1 2) $\frac{1}{3}$ 3) $\frac{2}{3}$ 4) 2

29. $\lim_{x\to 0}\frac{10^x-2^x-5^x+1}{x\tan x}$ **is**

1) $\log 2$ 2) $\frac{\log 2}{\log 5}$

3) $(\log 2)(\log 5)$ 4) $\log 10$

30. $\underset{n\to\infty}{Lim}\frac{(1+2+3+......+n\,terms)(1^2+2^2+......nterms)}{n(1^3+2^3+.....+nterms)}=$

1) $\frac{3}{2}$ 2) $\frac{2}{3}$ 3) 1 4) 0

31. $\underset{n\to\infty}{Lim}\left(\frac{1}{1.3}+\frac{1}{3.5}+....+\frac{1}{(2n-1)(2n+1)}\right)=$

1) 1 2) $\frac{1}{2}$ 3) $\frac{1}{3}$ 4) $\frac{1}{4}$

32. $\underset{n\to\infty}{Lim}\frac{1+\frac{1}{2}+\frac{1}{4}+....+\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{9}+....+\frac{1}{3^n}}=$

1) 4/3 2) 3/4 3) 1/2 4) 0

33. $\underset{x\to\infty}{Lim}\left(\frac{x^2\sin\left(\frac{1}{x}\right)-x}{1-|x|}\right)=$ **[EAM -2017]**

1) 0 2) 1 3) – 1 4) 2

34. **If** $o<h<q$ **then** $\lim_{n\to\infty}\left(q^n+h^n\right)^{\frac{1}{n}}=$

1) e 2) h 3) q 4) 0

35. $\underset{x\to\infty}{Lim}\left(\sqrt{x^2+ax+a^2}-\sqrt{x^2+a^2}\right)=$

1) 0 2) $\frac{a}{2}$ 3) $-\frac{a}{2}$ 4) a

36. $\lim\limits_{x\to\infty}\frac{x-\log x}{x+\log x}=$

1) 1 2) -1 3) 0 4) 2

37. $\lim\limits_{x\to\infty}\frac{2x+7\sin x}{4x+3\cos x}=$

1) 1 2) -1 3) 1/2 4) -1/2

38. $\lim\limits_{x\to\infty}\frac{8|x|+3x}{3|x|-2x}=$

1) 11 2) 8 3) 0 4) $\frac{1}{8}$

39. $\lim\limits_{x\to\infty}\frac{(x+1)^{10}+(x+2)^{10}+.....+(x+100)^{10}}{x^{10}+10^{10}}=$

1) 10 2) 100 3) 1000 4) 1

40. $\lim\limits_{n\to\infty}\frac{3.2^{n+1}-4.5^{n+1}}{5.2^{n}+7.5^{n}}=$

1) $-20/7$ 2) 20/7 3) 10/7 4) –10/7

41. If $\lim\limits_{x\to\infty}\left(1+\frac{\lambda}{x}+\frac{\mu}{x^2}\right)^{2x}=e^2$ **then**

1) $\lambda=1,\mu=2$ 2) $\lambda=2,\mu=1$

3) $\lambda=1,\mu=$ any real constant

4) $\lambda=\mu=1$

42. $\lim\limits_{x\to\infty}\left(\frac{x+a}{x+b}\right)^{x+b}$

1) 1 2) e^{b-a} 3) e^{a-b} 4) e^{b}

43. $\lim\limits_{x\to\pi}(1-4\tan x)^{\cot x}=$

1) e 2) e^4 3) e^{-1} 4) e^{-4}

44. $\lim\limits_{x\to\infty}\left(\frac{x^2+1}{x^2-1}\right)^{x^2}=$ [EAM -2016]

1) e 2) 1/e 3) e^2 4) e^{-2}

45. $\lim\limits_{x\to\infty}\left(\frac{3x-4}{3x+2}\right)^{\frac{x+1}{3}}=$

1) $e^{-2/3}$ 2) $e^{3/2}$ 3) $e^{2/3}$ 4) e

46. $\lim\limits_{x\to1}(2-x)^{\tan\left(\frac{\pi x}{2}\right)}=$

1) $e^{\frac{1}{\pi}}$ 2) $e^{\frac{2}{\pi}}$ 3) $-e^{\frac{2}{\pi}}$ 4) e

47. $\lim\limits_{x\to0+}(Sinx)^{\tan x}=$

1) e 2) e^2 3) -1 4) 1

48. $\lim\limits_{n\to\infty}\left(1+\sin\left(\frac{a}{n}\right)\right)^n=$

1) e 2) e^a 3) a^e 4) a

49. $\lim\limits_{h\to0}\frac{(2+h)\cos(2+h)-2\cos 2}{h}=$

1) $\cos 2-2\sin 2$ 2) $\cos 2+2\sin 2$

3) $\sin 2-2\cos 2$ 4) $\sin 2+2\cos 2$

50. $\lim\limits_{x\to0}\frac{x}{a}\left[\frac{b}{x}\right](a\neq0)$ [where [ ] denotes the G.I.F.) is equal to

1) a 2) b 3) $\frac{b}{a}$ 4) $1-\frac{b}{a}$

51. If $f(9)=9,\ f^1(9)=4,\ \lim\limits_{x\to9}\frac{\sqrt{f(x)}-3}{\sqrt{x}-3}=$

1) 4 2) $\frac{1}{4}$ 3) $\frac{1}{2}$ 4) $-\frac{1}{2}$

52. If $f(a)=2, f^1(a)=1, g(a)=-1, g^1(a)=2$, **then**

$\lim\limits_{x\to a}\frac{g(x)f(a)-g(a)f(x)}{x-a}=$

1) $\frac{1}{5}$ 2) 5 3) $-\frac{1}{5}$ 4) -5

53. $\lim\limits_{x\to0}\frac{1-\cos x\cos 2x\cos 3x}{\sin^2 2x}=$

1) $\frac{3}{2}$ 2) $\frac{5}{2}$ 3) $\frac{7}{4}$ 4) $\frac{9}{2}$

54 $\lim\limits_{x\to0}x^3\cos\frac{2}{x}=$

1) 0 2) 1 3) ∞

4) does not exist

KEY

01) 1	02) 3	03) 2	04) 3	05) 4	06) 2
07) 2	08) 1	09) 2	10) 3	11) 1	12) 4
13) 3	14) 2	15) 2	16) 1	17) 3	18) 3
19) 4	20) 2	21) 2	22) 1	23) 2	24) 4
25) 4	26) 1	27) 2	28) 4	29) 3	30) 2
31) 2	32) 1	33) 3	34) 3	35) 2	36) 1
37) 3	38) 1	39) 2	40) 1	41) 3	42) 3
43) 4	44) 3	45) 1	46) 2	47) 4	48) 2
49) 1	50) 3	51) 1	52) 2	53) 3	54) 1

SOLUTIONS

1. Using L-Hospital rule $= \lim_{x\to 2}\frac{14x-11}{6x-1} = \frac{17}{11}$

2. $\underset{x\to 0}{Lt}\frac{\sqrt[k]{1+x}}{x} = 1$ using L - hospital rule

$$\underset{x\to 0}{Lt}\frac{\frac{1}{k}(1+x)^{\frac{1}{k}-1}}{x} = \frac{1}{k}$$

3. Given limit

$$\lim_{x\to 0}\frac{(2x-3)(\sqrt{x}-1)}{(\sqrt{x}-1)(\sqrt{x}+1)(2x+3)} = \frac{(-1)}{2(5)} = \frac{-1}{10}$$

4. $\underset{x\to 0}{Lt}\frac{(1+\sin x)^{\frac{1}{3}}-(1-\sin x)^{\frac{1}{3}}}{x}$ this o/o form using L - hospital rule

$$= \underset{x\to 0}{Lt}\frac{\frac{1}{3}(1+\sin x)^{\frac{1}{3}-1}\cos x+\frac{1}{3}(1-\sin x)^{\frac{1}{3}-1}}{1}$$

$$= \frac{1}{3}+\frac{1}{3} = \frac{2}{3}$$

5. Using L-Hospital rule

6. Using $\lim_{x\to a}\frac{x^p-a^p}{x^q-a^q} = \frac{p}{q}a^{p-q}$

7. Use L-Hospital rule

$$\frac{d}{dx}(a^x) = a^x\log a,\ \frac{d}{dx}(x^a) = a.x^{a-1}$$

$$\frac{d}{dx}(x^x) = x^x(1+\log x)$$

8. On rationalizing given limit is

$$\lim_{x\to -1}\frac{(x+1)\left(\sqrt{x^2+3}+2\right)}{x^2-1} = -2$$

9. On rationalising given limit

$$= \lim_{x\to 0}\frac{x^3}{[\sin^{-1}(x^3)]}.\frac{1}{1+\sqrt{1-x^2}}$$

10. $\underset{x\to 0}{Lt}\frac{\cos 4x+a\cos 2x+b}{x^4}$ is finite

$\Rightarrow$ 1 + a + b = 0.......................1

using L Hospital rule

$$\underset{x\to 0}{Lt}\frac{-4\sin 4x-2a\sin 2x}{4x^3} = 10(say)$$

Again using L - Hospital

$$\Rightarrow \underset{x\to 0}{Lt}\frac{-16\cos 4x-4a\cos 2x}{12x^2} = K$$

$\Rightarrow$ -16 -4a = 0

a = -L1 substiting in ....(1)

b = -3

11. $\lim_{x\to 1}\left[\sec\left(\frac{\pi x}{2}\right)\log x\right]$

$$= \lim_{x\to 1}\frac{\log x}{\cos\left(\frac{\pi x}{2}\right)} = \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 1}\frac{\frac{1}{x}}{-\sin\left(\frac{\pi x}{2}\right).\frac{\pi}{2}} = -\frac{1}{\frac{\pi}{2}\sin\frac{\pi}{2}} = -\frac{2}{\pi}$$

12. $\underset{x\to 0}{Lt}\frac{\sin x}{(x)}$ this is of the form 0/0 using L - Hospital rule

$\underset{x\to 0}{Lt}\frac{\sin x}{-x}$ =-1, $\underset{x\to 0^+}{Lt}\frac{\sin x}{x}$ =1

$$\begin{pmatrix} x<0, & |x|=-x \\ x>0, & |x|=x \end{pmatrix}$$

$\underset{x\to 0^-}{Lt} f(x) \neq \underset{x\to 0^+}{Lt} f(x)$ does not exit.S

13. For

$$x \in (2,3), [x]=2, \frac{x}{3} \in \left(\frac{2}{3},1\right) \Rightarrow \left[\frac{x}{3}\right]=0$$

$$\therefore \lim_{x\to 2^+}\left(\frac{[x]^3}{3}-\left[\frac{x}{3}\right]^3\right)=\frac{1}{3}(2)^3-(0)^3=\frac{8}{3}$$

14. $\lim_{x\to 0}[\cos x]=\lim_{h\to 0}[\cos(0\pm h)]=\lim_{h\to 0}[\cosh]=0$

$(\because As\ h\to 0, \cosh \to 1)$

$$\therefore \lim_{x\to 0}\frac{\sin(\cos x)}{1+[\cos x]}=\frac{0}{1+0}=0$$

15. Given limit is $\lim_{x\to\frac{\pi}{4}}\dfrac{\tan\left(\frac{\pi}{4}-x\right)}{\frac{\pi}{4}-x}=1$

16. Given limit is $\lim_{x\to 0}\dfrac{(1/4)\sin 3x}{x}=\dfrac{3}{4}$

17. Given limit is

$$\lim_{(x-5)\to 0}\frac{\sin^2(x-5)}{(x-5)^2}\lim_{x\to 5}\frac{\tan(x-5)}{(x+5)}=0$$

18. Given limit is

$$\lim_{x\to 0}\frac{(1-\cos x)(1+\cos x+\cos^2 x)}{x\sin 2x}=\frac{3}{4}$$

19. $\lim_{x\to 1}\dfrac{(1-x)}{\cot\left(\frac{\pi x}{2}\right)}=\dfrac{2}{\pi}\lim_{x\to 1}\dfrac{-1}{-\cos ec^2\left(\frac{\pi x}{2}\right)}=\dfrac{2}{\pi}$

20. By L-Hospital rule

21. $\underset{\theta\to 0}{lim}\dfrac{\frac{\sin\theta}{\cos\theta}-\sin\theta}{\theta^3}=\underset{\theta\to 0}{Lim}\left(\dfrac{\frac{\sin\theta}{\cos\theta}}{\theta}\right)\left(\dfrac{1-\cos\theta}{\theta^2}\right)$

22. $\lim_{x\to 0}\dfrac{\sec 4x-\sec 2x}{\sec 3x-\sec x}=\dfrac{4^2-2^2}{3^2-1^2}=\dfrac{3}{2}$

23. $\lim_{x\to 0}\dfrac{4\sin^3\left(\frac{\pi}{200}x\right)}{x^3}=4\left(\dfrac{\pi}{200}\right)^3$

24. Use L-Hospital rule

25. $\underset{x\to 0}{Lt}\dfrac{e^{\alpha x}-e^{\beta x}}{\sin\alpha x-\sin\beta x}$ this is of the form 0/0

using L - Hospital rule.

$$\underset{x\to 0}{Lt}\frac{\alpha e^{\alpha x}-\beta e^{\beta x}}{\alpha\cos\alpha x-\beta\cos\beta x}=\frac{\alpha-\beta}{\alpha-\beta}=1$$

26. $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$

27. $\lim_{x\to 2}\dfrac{2^x+2^{3-x}-6}{\sqrt{2^{-x}}-2^{1-x}}$

$$=\lim_{x\to 2}\frac{(2^x)^2-6.2^x+2^3}{\sqrt{2^x}-2}$$

[Multiplying N^r and D^r by 2^x]

$$=\lim_{x\to 2}(2^x-2)(\sqrt{2^x}+2)=(2^2-2)(2+2)=8$$

28. $\lim_{x\to 0}\dfrac{e^x+\sin x-1}{\log(1+x)}=\lim_{x\to 0}\dfrac{e^x+\cos x}{\frac{1}{1+x}}=2$

29. $\lim_{x\to 0}\dfrac{(5^x-1)(2^x-1)}{x\tan x}$

30.

$$\underset{x\to\infty}{Lt}\frac{(1+2+3+.....nterms)(1^2+2^2+....nterms)}{n(1^3+2^3+......n\,terms)}=\frac{0}{0}$$

form

$$=\underset{x\to\infty}{Lt}\frac{\frac{n(n+1)}{2}.\frac{n(n+1)(2n+1)}{6}}{n.\frac{n^2(n+1)^2}{4}}=\underset{x\to\infty}{Lt}\frac{2n+1}{3n}=\frac{2}{3}$$

31. Given limit

$$=\frac{1}{2}\lim_{n\to\infty}\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+...+\frac{1}{(2n-1)}-\frac{1}{(2n+1)}\right)=\frac{1}{2}$$

32. Given limit (if r < 1 then sum of terms in G.P is $a\left(\frac{1-r^n}{1-r}\right)$

$$=\lim_{n\to\infty}\left(\frac{1+\frac{1}{2}+\frac{1}{4}+...+\infty}{1+\frac{1}{3}+\frac{1}{9}+...+\infty}\right)=\frac{4}{3}$$

33. $\underset{x\to\infty}{Lt}\frac{x^2\sin{}^1\!/\!_x-x}{1-|x|}$ divide by x

$$\underset{x\to\infty}{Lt}\frac{x\sin{}^1\!/\!_x-1}{{}^1\!/\!_x-1}=\underset{\frac{1}{x}\to\infty}{Lt}\frac{\sin{}^1\!/\!_x-1}{{}^1\!/\!_x-1}=\frac{1-1}{0-1}=0$$

34. Taking q^n common

35. $\underset{x\to\infty}{Lt}\left(\sqrt{x^2+ax+a^2}-\sqrt{x^2+a^2}\right)$ this is $\infty-\infty$ form

$$\underset{x\to\infty}{Lt}\frac{\left(\sqrt{x^2+ax+a^2}-\sqrt{x^2+a^2}\right)\left(\sqrt{x^2+ax+a^2}+\sqrt{x^2+a^2}\right)}{\sqrt{x^2+ax+a^2}+\sqrt{x^2+a^2}}=0$$

$$\underset{x\to\infty}{Lt}\frac{x^2+ax+a^2-x^2-a^2}{x\sqrt{1+\frac{a}{x}+\frac{a^2}{x^2}}+x\sqrt{1+\frac{a^2}{x^2}}}$$

$$\frac{a}{\sqrt{1+0+0}\sqrt{1+0}}=\frac{a}{2}$$

36. Use L-Hospital rule

37. $\underset{x\to\infty}{Lt}\frac{2x+7\sin x}{4x+3\cos x}=\frac{\infty}{\infty}$ from

$$\underset{x\to\infty}{Lt}\frac{x^{10}\left(\left(1+\frac{1}{x}\right)^{10}+\left(1+\frac{2}{x}\right)^{10}......\left(1+\frac{100}{x}\right)^{10}\right)}{x^{10}\left(1+\frac{10^{10}}{x^{10}}\right)}$$

=

$$=\frac{(1+0)^{10}+(1+0)^{10}+....(1+0)^{10}}{1+0}=\frac{100}{1}=100$$

38. $\underset{x\to\infty}{Lim}\frac{8|x|+3x}{3|x|-2x}=\underset{x\to\infty}{Lim}\frac{8x+3x}{3x-2x}=11$

39. Divide with x^{10}

40. Taking 5^n common and simplify

41. Given limit is

$$\Rightarrow e^{\lim_{x\to\infty}\left[1+\frac{\lambda}{x}+\frac{\mu}{x^2}-1\right].2x}=e^2\qquad\Rightarrow e^{\lim_{x\to\infty}\left(\lambda+\frac{\mu}{x}\right).2}=e^2$$

$$\Rightarrow e^{2\lambda}=e^2\Rightarrow\lambda=1,\mu\in R$$

42. Given limit $=e^{\lim_{x\to\infty}(x+b)\left[\frac{x+a}{x+b}-1\right]}$

43. $\lim_{x\to\pi}(1-4\tan x)^{\cot x}=\lim_{e^{x\to\pi}}\cot x(1-4\tan x-1)=e^{-4}$

44. $\underset{x\to\infty}{Lim}\left(\frac{x^2+1}{x^2-1}\right)^{x^2}=e^{\underset{x\to\infty}{Lim}x^2\left(\frac{x^2+1}{x^2-1}-1\right)}=e^2$

45. $\underset{x\to\infty}{Lim}\left(\frac{3x-4}{3x+2}\right)^{\frac{x+1}{3}}=\underset{x\to\infty}{Lim}\,e^{\frac{x+1}{3}\left(\frac{3x-4}{3x+2}-1\right)}=e^{\frac{-2}{3}}$

46. $\lim_{x\to1}(2-x)^{\tan\left(\frac{\pi x}{2}\right)}=e^{\lim_{x\to1}\tan\left(\frac{\pi x}{2}\right)(2-x-1)}=e^{\lim_{x\to1}\frac{(1-x)}{\cot\left(\frac{\pi x}{2}\right)}}$

and using L hospital rule.

47. $\lim_{x\to0}(\sin x)^{\tan x}=\lim_{x\to0}e^{\tan x\log(\sin x)}=e^0=1$

48. $e^{\lim_{n\to\infty}}n\left(1+\sin\left(\frac{a}{n}\right)-1\right)=e^a$

49. $\underset{h\to0}{Lt}\frac{(2+h)\cos(2+h)-2\cos^2}{h}=\frac{0}{0}$ form

$$\underset{h\to0}{Lt}\frac{-(2+h)\sin(2+h)+\cos(2+h)-0}{1}=$$

cos2 - 2sin2

50. $\frac{b}{x}-1<\left[\frac{b}{x}\right]\le\frac{b}{x}$

51. $\lim_{x\to 9}\frac{\frac{1}{2\sqrt{f(x)}}.f'(x)}{\frac{1}{2\sqrt{x}}}=\frac{f'(9)}{\sqrt{f(9)}}\times\sqrt{9}=\frac{4}{3}\times 3=4$

52. $f(a)=2, f'(a)=1, g(a)=-1, g'(a)=2$

$\lim_{x\to a}\frac{g(x)f(a)-g(a)f(x)}{x-a}$

By $L-H\ Rule$

$=g'(a)f(a)-g(a)f'(a)$

$=(2)(2)-(-1)(1)=5$

53. $\lim_{x\to 0}\frac{1-\left(1-\frac{x^2}{2}\right)\left(1-\frac{4x^2}{2}\right)\left(1-\frac{9x^2}{2}\right)}{4x^2}=\frac{7}{4}$

54. $-1\le\cos\frac{2}{x}\le 1\Rightarrow -x^3\le x^3\cos\frac{2}{x}\le x^3$ for $x>0$

and $x^3\le x^3\cos\frac{2}{x}\le -x^3$ for $x<0$.

where $\lim_{x\to 0}x^3\cos\frac{2}{x}=0$ (or) $0\times$ finite number between -1 and +1 = 0

EXERCISE - II

1. $\lim_{x\to 1}\frac{\left(\sum_{K=1}^{200}x^K\right)-200}{x-1}=$

1) 5050 2) 1000 3) 2010 4) 20100

2. $\lim_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}=$

1) $\frac{2}{\sqrt{3}}$ 2) $-\frac{1}{\sqrt{3}}$ 3) $\frac{2}{3\sqrt{3}}$ 4) $\frac{1}{\sqrt{3}}$

3. Let α and β be the roots of $ax^2+bx+c=0$, then

$\lim_{x\to\alpha}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}=$

1) $\frac{a^2(\alpha-\beta)^2}{2}$ 2) $\frac{a^2}{2(\alpha-\beta)^2}$

3) $\frac{a^2}{(\alpha-\beta)^2}$ 4) $-\frac{a^2}{2(\alpha-\beta)^2}$

4. If $\lim_{x\to\infty}\left(\frac{x^2-1}{x+1}-ax-b\right)=2$, **then**

1) $a=1$ and $b=-3$ 2) $a=1$ and $b=2$

3) $a=0$ and $b=-1$ 4) $a=2$ and $b=1$

5. If $f(x)=\frac{4-7x}{7x+4}$, $\lim_{x\to 0}f(x)=l$ **and**

$\lim_{x\to\infty}f(x)=m$ **the quadratic equation having roots as** $\frac{1}{l}$ **and** $\frac{1}{m}$ **is**

1) $x^2-1=0$ 2) $x^2+1=0$

3) 1/2 4) $x^3-1=0$

6. If $a=\min\{x^2+4x+5, x\in R\}$ **and**

$b=\lim_{\theta\to 0}\frac{1-\cos 2\theta}{\theta^2}$ **then the value of**

$\sum_{r=0}^{n}a^r b^{n-r}=$

1) $\frac{2^{n+1}-1}{4.2^n}$ 2) $2^{n+1}-1$

3) $\frac{2^{n+1}-1}{3.2^n}$ 4) 2^n-1

7. $\underset{x\to 0}{Lim}\frac{8}{x^8}\left(1-\cos\frac{x^2}{2}-\cos\frac{x^2}{4}+\cos\frac{x^2}{2}.\cos\frac{x^2}{4}\right)=$

1) $\frac{1}{16}$ 2) $\frac{1}{15}$ 3) $\frac{1}{32}$ 4) 1

8. **Arrange the following limits in the ascending order.**

1) $\underset{x\to 0}{Lim}\frac{\tan^4 x-\sin^4 x}{x^6}$ 2) $\underset{x\to 0}{Lim}\frac{\tan^8 x-\sin^8 x}{x^5\tan x^5}$

3) $\underset{x\to 0}{Lim}\frac{\tan^3 x-\sin^3 x}{x\sin^4 x}$ 4) $\underset{x\to 0}{Lim}\frac{\tan^5 x-\sin^5 x}{x^2.\sinh^3 x.\tan^2 x}$

1) 1, 2, 3, 4 2) 3, 1, 4, 2
3) 1, 2, 4, 3 4) 2, 1, 3, 4

9 . **If** $f:R\to R$ **defined by**

$$f(x)=\begin{cases}\frac{x-2}{x^2-3x+2} & if \quad x\in R-\{1,2\}\\ 2 & if \quad x=1\\ 1 & if \quad x=2\end{cases}$$

then $\underset{x\to 2}{Lim}\frac{f(x)-f(2)}{x-2}=$

1) 0 2) -1 3) 1 4) $-1/2$

10. $\underset{x\to o}{Lt}\frac{\sqrt{\frac{1}{2}(1-\cos x)}}{x}=$

1) 1 2) -1 3) 0
4) does not exist

11. $\underset{x\to\frac{\pi}{4}}{Lt}\frac{\sqrt{2}-cosx-sinx}{(4x-\pi)^2}=$ **[EAM -2020]**

1) $\frac{1}{16\sqrt{2}}$ 2) $\frac{1}{32\sqrt{2}}$ 3) $\frac{1}{16}$ 4) $\frac{1}{8}$

12. $\underset{x\to\pi}{Lt}\frac{\sqrt{2+\cos x}-1}{(\pi-x)^2}=$ ____

1) 0 2) $\frac{1}{4}$ 3) $\frac{1}{2}$ 4) 2

13. **The value of** $\lim_{x\to a}\frac{\log(x-a)}{\log(e^x-e^a)}$ **is**

1) 1 2) -1 3) 0 4) 2

14. $\lim_{x\to 0}\frac{(4^x-1)^3}{\sin\left(\frac{x}{4}\right)\log_e\left(1+\frac{x^2}{3}\right)}=$

1) $(\log_e 4)^3$ 2) $\log_e 4$
3) $12(\log_e 4)^3$ 4) $5(\log_e 4)^3$

15. $\underset{x\to\infty}{Lt}$ **x[log(x+1)-logx]=** [EAM -2016]

1) e^2 2) e 3)1 4) 1/e

16. $\underset{x\to 0}{Lim}\frac{27^x-9^x-3^x+1}{\sqrt{2}-\sqrt{1+\cos x}}=$

1) 0 2) $8\sqrt{2}$ $(\log 3)^2$ 3) 8 $(\log 3)^2$ 4)1

17. $\lim_{n\to\infty}\frac{1^3+2^3+3^3+....+n^3}{3n^4+5n^3+6}=$

1) 1/3 2) 1/5 3) 1/6 4)1/12

18. $\lim_{n\to\infty}\cos\left(\pi\sqrt{n^2+n}\right)$ **in equal to**

1) 0 2)1 3) 2
4) does not exist

19. **The value of**

$\underset{n\to\infty}{Lim}\frac{1.2+2.3+3.4+...+n.(n+1)}{n^3}$ **is**

1) 1 2) -1 3)1/3 4) -1/3

20. **If |x|<1, then**

$\underset{n\to\infty}{Lt}(1+x)(1+x^2)(1+x^4).....(1+x^{2n})=$

1) $\frac{1}{x}$ 2) $\frac{1}{1+x}$ 3) $\frac{1}{1-x}$ 4) $\frac{1}{x-1}$

21. $\underset{n\to\infty}{Lt}\frac{2.3^{n+1}-3.5^{n+1}}{2.3^n+3.5^n}=$

1) 5 2) 1/5 3) -5 4) 0

22. $\lim_{n\to\infty}\frac{1}{n^4}\left[1^2+(1^2+2^2)+...+(1^2+2^2+...+n^2)\right]=$

1) 1/6 2) 1/16 3) 1/12 4) 0

23. $\lim_{x\to 0}\left(\frac{1^x+2^x+3^x+...+n^x}{n}\right)^{\frac{1}{x}}=$

1) $(n!)^n$ 2) $(n!)^{1/n}$ 3) $n!$ 4) $\ln n!$

24. **If** $\lim_{x\to 0}\left[1+x\ln(1+b^2)\right]^{\frac{1}{x}}=2b\sin^2\theta\ ,b>0$

and $\theta\in(-\pi,\pi)$ **then the value of** θ **is**

1) $\pm\frac{\pi}{6}$ 2) $\pm\frac{\pi}{3}$ 3) $\pm\frac{\pi}{8}$ 4) $\pm\frac{\pi}{2}$

25. **If p and q are the roots of the quadratic equation** $ax^2+bx+c=0$ **then**

$\lim_{x\to p}\left(1+ax^2+bx+c\right)^{\frac{1}{x-p}}=$

1) $a(p-q)$ 2) $\log\left[a(p-q)\right]$

3) $e^{a(p-q)}$ 4) $e^{a(q-p)}$

27. $\underset{x\to\infty}{Lt}\left(\frac{x^2+5x+3}{x^2+x+2}\right)^x=$ [EAM -2015]

1) e^4 2) e^3 3) e^2 4) 2^4

28. $\underset{x\to\infty}{Lim}\left\{\left(\frac{x}{x+1}\right)^a+\sin\frac{1}{x}\right\}^x$ **is equal to**

1) e^{a-1} 2) e^{1-a} 3) e 4) 0

29. **A function** $f:R\to R$ **is such that** $f(1)=3$ **and** $f'(1)=6$**. Then** $\lim_{x\to 0}\left[\frac{f(1+x)}{f(1)}\right]^{1/x}=$

1) 1 2) e^2 3) $e^{1/2}$ 4) e^3

30. **If** $f^1(0)=3,$ **then**

$\lim_{x\to 0}\frac{x^2}{f(x^2)-6f(4x^2)+5f(7x^2)}=$

1) $\frac{1}{36}$ 2) $-\frac{1}{36}$ 3) $\frac{1}{34}$ 4) $\frac{1}{106}$

31. $\lim_{x\to 0}\frac{\sin^{-1}x-\sin x}{x^3}$ **[EAM -2018]**

1) $\frac{1}{2}$ 2) $\frac{1}{3}$ 3) $\frac{1}{4}$ 4) $\frac{1}{5}$

32. $\lim_{n\to\infty}\frac{\{x\}+\{2x\}+...+\{nx\}}{n^2}$

1) $\frac{1}{2}$ 2) 0 3) -1 4) 2

33. $\underset{x\to 0}{Lt}\left[\frac{\sin|x|}{|x|}\right]$ **Where [.] dentoes the greatest integer function.**

1) 0 2) 1 3) -1 4) does not exist

KEY

01) 4	02) 3	03) 1	04) 1	05) 1	06) 2
07) 3	08) 2	09) 4	10) 4	11) 1	12) 2
13) 1	14) 3	15) 3	16) 2	17) 4	18) 1
19) 3	20) 3	21) 3	22) 3	23) 2	24) 4
25) 3	26) 1	27) 2	28) 2	29) 1	31) 2
32) 2	33) 1				

SOLUTIONS

1. Use L-Hospital rule $\Rightarrow$ Given Limit

$=\lim_{x\to 1}\left(1+2x+3x^2+..+200.x^{199}\right)$

2. Using L-Hospital rule given limit is

$$\lim_{x\to a}\frac{\frac{1}{2\sqrt{a+2x}}(2)-\frac{1}{2\sqrt{3x}}(3)}{\frac{1}{2\sqrt{3a+x}}-2\frac{1}{2\sqrt{x}}}=\frac{2}{3\sqrt{3}}$$

3. $ax^2+bx+c=a(x-\alpha)(x-\beta)$

given limit is

$$\lim_{x\to\alpha}\frac{2\sin^2\left(\frac{a(x-\alpha)(x-\beta)}{2}\right)^2}{(x-\alpha)^2}=\frac{a^2(\alpha-\beta)^2}{2}$$

4. $\lim_{x\to\infty}\left(\frac{x^2-1-ax^2-ax-bx-b}{x+1}\right)=2$

$$\Rightarrow \lim_{x\to\infty}\left(\frac{(1-a)x^2-(a+b)x-(1+b)}{x+1}\right)=2$$

5. $l=1, m=-1$, $\frac{1}{l}=1, \frac{1}{m}=-1 \Rightarrow x^2-1=0$

6. $a=\frac{4ac-b^2}{4a}=\frac{4.5.1-16}{4}=1$

$b=\frac{2\sin 2\theta}{2\theta}=2$

$\sum_{r=0}^{n} a^r b^{n-r}=b^n+ab^{n-1}+a^2b^{n-2}+.....+a^n$

$=\frac{2^{n+1}-1}{2-1}$

7. Given limit is

$$\lim_{x\to 0}\frac{8}{x^8}\left(1-\cos\frac{x^2}{2}\right)\left(1-\cos\frac{x^2}{4}\right)=32\frac{1}{16}\frac{1}{64}=\frac{1}{32}$$

8. $\lim_{x\to 0}\frac{\tan^n ax-\sin^n ax}{x^{n+2}}=\frac{n}{2}a^{n+2}$

9. L.H.L$=\lim_{x\to 0^-}\frac{\{x\}}{\tan\{x\}}=\lim_{x\to 0^-}\frac{x-[x]}{\tan(x-[x])}$

$=\lim_{x\to 0^-}\frac{x+1}{\tan(x+1)}=\frac{1}{\tan 1}$

R.H.L.$=\lim_{x\to 0^+}\frac{x-[x]}{\tan(x-[x])}=\lim_{x\to 0}\frac{x}{\tan x}=1$

$\therefore$ L.H.L $\neq$ R.H.L.

10. $\lim_{x\to 0}\sqrt{\frac{1-\cos x}{2}}=\underset{x\to 0}{Lt}\frac{\sqrt{\sin^2 x/2}}{x}=\underset{x\to 0}{Lt}\frac{|\sin x/2|}{x}$

does not exist

11. $\underset{x\to\frac{\pi}{4}}{Lt}\frac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2}=\frac{0}{0}$ from using L - Hospital rule

$\underset{x\to\frac{\pi}{4}}{Lt}\frac{\sin x-\cos x}{2(4x-\pi)4}=\frac{0}{0}$ form again using L - Hospital rule

$\underset{x\to\frac{\pi}{4}}{Lt}\frac{\cos x+\sin x}{32}=\frac{\sqrt{2}}{32}=\frac{1}{16\sqrt{2}}$

12. $\lim_{x\to\pi}\frac{\sqrt{2+\cos x}-1}{(\pi-x)^2}$

$=\lim_{y\to 0}\frac{\sqrt{2+\cos(\pi-y)}-1}{y^2}=\lim_{y\to 0}\frac{\sqrt{2-\cos y}-1}{y^2}$

$=\lim_{y\to 0}\frac{2-\cos y-1}{y^2}\times\frac{1}{\sqrt{2-\cos y}+1}$

$=\lim_{y\to 0}\frac{2\sin^2\frac{y}{2}}{\left(\frac{y}{2}\right)^2}\times\frac{1}{4}\times\frac{1}{\sqrt{2-\cos y}+1}$

$=2\times 1\times\frac{1}{4}\times\frac{1}{1+1}=\frac{1}{4}$

13. $\lim_{x\to a}\frac{\log(x-a)}{\log(e^x-e^a)}=\lim_{x\to a}\frac{e^x}{(x-a)e^x+e^x}=\frac{e^a}{e^a}=1$

14. $\underset{x\to 0}{Lt}\frac{(4^x-1)^3}{\sin\left|\frac{x}{4}\right|\log\left(1+\frac{x^2}{3}\right)}=\frac{0}{0}$ form divide by x^3 on Nr and Dr.

$$\underset{x\to 0}{Lt}\frac{\left(\frac{4^x-1}{x}\right)^3}{\frac{\sin\frac{x}{4}}{x}\cdot\frac{1}{x^2}\log\left(1+\frac{x^2}{3}\right)}=$$

$$\underset{x\to 0}{Lt}\frac{\left(\frac{4^x-1}{x}\right)^3}{\frac{1}{12}\log\left(1+\frac{x^2}{3}\right)^{\frac{3}{x^2}}}=12(\log_e 4)^3$$

15. $\lim_{x\to\infty}\log_e\left[1+\frac{a}{x}\right]^x=a$

16. $\lim_{x\to 0}\frac{(9.3)^x-9^x-3^x+1}{\sqrt{2}-\sqrt{1+\cos x}}\times\frac{\sqrt{2}+\sqrt{1+\cos x}}{\sqrt{2}+\sqrt{1+\cos x}}$

$$=\lim_{x\to 0}\frac{(9^x-1)(3^x-1)}{x^2}\frac{x^2}{1-\cos x}\left(\sqrt{2}+\sqrt{1+\cos x}\right)$$

$$\left(\lim_{x\to 0}\frac{a^x-1}{x}=\log a, \lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}\right)$$

17. Given limit $=\lim_{n\to\infty}\frac{\frac{n^2(n+1)^2}{4}}{3n^4+5n^3+6}=\frac{1}{12}$

18. $\sqrt{n^2+n}=n\sqrt{1+\frac{1}{n}}$

$$=n\left(1+\frac{1}{n}\right)^{1/2}=n\left(1+\frac{1}{2n}\right)=\left(n+\frac{1}{2}\right)$$

$$\lim_{n\to\infty}\cos\left(\pi\left(n+\frac{1}{2}\right)\right)=\lim_{n\to\infty}\sin(n\pi)=0$$

19. $Lim_{n\to\infty}\frac{\sum_{r=1}^{n}r(r+1)}{n^3}$

$$=Lim_{n\to\infty}\frac{\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}}{n^3}$$

$$=Lim_{n\to\infty}\frac{n(n+1)(n+2)}{3n^3}=\frac{1}{3}$$

20. $\lim_{x\to\infty}1+x+x^2+....+x^{2n}=\frac{1}{1-x}$

21. $\lim_{x\to\infty}\frac{6.3^n-15.5^n}{2.3^n+3.5^n}$

22. $\lim_{n\to\infty}\frac{\Sigma\frac{n(n+1)(2n+1)}{6}}{n^4}$

23. $(1.2.3....n)^{1/n}$

24. $e^{ln(1+b^2)}=2b\sin^2\theta$

$$\Rightarrow\frac{1+b^2}{2b}=\sin^2\theta\Rightarrow\frac{1}{b}+b=2\sin^2\theta$$

$$\Rightarrow\theta=\pm\frac{\pi}{2}$$

25. Give = f¹(0) = 3

$$Lt_{x\to 0}\frac{x^3}{f(x^2)-6f(4x^2)+5f(7x^2)}=\frac{0}{0}\text{ form}$$

using L - Hospital rule

$$Lt_{x\to 0}\frac{2x}{f^1(x^2)2x-48f^1(4x^2)x+70xf^1(7x^2)}$$

$$Lt_{h\to 0}\frac{(2+h)\sin(2+h)+\cos(2+h)(1)-0}{1}=\cos 2-2\sin 2$$

26. $\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+2}\right)^x=1^\infty\Rightarrow e^{\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+2}-1\right)x}=e^4$

27. Given limit

$$=\lim_{x\to\infty}\left\{1+\left(1+\frac{1}{x}\right)^{-a}+\sin\frac{1}{x}-1\right\}$$

$$=\lim_{y\to 0}\{1+(1+y)^{-a}+\sin y-1\}^{1/y}$$

where $y=\frac{1}{x}$, $\quad =e^{\lim_{y\to 0}\frac{(1+y)^{-a}+\sin y-1}{y}}=e^{1-a}$

28. $\lim_{x\to 0}\left(\frac{f(1+x)}{f(1)}\right)^{\frac{1}{x}}=e^{\lim_{x\to 0}\frac{1}{x}\left\{\frac{f(1+x)}{f(1)}-1\right\}}=e^{\lim_{x\to 0}\frac{f'(1)}{f(1)}}=e^2$

29. Given f¹(0) = 3

$$Lt_{x\to 0}\frac{x^3}{f(x^2)-6f(4x^2)+5f(7x^2)}=\frac{0}{0}\text{ from}$$

using L - Hospital rule

$$\underset{x\to 0}{Lt}\frac{2x}{f^1\left(x^2\right)2x-48f^1\left(4x^2\right)+7cxf^1\left(7x^2\right)}$$

$$=\underset{x\to 0}{Lt}\frac{2x}{2x\left(f^1\left(x^2\right)2x-48f^1\left(4x^2\right)+7cxf^1\left(7x^2\right)\right)}$$

=

$$\frac{1}{f^1(0)-24f^1(0)+35f^1(0)}=\frac{1}{3-72+105}=\frac{1}{36}$$

30. $\lim_{x\to 0}\left(x+\frac{1^2x^3}{3!}+\frac{1^23^2x^5}{5!}....\right)-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\right)$

$$\lim_{x\to 0}\frac{\frac{2}{3!}x^3+\frac{8}{5!}x^5}{x^3}=\frac{1}{3}$$

31. $0\le\{x\}<1,\quad 0\le\{2x\}<1$

$$\frac{0}{n^2}\le\frac{\{x\}+\{2x\}+.....+\{nx\}}{n^2}<\frac{n}{n^2}$$

$$\lim_{n\to\infty}\frac{0}{n^2}\le\lim_{n\to\infty}\frac{\{x\}+\{2x\}+.....+\{nx\}}{n^2}\le\lim_{n\to\infty}\frac{1}{n}$$

$$\lim_{n\to\infty}f_1(x)=\lim_{n\to\infty}\frac{0}{n}=0$$

$$\lim_{n\to\infty}f_2(x)=\lim_{n\to\infty}\frac{1}{n}=0$$

$$\lim_{n\to\infty}\frac{\{x\}+\{2x\}+.....+\{nx\}}{n^2}=0$$

32. If $0<|x|<\frac{\pi}{2}$ then $|\sin x|<|x|<|\tan x|$

$$\therefore\frac{\sin|x|}{|x|}<1,\quad\left[\frac{\sin|x|}{|x|}\right]=0$$

EXERCISE - III

1. $\lim_{x\to 0}\left[\frac{100\tan x.\sin x}{x^2}\right]$ **where** $[.]$ **represents greatest integer function is**

1) 99 2) 100 3) 0 4) 98

2. If $[.]$ **denotes the greatest integer function then** $\lim_{x\to 0}\left[\frac{x^2}{\tan x.\sin x}\right]=$

1) 0 2) 1 3) -1
4) does not exist

3. The value of $\lim_{x\to 0}\left\{\left[\frac{100x}{\sin x}\right]+\left[\frac{99\sin x}{x}\right]\right\}$**, where** $[.]$ **represents the greatest integer function, is**

1) 199 2) 198 3) 0 4) 1

4. $\lim_{x\to 0}\left\{\left[\frac{a\sin x}{x}\right]+\left[\frac{b\tan x}{x}\right]\right\}=$ $a,b\in N$,

[where [ ] denotes G.I.F.]

1) $a+b$ 2) $a+b-1$ 3) 0 4) $\frac{a+b}{2}$

5. $\lim_{x\to 0^-}\frac{[x]+[x^2]+[x^3]+...+[x^{2n+1}]+n+1}{1+[x^2]+[x]+2x}$ $n\in N$

is equal to

1) $n+1$ 2) n 3) 1 4) 0

6. If $[.]$ **denotes the greatest integer function, then**

$$\lim_{x\to 0}\frac{\tan\left(\left[-2\pi^2\right]x^2\right)-x^2\tan\left(\left[-2\pi^2\right]\right)}{\sin^2 x}=$$

1) $-20+\tan 20$ 2) $20+\tan 20$
3) 20 4) $\tan 20$

7. If $\{x\}$ denotes fractional part of x then $\lim\limits_{x\to 1}\dfrac{x\sin\{x\}}{x-1}$

1) 0 2) -1 3) 1
4) does not exist

8. $\lim\limits_{x\to 0}\dfrac{e^{x^3}-1-x^3}{\sin^6(2x)}=$

1) $\frac{1}{128}$ 2) $\frac{2}{127}$ 3) $\frac{1}{126}$ 4) $\frac{1}{125}$

9. If $f(x)=\begin{vmatrix}\cos x & x & 1\\ 2\sin x & x^2 & 2x\\ \tan x & x & 1\end{vmatrix}$ **then** $\lim\limits_{x\to 0}\dfrac{f'(x)}{x}=$

1) 1 2) – 1 3) 2 4) – 2

10. $\lim\limits_{x\to 0}\left(\dfrac{\log_{\sec\left(\frac{x}{2}\right)}\cos x}{\log_{\sec x}\cos\left(x/2\right)}\right)=$

1) 14 2) 15 3) 16 4) 17

11. $\lim\limits_{n\to\infty}\dfrac{1}{n^4}\sum\limits_{r=1}^{n} r(r+2)(r+4)=$

1) $\frac{3}{4}$ 2) 0 3) $\frac{1}{8}$ 4) $\frac{1}{4}$

12. $\lim\limits_{n\to\infty}\left[\dfrac{7}{10}+\dfrac{29}{10^2}+\dfrac{133}{10^3}+........+\dfrac{5^n+2^n}{10^n}\right]=$

1) 3/4 2) 2 3) 5/4 4) 1/2

13. Suppose f(n+1) $=\dfrac{1}{2}\left\{f(n)+\dfrac{9}{f(n)}\right\}, n\in N.$ **If** $f(n)>0, \forall n\in N,$ **then** $\lim\limits_{n\to\infty}(f(n))=$

1) 3^{-1} 2) -3^{-1} 3) 3 4) –3

14. $\lim\limits_{x\to 0}\left[1^{1/\sin^2 x}+2^{1/\sin^2 x}+....+n^{1/\sin^2 x}\right]^{\sin^2 x}=$

1) ∞ 2) 0 3) $\frac{n+1}{2}$ 4) n

15. $\lim\limits_{n\to\infty}\dfrac{1-2+3-4+5-6+.....-2n}{\sqrt{n^2+1}+\sqrt{4n^2-1}}=$

1) $\frac{1}{3}$ 2) $-\frac{1}{3}$ 3) $-\frac{1}{5}$ 4) $\frac{1}{5}$

16. If $\lim\limits_{x\to 0}\left(x^{-3}\sin 3x+ax^{-2}+b\right)$ **exists and is equal to zero,then the value of** $a+2b=$

1) 3 2) 4 3) 0 4) 6

17. The graph of the function $y=f(x)$ has a unique tangent at the point $(e^a,0)$ through which the graph passes then

$\lim\limits_{x\to e^a}\dfrac{\log_e\{1+7f(x)\}-\sin f(x)}{3f(x)}$ **is**

1) 1 2) 2 3) 0 4) -1

18. The graph of $y=f(x)$ has unique tangent at the point $(a,0)$ through which the graph passes. Then $\lim\limits_{x\to a}\dfrac{\log\left[1+6f(x)\right]}{3f(x)}=$

1) 0 2) 1 3) 2 4) ∞

19. $\lim\limits_{x\to\infty}\left[\dfrac{1^2}{1-x^3}+\dfrac{3}{1+x^2}+\dfrac{5^2}{1-x^3}+\dfrac{7}{1+x^2}+...\right]=$

1) $\frac{-5}{6}$ 2) $\frac{-10}{3}$ 3) $\frac{5}{6}$ 4) $\frac{10}{3}$

20. Evaluate

$\lim\limits_{n\to\infty}\left\{\dfrac{1}{2}\tan\dfrac{x}{2}+\dfrac{1}{2^2}\tan\dfrac{x}{2^2}+...+\dfrac{1}{2^n}\tan\dfrac{x}{2^n}\right\}$

1) $x\tan\frac{x}{2}$ 2) $\frac{1}{x}\cot\frac{x}{2}$

3) $\frac{x-\cot x}{2}$ 4) $\frac{1}{x}-\cot x$

21. $\lim\limits_{x\to a^+}\dfrac{\{x\}\sin(x-a)}{(x-a)^2}=$ **where** $\{x\}$ **denotes fractional part of x and $a\in N$**

1) 0 2) 1 3) a 4) 5

22. $\lim\limits_{x\to -\pi}\dfrac{|x+\pi|}{\sin x}=$

1) 1 2) -1 3) π
4) does not exist

23. $\lim_{x\to 0}\frac{1-\sin[\cos x]}{[x]-[\sin x]}=$ **(where $[x]$ denotes greatest integral part of x)**

1) 0 2) 1 3) 2 4) ∞

24. $\lim_{x\to 0}\left[\frac{\sin(sgn(x))}{(sgn(x))}\right]=$

(where $[x]$ denotes integral part of x)

1) 0 2) 1 3) -1
4) does not exist

25. $\lim_{n\to\infty}\sum_{x=1}^{20}\cos^{2n}(x-10)=$

1) 0 2) 1 3) 19 4) 20

26. $\lim_{x\to 0}\left(1+x\left(1+\frac{f(x)}{kx^2}\right)\right)^{1/x}=e^3$ **and** $f(4)=64$ **then K has value**

1) 1 2) 2 3) 4 4) 5

27. $\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}=$

1) 1 2) e/2 3) –e/2 4) 2/e

28. The integer n for which $\lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$ **is a finite non-zero number is**

1) 1 2) 2 3) 3 4) 4

29. $\lim_{x\to 0}\frac{(1+x)^{1/x}-e\left(1-\frac{x}{2}\right)}{(1-\cos x)}$

1) $\frac{1}{2}e$ 2) $\frac{1}{4}e$ 3) $\frac{11}{12}e$ 4) $\frac{1}{12}e$

30. $\underset{n\to\infty}{Lim}\left(\frac{n}{n^2+1}+\frac{n}{n^2+2}+\frac{n}{n^2+3}+...+\frac{n}{n^2+n}\right)=$

1) 0 2) 1 3) ∞
4) does not exist

KEY

01) 1	02) 1	03) 2	04) 2	05) 4	06) 1
07) 4	08) 1	09) 4	10) 3	11) 4	12) 3
13) 3	14) 4	15) 2	16) 4	17) 2	18) 3
19) 2	20) 4	21) 2	22) 4	23) 4	24) 1
25) 2	26) 2	27) 3	28) 3	29) 3	30) 2

SOLUTIONS

1. We have $0<\frac{\tan x.\sin x}{x^2}<1$ in the nbd

$x=0\Rightarrow 0<\frac{100\tan x.\sin x}{x^2}<100$

2. $\sin x<x<\tan x$ in the nbd of 0

$\therefore 0<\frac{x^2}{\tan x\sin x}<1$

3. $\forall x>0;\frac{x}{\sin x}>1$ and $\forall x<0;\frac{\sin x}{x}<1$

$\frac{x}{\sin x}>1$ and $\frac{\sin x}{x}<1$

$\Rightarrow\frac{100x}{\sin x}>100$ and $\frac{99\sin x}{x}<99$

4. $\frac{\sin x}{x}<1\Rightarrow\frac{a\sin x}{x}<a$ but close to $a\Rightarrow\left[\frac{a\sin x}{x}\right]=a-1$

$\frac{\tan x}{x}>1\Rightarrow\frac{b\tan x}{x}>b$ but close to b$\Rightarrow\left[\frac{b\tan x}{x}\right]=b$

$\therefore\lim_{x\to 0}\left\{\left[\frac{a\sin x}{x}\right]+\left[\frac{b\tan x}{x}\right]\right\}=$

$a-1+b=a+b-1$

5. $\left[x^{2n+1}\right]=-1$ and $\left[x^{2n}\right]=0$ for $n=0,1,2,3,....$ Given limit

$=\lim_{x\to 0^-}\frac{(-1)+0+(-1)+0+...+0+(-1)+n+1}{1+0-1+2x}=0$

6. $\left[-2\pi^2\right]=-20$

7. $\lim_{x\to1^-}\frac{x\sin\{x\}}{x-1}=\lim_{x\to1^-}\frac{x\sin x}{x-1}=-\infty$

$=\lim_{x\to1^+}\frac{x\sin\{x\}}{x-1}=\lim_{x\to1^+}\frac{x\sin(x-1)}{x-1}=1$

8. $e^{x^3}=1+x^3+\frac{(x^3)^2}{2!}+\frac{(x^3)^3}{3!}+...\infty$

$\Rightarrow e^{x^3}-1-x^3=x^6\left[\frac{1}{2}+\frac{x^3}{6}+...+\infty\right]$

9. $f(x)=-x^2\cos x+x^2\tan x$

10. Use L hospital rule

11. $\lim_{n\to\infty}\frac{1}{n^4}\left[\sum n^3+6\sum n^2+8\sum n\right]=\frac{1}{4}$

12. $\lim_{n\to\infty}\left(\left(\frac{5}{10}\right)^n+\left(\frac{2}{10}\right)^n\right)$

13. $\lim_{n\to\infty}f(n+1)=\lim_{n\to\infty}f(n)=k$

$k=\frac{1}{2}\left[k+\frac{9}{k}\right]\Rightarrow k^2=9, k=3$

14. Given limit

$=\lim_{x\to0}n.\left[\left(\frac{1}{n}\right)^{\frac{1}{\sin^2x}}+\left(\frac{2}{n}\right)^{\frac{1}{\sin^2x}}+...+\left(\frac{n-1}{n}\right)^{\frac{1}{\sin^2x}}+1\right]$

$=n(0+0+...+0+1)=n$

15. Given $\lim_{n\to\infty}\frac{n^2-(n+n^2)}{\sqrt{n^2+1}+\sqrt{4n^2-1}}$ and divide with 'n'

16. $lim_{x\to0}\frac{\sin3x}{x^3}+\frac{a}{x^2}+b$

$=\lim_{x\to0}\frac{\sin3x+ax+bx^3}{x^3}$

Use L'Hospitals rule

17. $f(e^a)=0$

Given limit

$=\lim_{x\to e^a}\frac{\frac{1}{1+7.f(x)}7.f'(x)-\cos[f(x)].f'(x)}{3.f'(x)}$

$=\frac{\frac{7}{1+7f(e^a)}-\cos[f(e^a)]}{3}=\frac{7-1}{3}=2$

18. Use L hospitals rule

19. $\lim_{x\to\infty}\left[\frac{\sum_{k=1}^{x}(4k-3)^2}{1-x^3}+\sum_{k=1}^{x}\frac{4k-1}{1+x^2}\right]$

$=\frac{16}{-3}+\frac{4}{2}=\frac{-16}{3}+2=\frac{-10}{3}$

20. $\lim_{n\to\infty}\left\{\frac{1}{2}\tan\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}+...+\frac{1}{2^n}\tan\frac{x}{2^n}\right\}$

$=\lim_{n\to\infty}\left\{-\cot x+\left(\cot x+\frac{1}{2}\tan\frac{x}{2}\right)\right.$

$\left.+\frac{1}{2^2}\tan\frac{x}{2^2}+...+\frac{1}{2^n}\tan\frac{x}{2^n}\right\}$

. $=\lim_{n\to\infty}\left\{-\cot x+\left(\frac{1}{2}\cot\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}\right)\right.$

$\left.+.....+\frac{1}{2^n}\tan\frac{x}{2^n}\right\}$

$\left(\because \cot x+\frac{1}{2}\tan\frac{x}{2}=\frac{1}{2}\cot\frac{x}{2}\right)$

proceeding like this we get

$=\lim_{n\to\infty}\left\{-\cot x+\frac{1}{2^n}\cot\frac{x}{2^n}\right\}$

21. $f(x)=\frac{\{x\}\sin(x-a)}{(x-a)^2}=\frac{(x-[x])\sin(x-a)}{(x-a)^2}$

$\lim_{x\to a^+}f(x)=\lim_{h\to0}\frac{(a+h-[a+h])\sin(a+h-a)}{(a+h-a)^2}$

$\lim_{h\to0}\frac{(a+h-a)\sinh}{h^2}=1$

22. $\lim_{x\to -\pi^+} \frac{|x+\pi|}{\sin x} = \lim_{n\to 0} \frac{|-\pi+h+\pi|}{\sin(-\pi+h)}$

$= \lim_{h\to 0} \frac{|h|}{-\sin(\pi-h)} \quad = \lim_{h\to 0} \frac{h}{-\sinh} = -1$

$\lim_{x\to -\pi^-} \frac{|x+\pi|}{\sin x} = \lim_{n\to 0} \frac{|-\pi-h+\pi|}{\sin(-\pi-h)}$

$= \lim_{n\to 0} \frac{|-h|}{-\sin(\pi+h)} = \lim_{n\to 0} \frac{h}{\sinh} = 1$

$\therefore$ L.H.L $\neq$ R.H.L

$\Rightarrow$ limit does not exist

23. $\lim_{x\to 0^-} \frac{1-\sin[\cos x]}{[x]-[\sin x]} = \lim_{n\to 0} \frac{1-\sin[\cos(-h)]}{[-h]-[\sin(-h)]}$

$\lim_{n\to 0} \frac{1-\sin 0}{-1-(-1)} = \frac{1}{0} = \infty$

$\lim_{x\to 0^+} \frac{1-\sin[\cos x]}{[x]-[\sin x]} = \lim_{n\to 0} \frac{1-\sin[\cosh]}{[h]-[\sinh]}$

$= \lim_{h\to 0} \frac{1-\sin 0}{0-0} = \frac{1}{0} = \infty$

$\therefore \lim_{x\to 0} \frac{1-\sin[\cos x]}{[x]-[\sin x]} = \infty$

use $\cosh < 1 \Rightarrow [\cosh] = 0$

$h > 0, h \to 0, [h] = 0$

$[-h] = -1, \quad [\sinh] = 0, \quad [-\sinh] = -1$

$= 2\times 1^3 \times \frac{0}{8\times 1} = 0$

24. $\lim_{x\to 0^+} \left[\frac{\sin(\operatorname{sgn}(x))}{\operatorname{sgn}(x)}\right] = \lim_{x\to 0} \left[\frac{\sin 1}{1}\right] = 0$

$\lim_{x\to 0^-} \left[\frac{\sin(\operatorname{sgn}(x))}{\operatorname{sgn}(x)}\right] = \lim_{x\to 0} \left(\frac{\sin(-1)}{-1}\right)$

$\lim_{x\to 0} [\sin 1] = 0$

$\therefore \lim_{x\to 0} \left[\frac{\sin(\operatorname{sgn}(x))}{\operatorname{sgn}(x)}\right] = 0$

use $\operatorname{sgn}(x) = 1, x > 0$

$= -1, x < 0$

$= 0, x = 0$

25. $\lim_{n\to\infty} \cos^{2n} x = 1$ where $x = m\pi, m \in I$

$= 0$ where $x \neq m\pi, m \in I$

Here for $x = 10$, $\lim_{n\to\infty} \cos^{2n}(x-10) = 1$ and in all other cases it is zero

$\therefore \lim_{n\to\infty} \sum_{x=1}^{20} \cos^{2n}(x-10) = 1$

26. $\lim_{x\to 0} \left(1 + x\left(1 + \frac{f(x)}{kx^2}\right)\right)^{1/x} = e^{1+\frac{f(x)}{kx^2}}$

27. $(1+x)^{1/x} = e^{1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+......\infty}$

28. $\lim_{x\to 0} \frac{\left(\frac{-x^2}{2!} + \frac{x^4}{4!}.....\right)(-x-x^2+...)}{x^n}$

is non-zero if n =3

29. $\lim_{x\to 0} e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 + ...\right) - e\left(1 - \frac{x}{2}\right)$

$= \lim_{x\to 0} \frac{\frac{11}{24} e x^2}{2\sin^2\left(\frac{x}{2}\right)} = \frac{11e}{12}\left(\frac{\frac{x}{2}}{\sin\frac{x}{2}}\right) = \frac{11}{12}e$

30. Use Sandwich Theorem

JEE MAINS QUESTIONS

1. $\underset{x\to 0}{Lt}\dfrac{x\tan 2x-2x\tan x}{(1-\cos 2x)^2}$ [2018]

1) $\frac{1}{4}$ 2) 1 3) $\frac{1}{2}$ 4) $-\frac{1}{2}$

2. $\underset{x\to 0}{Lt}\dfrac{(27+x)^{\frac{1}{3}}}{9-(27+x)^{\frac{2}{3}}}$ equal. to [2018]

1) $\frac{1}{3}$ 2) $-\frac{1}{3}$ 3) $-\frac{1}{6}$ 4) $\frac{1}{6}$

3. For each t $\in$ R, Let [t] be the greatest integer less or equal to t. then

$\underset{x\to 0^+}{Lt}\left[\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+....\left[\frac{15}{x}\right]\right]$ [2018]

1) does not exist (inR) 2) is equal to 0
3) is equal to 15 4) is equal to 120

4. $\underset{y\to 0}{Lt}\dfrac{\sqrt{1+\sqrt{1+y^4}}-\sqrt{2}}{y^4}$ is equal to [2019]

1) exists and equals $\dfrac{1}{2\sqrt{2}}$

2) exists and equals $\dfrac{1}{4\sqrt{2}}$

3) exists and equals $\dfrac{1}{2\sqrt{2}\left(\sqrt{2}+1\right)}$

4) does not exist

5. For each x $\in$ R, let [x] be the greatest in teger less then or equal to x. Then

$\underset{x\to 1}{Lt}\dfrac{(1-|x|+|x|)\sin[x]}{|x|}$ = is equal to [2019]

1) 1 2) 0 3) -sin 1 4) does not exist

6. For each t $\in$ R, let [x] be the greatest in teger less than or equal to t. Then

$\underset{x\to 1}{Lt}\dfrac{(1-|x|+\sin|1-x|)\sin\left[\frac{\pi}{2}[1-x]\right]}{1|1-x|[1-x]}$ = [2019]

1) equal -1 2) equals 1

3) equal 0 4) does not exist

7. $\underset{x\to 0}{Lt}\dfrac{x\cot 4x}{\sin^2 x\cot^2(2x)}$ is equal to [2019]

1) 0 2) 4 3) 1 4) 2

8. $\underset{x\to 1}{Lt}\dfrac{\sqrt{\pi}-\sqrt{2\sin^{-2}x}}{\sqrt{1-x}}$ is equal to [2019]

1) $\sqrt{\frac{2}{\pi}}$ 2) $\frac{1}{\sqrt{2\pi}}$ 3) $\sqrt{\frac{\pi}{2}}$ 4) $\sqrt{\pi}$

9. $\underset{x\to 2}{Lt}\dfrac{3^x+3^{3-x}-12}{3^{-x/2}-3^{1-x}}$ is equal to .... [2020]

10. $\underset{x\to 0}{Lt}\left(\frac{3x^2+2}{7x^2+2}\right)^{\frac{1}{x^2}}$ [2020]

1) e^{-2} 2) e^2 3) $e^{2/7}$ 4) $e^{3/7}$

11. $\underset{x\to 0}{Lt}\frac{\int_0^x t\sin(lot)\,dt}{x}$ is equal to [2020]

1) 1 2) 10 3) 5 4) 0

12. $\underset{x\to 0}{Lt}\left(\tan\left(\frac{\pi}{4}+4\right)\right)^{\frac{1}{x}}$ is equal to [2020]

1) e^2 2) 1 3) e 4) 2

13. If $\underset{x\to 0}{Lt}\left\{\frac{1}{x^8}\left(1-\cos\frac{x^2}{4}+\cos\frac{x^2}{2}\cos\frac{x^2}{4}\right)\right\}=2^{-k}$

then the value of k is [2020]

14. $\underset{x\to 0}{Lt}\int_0^{(x-1)^2}\frac{t\cot\left(t^2\right)dt}{(x-1)\sin(x-1)}$ [2020]

1) does not exist 2) is equal to -1/2

3) is equal to 4 4) is equal to 1/2

15. If $\underset{x\to 1}{Lt}\frac{x+x^2+x^3+.....x^n-n}{x-1}$ = 820 (n(-n))

then the value of n is equal to [2020]

KEY

1) 3	2) 3	3) 4	4) 2
5) 3	6) 3	7) 3	8) 1
9) 3	10) 1	11) 4	12) 1
13) 5	14) 1	15) 40	

SOLUTIONS

1.

$$\underset{x\to 0}{Lt}\frac{x\tan 2x-2x\tan x}{(1-\cos 2x)^2}=\underset{x\to 0}{Lt}\frac{\frac{2x\tan x}{1-\tan^2 x}-2x\tan x}{\left(1-1+2\sin^2 x\right)^2}$$

$$=\underset{x\to 0}{Lt}\frac{2x\tan x}{1-\tan^2 x}\left(\frac{1-1+\tan^2 x}{4\sin^4 x}\right)$$

$$=\underset{x\to 0}{Lt}\,x\tan 2x\frac{\tan^2 x}{4\sin^4 x}=\underset{x\to 0}{Lt}\frac{\tan 2x}{x}.\frac{\tan^2 x}{x^2}.\frac{x^4}{4\sin^4 x}$$

$$= 2 \times 1 \times \frac{1}{4}=\frac{1}{2}.$$

2. $$\underset{x\to 0}{Lt}\frac{3\left[1+\frac{1}{3}\frac{x}{27}-1\right]}{9\left[1-1-\frac{2}{3}\frac{x}{27}\right]}=\underset{x\to 0}{Lt}\frac{1}{3}\left[\frac{\frac{x}{8-1}}{\frac{-2}{3}\frac{x}{27}}\right]=\frac{-1}{6}$$

3. $\underset{x\to0}{Lt}\, x\left(\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+\ldots\ldots\left[\frac{15}{x}\right]\right)$

$\underset{x\to0}{Lt}\, x\left(\left[\frac{1}{x}+\frac{2}{x}+\ldots\ldots\frac{15}{x}\right]-\left[\left(\left\{\frac{1}{x}\right\}+\left\{\frac{2}{x}\right\}\ldots\ldots+\left\{\frac{15}{x}\right\}\right)\right]\right)$

= 1 + 2 + 3 +......+15 - 0

= $\frac{12(15+1)}{y}=15x8=120$

4. $\underset{y\to0}{Lt}\frac{\sqrt{1+\sqrt{1+y^4}}-\sqrt{2}}{y^4}$ rationalising

$\Rightarrow \underset{y\to0}{Lt}\frac{\sqrt{1+\sqrt{1+y^4}}-\sqrt{2}}{y^4}x\frac{\sqrt{1+\sqrt{1+y^4}}+\sqrt{2}}{\sqrt{1+\sqrt{1+y^4}}+\sqrt{2}}$

$\Rightarrow \underset{y\to0}{Lt}\frac{\sqrt{1+y^4}-1}{2\sqrt{2}y^4}$ again rationalising

= $\underset{y\to0}{Lt}\frac{\sqrt{1+y^4}-1}{2\sqrt{2}y^4}+\frac{\sqrt{1+y^4}-1}{\sqrt{1+y^2}+1}$

= $\underset{y\to0}{Lt}\frac{1+y^4-1}{2\sqrt{2}y^4}x\frac{1}{1+1}=\frac{1}{4\sqrt{2}}$

5. $\underset{y\to0}{Lt}\frac{x([x]+|x|)\sin[x]}{|x|}$ put x = 0 - h, h $\to$0

$\underset{h\to0}{Lt}\frac{(0-h)([0-h]+|0-h|)\sin[0-h]}{|0-h|}$

=- (-1 + 0) sin (-1) = -sin 1

6. $\underset{x\to1}{Lt}\frac{(1-|x|+\sin|1-x|)\sin\left(\frac{\pi}{2}[1-x]\right)}{|1-x|[1-x]}$

put x = 1 + h, x $\to$0

= $\underset{h\to0}{Lt}\left(1-\frac{\sinh}{h}\right)x-1=1-1=0$

7. $\underset{x\to0}{Lt}\frac{x\cot4x}{\sin^2x\cot^2 2x}=\underset{x\to0}{Lt}\frac{x\tan^2 2x}{\tan4x.\sin^2x}$

= $\underset{x\to0}{Lt}\frac{\left(\frac{\tan2x}{2x}\right)^2x4}{\frac{\tan4x}{x}.\left(\frac{\sin x}{x}\right)^2}=\frac{4}{4x1}=1$

8. $\underset{h\to0}{Lt}=\frac{\sqrt{\pi}-\sqrt{2\sin^{-1}(1-h)}}{\sqrt{1-(1-h)}}$

= $\underset{h\to0}{Lt}=\frac{\sqrt{\pi}-\sqrt{2\sin^{-1}(1-h)}}{\sqrt{h}}$

rationalising

$\underset{h\to0}{Lt}=\frac{\sqrt{\pi}-\sqrt{2\sin^{-1}(1-h)}}{\sqrt{h}}x\frac{\sqrt{\pi}+\sqrt{2\sin^{-1}(1-h)}}{\sqrt{\pi}+\sqrt{2\sin^{-1}(1-h)}}$

$\underset{h\to0}{Lt}=\frac{\pi-2\sin^{-1}(1-h)}{\sqrt{h}\left(\sqrt{\pi}+\sqrt{2\sin^{-1}(1-h)}\right)}$

= $2\underset{h\to0}{Lt}=\frac{\frac{\pi}{2}-\sin^{-1}(1-h)}{\sqrt{h}2\sqrt{\pi}}$

= $\frac{1}{\sqrt{\pi}}.\underset{h\to0}{Lt}\frac{\cos^{-1}(1-h)}{\sqrt{h}}$ using L - Hos p i tal rule

= $\frac{1}{\sqrt{\pi}}.\underset{h\to0}{Lt}\frac{1}{\sqrt{1-(1-h)^2}}x2\sqrt{h}$

$= \frac{1}{\sqrt{\pi}} . \underset{h\to 0}{Lt} \frac{2\sqrt{h}}{-h^2+2h} = \frac{1}{\sqrt{\pi}} \frac{2}{\sqrt{2}} = \sqrt{\frac{2}{\pi}}$

9. Put $3^{\frac{x}{2}} = t$

$$\underset{x\to 3}{Lt} \frac{t^2+\frac{27}{t^2}-12}{\frac{1}{t}-\frac{3}{t^2}} = \underset{t\to 3}{Lt} \frac{t^4+27-12t^2}{t-3}$$

$$= \underset{x\to 3}{Lt} \frac{(t^2-3)(t+3)(t-3)}{t-3} = 6\,x\,6 = 36$$

10. $$L = \underset{x\to 0}{Lt} \left(\frac{3x^2+2}{7x^2+2}\right)^{\frac{1}{x^2}} = e^{\underset{x\to 0}{Lt} \frac{1}{x^2}\left(\frac{3x^2+2}{7x^2+2}-1\right)}$$

$$L = e^{\underset{x\to 0}{Lt} \frac{1}{x^2}\left(\frac{-4x^2}{7x^2+2}\right)} = e^{\frac{-4}{2}} = e^{-2}$$

11. Using L - Hospital rule $\underset{x\to 0}{Lt} \frac{x\sin 10x}{1} = 0$

12. $$\underset{x\to 0}{Lt} \left(\frac{1+\tan x}{1-\tan x}\right)^{\frac{1}{x}}$$

$$= e^{\underset{x\to 0}{Lt} \frac{1}{x}\left(\frac{1+\tan x}{1-\tan x}-1\right)}$$

$$= e^{\underset{x\to 0}{Lt} \frac{1}{x}\left(\frac{2\tan x}{1-\tan x}\right)} = e^{\frac{2}{1}} = e^2$$

13. $$\underset{x+0}{Lt} \frac{1}{x^3}\left(1-\cos\frac{x^2}{2}-\cos\frac{x^2}{4}+\frac{x^2}{2}.\cos\frac{x^2}{4}\right)$$

$$\Rightarrow \underset{x+0}{Lt} \frac{\left(1-\cos\frac{x^2}{2}\right)}{x^4} \frac{\left(1-\cos\frac{x^2}{4}\right)}{x^4} = 2^{-k}$$

$$\Rightarrow \underset{x+0}{Lt} \frac{2\sin^2\frac{x^2}{4}}{\frac{x^4}{16}x16} . \frac{2\sin^2\frac{x^2}{8}}{\frac{x^4}{64}x66} = 2^{-k}$$

$$\Rightarrow \frac{1}{8} x \frac{1}{32} = 2^{-k} \Rightarrow 2^{-8} = 2^{-k}$$

K = 8

14. $$\underset{x\to 1}{Lt} \frac{\int_0^{(x-1)^2} t\cos(t^2)\,dt}{(x-1)\sin(x-1)}$$ using L - Hospital rule

$$= \underset{x\to 1}{Lt} \frac{(x-1)^2\cos(x-1)^4\,2(x-1)}{(x-1)\cos(x-1)+\sin(x-1)(1)}$$

$$= \underset{x\to 1}{Lt} \frac{2(x-1)^2\cos(x-1)^4}{\cos(x-1)+\frac{\sin(x-1)}{x-1}} = \frac{2x0x1}{1+1} = \frac{0}{2} = 0$$

15. $$\underset{x\to 1}{Lt} \frac{x+x^2+x^3+....+x^n-n}{x-1} = 820 \quad \frac{0}{0} form$$

using L - Hospital rule

$$\underset{x\to 1}{Lt} \frac{1+2x+3x^2+....nx^{n-1}}{1} = 820$$

$\Rightarrow$ 1 + 2 + 3 + ... n = 820

$$\frac{n(n+1)}{2} = 820$$

n(n+1) = 1640

n(n+1) = 40 x 41

n = 40.

****

CONTINUITY

SYNOPSIS

Continuity at a point :

- A function f is said to be continuous at 'a' if f is defined in a neighbourhood of 'a' and $\underset{x \to a}{Lt} f(x) = f(a)$.

  i) If $\underset{x \to a^-}{Lt} f(x) = f(a)$ then $f(x)$ is left continuous at $x = a$.

  ii) If $\underset{x \to a^+}{Lt} f(x) = f(a)$ then $f(x)$ is right continuous at $x = a$.

- A function f is said to be continuous in an open interval (a,b) if it is continuous at each and every point in the interval (a,b).

- A function f is said to be continuous on $[a,b]$ if

  (i) f is continuous at each point of (a,b)

  (ii) f is right continuous at $x = a$

  (iii) f is left continuous at $x = b$

Discontinuity :

- If $f(x)$ is not continuous at $x = a$, we say that $f(x)$ is discontinuous at $x = a$.

  $f(x)$ will be discontinuous at $x = a$ in any of the following cases :

  i) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist but are not equal.

  (ii) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist and are equal but not equal to $f(a)$

  (iii) $f(a)$ is not defined

  (iv) At least one of the limit doesn't exist.

- If f and g are continuous functions of x at $x = a$, then the following functions are continuous at $x = a$.

  i) $f + g$ ii) $f - g$ iii) $f.g$ iv) cf if $c \in R$ v) $\frac{f}{g}$ if $g(a) \neq 0$.

Note: Even if f and g are not continuous at $x = a$, $f \pm g, f.g, \frac{f}{g}, gof$ may be continuous at $x = a$.

- If f is continuous at $x = a$ and g is continuous at $f(a)$,then (gof) is continuous at $x = a$.
- If f is contiunous in $[a,b]$ then it is bounded in $[a,b]$. i.e there exist k and m such that $k \leq f(x) \leq m, \forall x \in [a,b]$ where k and m are minimum and maximum values of $f(x)$ respectively in the interval $[a,b]$.

  In this case f takes every real value between k and m at least once. Thus range of f is $[k,m]$
- If f is continuous on $[a,b]$ such that $f(a)$ and $f(b)$ are of opposite signs then there exist at least one solution for the equation $f(x) = 0$ in the interval (a,b)

Types of discontinuity :

- **Discontinuity of first kind (or) Removable discontinuity :**

  If $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$ (or) $f(a)$ not difined then the f is said to have a removable discontinuity at $x = a$. It is also called discontinuity of the 1st kind. In this case we can redefine the function by making $\lim_{x \to a} f(x) = f(a)$ and make it continuous at $x = a$.

  Removable discontinuities are of two types

  1) Missing point discontinuity

  2) Isolated point discontinuity

- **Missing point discontinuity :**

$\lim_{x\to a} f(x)$ exists finitely and $f(a)$ is not defined.

Example: $f(x)=\dfrac{(2-x)(x^2-8)}{(2-x)}$ has a missing point discontinuity at $x=2$.

- **Isolated point discontinuity :**

$\lim_{x\to a} f(x)$ exists finitely and $f(a)$ is defined but $\lim_{x\to a} f(x)\neq f(a)$.

Example: $f(x)=[x]+[-x]$ has isolated point discontinuities at all integers.

- **Discontinuity of second kind (or) irremovable discontinuity :**

A function $f(x)$ is said to have a discontinuity of the second kind at $x=a$ if $\lim_{x\to a} f(x)$ does not exist.

Irremovable discontinuities are of three types
1) Finite discontinuity (or) jump discontinuity
2) Infinite discontinuity
3) Oscillatory discontinuity

- **Finite Discontinuity:**

$\lim_{x\to a^+} f(x)$, $\lim_{x\to a^-} f(x)$ are both finite and are not equal.

Example: $f(x)=\dfrac{1}{1+2^{\frac{1}{x}}}$ at $x=0$

- **Infinite Discontinuity :**

If at least one of the limits $\underset{x\to a^+}{Lt} f(x)$ and $\underset{x\to a^-}{Lt} f(x)$ be $\pm\infty$, then $f(x)$ has infinite discontinuity at $x=a$.

Example: $f(x)=\dfrac{\cos x}{x}$ at $x=0$.

- **Oscillatory Discontinuity :**

The Limit oscillates between two finite quantities

Example: $f(x)=\sin\dfrac{1}{x}$ at $x=0$.

- In case of discontinuity of the second kind, the absolute difference between the value of the RHL at $x=a$ and LHL at $x=a$ is called the Jump of Discontinuity. A function having a finite number of jumps in a given interval I is called a Piece wise continuous or Sectionally continuous function in this interval.
- All polynomials, Trigonometrical functions, exponential and Logarithmic functions are continuous in their respective domains.

Intermediate value theorem :

- Suppose $f(x)$ is continuous on a interval I, and a , b are any two points of I. If y_0 is a number between $f(a)$ and $f(b)$, then there exists a number *c* between *a* and *b* such that $f(c)=y_0$.

Single Point Continuity:

- Functions which are continuous only at one point are said to exhibit single point continuity behaviour.

Example1: $f(x)=\begin{cases} x, & if\ x\in Q \\ -x, & if\ x\notin Q \end{cases}$. is continuous only at $x=0$.

Example 2: $f(x)=\begin{cases} x & , if\ x\in Q \\ 1-x & , if\ x\notin Q \end{cases}$ is continuous only at $x=1/2$

EXAMPLES

1. **The function $f(x)=\dfrac{(3^x-1)^2}{\sin x\ .\ ln(1+x)}$, $x\neq 0$ is continuous at $x=0$. Then the value of $f(0)$ is**

Sol: Given $f(0)=\lim_{x\to 0}\dfrac{(3^x-1)^2}{\sin x.ln(1+x)}$

$$\Rightarrow f(0)=\lim_{x\to 0}\frac{\left(\frac{3^x-1}{x}\right)^2}{\left(\frac{\sin x}{x}\right).\left(\frac{ln(1+x)}{x}\right)}=(ln\,3)^2$$

2:

Let f be a continuous function on $[1,3]$.
If f takes only rational values for all x and $f(2)=10$ then $f(1.5)$ is equal to

Sol : f(x) is continuous function on [1,3] and takes only rational values then f(x) is constant function.

$\therefore f(2)=f(1.5)=10$

3: Let $f(x)$ be defined in the interval $[0,4]$ such that $f(x)=\begin{cases}1-x, & 0\le x\le 1\\ x+2, & 1<x<2\\ 4-x, & 2\le x\le 4\end{cases}$ then number of points where $f(f(x))$ is discontinuous is

Sol: $f(x)$ is discontinuous at x = 1 and x = 2

$\Rightarrow f(f(x))$ is discontinuous when

f(x) = 1 & 2

Now $1-x=1\Rightarrow x=0,$ where $f(x)$ is continuous

$x+2=1\Rightarrow x=-1\notin(1,2)$

$4-x=1\Rightarrow x=3\in[2,4]$

Now, $1-x=2\Rightarrow x=-1\notin[0,1]$

$x+2=2\Rightarrow x=0\notin(1,2]$

$4-x=2\Rightarrow x=2\in[2,4]$

Hence, $f(f(x))$ is discontinuous at two points, x = 2, 3.

4:

The jump of discontinuity of the function $f(x)=\dfrac{|2x-3|}{2x-3}$ is

Sol: $f\left(\frac{3}{2}+\right)=1, f\left(\frac{3}{2}-\right)=-1$

$\therefore$ Jump of discontinuity = 2

5:

If $y=\dfrac{1}{t^2+t-2}$ where $t=\dfrac{1}{x-1}$, then the number of points of discontinuous of $y=f(x), x\in R$ is

Sol: $t=\dfrac{1}{x-1}$ is discontinuous at x = 1. Also

$y=\dfrac{1}{t^2+t-2}$ is discountinuous at t = -2 and t = 1

When $t=-2,\ \dfrac{1}{x-1}=-2\Rightarrow x=\dfrac{1}{2},$

When $t=1,\ \dfrac{1}{x-1}=1\Rightarrow x=2,$

So, $y=f(x)$ is discontinuous at three points, $x=1,\ \frac{1}{2},\ 2$

EXERCISE - I

1. **The function** $f(x)=\dfrac{x\tan 2x}{\sin 3x.\sin 5x}$, **for** $x\neq 0$, **is continuous at** $x=0$, **then** $f(0)$

1) $\dfrac{2}{13}$ 2) $\dfrac{2}{17}$ 3) $\dfrac{2}{11}$ 4) $\dfrac{2}{15}$

2. **Let** $f(x)=\dfrac{x+x^2+....+x^n-n}{x-1}$, $x\neq 1$, **the value of** $f(1)$ **so that** f **is continuous at x=1 is**

1) n 2) $\dfrac{n+1}{2}$

3) $\dfrac{n(n+1)}{2}$ 4) $\dfrac{n(n-1)}{2}$

3. **The function**

$$f(x)=\begin{cases}\dfrac{\cos 3x-\cos 4x}{x^2}, for\ x\neq 0\\ \dfrac{7}{2} \quad , for\, x=0\end{cases}$$

at $x=0$ **is** **(EAM - 2017)**

1) continuous 2) discontinuous
3) left continuous 4) right continuous

4. **The function defined by**

$$f(x)=\begin{cases}x.\sin\dfrac{1}{x} & \text{for } x\neq 0\\ 0 & \text{for } x=0\end{cases}$$ **at x = 0 is**

1) continuous 2) right continuous
3) left continuous 4) can not be determined

5. **If the function f(x) =** $\dfrac{e^{x^2}-\cos x}{x^2}$ **for** $x\neq 0$ **is continuous at** $x=0$ **then** $f(0)=$

1) 1/2 2) 3/2 3) 2 4) 1/3

6. **The value of f(0) so that the function**

$f(x)=\dfrac{\log\left(1+\dfrac{x}{a}\right)-\log\left(1-\dfrac{x}{b}\right)}{x}, (x\neq 0)$ **is continuous at** $x=0$ **is**

1) $\dfrac{a+b}{ab}$ 2) $\dfrac{a-b}{ab}$ 3) $\dfrac{ab}{a+b}$ 4) $\dfrac{ab}{a-b}$

7. **If** $f(x)=x^{\frac{1}{x-1}}$ **for** $x\neq 1$ **and f is continuous at** $x=1$ **then** $f(1)=$

1) e 2) e^{-1} 3) e^{-2} 4) e^2

8. **If** $f(x)=\begin{cases}\dfrac{1-\sqrt{2}\sin x}{\pi-4x}, & if\ x\neq\dfrac{\pi}{4}\\ a, & if\ x=\dfrac{\pi}{4}\end{cases}$

is continuous at $x=\dfrac{\pi}{4}$ **then** $a=$

1) 4 2) 2 3) 1 4) 1/4

9. **The discontinuous points of** $f(x)=\dfrac{1}{\log|x|}$ **are**

1) $0,\pm 2$ 2) $1,\pm 2$ 3) $0,\pm 1$ 4) $0,\pm 3$

10. **Let** $f:R\to R$ **be defined by**

$$f(x)=\begin{cases}\alpha+\dfrac{\sin[x]}{x} & ,if\quad x>0\\ 2 & ,if\quad x=0\\ \beta+\left[\dfrac{\sin x-x}{x^3}\right] & ,if\quad x<0\end{cases}$$ **where** $[x]$

denotes the integral part of x. **If** f **continuous at** $x=0$, **then** $\beta-\alpha=$ **[EAM - 20]**

1) -1 2) 1 3) 0 4) 2

11. **If** $[x]$ **denotes a greatest integer not exceeding** x **and if the function** f **defined by**

$$f(x)=\begin{cases}\dfrac{a+2Cosx}{x^2}, & if\ x<0\\ b\tan\dfrac{a}{[x+4]}, & if\ x\geq 0\end{cases}$$

is continuous at x=0, then the ordered pair (a,b) **is** **[EAM - 2019]**

1) $(-2,1)$ 2) $(-2,-1)$

3) $\left(-1,\sqrt{3}\right)$ 4) $\left(-2,-\sqrt{3}\right)$

12. If $f:R\to R$ is defined by

$$f(x)=\begin{cases}\dfrac{x+2}{x^2+3x+2} & if \quad x\in R-\{-1,-2\}\\ -1 & if \quad x=-2\\ 0 & if \quad x=-1\end{cases}$$

then f is continuous on the set [EAM-18]

1) R 2) $R-\{-2\}$
3) $R-\{-1\}$ 4) $R-\{-1,-2\}$

13. If $f:[-2,2]\to R$ is defined by

$$f(x)=\begin{cases}\dfrac{\sqrt{1+cx}-\sqrt{1-cx}}{x}, & for \quad -2\le x<0\\ \dfrac{x+3}{x+1} & , for \quad 0\le x\le 2\end{cases}$$

is continuous on $[-2,2]$ then c = (EAM-16)

1) 3 2) $\dfrac{3}{2}$ 3) $\dfrac{3}{\sqrt{2}}$ 4) $\dfrac{2}{\sqrt{3}}$

14. If the function $f(x)=\begin{cases}\dfrac{2^{x+2}-16}{4^x-16}, & for\ x\ne 2\\ A & , for\ x=2\end{cases}$

is continuous at $x=2$, then $A=$

1) 2 2) 1/2 3) 1/4 4) 0

15. If $f(x)=\begin{cases}\dfrac{(e^{kx}-1).\sin kx}{x^2}, & for\ x\ne 0\\ 4 & , for\ x=0\end{cases}$

is continuous at $x=0$ then k =

1) ± 1 2) ± 2 3) 0 4) ± 3

16. The set of points of discontinuity of the function $f(x)=\dfrac{1}{x^2+x+1}$

1) ϕ 2) R 3) {0} 4) R^-

17. If $f:R\to R$ defined by

$$f(x)=\begin{cases}\dfrac{1+3x^2-\cos 2x}{x^2}, & for \quad x\ne 0\\ k & , for \quad x=0\end{cases}$$

is continuous at $x=0$, then $k=$ [EAM - 2017]

1) 1 2) 5 3) 6 4) 0

18. If $f:R\to R$ defined by

$$f(x)=\begin{cases}\dfrac{2\sin x-\sin 2x}{2x\cos x} & ,if \quad x\ne 0\\ a & ,if \quad x=0\end{cases}$$

then the value of a so that f is continuous at x=0 is [EAM - 2019]

1) 2 2) 1 3) -1 4) 0

19. If $f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x} & ,for\ -1\le x<0\\ 2x^2+3x-2 & ,for\ 0\le x\le 1\end{cases}$

is continuous at $x=0$ then $k=$

1) –4 2) -3 3) -2 4) -1

20. If $f(x)$ is a continuous function, then $\lim\limits_{x\to 0} f(x)\dfrac{|x|}{x}$ **exist if**

1) $f(x)$ is a polynomial 2) $f(x)=ax^2+bx+c$
3) $f(x)=ax^2+bx$ 4) $f(x)=ax+b$

21. If $f(x)=\begin{cases}\sin x,\ if\ x \text{ is rational}\\ \cos x,\ if\ x \text{ is irrational}\end{cases}$

then the function is

1) discontinuous at $x=n\pi+\dfrac{\pi}{4}$

2) continuous at $x=n\pi+\dfrac{\pi}{4}$

3) discontinuous at all x

4) continuous at all x

KEY

01) 4	02) 3	03) 1	04) 1	05) 2	06) 1
07) 1	08) 4	09) 3	10) 2	11) 2	12) 3
13) 1	14) 2	15) 2	16) 1	17) 2	18) 4
19) 3	20) 3	21) 2			

SOLUTIONS

1. $f(0)=k=\lim_{x\to 0}\frac{x\tan 2x}{\sin 3x\sin 5x}=\frac{2}{15}$

2. $f\left(1\right)=\lim_{x\to 1}\frac{x+x^2+....+x^n-n}{x-1}\left(\frac{0}{0}\text{ from}\right)$

$=\lim_{x\to 1}\frac{1+2x+....+nx^{n-1}}{1}=1+2+...+n$

$=\frac{n\left(n+1\right)}{2}$

3. $\lim_{x\to 0}\frac{\cos ax-\cos bx}{x^2}=\frac{b^2-a^2}{2}\ and\ \lim_{x\to 0}\frac{\sin x}{x}=1$

4. f(x) = r sin 1/x, x $\neq$ 0

$\underset{x\to 0}{Lt}\,f\left(x\right)=\underset{x\to 0}{Lt}\,x\sin\frac{1}{x}=0$ (finite value) =0

Given f(0) = 0

$\underset{x\to 0}{Lt}\,f\left(x\right)$= f(0) $\therefore$ f(x) is continuous at x = 0

$f(x)$ is continuous at $x=0$

5. Applying L-Hospital rule

$f(0)=\lim_{x\to 0}\frac{e^{x^2}.2x+\sin x}{2x}=\frac{3}{2}$

6. $\underset{x\to 0}{Lt}\,f\left(x\right)=f\left(0\right)$

$\underset{x\to 0}{Lt}\frac{\log\left(1+\frac{x}{a}\right)-\log\left(1-\frac{x}{b}\right)}{x}=f\left(0\right)$ using

L - Hospital rule

$\underset{x\to 0}{Lt}\frac{1}{1+\frac{x}{a}}.\frac{1}{a}-\frac{1}{1-\frac{x}{b}}.\frac{1}{b}=f\left(0\right)$

f(x) = $\frac{a+b}{ab}$.Use L-Hospital rule

7. $f\left(1\right)=e^{\lim_{x\to 1}\frac{1}{x-1}\left(x-1\right)}$

8. Use L-Hospital rule

9. $f(x)$ is discontinuous at $x=0,-1,1$

10. $\alpha+0=2=\beta-1$

$\alpha=\beta-1\Rightarrow\beta-\alpha=1$

11. f(x) is continuoud sy c = 0

$\underset{x\to 0}{Lt}\,f\left(x\right)=\underset{x\to 0}{Lt}\,f\left(x\right)=f\left(0\right)$

$\underset{x\to 0^-}{Lt}\frac{a+2\cos x}{x^2}=\underset{x\to 0^+}{Lt}\,b\tan\frac{\pi}{\left[x+4\right]}=6$

$\underset{x\to 0^-}{Lt}\frac{a+2\cos x}{x^2}=b$

$\underset{x\to 0}{Lt}\frac{2\sin x}{2x}=b\Rightarrow b=-1$

a + 2 = 0 ; a = -2

12. f(x) = $\frac{x+2}{x^2+3x+2},x\in R-\left\{-1,-2\right\}$

$\underset{x\to -1}{Lt}\,f\left(x\right)=\underset{x\to -1}{L}\frac{x-2}{\left(x+2\right)\left(x+1\right)}=\underset{x\to -1}{Lt}\frac{1}{x+1}=0$

Given f(-1) = 0

$\underset{x\to -1}{Lt}\,f\left(x\right)\pm f\left(-1\right)$

f(x) is not continuous on x = -1

13. f(x) = $\frac{\sqrt{1+cx}-\sqrt{1-cx}}{x}$ for -2 $\leq$ x < 0

$\frac{x+3}{x+1}$ 0 $\leq$ x $\leq$ 2

$\underset{x\to 0^-}{Lt}\,f\left(x\right)=\underset{x\to 0^+}{Lt}\,f\left(x\right)$

$\underset{x\to 0^-}{Lt}\frac{2cx}{x\sqrt{1+cx}+\sqrt{1-cx}}=\underset{x\to 0^+}{Lt}\frac{x+3}{x+1}$

$\Rightarrow\frac{2c}{c}=\frac{3}{1}\Rightarrow c=3$

14. $A=\lim_{x\to 2}f(x)$

$= \lim_{x\to 2} \frac{2^x.2^2-16}{4^x-16} = \lim_{x\to 2} \frac{2^2.2^x \log 2}{4^x.\log 4}$

$= \frac{4^2.\log 2}{4^2.\log 4} = \log_4 2 = \log_{2^2} 2^1 = \frac{1}{2}$

15. $f(0) = \lim_{x\to 0}\left(\frac{e^{kx}-1}{x}\right).\frac{\sin kx}{x} \Rightarrow 4 = k^2$

16. $f(x)$ is defined for any $x \in R$

17. Given f(x) is continuous at x = 0

$\underset{x\to 0}{Lt} f(x) = f(0)$

$\underset{x\to 0}{Lt} \frac{1+3x^2-\cos 2x}{x^2} = K$

$\Rightarrow \underset{x\to 0}{Lt}\left(3+\frac{1-\cos 2x}{x^2}\right) = K$

$3+\frac{2^2}{2} = k$; K = 5

18. Given f(x) is continuous at x = 0

$\underset{x\to 0}{Lt} f(x) = f(0)$

$\underset{x\to 0}{Lt} \frac{2\sin x - \sin 2x}{2x\cos x} = a$ using L - Hospital rule

$\underset{x\to 0}{Lt} \frac{2\cos x - 2\cos 2x}{2(-x\sin x+\cos x)} = a$

$\frac{0}{1} = a \Rightarrow a = 0$

19. $f(x)$ is continuous at 0

20. $\lim_{x\to 0^+}\frac{|x|}{x} = 1$ and $\lim_{x\to 0^-}\frac{|x|}{x} = -1$

Also $f(x)$ is continuous

$\therefore$ Given limit can exist only if

$\lim_{x\to 0} f(x) = 0$

$\therefore f(x) = ax^2+bx$ is the only choice

21. $\sin x = \cos x \Rightarrow x = n\pi + \frac{\pi}{4}$

EXERCISE - II

1. If $f(x) = \begin{cases} a^2\cos^2 x + b^2 \sin^2 x, x \le 0 \\ e^{ax+b}, x > 0 \end{cases}$

f(x) is continuous at x = 0 then

1) $2\log|a| = b$ 2) $2\log|b| = e$

3) $\log a = 2\log|b|$ 4) $a = b$

2. $f(x) = \begin{cases} \left(\frac{3}{x^2}\right)\sin 2x^2, & if\ x<0 \\ \frac{x^2+2x+c}{1-3x^2}, & if\ x \ge 0, x \ne \frac{1}{\sqrt{3}} \\ 0, & if\ x = \frac{1}{\sqrt{3}} \end{cases}$

then in order that f be continuous at $x = 0$, the value of c is

1) 2 2) 4 3) 6 4) 8

3. Let f be a continuous function on R such that

$f\left(\frac{1}{4n}\right) = \sin(e^n)e^{-n^2} + \frac{n^2}{n^2+1}$ **. Then the value of $f(0)$ is** [EAM -2018]

1) 1 2) 1/2 3) 0 4) 2

4. The function $f(x) = a[x+1]+b[x-1]$, $(a \ne 0, b \ne 0)$ where $[x]$ is the greatest integer function is continuous at $x = 1$ if

1) $a = 2b$ 2) $a = b$

3) $a+b = 0$ 4) $a+2b = 0$

5. Let

$f(x) = \begin{cases} \frac{(e^x-1)^{2n}}{\sin^n(x/a)(\log(1+(x/a)))^n}, & for\ x \ne 0 \\ 16^n, & for\ x = 0 \end{cases}$

and f is a continuous at x=0, then the value of a is

1) 16 2) 2 3) 8 4) 4

6. **Let** $f(x)=[2x^3-6]$ **when** $[x]$ **is greatest integer less than or equal to** x **then the number of points in (1,2) where** f **is discontinuous is**

1) 5 2) 7 3) 13 4) 12

7. **If** $f(x)=\begin{cases}\dfrac{\sin[x]}{[x]}, & for\ [x]\neq 0\\ 0 & ,for[x]=0\end{cases}$ **is**

1) continuous at $x=0$

2) discontinuous at $x=0$

3) L.H.L=0 4) R.H.L=1

8. **If** $f(x)=\begin{cases}(1+|\sin x|)^{\frac{a}{|\sin x|}}, & if\ -\frac{\pi}{6}<x<0\\ b & , if\ x=0\\ e^{\frac{\tan 2x}{\tan 3x}} & , if\ 0<x<\frac{\pi}{6}\end{cases}$ **is continuous at x = 0 then**

1) $a=e^{2/3}, b=2/3$ 2) $a=2/3, b=e^{2/3}$

3) $a=1/3, b=e^{1/3}$ 4) $a=e^{1/3}, b=e^{1/3}$

9. **If the function** $f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2} & , if\ x<0\\ a & , if\ x=0\\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & if\ x>0\end{cases}$ **is continuous at** $x=0$ **then** $a=$

1) 8 2) $\frac{1}{8}$ 3) -8 4) 0

10. **If** $f(x)=\begin{cases}\dfrac{72^x-9^x-8^x+1}{\sqrt{2}-\sqrt{1+\cos x}}, & if\ x\neq 0\\ K\log 2\log 3, & if\ x=0\end{cases}$ **is a continuous function, then K is equal to**

1) $\sqrt{2}$ 2) 24 3) $18\sqrt{3}$ 4) $24\sqrt{2}$

11. **The value of f(0) so that the function f(x)=**$\dfrac{1-\cos(1-\cos x)}{x^4}$ **is continuous everywhere is**

1) $\frac{1}{8}$ 2) $\frac{1}{2}$ 3) $\frac{1}{4}$ 4) $\frac{1}{3}$

12. **If** $x+2|y|=3y$**, where** $y=f(x)$**, then** $f(x)$ **is**

1) continuous everywhere
2) differentiable everywhere
3) discontinuous at $x=0$
4) Not differentiable at anywhere

13. **The function f(x)=**$[\cos x]$ **is**

1) continuous at $x=\frac{\pi}{2}$

2) discontinuous at $x=\frac{\pi}{2}$

3) L.H.L= -1 at $x=\frac{\pi}{2}$

4) R.H.L= 1 at $x=\frac{\pi}{2}$

14. **The function** $f(x)=\dfrac{1}{x^2-3|x|+2}$ **is discontinuous at the points**

1) $x=1, 2$ 2) $x=\pm 1, \pm 2$
3) R 4) $R-\{1,2\}$

15. $f(x)=\min\{x,x^2\}, \forall x\in R$ **then** $f(x)$ **is**

1) discontinuous at 0 2) discontinuous at 1
3) continuous on R 4) continuous at 0,1

16. **If** $f(x)=\dfrac{1}{1-x}$ **then the points of discontinuity of** $(f0f0f)(x)$ **is**

1) $\{0,1\}$ 2) $\{0,\pm 1\}$ 3) $\{1\}$ 4) $\{\pm 1\}$

KEY

01) 1	02) 3	03) 1	04) 3	05) 4	06) 3
07) 2	08) 2	09) 1	10) 4	11) 1	12) 1
13) 2	14) 2	15) 3	16) 1		

SOLUTIONS

1. $\underset{x\to 0^-}{Lt} f(x) = \underset{x\to 0^+}{Lt} f(x)$

$\underset{x\to 0^-}{Lt} a^2\cos^2 x + b^2\sin^2 x = \underset{x\to 0^+}{Lt} e^{ax+b}$

$\Rightarrow a^2 + 0 = e^b$

$b = \log_e a^2 \Rightarrow b = 2\log_e a$

2. $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-}\frac{3}{x^2}\sin 2x^2 = 6\lim\limits_{x\to 0}\frac{\sin 2x^2}{2x^2} = 6$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}\frac{x^2+2x+c}{1-3x^2} = \frac{c}{1} = c$

Hence for f to be continuous $c = 6$.

3. $f\left(\frac{1}{4n}\right) = \sin(e^n)e^{-n^2} + \frac{n^2}{n^2+1}$

$f\left(\frac{1}{4n}\right) = \sin(e^n)e^{-n^2} + \frac{\cancel{n^2}}{\cancel{n^2}\left(1+\frac{1}{n^2}\right)}$

$n \to \infty \Rightarrow \frac{1}{n} \to 0$

$f(0) = \underset{n\to\infty}{Lt} f\left(\frac{1}{4n}\right) = 0 + \frac{1}{1+0} = 1$

4. $f(1) = 2a,\ \lim\limits_{x\to 1^-} f(x) = a-b$ so

$a+b=0$ for f to be continuous at $x=1$.

5. $\lim\limits_{x\to 0} f(x) =$

$$\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^{2n}\frac{(x/a)^n}{\sin^n(x/a)}\times\frac{(x/a)^n}{\left(\log(1+(x/a))\right)^n}\cdot a^{2n}$$

$= a^{2n}$

since. $f(0) = \lim\limits_{x\to 0} f(x)$ so

$a^{2n} = 16^n = 4^{2n}$ thus $a = 4$.

6. $1 < x < 2 \Rightarrow 1 < x^3 < 8$

$\Rightarrow 2 < 2x^3 < 16 \Rightarrow -4 < 2x^3 - 6 < 10$

$\therefore [2x^3 - 6]$ is discontinuous at -3,-2,-1,0,1,2,3,4,5,6,7,8,9

7. $L.\text{H.L} = \lim\limits_{x\to 0^-}\frac{\sin(-1)}{-1} = \sin 1$

$\left(\because x \to 0^- \Rightarrow [x] = -1\right)$

$R.\text{H.L} = 0 \quad \left(\because x \to 0^+ \Rightarrow [x] = 0\right)$

8 . $\underset{n\to\infty}{Lt} f(x) = \underset{x\to 0^-}{Lt} f(x) = f(0)$

$\underset{n\to 0^+}{Lt} e^{\frac{\tan 2x}{\tan 3x}} = \underset{x\to 0^-}{Lt} \left(1+|\sin x|\right)^{\frac{a}{|\sin x|}} = b$

$\Rightarrow e^{\frac{2}{3}} = e^a \Rightarrow a = \frac{2}{3}$

$\Rightarrow e^{\frac{2}{3}} = b$

9. $a = \lim\limits_{x\to 0^-}\frac{1-\cos 4x}{x^2} = 8$

10. $K\log 2\log 3 = f(0)$

$= \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\frac{72^x - 9^x - 8^x + 1}{\sqrt{2}-\sqrt{1+\cos x}}$

$= \lim\limits_{x\to 0}\frac{(9^x-1)(8^x-1)}{x^2}\frac{x^2}{\sqrt{2}(1-\cos x/2)}$

$= \lim\limits_{x\to 0}\frac{9^x-1}{x}\cdot\frac{8^x-1}{x}\frac{16(x/4)^2}{\sqrt{2}\cdot 2\sin^2 x/4}$

$= \frac{16}{2\sqrt{2}}\log 9\log 8$

$= \frac{8}{\sqrt{2}}6\log 3\log 2 = 24\sqrt{2}\log 3\log 2$

Thus $K = 24\sqrt{2}$.

11. $\lim\limits_{x\to 0}\frac{1-\cos(1-\cos x)}{(1-\cos x)^2}\frac{(1-\cos x)^2}{x^4}$

$$= \lim_{x\to 0} \frac{1-\cos(1-\cos x)}{(1-\cos x)^2}\left(\frac{(1-\cos x)}{x^2}\right)^2$$

$$= \frac{1}{2}\times\frac{1}{4}\left(\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}\right)(formula)$$

$$= \frac{1}{8}$$

12. $x + 2|y| = 3y$

$\Rightarrow x + 2y = 3y, y \geq 0$ and

$x - 2y = 3y, y < 0$

$$\Rightarrow y = \begin{cases} x, & x \geq 0 \\ \frac{x}{5}, & x < 0 \end{cases}$$

Therefore, y is continuous everywhere but is not differentiable at $x = 0$.

13. $L.H.L = \lim_{x\to \frac{\pi^-}{2}} [\cos x] = 0$

$R.H.L = \lim_{x\to \frac{\pi^+}{2}} [\cos x] = -1$

14. $f(x)$ is discontinuous when

$x^2 - 3|x| + 2 = 0 \Rightarrow |x|^2 - 3|x| + 2 = 0$

$\Rightarrow |x| = 1, 2$

15. $x = x^2 \quad \Rightarrow \quad x = 0, x = 1$

$$f(x) = \begin{cases} x \text{ for } x < 0 \\ x^2 \text{ for } 0 \leq x \leq 1 \\ x \text{ for } x > 1 \end{cases}$$

16. $f(x) = \frac{1}{1-x}$

$$(fof)(x) = \frac{1}{1-\frac{1}{1-x}} = \frac{1-x}{-x} = \frac{x-1}{x}$$

$$(fofof)(x) = \frac{1}{1-\frac{x-1}{x}} = x$$

$(fofof)(x)$ is discontinuous at $x = 0, x = 1$.

EXERCISE - III

1. If $f(x) = \begin{cases} \frac{|x+2|}{\tan^{-1}(x+2)}, & if\ x \neq -2 \\ 2, & if\ x = -2 \end{cases}$ **then,** $f(x)$ **is**

1) continuous at $x = -2$
2) not continuous at $x = -2$
3) differentiable at $x = -2$
4) continuous but not diferentiable at $x = -2$

2. Let $f(x) = \frac{(256+ax)^{1/8} - 2}{(32+bx)^{1/5} - 2}$. **If** f **is continuous at** $x = 0$ **,then the value of** a/b **is**

1) $\frac{8}{5}f(0)$ 2) $\frac{32}{5}f(0)$ 3) $\frac{64}{5}f(0)$ 4) $\frac{16}{5}f(0)$

3. The values of a and b if f is continuous at $x = 0$**, where**

$$f(x) = \begin{cases} \left(1 + \frac{ax+bx^3}{x^2}\right)^{1/x}, & if\ x > 0 \\ 3, & if\ x = 0 \end{cases}$$

1) $a = 0, b = \log 3$ 2) $a = 1, b = \log 2$
3) $a = 2, b = \log 3$ 4) $a = 0, b = \log 2$

4. $If\ f(x) = \begin{cases} \frac{\sin(\cos x) - \cos x}{(\pi - 2x)^3}, & if\ x \neq \frac{\pi}{2} \\ k, & if\ x = \frac{\pi}{2} \end{cases}$

is continuous at $x = \frac{\pi}{2}$**, then k =**

1) 0 2) $-\frac{1}{6}$ 3) $-\frac{1}{24}$ 4) $-\frac{1}{48}$

5. If $f(x) = \lim_{n\to\infty} \frac{\log(2+x) - x^{2n}\sin x}{1+x^{2n}}$ **then** $f(x)$ **is discontinuous at**

1) $x = 1$ only 2) $x = -1$ only
3) $x = -1, 1$ only 4) no point

6. Let $f : R \to R$ **be given by**

$$f(x) = \begin{cases} 5x, & \text{if } x \in Q \\ x^2 + 6, & \text{if } x \in R - Q \end{cases}$$ **then**

1) f is continuous at $x = 2$ and $x = 3$

2) f is discontinuous at $x=2$ and $x=3$
3) f is contininuous at $x=2$ but not at $x=3$
4) f is continuous at $x=3$ but not at $x=2$

7. $If\ f(x)=\begin{cases}\dfrac{A+3\cos x}{x^2}, & if\ x<0\\ B\tan\left(\dfrac{\pi}{[x+3]}\right), & if\ x\geq 0\end{cases}$ **Where** $[.]$ **represents the greatest integer function, is continuous at** $x=0$ **Then**

1) $A=-3, B=-\sqrt{3}$ 2) $A=3, B=-\dfrac{\sqrt{3}}{2}$

3) $A=-3, B=-\dfrac{\sqrt{3}}{2}$ 4) $A=-\dfrac{\sqrt{3}}{2}, B=-3$

8. If $f(x)=\begin{cases}\dfrac{a|x^2-15x+56|}{x-8} & ,if\ x>9\\ b & ,if\ x=9\\ \dfrac{x-[x]}{x-8} & ,if\ x<9\end{cases}$,

where [.] denotes greatest integer function and the function is continuous then

1) $a=\dfrac{1}{2}, b=1$ 2) $a=0, b=1$

3) $a=\dfrac{-1}{2}, b=1$ 4) $a=\dfrac{-1}{2}, b=-1$

9. If $[.]$ **denotes the greatest integer function then the number of points where**

$f(x)=[x]+\left[x+\dfrac{1}{3}\right]+\left[x+\dfrac{2}{3}\right]$ **is**

discontinuous for $x\in(0,3)$ **are**

1) 2 2) 9 3) 8 4) 10

10. If the function

$f(x)=\left[\dfrac{(x-5)^3}{A}\right]\sin(x-5)+a\cos(x-2)$

where $[.]$ **denotes the greatest integer function, is continuous and differentiable in (7,9), then**

1) $A\in[8,64]$ 2) $A\in(0,8]$

3) $A\in[64,\infty)$ 4) $A\in[8,16]$

11. If $f(x)=\begin{cases}xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & if\ x\neq 0\\ 0 & ,\ if\ x=0\end{cases}$**, then** $f(x)$ **is**

1) continuous for all x, but is not differentiable
2) neither differentiable nor continuous
3) discontinuous everywhere
4) continuous as well as differentiable for all x

12. The function $f(x)=[x]^2-[x^2]$ **(where [y] is the largest integer** $\leq y$ **) is discontinuous at**

1) all integers
2) all integers except 0 and 1
3) all integers except 0
4) all integers except 1

13. If $f(x)=\dfrac{1}{x^2-17x+66}$ **then** $f\left(\dfrac{2}{x-2}\right)$ **is discontinuous at x is equal to**

1) $2,\dfrac{7}{3},\dfrac{25}{11}$ 2) $2,\dfrac{8}{3},\dfrac{24}{11}$

3) $2,\dfrac{7}{3},\dfrac{24}{11}$ 4) $2,6,11$

14. $f(x)=Sgn(x^3-x)$ **is discontinuous at** $x=$

1) 0 2) 1 3) - 1 4) 0, -1, 1

15. If $f(x)=Sgn(2\sin x+a)$ **is continuous for all** x **then the possible values of '** a **' are**

1) R 2) $a<-2$ or $a>2$
3) $(-2,2)$ 4) $(0,\infty)$

16. If $f:R\to R$ **is a function defined by** $f(x)=[x]\cos\left(\dfrac{2x-1}{2}\right)\pi$, **where [x] denotes the greatest integer function, then** f **is**

[AIEEE - 2012]

1) Continuous for every real x
2) discontinuous only at $x=0$
3) discontinuous only at non-zero integral values of x.
4) continuous only at $x=0$.

17. The values of p **and** q **for which the function**

$$f(x)=\begin{cases}\dfrac{\sin(p+1)x+\sin x}{x} & , \ if\ x<0 \\ q & , \ if\ x=0 \\ \dfrac{\sqrt{x+x^2}-\sqrt{x}}{x^{\frac{3}{2}}} & , \ if\ x>0\end{cases}$$

is continuous for all x in R, is [AIEEE - 2011]

1) $p=\frac{5}{2}, q=\frac{1}{2}$ 2) $p=-\frac{3}{2}, q=\frac{1}{2}$

3) $p=\frac{1}{2}, q=\frac{3}{2}$ 4) $p=\frac{1}{2}, q=-\frac{3}{2}$

KEY

01) 2	02) 3	03) 1	04) 4	05) 3	06) 1
07) 3	08) 1	09) 3	10) 3	11) 1	12) 4
13) 3	14) 4	15) 2	16) 1	17) 2	

SOLUTIONS

1. $\lim\limits_{x\to-2^-} f(x)=\lim\limits_{x\to-2^-}\dfrac{|x+2|}{\tan^{-1}(x+2)}$

$=\lim\limits_{x\to-2^-}\dfrac{-(x+2)}{\tan^{-1}(x+2)}=-1$

$\left[\because \lim\limits_{x\to0^-}\dfrac{\tan^{-1}x}{x}=1\right]$

$\lim\limits_{x\to-2^+} f(x)=\lim\limits_{x\to-2^+}\dfrac{|x+2|}{\tan^{-1}(x+2)}$

$=\lim\limits_{x\to-2^+}\dfrac{x+2}{\tan^{-1}(x+2)}=1$

$\therefore \lim\limits_{x\to-2} f(x)$ does not exist.

2. $\lim\limits_{x\to0}\dfrac{(256+ax)^{1/8}-2}{(32+bx)^{1/5}-2}=f(0)$

Use L-Hospital rule

$\Rightarrow \lim\limits_{x\to0}\dfrac{\frac{1}{8}(256+ax)^{\frac{-7}{8}}.a}{\frac{1}{5}(32+bx)^{\frac{-4}{5}}.b}=f(0)$

On simplify, $\dfrac{a}{b}=\dfrac{64}{5}.f(0)$

3. $\lim\limits_{x\to0^+} f(x)=f(0)$

$\Rightarrow \lim\limits_{x\to0^+}\left(1+\dfrac{ax+bx^3}{x^2}\right)^{1/x}=3$

$\Rightarrow \lim\limits_{x\to0}\left(1+\dfrac{a}{x}+bx\right)^{1/x}=3$

L.H.S. is exist when $a=0$

$\Rightarrow \lim\limits_{x\to0}(1+bx)^{1/x}=3$

$\Rightarrow e^6=3\Rightarrow b=\log_e^3$

4. $k=\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\sin(\cos x)-\cos x}{(\pi-2x)^3}$

$\Rightarrow k=\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\sin(\cos x)-\cos x}{(\cos x)^3}\times\dfrac{\sin^3\left(\frac{\pi}{2}-x\right)}{8\left(\frac{\pi}{2}-x\right)^3}$

$\Rightarrow k=-\dfrac{1}{6}\times\dfrac{1}{8}=-\dfrac{1}{48}$ $\left[\therefore \lim\limits_{x\to0}\dfrac{\sin x-x}{x^3}=-\dfrac{1}{6}\right]$

5. $f(x)$ is discontinuous at $x=1\ or\ x=-1$

6. f is continuous when $5x=x^2+6$

$\Rightarrow x^2-5x+6=0$

$\Rightarrow x=2,3$

7. $\lim\limits_{x\to0^-} f(x)=\lim\limits_{x\to0}\left(\dfrac{A+3}{x^2}-\dfrac{3}{2!}+\dfrac{3}{4!}x^2-\dfrac{3}{6!}x^4+.....\right)$

For this limit to exist, we must have $A+3=0$ and in that case, we have

$\lim\limits_{x\to0^-} f(x)=-\dfrac{3}{2}$ Now,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} B\tan\left(\frac{\pi}{[x+3]}\right) = B\tan\frac{\pi}{3} = B\sqrt{3}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

8. $\lim_{x\to 9^+} f(x) = \lim_{x\to 9^-} f(x) = f(9)$

$$\Rightarrow \lim_{x\to 9^+} \frac{a|x^2-15x+56|}{x-8} = \lim_{x\to 9^-} \frac{x-[x]}{x-8} = b$$

$$\Rightarrow \lim_{x\to 9^+} \frac{a(x-7)(x-8)}{x-8} = \lim_{x\to 9^-} \frac{x-8}{x-8} = b$$

$\Rightarrow 2a = 1 = b$

9. $f(x) = [3x]$ is discontinuous when $3x =$ an integer

$$\Rightarrow x = \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3}, \frac{7}{3}, \frac{8}{3} \in (0,3)$$

10. $[x]$ is not continuous & differentiable at integral values (points) so $f(x)$ continuous and differentiable in (7,9) if $\left[\frac{(x-5)^3}{A}\right] = 0$

$\Rightarrow A \geq (9-5)^3$

$\Rightarrow A \geq 64 \quad \therefore A \in [64, \infty)$

11. $|x|$ is not differentiable at $x = 0$

$|x|$ is continuous at $x = 0$

12. Clearly, $f(x) = 0$ for each integral value of x.

Also, if $0 < x < 1$, then $0 < x^2 < 1$,

$\Rightarrow [x] = 0$ and $[x^2] = 0$

$\therefore f(x) = 0$ for $0 < x < 1$

Again, if $1 \leq x < \sqrt{2}$ then $1 \leq x^2 < 2$

$\Rightarrow [x] = 1, [x^2] = 1$

11. $|x|$ is not differentiable at $x = 0$

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Again, if $1 \leq x < \sqrt{2}$ then $1 \leq x^2 < 2$

$\Rightarrow [x] = 1, [x^2] = 1$

However, at points x other than integers and not lying between 0 and $\sqrt{2}$, $f(x) \neq 0$

13. $u = \frac{2}{x-2}$ is discontinuous at $x = 2$

$$f(u) = \frac{1}{u^2 - 17u + 66} = \frac{1}{(u-11)(u-6)}$$ is

discontnuous at $u = 6, 11$

$$\therefore \frac{2}{x-2} = 6, 11 \Rightarrow x = \frac{7}{3}, \frac{24}{11}$$

14. $f(x)$ is discontinuous when $x^3 - x = 0$

$\Rightarrow x = 0, -1, 1$

15. Function continuous for all x

$$\Rightarrow 2\sin x + a \neq 0 \Rightarrow \sin x \neq \frac{-a}{2}$$

$$\Rightarrow \left|\frac{a}{2}\right| > 1 \Rightarrow a < -2 \,(\text{or})\, a > 2$$

16. $= [x]\sin \pi x$ is continuous for every real x.

17. Use L-Hospital's rule.

JEE MAINS QUESTIONS

1. The value of K for which the function

f(x) = $\begin{cases} \left(\frac{4}{5}\right)\frac{\tan 4x}{\tan 5x} & 0cx < \frac{\pi}{2} \\ k+\frac{2}{5} & x = \frac{\pi}{2} \end{cases}$ is continuous at

x = $\frac{\pi}{2}$ is [2017]

1) $\frac{17}{20}$ 2) $\frac{2}{5}$ 3) $\frac{3}{5}$ 4) - $\frac{2}{5}$

2. Let a, b $\in$ R (a $\neq$ 0) if the function f defined as

f(x) = $\begin{cases} \frac{2x^2}{a} & 0 \le x < 1 \\ a & 1 \le x < \infty \\ \frac{2b^2-4b}{x^3} & \sqrt{2} \le x < \infty \end{cases}$ is continuous

in (0, ∞) then (a, b) is [2016]

1) $\left(\sqrt{2}, 1-\sqrt{3}\right)$ 2) $\left(-\sqrt{2}, 1-\sqrt{3}\right)$

3) $\left(\sqrt{2}, -1-\sqrt{3}\right)$ 4) $\left(-\sqrt{2}, 1-\sqrt{3}\right)$

3. Let f(x) = $\begin{cases} (x-1)^{\frac{1}{2-x}} & x>1 \quad x \neq 2 \\ K & x = 2 \end{cases}$ the value of k for which f is continuous at x = 2 is [2018]

1) 1 2) e 3) e^{-1} 4) e^{-2}

4. If the function f defined as f(x) = $\frac{1}{x} - \frac{k-1}{e^{2x}-1} x \neq 0$ is continuous at x = 0, then the ordered pair (k, f(0)) = [2019]

1) (3, 2) 2) (3, 1) 3) (2, 1) 4) ($\frac{1}{3}$, 2)

5. If the function f defined on $\left(-\frac{1}{3}, \frac{1}{3}\right)$ by

f(x) = $\begin{cases} \frac{1}{x}\log\left(\frac{1+3X}{1-2X}\right) & x \neq 0 \\ k & x = 0 \end{cases}$ is continuous,

then k is equal to [2020]

6. Let [t] denofe the greatest integer $\le$ t and $\underset{x\to 0}{Lt}\, x\left[\frac{4}{x}\right] = A$. then the function

f(x) = [x^2] sin (πx) is discontinuous when x is equal [2020]

1) $\sqrt{A}$ 2) $\sqrt{A+1}$ 3) $\sqrt{A+5}$ 4) $\sqrt{A+21}$

7. Let f(x) = x.$\left[\frac{x}{2}\right]$ for - 10<x<10 where [t] denotes the greatest integer function.

Then the number of points of discontinuity of f is equal to [2020]

KEY

1) 3 2) 1 3) 3 4) 2

5) 5 6) 2 7) 8

SOLUTIONS

1) $\underset{x\to\frac{\pi}{2}}{Lt} f(x) = \underset{x\to\frac{\pi}{2}}{Lt}\left(\frac{4}{5}\right)^{\frac{\tan 4x}{\tan 5x}} = \left(\frac{4}{5}\right)^0 = 1$ given f(x) is continuous at x = $\frac{\pi}{2}$

$\underset{x\to\frac{\pi}{2}}{Lt} f(x) = f\left(\frac{\pi}{2}\right)$

1 = $k + \frac{2}{5} \Rightarrow = \frac{3}{5}$

2) Given f(x) is continuous at x = 1, $\sqrt{2}$

Key : 1

$\underset{x\to 1^-}{Lt} f(x) = \underset{x\to 1^+}{Lt} + f(x)$

$\underset{x\to 1^-}{Lt} \frac{2x^2}{a} = a \Rightarrow \frac{2}{a} = a \Rightarrow a^2 = 2, a = \sqrt{2}$

$\underset{x\to 1^-}{Lt} f(x) = \underset{x\to\sqrt{2}^+}{Lt} f(x)$

$a = \underset{x\to\sqrt{2}^+}{Lt} \frac{2b^2 - 4b}{x^3}$

$\sqrt{2} = \frac{2b^2 - 4b}{2\sqrt{2}}$

$\Rightarrow$ 4 = 2(b^2 - 2b)

$\Rightarrow$ b^2 - 2b - 2 = 0

$b = \frac{2 \pm \sqrt{4+8}}{2} = \frac{2 + 2\sqrt{3}}{2}$

b = 1 $\pm\sqrt{3}$.

3. Given f(x) is continuous at x = 2

$\underset{x\to 2}{Lt} f(x) = f(2) \Rightarrow \underset{x\to 2}{Lt} (x-1)^{\frac{1}{2-x}} = k$

$\Rightarrow k = \underset{x\to 2}{Lt} \frac{1}{2-x}(x-1-1)$

k = $e^{\underset{x\to 2}{Lt}\frac{1}{2-x}(x-1-1)}$

k = $e^{\underset{x\to 2}{Lt}\frac{-(2-x)}{2-x} \Rightarrow K = e^{-1}}$

4. Given f(x) is continuous at x = 0

$\underset{x\to 0}{Lt} f(x)$ = f(0) = $\underset{x\to 0}{Lt}\left(\frac{1}{x} - \frac{k-1}{e^{2x}-1}\right)$

$\Rightarrow f(0) = \underset{x\to 0}{Lt} \frac{e^{2x} - 1 - x(k-1)}{x(e^{2x}-1)}$

$$f(0) = \underset{x\to 0}{Lt} \frac{(2x) + \frac{(2x)^2}{2} + ...x(k-1)}{2x^2\left(\frac{e^{2x}-1}{2x}\right)}$$

$f(0) = \underset{x\to 0}{Lt} \frac{x(2-k+1) + 2x^2 + ...}{2x^2}$ is defined if

f(0) = 0 + 1 + 0..... 3 - k = 0, k = 3,

f(0) = 1

5. $\underset{x\to 0}{Lt} f(x) = \underset{x\to 0}{Lt} \frac{1}{x}\log\left(\frac{1+3x}{1-2x}\right)$

$= \underset{x\to 0}{Lt} \frac{\log(1+3x)}{x} - \underset{x\to 0}{Lt} \frac{\log(1-2x)}{x}$

$= \underset{x\to 0}{Lt} \frac{1}{1+3x}3 - \underset{x\to 0}{Lt} \frac{1}{1-2x}(-2)$

$= \frac{3}{1} + \frac{2}{1} = 5$

given f(x) is continuous at x = 0

$\underset{x\to 0}{Lt} f(x) = f(0)$, 5 = k.

6. $\underset{x\to 0}{Lt}\, x\left[\frac{4}{x}\right] = A \Rightarrow \underset{x\to 0}{Lt}\left(\frac{4}{x} - \left[\frac{4}{x}\right]\right) = A$

$\Rightarrow \underset{x\to 0}{Lt}\, 4 - x\left[\frac{4}{x}\right] = A$

$\Rightarrow$ 4 - 0 = A

check when

1) x = $\sqrt{A} \Rightarrow$ x = 2 $\Rightarrow$ continuous

2) x = $\sqrt{A+1} \Rightarrow x = \sqrt{5} \Rightarrow$ discontinuous

3) x = $\sqrt{A+5} \Rightarrow x = 3 \Rightarrow$ continuous

4) x = $\sqrt{A+21} \Rightarrow x = 5 \Rightarrow$ continuous

7. f(1) = r$\left[\frac{x}{2}\right]$ may be discontinous where $\frac{x}{2}$ is on integer. So possible points of discon tinuity are x = ± 2, ± 4, ± 6, ± 8 and 0

but at x = 0

$\underset{x\to 0}{Lt} f(x) = 0 = f(0) = \underset{x\to 0^-}{Lt} f(x)$

so f(x) will be distentinuous at x = ± 2, ± 4, ± 6, ± 8

number of points = 8

****

DIFFERENTIABILITY

SYNOPSIS

Differentiability at a point :

✈ (i) A function f(x) is differentiable at a point x = a, if $\underset{x\to a}{Lt}\dfrac{f(x)-f(a)}{x-a}$ exists finitely and it is denoted by $f'(a)$

i.e $f'(a)=\underset{x\to a}{Lt}\dfrac{f(x)-f(a)}{x-a}$

(ii)The right hand derivative of f(x) at x = a is denoted by $f'(a+)$ and is defined as

i.e $f'(a+)=\underset{h\to 0}{Lt}\dfrac{f(a+h)-f(a)}{h}$

(iii) The left hand derivative of $f(x)$ at x = a is denoted by $f'(a-)$ and is defined as

i.e $f'(a-)=\underset{h\to 0}{Lt}\dfrac{f(a-h)-f(a)}{-h}$

(iv)If f is differentiable at x = a then f is also continous at $x=a$. However the converse need not be true.

(v) If f is not continous at x = a then f is not differentiable at x = a

Differentiability of a function over an interval :

✈ i) A function $f(x)$ defined on an (a,b) is said to be differentiable in (a,b) if it is differentiable at each point of (a,b)

ii)A function $f(x)$ defined on [a,b] is said to be differentiable or derivable if

a) f is differentiable from the right at a.

b) f is differentiable at every point on (a,b)

c) f is differentiable from the left at b.

iii)A function f is said to be a differentiable function, if it is differentiable at every point on its domain.

iv)Exponential, logarithemic, trigonometric, inverse trigonometric functions are differentiable in their domain.

v)Polynomial, constant functions are differentiable at each point 'x', where $x\in R$

Standard Results :

$f(x)$	$g(x)$	$f(x)\pm g(x)$	$f(x).g(x)$
Differentiable	Differentiable	Differentiable	Differentiable
Differentiable	Non Differentiable	Non Differentiable	May bo or not
Non Differentiable	Non Differentiable	May bo or not	May bo or not

i) $|x-a|$ is not diffrentiable at x = a

ii) $(x-a)^n|x-a|$ is diffrentiable when $n\geq 1$ and is not differentiable when n < 1

iii) $Sgn(x-a)$ is not differentiable at x = a

iv) $x^n\sin\frac{1}{x}, x^n\cos\frac{1}{x}$ are differentiable when n>1 and are not differentiable when $n\leq 1$

v) $\{x\},[x]$ are not differentiable at all intergral points of x

vi) $\dfrac{d}{dx}\int_{\alpha(x)}^{\beta(x)}f(t)dt=f(\beta(x))\dfrac{d}{dx}\beta(x)-$

$$f(\alpha(x))\frac{d}{dx}\alpha(x)$$

Differentiability of Functional Equations

✈ (i) if f(x + y) = f(x).f(y) then

$$f'(x)=f'(0).f(x)$$

Proof: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \dfrac{f(x)(f(h)-1)}{h}$

$= f(x).f'(0)$

(ii) Funtional equation relations.

a) $f(x+y) = f(x).f(y) \forall x, y$

$\Rightarrow f(x) = a^x (a > 0)$

b) $f(x+y) = f(x) + f(y)\ \forall x, y \in R$

$\Rightarrow\ f(x) = kx$

c) $f(xy) = f(x).f(y) \forall x, y \in R$

$\Rightarrow\ f(x) = x^n$

d) $f(xy) = f(x) + f(y) \forall x, y \in R^+$

$\Rightarrow f(x) = k \log x (x > 0)$

e) $f(x) f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) \forall x \in R - \{0\}$

$\Rightarrow f(x) = 1 \pm x^n$

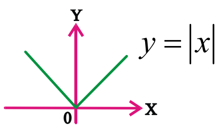
(f) $f\left(\dfrac{mx+ny}{m+n}\right) = \dfrac{m f(x) + n f(y)}{m+n}$, $m+n \neq 0$

$\Rightarrow f(x) = ax + b$

EXAMPLES

1. **Examine the continuity and differentiability of $f(x) = |x|$ at x = 0**

Sol:



Hence $y = |x|$ is continuous everywhere but not differentiable at x = 0

($\because$ sharp corner at x = 0)

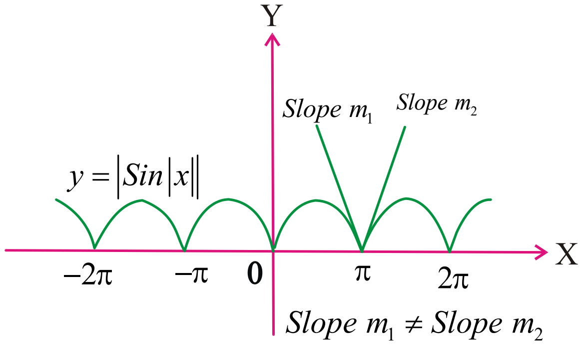
2:

Examine the continuity and differentiability of $y = |\sin|x||$ at $x = n\pi, n \in I$

Sol: It is clear from the graph that $y = |\sin|x||$ is continuous everywhere but not differentiable at $x = ... -2\pi, -\pi, 0, \pi, 2\pi, ...$

i.e., $x = n\pi, n \in I$

We observe that at all integral values of π, f has a sharp corner



3:

The differentiablity of

$$f(x) = \begin{cases} x\left(\dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$ **at x = 0**

Sol: We have $f'(0^-) = \lim_{h \to 0} \dfrac{f(0-h) - f(0)}{-h}$

$= \lim_{h \to 0} \left(\dfrac{e^{-2/h} - 1}{e^{-2/h} + 1}\right) = \dfrac{0-1}{0+1} = -1$

Similarly $f'(0^+) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = 1$

$\therefore$ LHD $\neq$ RHD

$\therefore$ f is not differentiable at x = 0

4:

If f(x+y) = f(x).f(y) $\forall$ x, y $\in$ R, f(5) = 2, $f'(0) = 3$ then $f'(5) =$ [AIEEE 2002]

Sol: $f'(x) = f'(0).f(x)$

$f'(5) = f'(0).f(5) = (3)(2) = 6$

5:

Let f:R $\to$ R is differentiable function & f(1)= 4 then $g(x) = \lim_{x \to 1} \int_4^{f(x)} \dfrac{2t}{x-1} dt$ =

Sol : $g(x)=\lim_{x\to 1}\int_4^{f(x)}\frac{2t}{x-1}dt=\lim_{x\to 1}\frac{\int_4^{f(x)}2t\,dt}{x-1}$

Apply L-Hospital rule

$= 2\lim_{x\to 1}\frac{f(x)f'(x)-4.0}{1}$

$=\lim_{x\to 1}2f(x)f'(x)=2f(1)f'(1)=8f'(1)$

EXERCISE - I

1. Let f(x)=|x-1|+|x+1|

1) f(x) is differentiable at $x=\pm 1$

2) f(x) is not differentiable at $x=\pm 1$

3) f(x) is neither continuous nor differentiable at $x=\pm 1$

4) f(x) is not continuous at x=0

2. **Let** $f(x)=\begin{cases}\frac{x}{1+2^{1/x}}, & x\neq 0\\ 0 & x=0\end{cases}$ **then**

1) LHD f(x) at x=0 is 1

2) RHD of f(x) at x=0 is not equal to zero

3) f(x) is differentiable at x=0

4) $\underset{x\to 0}{Lt}\, f(x)=1$

3. If $f(x)=|x|e^{x}$**, then at x = 0**

1) f is continuous

2) f is continuous but not differentiable

3) f is differentiable

4) the derivative is 1

4. The set of all points where $f(x)=2x|x|$ **is differentiable**

1) $(-\infty,\infty)$ 2) $(-\infty,\infty)-\{0\}$

3) $(0,\infty)$ 4) $[0,\infty)$

5. Let f(x)= $x^{3/2}-\sqrt{x^3+x^2}$ **then**

1) LHD at *x=0* exist but RHD at *x=0* does not exist

2) *f(x)* is not differentiable at *x=0*

3) RHD at *x=0* exist but LHD at *x=0* does not exist

4) *f(x)* is differentiable and continuous at *x=0*

6. If $f(x)=x.\left(\frac{a^{1/x}-a^{-1/x}}{a^{1/x}+a^{-1/x}}\right)$**,** $x\neq$ **0(a>0), f(0)=0 then**

1) *f* is differentiable at *x=0*

2) *f* is not differentiable at *x=0*

3) *f* is not continuous at *x=0*

4) $\lim_{x\to 0}f(x)$ does not exists

7. **Let f(x)=$\begin{cases} x^2, \text{ if } x \le x_0 \\ ax+b, \text{if } x > x_0 \end{cases}$. If f is differentiable at x_0 then**

1) $a = x_0, b = -x_0$ 2) $a = 2x_0, b = -x_0^2$

3) $a = 2x_0, b = x_0^2$ 4) $a = x_0, b = x_0^2$

8. **The left-hand derivative of f(x)=[x]sin π x at x = k, k is an integer is**

1) $(-1)^k(k-1)\pi$ 2) $(-1)^{k-1}(k-1)\pi$

3) $(-1)^k k\pi$ 4) $(-1)^{k-1}k\pi$

9. **Let** $f(x)=\begin{cases} x & x<1 \\ 2-x & 1\le x\le 2 \\ -2+3x-x^2 & x>2 \end{cases}$ **then f(x) is**

1) differentiable at $x=1$
2) differentiable at $x=2$
3) differentiable at $x=1$ *and* $x=2$
4) not differentiable at $x=0$

10. **Let f(x)=asin|x|+be$^{|x|}$ is differentiable when**

1) a = -b 2) a = b 3) a = 0 4) b = 0

11. $f(x)=\begin{cases} |x-4| & for\ x \ge 1 \\ x^3/2 - x^2 + 3x + 1/2 & for\ x<1 \end{cases}$, **then**

1) f(x) is continuous at x=1 and x=4
2) f(x) is differentiable at x=4
3) f(x) is continuous and differentiable at x=1
4) f(x) is only continuous at x=1

12. **The set of all points where the function** $f(x)=\sqrt{1-e^{-x^2}}$ **is differentiable is**

1) $(0,\infty)$ 2) $(-\infty,\infty)$

3) $(-\infty,\infty)-\{0\}$ 4) $(-\infty,\infty)-\{0,1,2\}$

13. **If f(x)=|x - a |+ | x + b|, x$\in$ R,b>a>0. Then**

1) $f'(a+)=1$ 2) $f'(a+)=0$

3) $f'(-b+)=0$ 4) $f'(-b+)=1$

14. **If [.] denote the greatest integer function and** $f(x)=[\tan^2 x]$, **then**

1) $\lim_{x\to0} f(x)$ does not exist
2) f is not continuous at x = 0
3) $f(x)$ is differentiable at x = 0
4) $f'(0)=1$

KEY

01) 2	02) 1	03) 2	04) 1	05) 1	06) 2
07) 2	08) 1	09) 2	10) 1	11) 1	12) 3
13) 3	14) 3				

SOLUTIONS

1. $f(x)=\begin{cases} -2x, \text{ if } x<-1 \\ 2, \text{ if } -1\le x<1 \\ 2x, \text{ if } x\ge1 \end{cases} \Rightarrow f'(x)=\begin{cases} -2, \text{ if } x<-1 \\ 0, \text{ if } -1\le x<1 \\ 2, \text{ if } x\ge1 \end{cases}$

$f'(-1^-)=-2,\ f'(-1^+)=0,\ f'(1^-)=0,\ f'(1^+)=2$

$\therefore$ f(x) is not differentiable at x=$\pm$ 1

2. $f'(0-)=\lim_{x\to0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0^-}\frac{\frac{x}{1+2^{1/x}}-0}{x}$

$=\frac{1}{1+2^{1/0-}}=\frac{1}{1+2^{-\infty}}=1$

$f'(0+)=\lim_{x\to0+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0+}\frac{\frac{x}{1+2^{1/x}}-0}{x}$

$=\frac{1}{1+2^{1/0+}}=\frac{1}{1+2^{\infty}}=0$

3. $L.H.D=\lim_{x\to0-}\frac{-xe^x}{x}=-1$

$R.H.D=\lim_{x\to0^+}\frac{xe^x}{x}=1$

4. $f(x)=\begin{cases} 2x^2 & x\ge0 \\ -2x^2 & x<0 \end{cases}$ is differentiable everywhere.

5. $f(x)=x^{3/2}-\sqrt{x^3+x^2}$

$\therefore$ L.H.D at $x=0$ does not exist as $D_f=[0,\infty)$

6. $\lim_{x\to0+} f(x)=\lim_{x\to0+}\frac{x(a^{1/x}-a^{-1/x})}{a^{1/x}+a^{-1/x}}=\lim_{x\to0+}\frac{x(1-a^{-2/x})}{1+a^{-2/x}}=0$

also, $\lim_{x\to0-} f(x)=\lim_{x\to0-} x\left(\frac{a^{2/x}-1}{a^{2/x}-1}\right)=0$

so f is continuous at $x=0$

$f'(0+)=\lim_{x\to0+}\frac{h(a^{1/h}-a^{-1/h})}{h(a^{1/h}+a^{-1/h})}=\lim_{x\to0+}\frac{1-a^{-2/h}}{1+a^{-2/h}}=1$

similarly $f'(0-)=-1$ hence ,f is not differentiable at $x=0$

7. Since f is differentiable so it is continuous also,

$=\lim_{h\to0}\frac{a+b-h+h-a-b}{h}=0$

therefore $x_0^2 = f(x_0) = \lim_{x \to x_0^+} f(x) = ax_0 + b$

also, $\lim_{h \to 0+} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0+} \frac{a(x_0+h) + b - x_0^2}{h}$

$= \lim_{h \to 0+} \frac{x_0^2 + ah - x_0^2}{h}$ $=a(\because x_0^2 = ax_0 + b)$

thus $a = f'(x_0-) = \lim_{h \to 0+} \frac{(x_0+h)^2 - x_0^2}{h} = 2x_0$

hence $x_0^2 = 2x_0^2 + b$ then $b = -x_0^2$

8. Clearly f(k)=0, so the left hand derivative is equal to $\lim_{h \to 0-} \frac{f(k+h) - f(k)}{h}$

$= \lim_{h \to 0-} \frac{[k+h]\sin(k+h)\pi}{h} = \lim_{h \to 0-} \frac{(k-1)\sin(k\pi + h\pi)}{h}$

$= \lim_{h \to 0-} \frac{(k-1)(-1)^k \sinh\pi}{h}$ (since $h < 0$) $= (k-1)(-1)^k \pi$

9. $f'(x) = \begin{cases} 1, & \text{if } x < 1 \\ -1, & \text{if } 1 \le x \le 2 \\ 3 - 2x, & \text{if } x > 2 \end{cases}$

$f'(1-) = 1,\ f'(1+) = -1,\ f'(2-) = -1,\ f'(2+) = -1$

$f'(2-) = f'(2-)$

10. f(x)=$\begin{cases} -a\sin x + be^{-x} & \text{if } x < 0 \\ a\sin x + be^{x} & \text{if } x > 0 \end{cases}$; $f'(x) = \begin{cases} -a\cos x - be^{-x} & \text{if } x < 0 \\ a\cos x + be^{x} & \text{if } x > 0 \end{cases}$

then $f'(0-) = -a - b$ and $f'(0+) = a + b$

If a=-b, then $f'(0-) = f'(0+)$

11. Since g(x)=|x| is a continuous function and $\lim_{x \to 1+} f(x) = 3 = \lim_{x \to 1-} f(x)$, so f is continous function. In particular f is continuous at x=1 and x=4) f is clearly not differentiable at x=4) Since g(x)=|x| is not differentiable at x=0. Now

$f'(1+) = \lim_{h \to 0+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0+} \frac{|-3+h| - 3}{h} = -1$

$f'(1-) = \lim_{h \to 0-} \frac{(1/2)(1+h)^3 - (1+h)^2 + 3(1+h) + (1/2) - 3}{h}$

$= \lim_{h \to 0-} \frac{(1/2)(h^3 + 3h^2 + 3h) - (h^2 + 2h) + 3h}{h} = \frac{5}{2}$

12. $f'(x) = \frac{xe^{-x^2}}{\sqrt{1 - e^{-x^2}}}$ is not differentiable only at x = 0

13. $f'(x) = \begin{cases} -2, & \text{if } x < -b \\ 0, & \text{if } -b \le x \le a \\ 2, & \text{if } x > a \end{cases}$; $f'(a+) = 2,\ f'(-b+) = 0$

14. $0 < x < \pi/4$, $[\tan^2 x] = 0$. Also $\tan^2 x$ is an even function

$\lim_{x \to 0+} \left(\frac{f(0+x) - f(0)}{x} \right) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = 0$

$\therefore$ f is continuous at x = 0 and differentiable x = 0. Also $f'(0) = 0$

EXERCISE - II

1. If f(x)=p|sin x| + qe$^{|x|}$+r|x|^3 and f(x) is differentiable at x=0, then

1) q+r=0; p is any real number
2) p+q=0; r is any real number
3) q=0, r=0; p is any real number
4) r=0,p=0; q is any real number

2. Let f(x)=$\frac{\sin 4\pi[x]}{1+[x]^2}$,where $[x]$ is the greatest integer less than or equal to x, then

1) f(x) is not differentiable at some points
2) f(x) exists but is different from zero
3) LHD (at x = 0) = 0, RHD (at x = 1) = 0
4) $f'(x)$=0 but f is not a constant function

3. If $f(x) = \begin{cases} -3x + 2, & x < 1 \\ \frac{1}{2}x^2 + 7, & x \ge 1 \end{cases}$, then which of the following is not true

1) $f'(1+) = 1$ 2) $f'(1-) = -3$

3) $f'(1-) = f'(1+) = 1$

4) f is not differentiable at x = 1

4. If $f(x) = \begin{cases} -\frac{1}{2}x^2, & \text{for } x < 1 \\ \frac{3}{2}x^2 + 1, & \text{for } x \ge 1 \end{cases}$, then

1) f is differentiable everywhere on R
2) $f'(1-) = -1$ and $f'(1+) = 0$
3) $f'(1-) = -1$ and $f'(1+) = 3$
4) $f'(1-) = 1$ and $f'(1+) = -1$

5. **The function given by $y=||x|-1|$ is differentiable for all real numbers except the points**

1) $\{0,1,-1\}$ 2) ± 1 3) 1 4) -1

6. If $f(x)=|x|+|\sin x|$ for $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, then its left hand derivative at x = 0 is **(Eam-2011)**

1) 0 2) -1 3) -2 4) -3

7. **Let** $f(x)=\begin{cases}\dfrac{\sin|x^2-5x+6|}{x^2-5x+6}, & x\neq 2,3\\ 1, & x=2\, or\, 3\end{cases}$

the set of all points where f is differentiable is

1) $(-\infty,\infty)$ 2) $(-\infty,\infty)\sim\{2\}$

3) $(-\infty,\infty)\sim\{3\}$ 4) $(-\infty,\infty)\sim\{2,3\}$

8. $f(x)=|\cos x|$ **is not differentiable for the points given by** $x=$

1) $\frac{\pi}{2}$ 2) $(2n+1)\pi,\ \forall n\in I$

3) $(2n+1)\frac{\pi}{2}\forall n\in I$ 4) 0

9. **Let h(x)=min{x,x^2} for x $\in\mathbb{R}$. Then which of the following is correct**

1) h is continuous for allx

2) h is differentiable for all x

3) h$'$(x)=1 for all x>1

4) h is not a differentiable at 2 values of x

10. **Let** $f(x)=\begin{cases}3^x & \forall|x|\le 1\\ 7-x & \forall 1<x<7\end{cases}$ **then f(x) is**

1) continuous $\forall$ 1 $\le$ x $\le$ 7 but not differentiable at x=1
2) continuous -1 $\le$ x<7 & differentiable at x=1
3) neither continuous in [-1,7)nor differentiable at x=1
4) continuous & differentiable at x=1

11. **If** $f(x+y)=2f(x)f(y)$ **all** $x,y\in R$ **where** $f'(0)=3$ **and** $f(4)=2$**, then** $f'(4)$ **is equal to**

1) 6 2) 12 3) 4 4)3

12. **Let** f(x+y)=f(x)f(y) and f(x)=1+(sin2x)g(x) **where g(x) is continuous. Then** f^1(x) **equals**

1) f(x)g(0) 2) 2f(x)g(0) 3) 2g(0) 4) 2f(0)

13. **Let a function y = f(x) be difined as** $x=2t-|t|,\ y=t^2+t|t|$ **,Where** $t\in R$ **then f(x) is**

1) Continuous and differentiable in $[-1,1]$

2) Continuous but not differentiable in $[-1,1]$

3) Continuous $[-1,1]$ and differentiable in (-1,1) only

4) Discontinuous on $[-1,1]$

14. **If f(x)=**$\begin{cases}[\cos\pi x], & x<1\\ |x-2|, & 1\le x<2\end{cases}$**, then f(x) is**

1) discontinuous and non-differentiable at x = -1 and x=1

2) continuous and differentiable at x=0

3) not differentiable at x= 1/2

4) continuous but not differentiable at x=0

15. **Let f(x)=**$\begin{cases}-1, & -2\le x<0\\ x^2-1, & 0<x\le 2\end{cases}$ **and g(x)=|f(x)|+f|x|**

then the number of points which g(x) is non differentiable, is

1) at most one point 2) 2
3) exactly one point 4) infinite

16. **Let f(x+y)=f(x)f(y) and f(x)=1+xg(x)G(x), where** $\lim_{x\to 0}g(x)=a$ **and** $\lim_{x\to 0}G(x)=b$**. Then** $f'(x)$ **is equal to**

1) 1+ab 2) ab 3) f(x) 4) abf(x)

17. **Let f be a differentiable function satisfying the condition** $f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)}$, **for all** $x,y(\neq 0)\in R$ **and** $f(y)\neq 0$**. If** $f'(1)=2$**, then** $f'(x)$ **is equal to**

1) $2f(x)$ 2) $\frac{f(x)}{x}$ 3) $2xf(x)$ 4) $\frac{2f(x)}{x}$

18. If $f : R \to R$ be a differentiable function, such that $f(x+2y) = f(x) + f(2y) + 4xy$ for all $x, y \in R$ then

1) $f'(1) = f'(0) + 1$ 2) $f'(1) = f'(0) - 1$

3) $f'(0) = f'(1) + 2$ 4) $f'(0) = f'(1) - 2$

KEY

01) 2	02) 3	03) 3	04) 3	05) 1	06) 3
07) 4	08) 3	09) 4	10) 3	11) 2	12) 2
13) 1	14) 3	15) 3	16) 4	17) 4	18) 4

SOLUTIONS

1. For $-\frac{\pi}{2} < x \le 0$, f(x)= $-p\sin x + qe^{-x} - rx^3$, so

$= -p - q$

For $0 < x < \frac{\pi}{2}$, f(x)=p sin x+qe^x+rx^3,

$f'(0+) = \lim_{x\to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0+}\left[\frac{p\sin x}{x} + q\left(\frac{e^x-1}{x}\right) - rx^2\right]$ =p+q

For f to be diffentiable at x=0,we must have p+q = - p -q$\Rightarrow$ p + q =0.

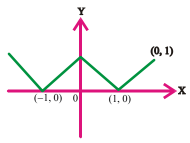
2. We have $\frac{\sin 4\pi[x]}{1+[x]^2} = 0, \ \forall x$

[$\because 4\pi[x]$ is an integral multiple of 'π']

$\Rightarrow f(x) = 0$ for all x

3. $f'(x) = \begin{cases} -3, & \text{if } x < 1 \\ x, & \text{if } x \ge 1 \end{cases}$

4. $f'(x) = \begin{cases} -x & \text{if } x < 1 \\ 3x & \text{if } x \ge 1 \end{cases}$

5. 

6. $x \to 0^-, f(x) = -x - \sin x$

7. The function is clearly differentiable except possible at x =2,3

$f'(2+) = \lim_{h\to 0+} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0+} \frac{\sin h(1-h) + h(1-h)}{h^2(-1+h)}$

$= -\lim_{h\to 0+}\left(\frac{\sin h(1-h)}{h(1-h)} + 1\right)\frac{1}{h}$ which is does not exist

8. f is not differentiable at all points where cosx=0

9. $h(x) = \begin{cases} x, & x \ge 1 \\ x^2, & 0 \le x < 1 \\ x, & x < 0 \end{cases}$

From the graph it is clear that h is continuous. Also h is differentiable except possible at x=0 and 1

$h'(x) = \begin{cases} 1, & x > 1 \\ 2x, & 0 < x < 1 \\ 1 & x < 0 \end{cases}$

for x=1, $h'(1+) = \lim_{t\to 0+} \frac{h(1+t)-h(1)}{t} = \lim_{t\to 0+} \frac{1+t-1}{t} = 1$

but $h'(1-) = \lim_{t\to 0+} \frac{h(1-t)-h(1)}{t} = \lim_{t\to 0+} \frac{(1-t)^2-1}{t} = -2$

so h is not differentiable at 1

similarly $h'(0+) = 0$ but $h'(0-) = 1$

10. $f(x) = \begin{cases} 3^x & \forall |x| \le 1 \\ 7-x & 1 < x < 7 \end{cases}$

$f'(1^-) = 3\log 3$, $f'(1+) = 0-1 = -1$ $f'(1-) \ne f'(1+)$

$\therefore$ f(x) is not differentiable at x=1

Hence f is not differentiable at x=1

11. $f'(4) = \lim_{h\to 0} \frac{f(4+h)-f(4)}{h}$

$\Rightarrow f'(4) = \lim_{h\to 0} \frac{f(4+h)-f(4+0)}{h}$

$\Rightarrow f'(4) = \lim_{h\to 0} \frac{2f(4)f(h)-2f(4)f(0)}{h}$

$\Rightarrow f'(4) = \lim_{h\to 0} 2f(4)\left[\frac{f(h)-f(0)}{h-0}\right]$

$f'(4) = 4\lim_{h\to 0} \frac{f(h)-f(0)}{h-0}$

$= 4f'(0) = 4 \times 3 = 12$

12. $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f(x)f(h)-f(x)}{h}$

$$=f(x)\lim_{h\to0}\frac{f(h)-1}{h}=f(x)\lim_{h\to0}\frac{1+(\sin 2h)g(h)-1}{h}$$

$$=f(x)\lim_{h\to0}\frac{\sin 2h}{h}.\lim_{h\to0}g(h)=2f(x)g(0)$$

13. When $t\ge0$,

we have $x=2t-t$ and

$y=t^2+t^2=2t^2 \Rightarrow y=2x^2, x\ge0$

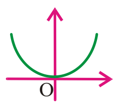
When t<0, we have

x = 2t +t = 3t and y = t^2 - t^2 = 0 $\Rightarrow$ y =0 for all x <0

14. We have, $f(x)=\begin{cases}[\cos \pi x] & x<1\\ |x-2| & 1<x<2\end{cases}$

$$=\begin{cases}2-x, & 1\le x<2\\ -1, & 1/2<x<1\\ 0, & 0<x\le 1/2\\ 1, & x=0\\ 0, & -1/2\le x<0\\ -1, & -1/2<x<-1/2\end{cases}$$

It is evident from the definition that f(x) is discontinuous at x=1/2

15. 

$|f(x)|=\begin{cases}1, & -2\le x\le 0\\ 1-x^2, 0<x\le1\\ x^2-1, 1<x\le2\end{cases}$

and $f|(x)|=x^2-1, -2\le x\le 2$

$$\therefore g(x)=\begin{cases}x^2, & -2\le x\le0\\ 0, & 0<x\le1\\ 2(x^2-1), 1<x\le2\end{cases}$$

(by adding the function in proper domains)

$\therefore g(x)$ is differentiable everywhere except at x=1

16. $f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$

$[\because f(x+y)=f(x)f(y)]$

$$=f(x)\lim_{h\to0}\frac{f(h)-1}{h}=f(x)\lim_{h\to0}\frac{1+hg(h)G(h)-1}{h}$$

$$=f(x)\lim_{h\to0}g(h)G(h)=f(x)\lim_{h\to0}G(h)\lim_{h\to0}g(h)=abf(x)$$

17. $f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)}$, replacing x and y both by 1,

we get

$$f(1)=\frac{f(1)}{f(1)}\Rightarrow f(1)=1$$

Now $f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$

$$\Rightarrow f'(x)=f(x)\lim_{h\to0}\left\{\frac{\frac{f(x+h)}{f(x)}-1}{h}\right\}$$

$$\Rightarrow f'(x)=f(x)\lim_{h\to0}\left\{\frac{f\left(\frac{x+h}{x}\right)-1}{h}\right\}$$

$$\Rightarrow f'(x)=\frac{f(x)}{x}\lim_{h\to0}\frac{f\left(1+\frac{h}{x}\right)-f(1)}{\frac{h}{x}}$$

$$\Rightarrow f'(x)=\frac{f(x)}{x}f'(1)=\frac{2f(x)}{x}$$

18. $f(x+2y)=f(x)+f(2y)+4xy$ for all $x,y\in R$ putting x = y = 0, we get $f(0)=0$

Now, $f(x+2y)=f(x)+f(2y)+4xy$

$$\Rightarrow\frac{f(x+2y)-f(x)}{2y}=2x+\frac{f(2y)}{2y}$$

$$\Rightarrow\lim_{y\to0}\frac{f(x+2y)-f(x)}{2y}$$

$$=\lim_{y\to0}\left\{2x+\frac{f(2y)-f(0)}{2y}\right\}$$

$\Rightarrow f'(x)=2x+f'(0)$ for all x

$\Rightarrow f'(1)=2+f'(0)$

EXERCISE - III

1. **Let f(x)=$\begin{cases} x^n \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then f(x) is continuous but not differentiable at x=0 if**

1) $n \in (0,1]$ 2) $n \in [1,\infty)$
3) $n \in (-\infty,0)$ 4) n=0

2. **The values of a and b such that the function of defined as $f(x)=\begin{cases} ax^2-b, & |x|<1 \\ -\frac{1}{|x|}, & |x| \geq 1 \end{cases}$ is differentiable are**

1) a=1,b=-1 2) a=$\frac{1}{2}$,b=$\frac{1}{2}$
3) a=$\frac{1}{2}$,b=$\frac{3}{2}$ 4) a=$\frac{3}{2}$,b=$\frac{3}{2}$

3. **Let f(x) be defined by $f(x)=\begin{cases} \sin 2x & \text{if } 0<x\leq \pi/6 \\ ax+b & \text{if } \pi/6<x\leq 1 \end{cases}$**

The values of a and b such that f and f' are continuous, are

1) a = 1, b = $\frac{1}{\sqrt{2}}+\frac{\pi}{6}$ 2) a = $\frac{1}{\sqrt{2}}$, b = $\frac{1}{\sqrt{2}}$
3) a = 1, b = $\frac{\sqrt{3}}{2}-\frac{\pi}{6}$ 4) a = $\frac{\sqrt{3}}{2}$, b = $\frac{\sqrt{3}}{2}+\frac{\pi}{6}$

4. $f(x)=\begin{cases} b\sin^{-1}\left(\frac{x+c}{2}\right), & -\frac{1}{2}<x<0 \\ \frac{1}{2} & at\, x=0 \\ \frac{e^{\frac{ax}{2}}-1}{x} & 0<x<\frac{1}{2} \end{cases}$

If f(x) is differentiable at x =0 and $|c|<\frac{1}{2}$ then

1) a = 1 and $64b^2 + c^2 = 4$
2) a = 0 and $64b^2 + c^2 = 2$
3) a = 2 and $64b^2 + c^2 = 1$
4) a = 3 and $64b^2 + c^2 = 3$

5. **If f(x)=$\begin{cases} \frac{|x+2|}{\tan^{-1}(x+2)}, & x\neq -2 \\ 2, & x=-2 \end{cases}$, then f(x) is**

1) continuous at x = -2
2) not coninuous at x = -2
3) differentiable at x = -2
4) continuous but not derivable at x = -2

6. **Let** $f(x)=\begin{cases} (x-1)\sin\frac{1}{x-1}, & if\ x\neq 1 \\ 0 & ,if\ x=1 \end{cases}$
Then which one of the following is ture? [AIEEE 2008]

1) f is neither differentiable at $x=0$ nor at $x=1$
2) f is differentiable at $x=0$ and at $x=1$
3) f is differentiable at $x=0$ but not at $x=1$
4) f is differentiable at $x=1$ not at $x=0$

7. **If x+4|y|=6y, then y as a function of x is**

1) continuous at x=0 2) derivable at x=0
3) $\frac{dy}{dx}=\frac{1}{2}$ for all x 4) $\frac{dy}{dx}$=0 for all x

8. **If the function f(x)=$\left[\frac{(x-5)^3}{A}\right]$sin(x-5) +acos(x-2), where [.] denotes the greatest integer function and a$\in$R ,is continuous and differentiable in (7,9) then**

1) A$\in$[8,64] 2) A$\in$(0,8)
3) A$\in$[64,∞) 4) A$\in$(0,0)

9. **Let f(x)=$[x]^2+\sqrt{\{x\}}$,where [ ] & { } respectively denotes the greatest integer and fractional part of functions, then**

1) f(x) is continuous at all integral points
2) f(x) is not differentiable $\forall x\in I$
3) f(x) is discontinuous as $x\in$ I-{1}
4) f(x) is continuous & differentiable at x=0

10. **Suppose $f(x)$ is differentiable at $x=1$ and $\lim_{h\to 0}\frac{1}{h}f(1+h)=5$, then $f'(1)$ equals**

1) 3 2) 4 3) 5 4) 6

11. **The set of poits where $f(x)=\frac{x}{1+|x|}$ is differentiable is**

1) $(-\infty,0)\cup(0,\infty)$ 2) $(-\infty,-1)\cup(-1,\infty)$
3) $(-\infty,\infty)$ 4) $(0,\infty)$

12. **Let $f:R\to R$ be a funtction defined by $f(x)=\min\{x+1,|x|+1\}$, Then which of the following is true? [AIEEE - 2007]**

1) $f(x)$ is differentiable everywhere
2) $f(x)$ is not diffferentiable at $x=0$
3) $f(x)\geq 1$ for all $x\in R$

4) $f(x)$ is not differentaible at $x=1$

13. If funcion $f(x)$ is differentiable at $x=a$, then $\lim_{x\to a}\frac{x^2 f(a)-a^2 f(x)}{x-a}$ is :

[AIEEE- 2011]

1) $-a^2 f'(a)$ 2) $a\ f(a)-a^2 f'(a)$

3) $2a\ f(a)-a^2 f'(a)$ 4) $2a\ f(a)+a^2 f'(a)$

14. If $f:(-1,1)\to R$ be a differentiable function with $f(0)=-1$ and $f'(0)=1$. Let $g(x)=\left[f(2f(x)+2)\right]^2$, then $g'(0)$ is equal to (AIEEE-2010)

1) 4 2) -4 3) 0 4) -2

15. Let f(x)=$\begin{cases} A+\sin^{-1}(x+B), & \forall x\geq 1 \\ x, & \forall x<1 \end{cases}$ is differentiable then

1) A=-1,B=-1 2) A=1,B=-1

3) A=B=1 4) A=0,B=1

16. Let f(x) be differentiable function such that $f\left(\frac{x+y}{1-xy}\right)=f(x)+f(y)$ $\forall$ x and y. If $\underset{x\to 0}{\text{lt}}\frac{f(x)}{x}=\frac{1}{3}$ then $f'(1)$ equals

1) $\frac{1}{4}$ 2) $\frac{1}{6}$ 3) $\frac{1}{12}$ 4) $\frac{1}{8}$

17. if $f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}$ $\forall x,y\in R$ and $f'(0)=-1, f(0)=1,$ *then* $f(2)=$

1) $\frac{1}{2}$ 2) $-\frac{1}{2}$ 3) 1 4) -1

18. Suppose that f is a differentiable function with the property that f(x+y) =f(x)+f(y)+xy and $\lim_{h\to 0}\frac{1}{h}f(h)=3$, then

1) f is a linear function 2) f(x)=3x+x^2

3) f(x)=3x+$x^2/2$ 4) f(x)=3x-$x^2/2$

19. If f(x+y+z) = f(x).f(y).f(z) for all x,y,z and f(2)=5, $f(0)$=3, then $f'(2)$ equals

1) 15 2) 9 3) 16 4) 6

20. Given that f(x) is a differentiable function of x and that f(x).f(y)=f(x)+f(y)+f(xy)-2 and that f(2)=5. Then $f^1(3)$ is equal to

1) 6 2) 24 3) 15 4) 19

21. If $f(x)=|\cos x-\sin x|$, then $f'\left(\frac{\pi}{4}\right)=$

1) $\sqrt{2}$ 2) $-\sqrt{2}$ 3) 0

4) does not exists

22. Let f(x) be a polynomial of degree two which is positive for all x $\in$ R.

$g(x)=f(x)+f'(x)+f''(x)+f'''(x)+x^2 f^{iv}(x)$,

then for any real x

1) g(x)<0 2) g(x)>0 3) g(x)=0 4) g(x) $\geq$ 0

23. Which of the following function is differentiable at *x=0*

1) $cos(|x|)+|x|$ 2) $cos(|x|)-|x|$

3) $sin(|x|)+|x|$ 4) $sin(|x|)-|x|$

24. The function f(x)=|x³| is

1) differentiable everywhere

2) continuous but not differentiable at x=0

3) not a continuous function

4) a function with range $(0,\infty)$

KEY

01) 1	02) 3	03) 3	04) 1	05) 2	06) 3
07) 1	08) 3	09) 3	10) 3	11) 3	12) 1
13) 3	14) 2	15) 2	16) 2	17) 4	18) 3
19) 1	20) 1	21) 4	22) 2	23) 4	24) 1

SOLUTIONS

1. since f(x) is continuous at x=0, therefore

$\lim_{x\to 0} f(x) = f(0) = 0 \Rightarrow \lim_{x\to 0} x^n \sin\left(1/x\right) = 0 \Rightarrow n > 0$

f(x) is differentiable at x=0 if

$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$ exists finitely

$\Rightarrow \lim_{x\to 0}\frac{x^n \sin(1/x)-0}{x}$ exists finitely

$\Rightarrow \lim_{x\to 0} x^{n-1}\sin(1/x)$ exists finitely $\Rightarrow n-1>0 \Rightarrow n>1$

If $n \le 1$, then $\lim_{x\to 0} x^{n-1}\sin(1/x)$ does not exist and hence f(x) is not differentiable at x=0

hence f(x) is continuous but not differentiable at x=0 for $0 < n \le 1$, i.e $n \in (0,1]$

2. Since every differentiable function is continuous, so we must have

$\lim_{x\to 1-} f(x) = f(1) \Rightarrow a-b=-1$

for f to be differentiable, $f'(1-) = f'(1+)$

$\Rightarrow \lim_{h\to 0-}\left[\frac{a(1+h)^2-b+1}{h}\right] = \lim_{h\to 0-}\left[\frac{-1|1+h|+1}{h}\right]$

$= \lim_{h\to 0-}\left[\frac{a(2h+h^2)}{h}\right] = \lim_{h\to 0+}\frac{h}{h(1+h)}$ (as a-b=-1)

$\Rightarrow 2a = 1$, Hence $a = 1/2$ and $b = 3/2$

3. $f(x)=\begin{cases}\sin 2x & \text{if } 0<x\le \pi/6\\ ax+b & \text{if } \pi/6<x\le 1\end{cases}$,

then $f'(x)=\begin{cases}2\cos 2x & \text{if } 0<x\le \pi/6\\ a & \text{if } \pi/6<x\le 1\end{cases}$

f and f' are continuous

$\lim_{x\to \pi/6-} f(x) = \lim_{x\to \pi/6+} f(x) \Rightarrow \lim_{x\to \pi/6}\sin 2x = \lim_{x\to \pi/6} ax+b$

$\frac{\sqrt{3}}{2} = \frac{a\pi}{6}+b \Rightarrow b = \frac{\sqrt{3}}{2}-\frac{\pi}{6}$

$\lim_{x\to \pi/6-} f'(x) = \lim_{x\to \pi/6+} f'(x)$ then $\lim_{x\to \pi/6} 2\cos 2x = \lim_{x\to \pi/6} a$

i.e. a=1

4. Find $f'(x)$ and $f'(0^+) = f'(0^-)$

5. $\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0}\frac{|-2-h+2|}{\tan^{-1}(-2-h+2)}$

$= \lim_{h\to 0}\frac{h}{\tan^{-1}(-h)} = \lim_{h\to 0}\frac{-h}{\tan^{-1}(h)} = -1$

and $\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(-2+h) = \lim_{h\to 0}\frac{|-2+h+2|}{\tan^{-1}(-2+h+2)}$

$= \lim_{h\to 0}\frac{h}{\tan^{-1}(h)} = 1 \qquad \therefore \lim_{x\to 2-} f(x) \ne \lim_{x\to 2+} f(x)$

so, f is neither continuous nor differentiable at x = – 2

6. $Lf'(1) = \lim_{h\to 0}\frac{(1-h-1)\sin\left(\frac{1}{1-h-1}\right)-0}{-h}$

$= -\lim_{h\to 0}\sin\frac{1}{h}$ similiraly $Rf'(1) = \lim_{h\to 0}\sin\frac{1}{h}$

'f' is not differentiable at x=1, Clearly 'f' is differentiable at x =0.

As $Lf'(0) = Rf'(0) = Cos1 - Sin1$

7. We have, x+4|y|=6y

$\Rightarrow \begin{cases}x-4y=6y, & \text{if } y<0\\ x+4y=6y, & \text{if } y\ge 0\end{cases}$

$\Rightarrow y=\begin{cases}\frac{1}{2}x, & \text{if } x\ge 0\\ \frac{1}{10}x, & \text{if } x<0\end{cases} \Rightarrow y=f(x)=\begin{cases}\frac{1}{2}x, & \text{if } x\ge 0\\ \frac{1}{10}x, & \text{if } x<0\end{cases}$

clearly, $y=f(x)$ is continuous at x=0 but it is not differentiable at x=0

8. [x] is not conitnuous and differentiable at integral values (points)

So f(x) is continuous and differentiable in (7,9)

if $\left[\frac{(x-5)^3}{A}\right]=0 \Rightarrow A \ge (9-5)^3 \Rightarrow A \ge 64 \therefore A \in [64,\infty)$

9. If $k \in I$

$\mathop{Lt}_{x\to k+} f(x) = k^2 - 0 \Rightarrow \mathop{Lt}_{x\to k-} f(x) = (k-1)^2+1$

Again $\mathop{Lt}_{x\to k-} f(x) = \mathop{Lt}_{x\to k+} f(x) = f(k)$

$k^2=(k-1)^2+1 \Rightarrow 2k=2$ then k=1

i.e., f(x) is continuous at k=1 and no other integral point.

So f(x) is discontinuous for all integral points except x=1

10. $f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$

given that $\lim_{h\to 0}\frac{f(1+h)}{h} = 5$ and hence

$f(1)=0$

$f^1(1) = \lim_{h\to 0}\frac{f(1+h)}{h} = 5$

11. $f'(x)=\begin{cases}\dfrac{x}{(1-x)^2}; x<0\\ \dfrac{x}{(1+x)^2}; x\ge 0\end{cases}$

12. $f(x)=x+1, \forall x\in R$

13. $\lim_{x\to a}\frac{x^2f(a)-a^2f(x)}{x-a}$

$\lim_{x\to a}\frac{2xf(a)-a^2f'(x)}{1}$

$=2af(a)-a^2f'(a)$

14. $g'(x)=2[f(2f(x)+2)]\times f'((2f(x))+2)\times 2f'(x)$

$g'(0)=[f(2f(0)+2)]\times f'((2f(0))+2)\times 2f'(0)$

$=2f(0)\times f'(0)\times 2f'(0)=2(-1)\times 1\times 2\times 1=-4$

15. As $x\ge 1$ $\sin^{-1}(x+B)$ is defined when B = -1 from the options.f is differentiable $\Rightarrow$ f is continuous.

16. $f\left(\frac{x+y}{1-xy}\right)=f(x)+f(y)\Rightarrow f(x)=A\tan^{-1}x$

Now, $\underset{x\to 0}{lt}\frac{f(x)}{x}=A\underset{x\to 0}{lt}\frac{\tan^{-1}x}{x}$, $\therefore A=\frac{1}{3}$ $\therefore f(x)=\frac{1}{3}\tan^{-1}x$

17. Take $f(x) = ax+b$

18. $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f(x)+f(h)+xh-f(x)}{h}$

$= =\lim_{h\to 0}\frac{1}{h}f(h)+x=3+x$

Hence f(x)=3x+$x^2/2$+c. Putting x=y=0 in the given equation, we have f(0)=f(0)+f(0)+0 $\Rightarrow$ f(0)=0. Thus c=0 and f(x)=3x+$x^2/2$

19. We have, $f(x+y+z)=f(x)f(y)f(z)$ for all x,y,z

$\Rightarrow f(0)=f(0)f(0)f(0)$ (putting x=y=z=0)

$\Rightarrow f(0)\{1-(f(0))^2\}=0\Rightarrow f(0)=1$

($\because f(0)=0\Rightarrow f(x)=0$ for all x)

Putting z=0 and y=2, we get

$f(x+2)=f(x)f(2)f(0)\Rightarrow f(x+2)=5f(x)$ for all x

$\Rightarrow f'(2)=5f'(0)=5\times 3=15$

hence f is differentiable everywhere

20. We have, $f(x).f(y)=f(x)+f(y)+f(xy)-2$

$\Rightarrow f(x).f(1/x)=f(x)+f(1/x)+f(1)-2$

$\Rightarrow f(x).f(1/x)=f(x)+f(1/x)$

(since $f(1)=2$ putting x=y=1)

$\Rightarrow f(x)=x^n+1\Rightarrow f(2)=x^2+1$ (since f(2)=5)

$\Rightarrow n=2$

$\therefore f(x)=x^2+1\Rightarrow f(3)=10$

21. $f(x)=\begin{cases}\cos x-\sin x \text{ for } x\in(0,\pi/4)\\ \sin x-\cos x \text{ for } x\in(\pi/4,\pi/2)\end{cases}$

22. Let f(x)=ax^2+bx+c. As f(x)>0 for all x $\in$ R, we must have, a>0 and b^2-4ac<0

$g(x)=ax^2+bx+c+(2ax+b)+2a+0+(x^2).0$

$=ax^2+(b+2a)x+b+c+2a$

Discriminant of g(x)=$(b+2a)^2$-4a(b+c+2a)

= $-4a^2+(b^2-4ac)<0$

Thus g(x)>0 for x $\in$ R

23. $\cos|x|=\cos x$ is differentialbe at x=0, but $|x|$ is not differentiable at x=0, Hence 1 & 2 options are not correct.

$f(x)=\sin|x|-|x|=\begin{cases}-\sin x+x & x<0\\ \sin x-x & x\ge 0\end{cases}$

is differentiable at x=0

24. f(x)=$|x^3|$ then $f(x)=\begin{cases}-x^3 \text{ if } x<0\\ 0 \text{ if } x=0\\ x^3 \text{ if } x>0\end{cases}$

$f'(0-)=0; f'(0+)=0\Rightarrow f'(0-)=f'(0+)$

then f is differentiable at x=0

i.e f is continuous at x=0

JEE MAINS QUESTIONS

1. Let S = {t ε R : f(x) = |x - π|. ($e^{|x|}$ - 1) sin |x| is not differentable at t}. Then the set S is equal to [2018]

1) {0, π} 2) ϕ 3) {0} 4) {π}

2. Let f(x) = $f(x) = \begin{cases} -1, & -2 \le x < 0 \\ x^2 - 1, & 0 \le x \ge 2 \end{cases}$ and g(x) = $|f(x)| + f(|x|)$. then in the interval (-2, 2), g is [2019]

1) non continuous

2) differentiable at all points

3) not differentiable at two points

4) non differentiable at one point

3. Let k be the set of all realvalues of x where the function f(x) = sin |x| - |x|+ 2(x - π) cos |x| is n of differentiable. Then the set k is equal to [2019]

1) {0, π} 2) ϕ 3) {π} 4) {0}

4. Let S be the set of all points in (- π, π) at which the function, f(x) = min{sinx, cosx} is not difterentiable . Then s is a sub set of which of the following [2019]

1) $\{\frac{-\pi}{2}, \frac{-\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}\}$ 2) $\{\frac{-3\pi}{4}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{4}\}$

3) $\{\frac{-\pi}{4}, 0, \frac{\pi}{4}\}$ 4) $\left\{\frac{-3\pi}{4}, \frac{-\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{4}\right\}$

5. Let s be the set of points where the function f(x) = |2-|x-3|| x ε R is not differentable then $\sum_{x \varepsilon s} f(f(x))$ is equal to [2020]

6. Suppose a differentiable function f(x) satisfies the identity

f(x+y) = f(x) +f(y) +xy^2 + x^2y for all real x and y. If $\underset{1 \to x}{Lt} \frac{t^2 f^2(x) - x^2 f(t)}{t - x} = 0$ if f(x) =1 then x is equal to [2020]

7. Let f:(0, ∞) $\to$(0, ∞) be a differensiable function such that f(1) = e and

$\underset{t \to x}{Lt} \frac{t^2 f^2(x) - x^2 F(t)}{t - x} = 0$. If f(x) = 1 then x is equal to [2020]

1) 2e 2) e 3) $\frac{1}{2e}$ 4) $\frac{1}{e}$

KEY

1) 2 2) 4 3) 2 4) 4 5) 3

6) 10 7) 4

SOLUTIONS

1) f(x) = $|x-\pi|\left(e^{|x|}-1\right)\sin|x|$ at x = 0, π

f(0^+) = $\underset{x\to 0^+}{Lt}\left(\frac{|h-\pi|\left(e^{|h|}-1\right)\sin|h|}{h}\right)$

= $\underset{h\to 0^+}{Lt}\left(\frac{|h-\pi|\left(e^{h}-1\right)\sin h}{h}\right)$ = 0

$$f\left(0^-\right)=\underset{h\to 0^+}{Lt}\left(\frac{|-h-\pi|\left(e^{|h|}-1\right)\sin|-h|}{-h}\right)=0$$

$$f\left(\pi^+\right)=\underset{h\to 0^+}{Lt}\left(\frac{|h|\left(e^{|h+\pi|}-1\right)\sin|\pi+h|}{h}\right)$$

$$=\underset{h\to 0^+}{Lt}\left(\frac{-h\left(e^{h+\pi}-1\right)\sinh}{h}\right)=0$$

$$f\left(\pi^-\right)=\underset{h\to 0^+}{Lt}\left(\frac{|-h|\left(e^{|h-\pi|}-1\right)\sinh}{h}\right)=0$$

$\therefore$ f(x) is differentiable for all x $\in$ R

2. $\left|f(x)\right|=\begin{cases}1 & -2\le x<0\\ 1-x^2 & 0\le x\le 1\\ x^2-1 & 1\le x\le 2\end{cases}$ if |x| = x^2 - 1

x $\in$[-2, 2]

g(x) = $=\begin{cases}x^2 & x\in[-2,0]\\ 0 & x\in[0,1)\\ 2\left(x^2-1\right) & x\in[1,2]\end{cases}$

g^1(x) = $=\begin{cases}2x & x\in[-2,0]\\ 0 & x\in[0,1)\\ 4x & x\in[1,2]\end{cases}$

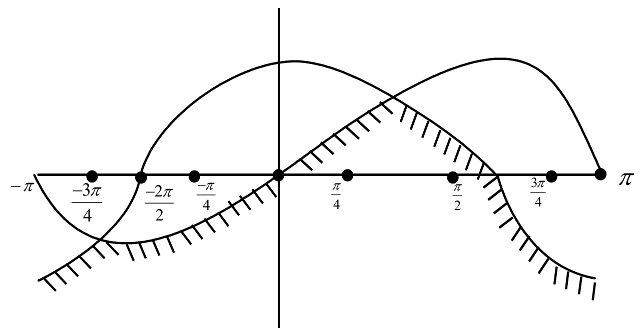
not differentiable at x = 1

3. f(x) - sin |x| -|x| + 2(x-π) cosx

$\therefore$ sin |x| -|x| is differentiable function at x = 0

$\therefore$ k = ϕ

4.



non - differentiable at x = $\frac{\pi}{4}$, $\frac{-3\pi}{4}$

5. $\therefore$ f(x) is non differentiable at x = 1, 3, 5

$$\sum f\left(f(x)\right)=f\left(f(1)+f\left(f(3)\right)\right)+f\left(f(5)\right)$$

= 1 + 1 + 1 = 3

6. $f(x + y) = f(x)+f(y) + xy^2 + x^2y$

Differentiate w.r.t. x

$f^1(x+y) = f^1(x) + y^2+2xy$

put y = -x

$f^1(0) = f^1(x)+x^2 - 2x^2 \Rightarrow f^1(0) = f^1(x)-x^2$ ......[1]

$\underset{x\to 0}{Lt}\frac{f(x)}{x} = 1 \Rightarrow f^1(0) = 1$ ....[2]

from (1) and (2) $f^1(x) = 1 + x^2$

$f^1(3) = 1+9 = 10$

7. $\underset{t\to x}{Lt}\frac{t^2 f^2(x) - x^2 f^2(t)}{t-x}$ = 0 using L - M

opointal rule

$\underset{t\to x}{Lt}\frac{2t + f^2(x) - 2x^2 f(t) f^1(t)}{1} = 0$

$\Rightarrow 2x f^2(x) - 2x^2 f(x) f^1(x) = 0$

$f(x) = xf^1(x) \Rightarrow \frac{f^1(x)}{f(x)} = \frac{1}{x}$

Integrating on both sides we get

log [f(x)\ = log x + logc

f(x) = xc

$\therefore$ f(1) = c $\Rightarrow$ c = e so f(x) = ex

when f(x) = 1 = ex

$x = \frac{1}{e}$.

LIMITS, CONTINUTITY & DIFFERENTIABILITY

ADVANCED LEVEL QUESTIONS

SINGLE ANSWER TYPE QUESTIONS

1. **The integral value of n for which**

$$\lim_{x\to 0}\frac{\cos^2 x - \cos x - e^x \cos x + e^x - \left(\frac{x^3}{2}\right)}{x^n}$$ **is finite and non zero is**

A) 2 B) 4 C) 5 D) 6

2. **The integer n for which** $\lim_{x\to 0}\frac{(\cos x - 1)(\cos x - e^x)}{x^n}$ **is a finite nonzero number is** **[IIT - 2002]**

A) 1 B) 2 C) 3 D) 4

3. **If** $A_i = \frac{x-a_i}{|x-a_i|},\ i = 1,2,3,....,n$ **and** $a_1 < a_2 < a_3 < ..... < a_n$,

then $\lim_{x\to a_m}(A_1A_2....A_n), 1\le m\le n$

A) is equal to $(-1)^m$ B) is equal to $(-1)^{m+1}$

C) is equal to $(-1)^{m-1}$ D) Does not exist

4. **The value of** $\lim_{x\to\infty}\left\{\frac{x}{x+\frac{\sqrt[3]{x}}{x+\frac{\sqrt[3]{x}}{x+\sqrt[3]{x}}...\infty}}\right\}$ **is**

A) 1 B) 0 C) 2 D) $\frac{1}{2}$

5. **If** $\lim_{x\to 0}\left(x^{-3}\sin 3x + ax^{-2} + b\right)$ **exists and is equal to 0, then**

A) $a = -3$ and $b = \frac{9}{2}$

B) $a = 3$ and $b = \frac{9}{2}$

C) $a = -3$ and $b = -\frac{9}{2}$

D) $a = 3$ and $b = -\frac{9}{2}$

6. **Let** $\alpha(a)$ **and** $\beta(a)$ **be the roots of the equation**

$$\left(\sqrt[3]{1+a}-1\right)x^2+\left(\sqrt{1+a}-1\right)x+\left(\sqrt[6]{1+a}-1\right)=0$$

where $a > -1$. **The** $\lim_{a\to 0^+}\alpha(a)$ **and** $\lim_{a\to 0^+}\beta(a)$ **are** **[IIT-2012]**

A) $-\frac{5}{2}$ and 1 B) $-\frac{1}{2}$ and -1

C) $-\frac{7}{2}$ and D) $-\frac{9}{2}$ and 3

7. Which of the following is differentiable at $x = 0$? **[IIT - 2000]**

A) $\cos(|x|)+|x|$ B) $\cos(|x|)-|x|$

C) $\sin(|x|)+|x|$ D) $\sin(|x|)-|x|$

8. **Let** $f\left[0,\frac{\pi}{2}\right]\to R$ **be a function defined by f(x)** $= \max\left\{\sin x, \cos x, \frac{3}{4}\right\}$**, then number of points where f(x) is non differentiable is**

A) 1 B) 2 C) 3 D) 0

9. **Let** $f(x) = [3+2\cos x], x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$**, where [.] denotes the greatest integer function. Then number of points of discontinuity of** $f(x)$ **is**

(A) 3 (B) 2 (C) 5 (D) 6

10. **A function f : R ® R satisfies the equation f(x) f(y) – f(xy) = x + y $\forall$ x, y Î R and f(1) > 0, then**

A) $f(x)\,f^{-1}(x) = x^2 - 4$ B) $f(x)\,f^{-1}(x) = x^2 - 6$

C) $f(x)\,f^{-1}(x) = x^2 - 1$ D) $f(x)\,f^{-1}(x) = x^2 + 6$

11. **The function f(x) = $[x]^2$ - $[x^2]$ (where [x] is the greatest integer less than or equal to x), is discontinuous at : [IIT - 1999]**

A) all integers

B) all integers except 0 and 1

C) all integers except 0

D) all integers except 1

12. **(i) The left hand derivative of,**

f(x) = $[x]\sin(\pi x)$ at x = k, k an integer ([.] denotes G.I.F) is [IIT - 2000]

A) $(-1)^k(k-1)\pi$ B) $(-1)^{k-1}(k-1)\pi$

C) $(-1)^k k\pi$ D) $(-1)^{k-1}k\pi$

13. **The domain of the derivative of the function**

f(x) = $\begin{cases} \tan^{-1} x & \text{if } |x| \le 1 \\ \frac{1}{2}(|x|-1) & \text{if } |x| > 1 \end{cases}$ **is [IIT - 2002]**

A) R - {0} B) R - {1}

C) R - {-1} D) R - {-1, 1}

14. **Let f(x) = | | x | – 1 | , then points where f(x) is differentiable is (are) [IIT - 2005]**

A) 0, ±1 B) ±1 C) 0 D) 1

15. **f ¢¢(x) = – f(x) where f(x) is a continuous double differentiable function & g(x) = f ¢(x). If**

F(x) = $\left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$ **and**

F(5) = 5, then F(10) is [IIT - 2006]

A) 0 B) 5 C) 10 D) 25

16. **Number of points, where the function $f(x) =$**

Max $\left\{\operatorname{sgn}(x), -\sqrt{(9-x^2)}, x^3\right\}$ **is continuous**

but not differentiable is:

A) 6 B) 5 C) 4 D) 3

17. **Let** $f(x) = \underset{n\to\infty}{Lim} \dfrac{(x^2+2x+3+\sin \pi x)^n - 1}{(x^2+2x+3+\sin \pi x)^n + 1}$,

then

A) $f(x)$ is continuous and differentiable for all $x \in R$

B) $f(x)$ is continuous but not differentiable for all $x \in R$

C) $f(x)$ is discontinuous at infinite number of points.

D) $f(x)$ is discontinuous at finite number of points.

18. **Let** $g(x) = \begin{cases} \dfrac{x^2 + x\tan x - x\tan 2x}{ax + \tan x - \tan 3x}; & x \ne 0 \\ 0 & ; \ x = 0 \end{cases}$ **. If**

$g'(0)$ **exists and is equal to non zero value b, then** $\frac{b}{a}$ **is equal to**

A) $\frac{7}{13}$ B) $\frac{7}{26}$ C) $\frac{7}{52}$ D) $\frac{5}{52}$

19. **Let**

$$f(x) = \frac{e^{\tan x} - e^x + \ln(\sec x + \tan x) - x}{\tan x - x}$$

be a continuous function at x = 0. The value of $f(0)$ equals

A) $\frac{1}{2}$ B) $\frac{2}{3}$ C) $\frac{3}{2}$ D) 2

20. **The value of**

$$\underset{x\to\frac{\pi}{2}}{\text{Lim}} \left(2^{\frac{1}{\cos^2 x}} + 3^{\frac{1}{\cos^2 x}} + 4^{\frac{1}{\cos^2 x}} + 5^{\frac{1}{\cos^2 x}} + 6^{\frac{1}{\cos^2 x}}\right)^{2\cos^2 x}$$ **is**

A) 1 B) 6 C) 36 D) $\frac{1}{36}$

21. $f(x)=\max\left\{\frac{x}{n},|\sin\pi x|, n\in N\right\}$ **has maximum points of non-differentiabilityu for $x\in(0,4)$, then**

A) maximum value of n in more than 4.5

B) least value of n is more that 3.5

C) maximum value of n is less than 4.5

D) least value of n is less than 3.5

22. Let x_n be defined as $\left(1+\frac{1}{n}\right)^{n+x_n}=e$, then $\lim_{n\to\infty} x_n$ equals

A) 1 B) $\frac{1}{2}$ C) $\frac{1}{e}$ D) 0

KEY

01) B	02) C	03) D	04) A	05) A	06) B
07) D	08) B	09) A	10) C	11) D	12) A
13) D	14) A	15) B	16) B	17) A	18) C
19) C	20) C	21) B	22) B		

SOLUTIONS

1. Given $\lim_{x\to0}\dfrac{\cos^2 x-\cos x-e^x\cos x+e^x-\frac{x^3}{2}}{x^n}$

$$=\lim_{x\to0}\frac{(\cos x-1)(\cos x-e^x)-\frac{x^3}{2}}{x^n}$$

$$=\lim_{x\to0}\frac{\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+....-1\right)}{x^n}$$

$$\times\frac{\left[\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-...\right)-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}-...\right)\right]-\frac{x^3}{2}}{x^n}$$

$$\lim_{x\to0}\frac{\left(-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}...\right)\left[\left(-x-x^2-\frac{x^3}{3!}-\frac{2x^5}{5!}-...\right)\right]-\frac{x^3}{2}}{x^n}$$

$$=\lim_{x\to0}\frac{\left(\frac{x^3}{2}+\frac{x^4}{2}+\frac{x^5}{12}-\frac{x^5}{24}+.0..\right)-\frac{x^3}{2}}{x^n}$$

= non zero if n= 4

2. given that,

$\lim_{x\to0}\dfrac{(\cos x-1)(\cos x-e^x)}{x^n}=$ finite non zero number

$$=\lim_{x\to0}\frac{(\cos x-1)(1+\cos x)(e^x-\cos x)}{x^n(1+\cos x)}$$

$$=\lim_{x\to0}\left(\frac{\sin^2 x}{x^2}\right)\cdot\left(\frac{e^x-\cos x}{x^{n-2}}\right)\cdot\left(\frac{1}{1+\cos x}\right)$$

$$=1.\frac{1}{2}\lim_{x\to0}\frac{\left[1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+...\infty\right]-\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+...\infty\right]}{x^{n-2}}$$

$$=\frac{1}{2}\lim_{x\to0}\frac{\left(1+x+\frac{x^2}{3!}+\frac{2x^3}{4!}+...\infty\right)}{x^{n-3}}$$

for this limit to be finite $n-3=0 \Rightarrow n=3$

3. $A_i=\dfrac{x-a_i}{|x-a_i|}, i=1,2,3,....,n$

$a_1<a_2<a_3<....a_n$.

If x is in the left neghbourhood of

$a_1<a_2<....a_{m-1}<x<a_m<a_{m+1}<....<a_n$

$A_i=\dfrac{x-a_i}{x-a_i}=1, i=1,2,....,m-1$

$A_i=\dfrac{x-a_i}{(a_i-x)}=-1,\qquad i=m,m-1,...,n$

$\therefore\ A_1A_2....A_n=(-1)^{n-m+1}$

If x is in the right neighbourhood of a_m

$a_1<a_2<....a_{m-1}<a_m<x<a_{m+1}<....<a_n$

$A_i=\dfrac{x-a_i}{x-a_i}=1, i=1,2,....,n$

$\therefore\ A_1A_2....A_n=(-1)^{n-m}$

$\therefore\ \lim_{x\to a_m^+}(A_1A_2.....A_n)=(-1)^{n-m}$

$\therefore$ LHL $\neq$ RHL

Hence, $\lim_{x\to a_m}(A_1A_2.....A_n)$ does not exist.

4. Let $y = \cfrac{x}{x + \cfrac{\sqrt[3]{x}}{x + \sqrt[3]{x}} \ldots \infty}$

$= \dfrac{x}{x + \dfrac{1}{x^{2/3}} \times \dfrac{x}{x + \sqrt[3]{x} \ldots \infty}} = \dfrac{x}{x + \dfrac{y}{x^{2/3}}}$

$\Rightarrow y = \dfrac{x^{5/3}}{x^{5/3} + y} \Rightarrow y^2 + \left(x^{5/3}\right) y - x^{5/3} = 0$.

$\therefore \Rightarrow y = \dfrac{-x^{5/3} \pm \sqrt{x^{10/3} + 4x^{5/3}}}{2}$

$= \dfrac{-x^{5/3} + \sqrt{x^{10/3} + 4x^{5/3}}}{2}$ $(\because y > 0)$

$= \dfrac{4x^{5/3}}{2\sqrt{\left(x^{10/3} + 4x^{5/3}\right)} + x^{5/3}} = \dfrac{2}{\sqrt{\left(1 + \dfrac{4}{x^{5/3}}\right)} + 1}$

Hence, $\lim\limits_{x \to \infty} = \dfrac{2}{\sqrt{1+0} + 1} = \dfrac{2}{2} = 1$

5. For existence of limit, (3+a) = 0, a = –3

given limit $= \lim\limits_{x \to 0} \dfrac{\sin 3x - 3x + bx^3}{x^3}$, $b = \dfrac{9}{2}$

6. Let $1 + a = y$

$\Rightarrow \left(y^{1/3} - 1\right)x^2 + \left(y^{1/2} - 1\right)x + y^{1/6} - 1 = 0$

$\Rightarrow \left(\dfrac{y^{1/3} - 1}{y - 1}\right)x^2 + \left(\dfrac{y^{1/2} - 1}{y - 1}\right)x + \dfrac{y^{1/6} - 1}{y - 1} = 0$

Now taking $\lim\limits_{y \to 1}$ on both the sides

$\Rightarrow \dfrac{1}{3}x^2 + \dfrac{1}{2}x + \dfrac{1}{6} = 0 \qquad \Rightarrow 2x^2 + 3x + 1 = 0$

$x = -1, -\dfrac{1}{2}$.

7. At x = 1

L.H.L. $= \lim\limits_{h \to 0} f(1-h) = \lim\limits_{h \to 0} (1-h)[1-h] = 0$

R.H.L. $\lim\limits_{h \to 0} f(1+h) = \lim\limits_{h \to 0} (1+h)[1+h] = 1$

and $f(1) = 1$

$\therefore$ f(x) is discontinuous function at x = 1 obviously it is not differentiable at x = 1

At x = 2,

L.H.L. $= \lim\limits_{h \to 0} f(2-h) = \lim\limits_{h \to 0} (2-h)[2-h] = 2$

R.H.L. $= \lim\limits_{h \to 0} (1+h)[2+h] = 2$

$\lim\limits_{x \to 2} f(x) = 2$, $\therefore$ f(x) is continuous at x = 2

L.H.D. $= \lim\limits_{h \to 0} \dfrac{f(2-h) - f(2)}{-h} = \lim\limits_{h \to 0} \dfrac{(2-h) - 2}{-h} = 1$

R.H.D. $= \lim\limits_{h \to 0} \dfrac{f(2+h) - f(2)}{h} = \lim\limits_{h \to 0} \dfrac{(2+h-1)2 - 2}{h} = 2$

$\therefore$ f(x) is not differentiable at x = 2

8. By graph of y = sinx, y = cosx, y = 3/4, we get graph of f(x) and then we get two points of non differentiability.

9. $3 \le 3 + 2\cos x \le 5$ for $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$f(x) = [3 + 2\cos x]$ is discontinuous at those ponts whre 3+2cosx is an integer.

Now, 3+2cosx = 3, if cosx = 0. So, $x = -\dfrac{\pi}{2}, \dfrac{\pi}{2}$ (Not possible)

3+cosx = 4, if $\cos x = \dfrac{1}{2}$.

So, x have two values $\dfrac{\pi}{3}$ and $-\dfrac{\pi}{3}$

3+2cosx = 5, if cosx = 1. so, x = 0

The number of values of x = 2+1 = 3

Hence, (A) is correct.

10. Taking x = y = 1, we get $f(1)f(1) - f(1) = 2$

Þ $f^2(1) - f(1) - 2 = 0$ Þ $(f(1) - 2)(f(1) + 1) = 0$

Þ $f(1) = 2$ (as f(1) > 0)

Taking y = 1, we get

f(x). f(1) – f(x) = x + 1

$\Rightarrow$ f(x) = x + 1 Þ $f^{-1}(x) = x - 1$

\ $f(x).f^{-1}(x) = x^2 - 1$

11. $f(x) = [x]^2 - [x^2]$

Let x=m, $m \in I$

$\lim\limits_{x \to m^-} f(x) = (m-1)^2 - (m^2 - 1) = 2 - 2m$

$\lim\limits_{x \to m^+} f(x) = (m^2 - m^2) = 0$

$\therefore$ f is continuous only at m=1.

12. (i) $\underset{\text{at } x=k}{LD} = \lim_{h\to 0}\frac{f(k)-f(k-h)}{h}$

(k = integer)

$= \lim_{h\to 0}\frac{[k]\sin k\pi - [k-h]\sin(k-h)\pi}{h}$

$= \lim_{h\to 0}\frac{-(k-1)\sin(k-h)\pi}{h}$ [$\because \sin k\pi = 0$]

$= \lim_{h\to 0}\frac{-(k-1)\sin(k\pi - h\pi)}{h}$

[sin (k π - q) = $(-1)^{k-1}$ sinq]

$= \lim_{h\to 0}\frac{-(k-1)(-1)^k \sin h\pi}{h\pi}\times\pi$

= p(k – 1) $(-1)^{k-1}$

(ii) $f(x) = \cos|x| + |x| = \begin{cases}\cos x - x, & x<0\\ \cos x + x, & x\ge 0\end{cases}$

At x = 0

$f'(x) = \begin{cases}-\sin x - 1, & x<0\\ -\sin x + 1, & x>0\end{cases}$

LHD = –1 , RHD = 1

∴ Not differentiable

$f(x) = \cos|x| - |x| = \begin{cases}\cos x + x, & x<0\\ \cos x - x, & x\ge 0\end{cases}$

Not differentiable at x = 0

$f(x) = \sin|x| + |x| = \begin{cases}-\sin x - x, & x<0\\ +\sin x + x, & x\ge 0\end{cases}$

Not differentiable at x = 0

$f(x) = \sin|x| - |x| = \begin{cases}-\sin x + x, & x<0\\ +\sin x - x, & x\ge 0\end{cases}$

At x = 0

$f'(x) = \begin{cases}-\cos x + 1, & x<0\\ +\cos x - 1, & x\ge 0\end{cases}$

LHD = 0, RHD = 0

∴ f is differentiable at x = 0

13. The given function is

$$f(x) = \begin{cases}\tan^{-1}x & \text{if } |x|\le 1\\ \frac{1}{2}(|x|-1) & \text{if } |x|>1\end{cases}$$

$$\Rightarrow f(x) = \begin{cases}\frac{1}{2}(-x-1) & \text{if } x<-1\\ \tan^{-1}x & \text{if } -1\le x\le 1\\ \frac{1}{2}(x-1) & \text{if } x>1\end{cases}$$

Clearly L.H.L. at (x = –1) = $\lim_{h\to 0} f(-1-h)$

R.H.L. at (x = –1) = $\lim_{h\to 0} f(-1+h)$

$= \lim_{h\to 0}\tan^{-1}(-1+h) = -3\pi/4$

∴ L.H.L. ≠ R.H.L. at x = –1

∴ f(x) is discontinuous at x = –1

Also we can prove in the same way, that f(x) is discontinuous at x = 1

∴ f(x) can not be found for x = ± 1 or domain of f¢(x) = R – {–1, 1}

14. Given function is y = | | x | –1 | or

$y = \begin{cases}-|x|+1 & \text{if } |x|<1\\ |x|-1 & \text{if } |x|\ge 1\end{cases}$

$= \begin{cases}-|x|+1 & \text{if } -1<x<1\\ |x|-1 & \text{if } x\le -1 \text{ or } x\ge 1\end{cases}$

$= \begin{cases}-x-1 & \text{if } x\le -1\\ x+1 & \text{if } -1<x<0\\ -x+1 & \text{if } 0\le x<1\\ x-1 & \text{if } x\ge 1\end{cases}$

Here Ly¢ (–1) = –1 and Ry¢ (–1) = 1

Ly¢ (0) = 1 and Ry¢ (0) = –1 and Ly¢ (1) = –1 and Ry¢ (1) = 1

⇒ y is not differentiable at x = –1, 0, 1 \

15. $f''(x) = -f(x) \Rightarrow \frac{d}{dx}f'(x) = -f(x)$

$g'(x) = -f(x)$ & $f'(x) = g(x)$

$F(x) = \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$

$F'(x) = 0 \Rightarrow F(x) = C \Rightarrow F(10) = 5$

$0 + e^{\lim_{x\to 0}\sin x \ln\left(\frac{1}{x}\right)} = 1.$

16. Let $g(x) = -\sqrt{(9-x^2)}$ is defined for

$x\in[-3,3]$

so, $f(x)$ is defined on $[-3,3]$

It is clear from the graph that $f(x)$ is continuous but not differentiable at A,B and C.
It is note that at point P, right hand derivative $-\infty$ and Q, left hand derivative is $+\infty$.

So, $f(x)$ is not differentible at P and Q.

17. $x^2+2x+3+\sin\pi x=(x+1)^2+2+\sin\pi x>1$

$\therefore\ f(x)=1\ \forall\ x\in R$

18. $g'(0)=b=\lim\limits_{x\to0}\dfrac{x^2+x\tan x-x\tan 2x}{x(ax+\tan x-\tan 3x)}=\lim\limits_{x\to0}\dfrac{x+\tan x-\tan 2x}{ax+\tan x-\tan 3x}$

$$=\lim_{x\to0}\frac{x+\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+.....\infty\right)-\left(2x+\frac{8x^3}{3}+\frac{2}{15}.32x^5+.....\infty\right)}{ax+\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+.....\infty\right)-\left(3x+\frac{27x^3}{3}+\frac{2}{15}.243x^5+.....\infty\right)}$$

On simplifying a = 2, b= $\dfrac{7}{26}$.

19. For continuity of f at x = 0, we have

$$k=f(0)=\lim_{x\to0}f(x)=\lim_{x\to0}\frac{e^{\tan x}-e^x}{\tan x-x}+\lim_{x\to0}\frac{\ln(\sec x+\tan x)-x}{\left(\frac{\tan x-x}{x^3}\right)x^3}$$

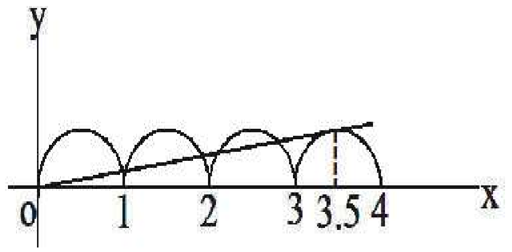
$$=\lim_{x\to0}\frac{e^x\left(e^{\tan x-x}-1\right)}{\tan x-x}+3\lim_{x\to0}\frac{\ln(\sec x-\tan x)-x}{x^3}$$

$=1+3\lim\limits_{x\to0}\dfrac{\sec x-1}{3x^2}$ (Using LH Rule)

$=1+\dfrac{1}{2}=\dfrac{3}{2}$

20. $$\left(6^{\frac{1}{\cos^2 x}}\right)^{2\cos^2 x}.\lim_{x\to\frac{\pi}{2}}\left(\left(\frac{1}{3}\right)^{\frac{1}{\cos^2 x}}+\left(\frac{1}{2}\right)^{\frac{1}{\cos^2 x}}+\left(\frac{2}{3}\right)^{\frac{1}{\cos^2 x}}+\left(\frac{5}{6}\right)^{\frac{1}{\cos^2 x}}+1\right)=36$$

21. $f(x)=\max\left\{\dfrac{x}{n},|\sin\pi x|\right\}$



Thus, for the maximum points of non differentiability, graphs of $y=\dfrac{x}{7}$ and $y=|\sin\pi x|$ must intersect at maximum number of points which occurs when $n>3.5$.
Hence, the least value of n is 4.

22. Given $\left(1+\dfrac{1}{n}\right)^{n+x_n}=e$

taking log

$$(n+x_n)\ln\left(1+\frac{1}{n}\right)=1\Rightarrow n+x_n=\frac{1}{\ln\left(1+\frac{1}{n}\right)}$$

$$\Rightarrow x_n=\frac{1}{\ln\left(1+\frac{1}{n}\right)}-n \quad .....(1)$$

let $\dfrac{n+1}{n}=u \Rightarrow nu=n+1 \Rightarrow n=\dfrac{1}{u-1}$

$$x_n=\lim_{u\to1}\left(\frac{1}{\ln u}-\frac{1}{u-1}\right)=\lim_{u\to1}\frac{(u-1)-\ln u}{(u-1)\ln u}$$

$$=\lim_{u\to1}\frac{1-\frac{1}{u}}{\frac{u-1}{u}+\ln u}=\lim_{u\to1}\frac{\frac{1}{u^2}}{\frac{1}{u^2}+\frac{1}{u}}=\frac{1}{2}$$

MULTIPLE ANSWER TYPE QUESTIONS

1. The function f(x) = | |2x – 3| - 10| is non differentiable at

A) x $\in\left\{\dfrac{-7}{2},\dfrac{13}{2}\right\}$ B) x $\in\left\{\dfrac{-7}{2},\dfrac{13}{2},\dfrac{3}{2}\right\}$

C) x $\in\left\{\dfrac{3}{2}\right\}$ D) x $\in\left\{-\dfrac{3}{2}\right\}$

2. **If** $f(x)=\begin{cases}|x|-3, & x<1\\ |x-2|+a, & x\geq 1\end{cases}$ **and**

$g(x)=\begin{cases}2-|x|, & x<2\\ sgn(x)-b, & x\geq 2\end{cases}$ **and**

$h(x)=f(x)+g(x)$ **is discontinuous at exactly one point, then which of the following values of a and b are possible:**

A) $a=-3, b=0$ B) $a=2, b=1$

C) $a=2, b=0$ D) $a=-3, b=1$

3. **Let [x] denote the greatest integer less than or equal to x. If f(x) =** $[x \sin \pi x]$**, then f(x) is**

A) continuous at x = 0

B) continuous in (-1, 0)

C) differentiable at x = 1

D) differentiable in (-1, 1)

4. **If f(x) = min {1, x^2, x^3}, then**

A) f(x) is continuous everywhere

B) f(x) is continuous and differentiable everywhere

C) f(x) is not differentiable at two points

D) f(x) is not differentiable at one point

5. **For a function**

$f(x)=\dfrac{ln(\{\sin x\}\{\cos x\}+1)}{\{\sin x\}\{\cos x\}}$, **where** $\{.\}$ **denotes fractional part function, then**

A) $f(0^-)=f\left(\dfrac{\pi}{2}^+\right)$ B) $f(0^+)=f\left(\dfrac{\pi}{2}^-\right)$

C) $\underset{x\to 0}{Lt} f(x)=1$ D) $\underset{x\to \pi/2}{Lt} f(x)=1$

6. **Let** $f(x)=\dfrac{1}{[\sin x]}$, **(where** $[.]$ **denotes the**

A) domain of $f(x)$ is $(2n\pi+\pi, 2n\pi+2\pi)\cup\{2n\pi+\pi/2\}$, where $n\in I$

B) $f(x)$ is continuous, when $x\in(2n\pi+\pi, 2n\pi+2\pi)$, where $n\in I$

C) f(x) is differentiable at $x=\pi/2$

D) none of these

7. **Let** $f(x)=\dfrac{\sin^{-1}(1-\{x\}).\cos^{-1}(1-\{x\})}{\sqrt{2}\{x\}.(1-\{x\})}$,

where $\{x\}$ **denotes the fractional part of x. Then**

A) $\lim\limits_{x\to 0+} f(x)=\infty$ B) $\lim\limits_{x\to 0-} f(x)=\dfrac{\pi}{2\sqrt{2}}$

C) $\lim\limits_{x\to 0+} f(x)=-\dfrac{\pi}{2}$ D) $\lim\limits_{x\to 0-} f(x)=0$

8. **The function, f(x) =**

$\max\{(1-x), (1+x), 2\}, x\in(-\infty, \infty)$ **is**

[IIT - 1995]

A) continuous at all points

B) differentiable at all points

C) differentiable at all points except at x = 1 and x = -1

D) continuous at all points except at x = 1 and x = -1, where it is discontinuous

9. **Let** $L=\lim\limits_{x\to 0}\dfrac{a-\sqrt{a^2-x^2}-\dfrac{x^2}{4}}{x^4}$, **a > 0. If L is finite, then** **[IIT - 2009]**

A) a = 2 B) a = 1

C) L = 1/64 D) L = 1/34.

10. **Let** $f: R\to R$ **be a functin such that** $f(x+y)=f(x)+f(y),\ \forall\ x, y\in R$**. If R. If** $f(x)$ **is differentialbe at x = 0, then**

[IIT-2011]

A) $f(x)$ is differentiable only in a finite interval containing zero

B) $f(x)$ is continuous $\forall\ x\in R$

C) $f'(x)$ is continuous $\forall\ x\in R$

D) $f(x)$ is differentiable except atg finitely many points

11. **If** $f(x)=\begin{cases}-x-\dfrac{\pi}{2}, & x\leq-\dfrac{\pi}{2}\\ -\cos x, & -\dfrac{\pi}{2}<x\leq 0\\ x-1, & 0<x\leq 1\\ \ln x, & x>1\end{cases}$**, then**

[IIT-2011]

A) $f(x)$ is continuous at $x=-\frac{\pi}{2}$

B) $f(x)$ is not differentialbe at x = 0

C) $f(x)$ is differentialbe at x = 1

D) $f(x)$ is differentialbe at x = –3/2

12. For every integer n, let a_n and b_n be real numbers. Let function $f: IR$ be given by

$$f(x)=\begin{cases} a_n+\sin\pi x, & for\ \ x\in[2n,2n+1] \\ b_n+\cos\pi x, & for\ \ x\in(2n-1,2n) \end{cases}$$ **, for**

all integers n. If f is continuous, then which of the following hold(s) for all n [IIT-2012]

A) $a_{n-1}-b_{n-1}=0$ B) $a_n-b_n=1$

C) $a_n-b_{n+1}=1$ D) $a_{n-1}-b_n=-1$

13. Which of the following function(s) not defined at $x=0$ has/have removable discontinuity at $x=0$?

A) $f(x)=\frac{1}{1+2^{\cot x}}$

B) $f(x)=\cos\left(\frac{|\sin x|}{x}\right)$

C) $f(x)=x\sin\left(\frac{\pi}{x}\right)$ D) $f(x)=\frac{1}{\ln|x|}$

14. f(x) = min {1, cos x, 1-sin x}, $-\pi \le x \le \pi$, then

A) f(x) is not differentiable at 0

B) f(x) is differentiable at $\pi/2$

C) f(x) has local maxima at 0

D) f(x) local maximum at x= $\pi/2$

15. The function f(x) = $\left||e^x-1|-1\right|$ is

A) continuous for all x

B) differentiable for all x

C) not continuous at x = 0, ln 2

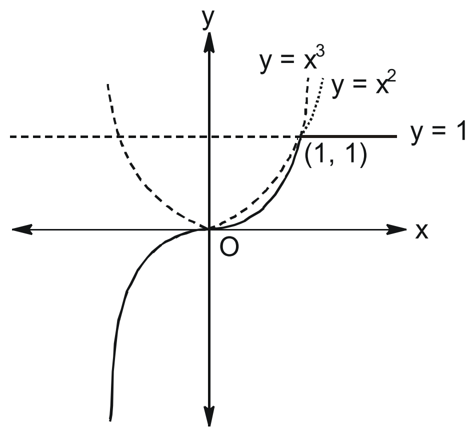
D) not differentiable at $x=\ln 2$

KEY

01) A,B,C	02) A,B,C	03) A,B,D
04) A,D	05) A,B	06) A,B
07) A,B08) A,C	09) A,C	10) B,C
11) A,B,C,D	12) B,D	13) B,D
14) A,C	15) A,D	

SOLUTIONS

1. By graph of f(x)
2. $f(x)$ is continuous for all x if it is continuous at $x=1$ for which $|1|-3=|1-2|+a \Rightarrow a=-3$ and $g(x)$ is continuous for all x if it is continuous at $x=2$ for which $2-|2|=\text{sgn}(2)-b \Rightarrow 0=1-b \Rightarrow b=1$
   Thus, $h(x)=f(x)+g(x)$ is continuous for all x if $a=-3, b=1$
   Hence, $h(x)=f(x)+g(x)$ is discontinuous at exactly one point for options $(a),(b)$ and (c).
3. We have, for $-1 \le x \le 1 \Rightarrow 0 \le x \sin \pi x \le 1/2$ $\therefore$ f(x) = [x sin π x] = 0
   Also x sin px becomes negative and numerically less than 1 when x is slightly greater than 1 and so by definition of [x].
   f(x) = [x sin π x] = –1 when 1 < x < 1 + h thus f(x) is constant and equal to 0 in the closed interval [–1, 1] and so f(x) is continuous and differentiable in the open interval (–1, 1).
   At x = 1, f(x) is clearly discontinuous, since f(1 – 0) = 0 and f(1 + 0) = –1 and f(x)is non-differentiable at x = 1.
4. from graph f(x) is continuous every where but not differentiate at x = 1.



6. We have $f(x)=\dfrac{1}{[\sin x]}$

$f(x)$ is defined, when $-1\le \sin x<0$ and $\sin x=1$

$\therefore \quad x\in\left((2n+1)\pi,(2n+2)\pi\right)\cup\left\{2n\pi+\dfrac{\pi}{2}\right\},$

where $n\in I$

$\therefore\ f(x)$ is continuous function.

Hence, $f(x)$ is continuous in $\left((2n+1)\pi,(2n+2)\pi\right)$, where $n\in I$.

and $Lf'(\pi/2)=\lim\limits_{h\to 0}\dfrac{f(\pi/2-h)-f(\pi/2)}{-h}$

$$=\lim_{h\to 0}\frac{\dfrac{1}{[\sin(\pi/2-h)]}-\dfrac{1}{\sin\pi/2}}{-h}$$

$$=\lim_{h\to 0}\frac{\dfrac{1}{[\cosh]}-1}{-h}=-\infty$$

Hence, $f(x)$ is differentiable at $x=\pi/2$.

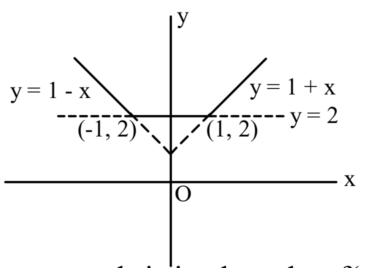
7. $\lim\limits_{x\to 0+}f(x)=\lim\limits_{h\to 0}f(0+h)$

$$=\lim_{h\to 0}\frac{\sin^{-1}(1-h)}{(1-h)}\lim_{h\to 0}\frac{\cos^{-1}(1-h)}{\sqrt{2h}}=\infty$$

$\lim\limits_{x\to 0-}f(x)=\lim\limits_{h\to 0}f(0-h)$

$$=\lim_{h\to 0}\frac{\sin^{-1}(1+h-1).\cos^{-1}(1+h-1)}{\sqrt{2}(-h+1).(1+h-1)}$$

$$=1.\frac{\pi/2}{\sqrt{2}}=\frac{\pi}{2\sqrt{2}}$$

8. 

From graph it is clear that f(x) is continuous everywhere and also differentiable everywhere expect at x = 1 and –1.

9. $L=\lim\limits_{x\to 0}\dfrac{a-\sqrt{a^2-x^2}-\dfrac{x^2}{4}}{x^4}\quad a>0$

$$\lim_{x\to 0}\frac{a-(a^2-x^2)^{1/2}-\dfrac{x^2}{4}}{x^4}$$

$$=\lim_{x\to 0}\frac{a-a\left(1-\left(\dfrac{x}{a}\right)^2\right)^{1/2}-\dfrac{x^2}{4}}{x^4}$$

$$\lim_{x\to 0}\frac{a-a\left(1-\dfrac{1}{2}\cdot\dfrac{x^2}{a^2}-\dfrac{1}{2}\cdot\dfrac{1}{2}\cdot\dfrac{x^4}{2a^4}\right)-\dfrac{x^2}{4}}{x^4}$$

$$=\lim_{x\to 0}\frac{\dfrac{1}{2}\dfrac{x^2}{a}+\dfrac{1}{8}\cdot\dfrac{x^4}{a^4}-\dfrac{x^2}{4}}{x^4}\qquad\frac{x^2\left(\dfrac{1}{2a}-\dfrac{1}{4}\right)+\dfrac{1}{8}\cdot\dfrac{x^4}{a^3}}{x^4}$$

If $\dfrac{1}{2a}-\dfrac{1}{4}=0\ \Rightarrow a=2$,

if a = 2, $L=\dfrac{1}{8}\cdot\dfrac{1}{8}=\dfrac{1}{64}$.

10. $\because f(0)=0$ and $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{f(h)}{h}=f'(0)=k(say)$$

$$\Rightarrow f(x)=kx+c\Rightarrow f(x)=kx\left(\because f(0)=0\right)$$

11. $\lim\limits_{x\to\frac{\pi}{2}^-}f(x)=0=f\left(-\dfrac{\pi}{2}\right)$

$\lim\limits_{x\to-\frac{\pi}{2}^+}f(x)=\cos\left(-\dfrac{\pi}{2}\right)=0$

$$f'(x)=\begin{cases}-1, & x\le-\dfrac{\pi}{2}\\ \sin x, & \dfrac{-\pi}{2}<x\le 0\\ 1, & 0<x\le 1\\ \dfrac{1}{x}, & x>1\end{cases}$$

Clearly, $f(x)$ is not differentiable at x = 0 as

$f'(0^-)=0$ and $f'(0^+)=1$.

$f(x)$ is differentiable at x = 1 as

$f'(1^-)=f'(1^+)=1$.

12. At $x=2n$

LHL $=\lim_{h\to 0}\left(b_n+\cos\pi(2n-h)\right)=b_n+1$

RHL $=\lim_{h\to 0}\left(a_n+\sin\pi(2n-h)\right)=a_n$

$f(2n)=a_n$

For continuity $b_{n+1}=a_n$

At $x=2n+1$

LHL $=\lim_{h\to 0}\left(a_n+sin\pi(2n+1-h)\right)=a_n+1$

RHL $=\lim_{h\to 0}\left(b_{n+1}+\cos\pi(2n+1-h)\right)=b_{n+1}-1$

$\lim_{h\to 0}\left(an+\sin f(2n+1)\right)=a_n$

for continuity

$a_n=b_{n+1}-1$, $a_{n-1}-b_n=-1$.

13. (b,c,d)

(a) $f(0^-)=\lim_{h\to 0}f(o-h)=\lim_{h\to 0}\frac{1}{1+2^{\cot(o-h)}}$

$=\lim_{h\to 0}\frac{1}{1+2^{-\text{coth}}}=\frac{1}{1+2^{-\infty}}=\frac{1}{1+0}=1$

$\because f(0^-)\neq f(0^+)$

(b)

$f(0^-)=\lim_{h\to 0}f(0-h)=\lim_{h\to 0}\cos\left(\frac{|\sin(0-h)|}{0-h}\right)$

$=\lim_{h\to 0}\cos\left(\frac{\sinh}{-h}\right)$

$=\cos\left(-\lim_{h\to 0}\cos\frac{\sinh}{-h}\right)=\cos(-1)=\cos 1$

and

$f(0^+)=\lim_{h\to 0}f|0+h|=\lim_{h\to 0}\cos\left(\frac{|\sin(0-h)|}{0-h}\right)$

$=\lim_{h\to 0}\cos\left(\frac{\sinh}{h}\right)=\cos\left(\lim_{h\to 0}\frac{\sinh}{h}\right)=\cos 1$

$\therefore f(0^-)=f(0^+)\neq f(0)$

(C)

$f(0^-)=\lim_{h\to 0}f(0-h)=\lim_{h\to 0}(-h)\sin\left(\frac{\pi}{-h}\right)$

$=\lim_{h\to 0}hh\sin\left(\frac{\pi}{h}\right)$

$=0\times\sin\infty$

$=0\times$(lies between - 1+to 1)

and

$f(0^+)=\lim_{h\to 0}f(0+h)=\lim_{h\to 0}h\sin\left(\frac{\pi}{h}\right)$

$=0\times\sin\infty=0\times$(lies between -1 to 1)$=0$

$\therefore f(0^-)=f(0^+)\neq f(0)$

(d): $f(0^-)=\lim_{h\to 0}f(0-h)=\lim_{h\to 0}\frac{1}{\ln|0-h|}$

$=\lim_{h\to 0}\frac{1}{\ln h}$

$=\frac{1}{\ln 0}=\frac{1}{-\infty}=0$

and

$f(0^+)=\lim_{h\to 0}f(0+h)=\lim_{h\to 0}\frac{1}{\ln|0+h|}=\lim_{h\to 0}\frac{1}{\ln h}$

$=\frac{1}{-\infty}=0$

$\therefore f(0^-)=f(0^+)\neq f(0)$

14. We have, $\text{f}(\text{x})=\min\{1,\cos\text{x},1-\sin\text{x}\}$

$\therefore$ f(x) can be rewritten as

$$\text{f}(\text{x})=\begin{cases}\cos\text{x}, & -\frac{\pi}{2}\le\text{x}\le 0\\ 1-\sin\text{x}, & 0<\text{x}\le\frac{\pi}{2}\\ \cos\text{x}, & \frac{\pi}{2}<\text{x}\le\pi\end{cases}$$

$$\Rightarrow f'(x)=\begin{cases}-\sin x, -\frac{\pi}{2}\le x\le 0\\ -\cos x, 0<x\le\frac{\pi}{2}\\ -\sin x, \frac{\pi}{2}<x\le\pi\end{cases}$$

$\because\ f'(0)=0$

hence, f(x) has local maxima at 0 and f(x) is not differentiable at x = 0.

15. $f(x)=\begin{cases}e^x & x<0\\ 2-e^x & 0\le x<\ln 2\\ e^x-2 & x\ge\ln 2\end{cases}$

f is continuous $\forall x\in R$, but is not differentiable at $x=0,\ln 2$

COMPREHENSION TYPE QUESTIONS

Passage - 1

Let $f(x)=\begin{cases}x+2, & 0\le x<2\\ 6-x, & x\ge 2\end{cases}$,

$$g(x)=\begin{cases}1+\tan x, & 0\le x<\frac{\pi}{4}\\ 3-\cot x, & \frac{\pi}{4}\le x<\pi\end{cases}$$

1. $f(g(x))$ **is**

(A) discontinuous at $x=\frac{\pi}{4}$

(B) differentiable at $x=\frac{\pi}{4}$

(C)continuous but non differentialbe at $x=\frac{\pi}{4}$

(D) differentiable at $x=\frac{\pi}{4}$, but derivative is not continuous.

2. The number of points of non differentiability of $h(x)=|f(g(x))|$ **is**

A) 1 B) 2 C) 3 D) 4

3. The range of $h(x)=f(g(x))$ **is**

A) $(-\infty,\infty)$ B) $(4,\infty)$

C) $(-\infty,4]$ D) $[4,\infty)$

Passage - 2

Let $f(x)=\dfrac{\sin^{-1}(1-\{x\}).\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}.(1-\{x\})}$,

where $\{.\}$ denotes the fractional part function.

4. If $R=\lim_{x\to 0^+}f(x)$**, then the value of** $\cos(100R)$ **is :**

A) -1 B) 0 C) 1/2 D) 1

5. If L = $\lim_{x\to 0^-}f(x)$**, then the value of** $\sin\left(99\sqrt{2}L\right)$ **is :**

A) -1 B) 0 C) 1/2 D) 1

6. The value of $\left[2R^2+4L^2\right]$ **is [where [.] denotes the greatest integer function]** :

A) 3 B) 6 C) 9 D) 12

Passage - 3

Suppose f, g and h be three real valued function defined on R.

Let $f(x)=2x+|x|$, $g(x)=\frac{1}{3}(2x-|x|)$ **and** $h(x)=f(g(x))$

7. The range of the function $k(x)=1+\frac{1}{\pi}\left(\cos^{-1}(h(x))+\cot^{-1}(h(x))\right)$ **is equal to**

A) $\left[\frac{1}{4},\frac{7}{4}\right]$ B) $\left[\frac{5}{4},\frac{11}{4}\right]$ C) $\left[\frac{1}{4},\frac{5}{4}\right]$ D) $\left[\frac{7}{4},\frac{11}{4}\right]$

8. The domain of definition of the function $l(x)=\sin^{-1}(f(x)-g(x))$ **is equal to**

A) $\left[\frac{3}{8},\infty\right)$ B) $(-\infty,1]$

C) $[-1,1]$ D) $\left(-\infty,\frac{3}{8}\right]$

9. The function

$T(x)=f(g(f(x)))+g(f(g(x)))$ **is**

A) continuous and diferentiable in $(-\infty,\infty)$

B) continuous but not derivable $\forall x\in R$

C) neither continuous nor derivable $\forall x\in R$

D) an odd function

KEY

01) C 02) B 03) C 04) D 05) A 06) C
07) B 08) D 09) B

SOLUTIONS

Passage - 1
1 to 3

For $0\le x\le\frac{\pi}{4}$, $g(x)=1+\tan x$

$x\in\left[0,\frac{\pi}{4}\right)\Rightarrow 1+\tan x\in[1,2)$

so $f(g(x))=f(1+\tan x)=1+\tan x+2$

and for $x\in\left[\frac{\pi}{4},\pi\right)$, $g(x)=3-\cot x$

$x\in\left[\frac{\pi}{4},\pi\right)\Rightarrow 3-\cot x\in[2,\infty)$

so $f(g(x))=f(3-\cot x)=6-(3-\cot x)$

Let $h(x)=f(g(x))=\begin{cases}3+\tan x\ ,\ 0\le x<\frac{\pi}{4}\\ 3+\cot x\ ,\ \frac{\pi}{4}\le x<\pi\end{cases}$

clearly, $f(g(x))$ is continuous in $[0,\pi)$

Now $h'\left(\frac{\pi}{4}^{+}\right)=\lim_{x\to\frac{\pi}{4}^{+}}\left(-\operatorname{cosec}^2 x\right)=-2$

$h'\left(\frac{\pi}{4}^{-}\right)=\lim_{x\to\frac{\pi}{4}^{-}}\left(\sec^2 x\right)=2$

So $f(g(x))$ is differentiable every where in $(0,\pi]$ other than at $x=\frac{\pi}{4}$

$|f(g(x))|=\begin{cases}|3+\tan x|,\ 0\le x<\frac{\pi}{4}\\ |3+\cot x|,\frac{\pi}{4}\le x<\pi\end{cases}$

which is non differentiable at $x=\frac{\pi}{4}$ and where $3+\cot x=0$ or $x=\cot^{-1}(-3)$

For $x\in\left[0,\frac{\pi}{4}\right),3+\tan x\in[3,4)$

For $x\in\left[\frac{\pi}{4},\pi\right),3+\cot x\in(-\infty,4]$

Hence the range is $(-\infty,4]$

Passage - 2
4 to 6

We have

$$f(x)=\frac{\sin^{-1}(1-\{x\}).\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}(1-\{x\})}$$

1. (D) $R=\lim_{x\to0^{+}}f(x)=\lim_{h\to0}f(0+h)$

$$=\lim_{h\to0}\frac{\sin^{-1}(1-\{0+h\}).\cos^{-1}(1-\{0+h\})}{\sqrt{2\{0+h\}}(1-\{0+h\})}$$

$$=\lim_{h\to0}\frac{\sin^{-1}(1-h).\cos^{-1}(1-h)}{\sqrt{2h}(1-h)}$$

$$=\lim_{h\to0}\frac{\sin^{-1}(1-h)}{(1-h)}\lim_{h\to0}\frac{\cos^{-1}(1-h)}{\sqrt{2h}}$$

in second limit put $1-h=\cos\theta$

$$=\lim_{h\to0}\frac{\sin^{-1}(1-h)}{(1-h)}\lim_{\theta\to0}\frac{\cos^{-1}(\cos\theta)}{\sqrt{2(1-\cos\theta)}}$$

$$= \lim_{h\to 0} \frac{\sin^{-1}(1-h)}{(1-h)} \lim_{\theta\to 0} \frac{\theta}{2\sin(\theta/2)}$$

$$= \sin^{-1} 1.1 = \pi/2 \Rightarrow 100R = 50\pi$$

$$\therefore \cos(100R) = \cos 50\pi = (-1)^{50} = 1$$

2. (a): $L = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$

$$= \lim_{h\to 0} \frac{\sin^{-1}(1-\{0-h\}).\cos^{-1}(1-\{0-h\})}{\sqrt{2\{0-h\}}.(1-\{0-h\})}$$

$$= \lim_{h\to 0} \frac{\sin^{-1}(1+h-1).\cos^{-1}(1+h-1)}{\sqrt{2(-h+1)}.(1-h-1)}$$

$$= \lim_{h\to 0} \frac{\sin^{-1} h}{h}.\lim_{h\to 0} \frac{\cos^{-1} h}{\sqrt{2(1-h)}}$$

$$= 1.\frac{\pi/2}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}} \Rightarrow 99\sqrt{2}L = \frac{99\pi}{2}$$

$$\therefore \sin(99\sqrt{2}L) = \sin\left(\frac{99\pi}{2}\right) = (-1)^{\frac{99-1}{2}} = (-1)^{49} = -1$$

21 (C). $\because R = \frac{\pi}{2}$ and $L = \frac{\pi}{2\sqrt{2}}$

$$\therefore 2R^2 + 4L^2 = \pi^2$$

$$\Rightarrow [2R^2 + 4L^2] = [\pi^2] = [9.87] = 9.$$

Passage - 3
7 to 9

We have $f(x) = \begin{cases} 3x, x \ge 0 \\ x, \ x < 0 \end{cases}$ and

$$g(x) = \begin{cases} \frac{x}{3}, x \ge 0 \\ x, \ x < 0 \end{cases}$$

Clearly f and g are inverse of each other

Now, $h(x) = f(g(x)) = \begin{cases} 3\left(\frac{x}{3}\right) = x, \ x \ge 0 \\ x, \quad x < 0 \end{cases}$

(i) As $h(x) = x \forall x \in R$

$$\Rightarrow k(x) = 1 + \frac{1}{\pi}(\cos^{-1} x + cot^{-1}x)$$

Domain of $k(x) = [-1,1]$ and $k(x)$ is decreasing function on $[-1,1]$

As $k(x)$ is continuous function on $[-1,1]$

Now, $k_{\min}(x=1) = 1 + \frac{1}{\pi}(\cos^{-1} 1 + \cot^{-1} 1)$

$$= 1 + \frac{1}{\pi}\left(0 + \frac{\pi}{4}\right) = 1 + \frac{1}{4} = \frac{5}{4}$$

$$k_{\max}(x=-1) = 1 + \frac{1}{\pi}(\cos^{-1}(-1) + \cot^{-1}(-1))$$

$$= 1 + \frac{1}{\pi}\left(\pi + \frac{3\pi}{4}\right) = 1 + \frac{7}{4} = \frac{11}{4}$$

$\Rightarrow$ Range of $k(x) = \left[\frac{5}{4}, \frac{11}{4}\right]$

(ii) We have

$$f(x) - g(x) = (2x + |x|) - \frac{1}{3}(2x - |x|)$$

$$= \frac{4x}{3} + \frac{4}{3}|x| = \begin{cases} \frac{8}{3}x; \ x \ge 0 \\ 0 \ ; \ x < 0 \end{cases}$$

$\therefore$ For domain of function,

$$0 \le \frac{8x}{3} \le 1 \Rightarrow 0 \le x \le \frac{3}{8}$$

$\Rightarrow$ Domain of $l(x) = \left(-\infty, \frac{3}{8}\right]$

Note : Range of function $l(x) = \left[0, \frac{\pi}{2}\right]$

(iii) As f and g are ianverse of each other, so $T(x) =$

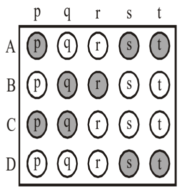
$$T0x = f(g(f(x))) + g(f(g(x)))$$

$$= f(x) + g(x) = (2x + |x|) + \frac{1}{3}(2x - |x|)$$

$\Rightarrow T(x) = \begin{cases} \frac{10x}{3}, x \ge 0 \\ 2x, \ x < 0 \end{cases}$ Clearly, $T(x)$ is continuous but non-derivable at x $= 0$

MATRIXMATCHING TYPE QUESTIONS

The statements in Column I are labelled A, B, C and D, while the statements in Column II are labelled p, q, r, s and t. Any given statement in Column I can have correct matching with ONE OR MORE statements(s) in Column II. The appropriate bubbles corresponding to the answers to these equations have to be darkened as illustrated in the following exampel. If the correct matches are A - p, s and t; B - q and r; C - p and q; and D - s and t; then the correct darkening of bubbles will look like the following.



1. Column I (functions)

(A) f(x) = $|x|$

(B) $f(x) = x^n|x|, n \in N$

(C) $f(x) = \begin{cases} x\ln|\sin x|, & x \neq 0 \\ 0 & , x = 0 \end{cases}$

(D) $f(x) = \begin{cases} xe^{1/x}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$

Column II (properties)

(P) continuous at x = 0
(Q) Discontinuous at x = 0
(R) differentiable at x = 0
(S) non-differentiable at x = 0

2. Column- I

(A) Let $f(x) = \begin{cases} x^6, & x > 1 \\ x^3, & x \leq 1 \end{cases}$ Then f(x) is differentiable at x =

(B) Number of points of non-differentiability of $f(x) = \min\{2, x^2, x^3\}$ is

(C) $f(x) = x^2 \sin\left(\frac{1}{x}\right), x \neq 0.\ f(0) = 0$ then $f'(0^-)$ is

(D) $f(x) = |x-1| + |x| + |x+1|$, then $f'(0^+)$ is

Column -II

(P) 0
(Q) 1
(R) 2
(S) 3
(T) 5

3. Column I (functions) **[IIT-2007]**

(A) $x|x|$

(B) $\sqrt{|x|}$

(C) $x + [x]$, $[.] \to$ G.I.F.

(D) $|x-1| + |x+1|$

Column II (properties)

(P) continuous in (-1, 1)
(Q) differentiable in (-1, 1)
(R) strictly increasing in (-1, 1)
(S) not differentiable at least at one point in (-1, 1)

KEY

01) (A – p,s), (B – p,r), (C – p,s), (D – q,s)
02) A-p, B-r, C-p, D- q
03) (A – p,q,r), (B – p,s), (C – r,s), (D – p,q)

SOLUTIONS

1. Conceptual
2. Conceptual
3. (A) $f(x) = x|x| = \begin{cases} -x^2, & x < 0 \\ x^2, & x > 0 \end{cases}$ is continuous and differentiable everywhere. also increasing.

(B) $f(x) = \sqrt{|x|} = \begin{cases} \sqrt{-x}, & x < 0 \\ \sqrt{x}, & x \geq 0 \end{cases}$

$f¢(x) = \begin{cases} -\frac{1}{2\sqrt{-x}}, & x < 0 \\ \frac{1}{2\sqrt{x}}, & x > 0 \end{cases}$

Continuous everywhere differentiable everywhere except at x = 0. Not increasing.

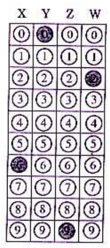
(C) f(x) = x + [x] At integral point x = I, LHL = I + (I – 1) = 2I – 1, RHL = I + I = 2I = f(I) , So not continuous hence not differentiable at integral points but increasing.

(D) $f(x) = |x-1| + |x+1| = \begin{cases} -2x & , \quad x < -1 \\ 2 & , \quad -1 \le x < 1 \\ 2x & , \quad 1 \le x \end{cases}$

Continuous everywhere, differentiable everywhere but not increasing in (1,1).

INTEGER TYPE QUESTIONS

This section contains 9 questions. The answer to each of the questions is a single-digit integer, ranging from 0 to 9. The appropriate bubbles below the respective question numbers in the ORS have to be darkened. For example, if the correct answers to question numbers X, Y, Z and W (say) are 6, 0, 9 and 2, respectively, then the correct darkening of bubbles will look like the following.



1. $\dfrac{f(x+2y)}{3} = \dfrac{f(x)+2f(y)}{3} \quad \forall x, y \in R$.

   If $f'(0) = 1$, f(0) = 2, then f(2) is

2. **The number of points of discontinuity of $f(x) = [2\cos x], x \in (0, 2\pi)$,([.] represents the greatest integer function) are ...... (where [.] represents greatest integer function)**

3. **Let p(x) be a polynomial of degree 4 having extremum at x = 1, 2 and $\lim\limits_{x\to 0}\left(1+\dfrac{p(x)}{x^2}\right)$ = 2. Then the value of p(2) is [IIT - 2009]**

4. **If the function $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$, then $g'(1)$ is .... [IIT - 2009]**

5. **The largest value of the non-negative integer a for which**

$$\lim_{x\to 1}\left\{\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4} \text{ is}$$

6. **If $\lim\limits_{x\to 0}\sin\left(\dfrac{\pi\left(1-\cos^m x\right)}{x^n}\right)$ exists, where $m, n \in N$, then the sum of all possible values of n is __________.**

7. **If $\lim\limits_{x\to\infty}\left(e^{2x}+e^x+x\right)^{\frac{1}{x}} = e^a$, then the value of a is**

8. **If a and b are the numbers of points of non differentiability of $f(x) = \left[\sin^{-1} x\right]$ and $f(x) = \left[\dfrac{2}{1+x^2}\right]$, $x \ge 0$, (where [.] represents greatest integer function) respectively, then the value of $a+b$ is**

9. **The least integral value of a for which the function $f(x) = \left[(x-2)^3 / a\right]\sin(x-2) + a\cos(x-2)$, where [.] denotes the greatest integer function, is continuous in [0,2] is _____**

KEY

01) 4	02) 6	03) 0	04) 2
05) 2	06) 3	07) 2	08) 5
09) 9			

SOLUTIONS

1. $\dfrac{f(x+2y)}{3} = \dfrac{f(x)+2f(y)}{3}$

$\dfrac{1}{3}f'\dfrac{(x+2y)}{3} = \dfrac{f'(x)}{3}$ ... (1)

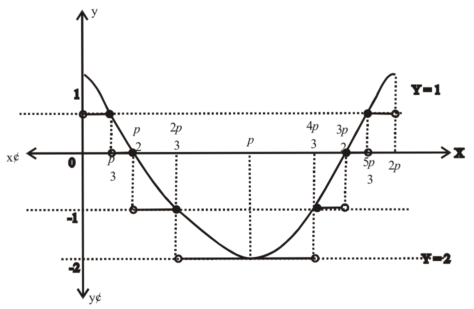
$\dfrac{2}{3}f'\left(\dfrac{x+2y}{3}\right) = \dfrac{2f'(x)}{3}$ ... (ii)

for (i & (ii) f'(x) = f'(y) $\Rightarrow$ f'(x) = C = 1,

f(x) = x + d , As f(0) = 2

f(x) = x + 2 , f(2) = 2 + 2 = 4

2. $f(x) = [2\cos x]$



clearly, from the graph, it can be seen that

$f(x)$ is discontinuous at $x=\frac{\pi}{3},\frac{\pi}{2},\frac{2\pi}{3},\frac{4\pi}{3},\frac{3\pi}{2},\frac{5\pi}{3}$

3. $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ Given

$$\lim_{x\to 0}\left(1+\frac{P(x)}{x^2}\right)=2$$

Limit exist only if, $d = e = 0$

$\lim_{x\to 0}[1+ax^2+bx+c]=2$ π $c + 1 = 2$

$\Rightarrow$ $c = 1$

$P(x) = ax^4 + bx^3 + x^2$

$P'(x) = 4ax^3 + 3bx^2 + 2x = x(4ax^2 + 3bx + 2)$

Note: $4ax^2 + 3bx + 2 \equiv \lambda(x - 1)(x - 2)$

$= \lambda(x^2 - 3x + 2)$

$\Rightarrow \lambda = 1,\ a = \frac{1}{4},\ b = -1 \Rightarrow$

$$P(x)=\frac{1}{4}x^4-x^3+x^2$$

$$\therefore P(2)=\frac{1}{4}2^4-2^3+2^2 = 4-8+4=0$$

4. $f(x) = x^3 + e^{x/2}$, $g(x) = f^{-1}(x)$

$f(g(x)) = x$ $f(0) = 1 \Rightarrow f^{-1}(1) = 0 \Rightarrow g(1) = 0$

$f'(g(x)).g'(x)=1$, $g'(x)=\frac{1}{f'(g(x))}$

$$g'(1)=\frac{1}{f'(g(1))}=\frac{1}{f'(0)}$$

$f'(x)=3x^2+\frac{1}{2}e^{x/2}$ $f'(0)=\frac{1}{2}$. So, $g'(1)=2$.

5. $$\lim_{x\to 1}\left[\frac{-ax+\sin(x-1)+a}{x+\sin(x-1)-1}\right]^{\frac{1-x}{1-\sqrt{x}}}=\frac{1}{4}$$

$$\lim_{x\to 1}\left[\frac{\frac{\sin(x-1)}{(x-1)}-a}{\frac{\sin(x-1)}{(x-1)}+1}\right]=\frac{1}{4}\Rightarrow\left(\frac{1-a}{2}\right)^2=\frac{1}{4}$$

$\Rightarrow a=0, a=2 \Rightarrow a=2$

6. $$\lim_{x\to 0}\sin\left(\frac{\pi\left(1-\cos^m x\right)}{x^n}\right)$$

$$=\sin\left(\lim_{x\to 0}\frac{\pi\left(1-\cos^m x\right)}{x^n}\right)$$

$$=\sin\left(\lim_{x\to 0}2\pi m\frac{\sin^2 x/2}{x^n}\right)$$

$\Rightarrow m\in N$ and $n=1$ or 2

7. $$L=e^{\lim_{x\to\infty}\frac{\ln\left(e^{2x}+e^x+x\right)}{x}}$$

using L Hospital's rule

$$L=e^{\lim_{x\to\infty}\frac{2e^{2x}+e^x+1}{e^{2x}+e^x+x}}=e^{\lim_{x\to\infty}\frac{2+e^{-x}+e^{-2x}}{1+e^{-x}+xe^{-2x}}}=e^2$$

8. $\sin^{-1}x$ is a monotonically increasing function.

Hence, $f(x)=\left[\sin^{-1}x\right]$ is discontinuous, where $\sin^{-1}x$ is an integer.

$\Rightarrow \sin^{-1}x=-1,0,1$ or $x=-\sin 1, 0, \sin 1$

$\frac{2}{1+x^2}, x\ge 0$, is a monotonically decreasing function.

Hence, $f(x)=\left[\frac{2}{1+x^2}\right], x\ge 0$ is discontinuous

when $\frac{2}{1+x^2}$ is an integer.

$\Rightarrow \frac{2}{1+x^2}=1,2 \Rightarrow x-1,0$

9. $\sin(x-2)$ and $\cos(x-2)$ are continuous for all x. Since $\left[x^3\right]$ is not continuous at integral values of x^3, $f(x)$ is continous in $[0,2]$ if

$\left[\frac{(x-2)^3}{a}\right]=0$, $\forall x\in[0,2]$.

Now, $(x-2)^3\in[0,8]$ for $x\in[4,6]$

$\Rightarrow a>8$ for $\left[\frac{(x-2)^3}{a}\right]=0$

DIFFERENTIATION

SYNOPSIS

Derivative :

- (i) A function y = f(x) is said to be differentiable if $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ exists finitely. This limit is usually denoted by $f^1(x)$ or $\frac{dy}{dx}$.

  (ii) Let y = f(x) be a function and if $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists, then f is said to be differentiable at 'a' and the limit is called the derivative of f at a . The derivative of 'f' at a is denoted by any one of the forms $\left(\frac{dy}{dx}\right)_{x=a}$ (or) $f^1(a)$

Sum or difference rule :

- $\frac{d}{dx}\left(c_1 f(x)\pm c_2 g(x)\right)=c_1\frac{d}{dx}\left(f(x)\right)\pm c_2\frac{d}{dx}\left(g(x)\right)$

  where c_1, c_2 are constants.

Product Rule :

- i) If f(x), g(x) are two differentiable functions of 'x' then

  $$\frac{d}{dx}\left(f(x).g(x)\right)=f(x)\frac{d}{dx}\left(g(x)\right)+g(x)\frac{d}{dx}\left(f(x)\right)$$

  ii) If f(x), g(x) and h(x) are three differentiable functions of 'x' then

  $$\frac{d}{dx}\left(f(x).g(x).h(x)\right)=g(x).h(x)\frac{d}{dx}\left(f(x)\right)+$$

  $$f(x).h(x)\frac{d}{dx}\left(g(x)\right)+f(x).g(x)\frac{d}{dx}\left(h(x)\right)$$

Quotient Rule :

- If f(x), g(x) are two differentiable functions of 'x' then $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\frac{d}{dx}\left(f(x)\right)-f(x)\frac{d}{dx}\left(g(x)\right)}{\left(g(x)\right)^2}$

Chain Rule :

- If 'y' is a function of 't' and 't' is a function of 'x' i.e., $y=f(t)$ and $t=g(x)$ then $\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}$

  similarly, if $y=f(u)$, where $u=f(v)$, $v=h(x)$ then $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx}$.

  Note : We can extend this rule to any number of functions.

Derivatives of Basic elementary functions :

- i) $\frac{d}{dx}(\text{Constant})=0$

  ii) $\frac{d}{dx}\left(x^n\right)=n.x^{n-1}$

  iii) $\frac{d}{dx}\left(e^x\right)=e^x$

  iv) $\frac{d}{dx}\left(a^x\right)=a^x.\log a$

  v) $\frac{d}{dx}(|x|)=\frac{|x|}{x}; if\ x\neq 0$

  vi) $\frac{d}{dx}\left(\log_e|x|\right)=\frac{1}{x}$

  vii) $\frac{d}{dx}\left(\log_a|x|\right)=\frac{1}{x\log a}$

  viii) $\frac{d}{dx}\left(\sqrt{x}\right)=\frac{1}{2\sqrt{x}}$

  ix) $\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{-1}{x^2}$

  x) $\frac{d}{dx}\left(\frac{1}{x^n}\right)=\frac{-n}{x^{n+1}}$

  xi) $\frac{d}{dx}([x])=\begin{cases}0, & \forall x\notin z\\ does\ not\ exist, & \forall x\in z\end{cases}$

  where [.] stands for greatest integer function.

Derivatives of Trigonometric functions :

- i) $\frac{d}{dx}(\sin x) = \cos x$

  ii) $\frac{d}{dx}(cosx) = -sinx$

  iii) $\frac{d}{dx}(\tan x) = \sec^2 x$

  iv) $\frac{d}{dx}(\cot x) = -\cos ec^2 x$

  v) $\frac{d}{dx}(\sec x) = \sec x.\tan x$

  vi) $\frac{d}{dx}(\cos ecx) = -\cos ecx.\cot x$

Derivatives of Inverse Trignometric functions :

- i) $\frac{d}{dx}\left(Sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}}, x \in (-1,1)$

  ii) $\frac{d}{dx}\left(Cos^{-1}x\right) = \frac{-1}{\sqrt{1-x^2}}, x \in (-1,1)$

  iii) $\frac{d}{dx}\left(Tan^{-1}x\right) = \frac{1}{1+x^2}, x \in R$

  iv) $\frac{d}{dx}\left(Cot^{-1}x\right) = \frac{-1}{1+x^2}, x \in R$

  v) $\frac{d}{dx}\left(Sec^{-1}x\right) = \frac{1}{|x|.\sqrt{x^2-1}}, x < -1 \ or \ x > 1$

  vi) $\frac{d}{dx}(Co\sec^{-1} x) = \frac{-1}{|x|.\sqrt{x^2-1}}, x < -1 \ or \ x > 1$

Derivatives of Hyperbolic functions :

- i) $\frac{d}{dx}(\sinh x) = \cosh x$

  ii) $\frac{d}{dx}(\cosh x) = \sinh x$

  iii) $\frac{d}{dx}(\tanh x) = \sec h^2 x$

  iv) $\frac{d}{dx}(\coth x) = -\cos ech^2 x$

  v) $\frac{d}{dx}(\sec hx) = -\sec hx \tanh x$

  vi) $\frac{d}{dx}(\cos echx) = -\cos echx.\coth x$

Derivatives of Inverse Hyperbolic functions :

- i) $\frac{d}{dx}\left(Sinh^{-1}x\right) = \frac{1}{\sqrt{1+x^2}}$

  ii) $\frac{d}{dx}(Cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}}$, for $x \notin (-1,1)$

  iii) $\frac{d}{dx}\left(Tanh^{-1}x\right) = \frac{1}{1-x^2}, for \ x \in (-1,1)$

  iv) $\frac{d}{dx}\left(Coth^{-1}x\right) = \frac{1}{1-x^2}, for \ x \in (-\infty,-1) U (1,\infty)$

  v) $\frac{d}{dx}\left(Sech^{-1}x\right) = \frac{-1}{x\sqrt{1-x^2}}, for \ x \in (0,1)$

  vi) $\frac{d}{dx}\left(Co\sec h^{-1}x\right) = \frac{-1}{|x|\sqrt{1+x^2}}$

  for x $\in$ (-∞ ,0)U(0, ∞)

Derivative of a Determinant :

- If $y = \begin{vmatrix} u(x) & v(x) & w(x) \\ p(x) & q(x) & r(x) \\ \lambda(x) & \mu(x) & \gamma(x) \end{vmatrix}$ then $\frac{dy}{dx} =$

$$\begin{vmatrix} u^1(x) & v^1(x) & w^1(x) \\ p(x) & q(x) & r(x) \\ \lambda(x) & \mu(x) & \gamma(x) \end{vmatrix} + \begin{vmatrix} u(x) & v(x) & w(x) \\ p^1(x) & q^1(x) & r^1(x) \\ \lambda(x) & \mu(x) & \gamma(x) \end{vmatrix}$$

$$+ \begin{vmatrix} u(x) & v(x) & w(x) \\ p(x) & q(x) & r(x) \\ \lambda^1(x) & \mu^1(x) & \gamma^1(x) \end{vmatrix}$$ or similarly column wise

Parametric Differentiation :

- If x=f(t) and y=g(t) are the parametric equations of a curve then

  i) $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{g^1(t)}{f^1(t)}$

  ii) $\frac{d^n y}{dx^n} = \frac{d}{dt}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)\left(\frac{dt}{dx}\right)$

Derivative of Implicit Functions :

- If $f(x,y) = 0$, then differentiate each term w.r.t. x regarding y as a function of x and then collect the terms of $\frac{dy}{dx}$ together on left hand side and remaining terms on right handside and then find $\frac{dy}{dx}$.

Alternative method : If $f(x,y)=0$

then $$\frac{dy}{dx}=-\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}$$

$$=-\frac{\text{Partial derivative of } f \text{ w.r.t. } x}{\text{Partial derivative of } f \text{ w.r.t. } y}$$

Logarithemic Differentiation :

✈ If $y=\{f_1(x)\}^{f_2(x)}$ or

$y=f_1(x).f_2(x).f_3(x).....$ or

$$y=\frac{f_1(x).f_2(x).f_3(x)....}{g_1(x).g_2(x).g_3(x)....}$$

then take logarithm on both sides and differentiate both sides w.r.t. x

Derivative of Composite Function :

✈ $y=(gof)(x)\Rightarrow y=g(f(x))$

$$\Rightarrow\frac{dy}{dx}=g^1(f(x)).f^1(x)$$

Standard Derivatives :

✈ i) If $y=\sqrt[n]{f(x)+\sqrt[n]{f(x)+\sqrt[n]{f(x)+.........to\ \infty}}}$ then

$$\frac{dy}{dx}=\frac{f^1(x)}{ny^{n-1}-1}.$$

ii) If $y=\sqrt{f(x)+\sqrt{f(x)+\sqrt{f(x)+.........to\infty}}}$ then

$$\frac{dy}{dx}=\frac{f^1(x)}{(2y-1)}.$$

iii) Let f(x), g(x) be the two differentiable functions of 'x' and if $y=f(x)^{g(x)}$ then

$$\frac{dy}{dx}=y\left(g^1(x)\log f(x)+g(x)\frac{f^1(x)}{f(x)}\right).$$

iv) If $y=f(x)+\frac{1}{y}$ then $\frac{dy}{dx}=\frac{y^2f^1(x)}{y^2+1}$

v) If $f(x+y)=f(x).f(y)$ $\forall$ $x,y\in R$ and $f(x)\neq 0$ then $f^1(x)=f^1(0).f(x)$.

vi) If $f(x)=|x|$, then $f^1(0)$ does not exist.

vii) $\frac{d}{dx}\left(\frac{ax+b}{cx+d}\right)=\frac{ad-bc}{(cx+d)^2}$

viii) $\frac{d}{dx}\left(\frac{af(x)+b}{cf(x)+d}\right)=\frac{(ad-bc)f^1(x)}{\{cf(x)+d\}^2}$

ix) $\frac{d}{dx}\{\log_e f(x)\}=\frac{f^1(x)}{f(x)}$

x) $\frac{d}{dx}\{\sqrt{f(x)}\}=\frac{f^1(x)}{2\sqrt{f(x)}}$

xi) $\frac{d}{dx}\left\{\frac{1}{f(x)}\right\}=\frac{-f^1(x)}{(f(x))^2}$

xii) $\frac{d}{dx}\{(f(x))^n\}=n\{f(x)\}^{n-1}.f^1(x)$

xiii) If $y=\operatorname{Tan}^{-1}\left(\frac{f(x)\pm g(x)}{1\mp f(x).g(x)}\right)$ then

$$\frac{dy}{dx}=\frac{f^1(x)}{1+\{f(x)\}^2}\pm\frac{g^1(x)}{1+\{g(x)\}^2}$$

xiv) If $y=f(x)$ and $z=g(x)$ then $\frac{dy}{dz}=\frac{f^1(x)}{g^1(x)}$.

xv) If $y=f(x)^y$ then $\frac{dy}{dx}=\frac{y^2.f^1(x)}{f(x)\{1-y\log_e f(x)\}}$

Substitutions :

✈ While differentiating the given function using trigonometric transformation, observe the following points.

i) If the function involve the term $\sqrt{a^2-x^2}$, then put $x=a\sin\theta$ (or) $x=a\cos\theta$

ii) If the function involve the term $\sqrt{a^2+x^2}$, then put $x=a\tan\theta$ (or) $x=a\cot\theta$

iii) If the function involve the term $\sqrt{x^2-a^2}$, then put $x=a\sec\theta$ (or) $x=a\operatorname{cosec}\theta$

iv) If the function involve the term

$\sqrt{\frac{a-x}{a+x}}$ (or) $\sqrt{\frac{a+x}{a-x}}$, then

put $x=a\cos\theta$ (or) $x=a\cos 2\theta$

Higher Order Derivatives of functions :

✈ **i)** $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$

ii) $\frac{d^3y}{dx^3}=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$

EXAMPLES

1. $\frac{d}{dx}\left[(x+1)(x^2+1)(x^4+1)(x^8+1)\right]=$

$(15x^p-16x^q+1)(x-1)^{-2}\Rightarrow(p,q)=$

Sol : $f(x)=(x+1)(x^2+1)(x^4+1)(x^8+1)=\frac{x^{16}-1}{x-1}$

$\Rightarrow f^1(x)=\frac{16x^{15}(x-1)-(x^{16}-1)}{(x-1)^2}$

$p=16,q=15$

2:

$\frac{d}{dx}\left(\sin^2\left(\log\sqrt{x}\right)\right)=$

Sol: $2\sin\left(\log\sqrt{x}\right)\cos\left(\log\sqrt{x}\right)\frac{1}{\sqrt{x}}.\frac{1}{2\sqrt{x}}$

3:

For a real number 'y', Let [y] denote the integral part of 'y'. Then derivative of the function

$f(x)=\frac{\tan[x-\pi]\pi}{1+[x]^2}$ ***is***

Sol : $[(x-\pi)]\pi$ is an integral multiple of π , hence

$f(x)=0\Rightarrow f^1(x)=0$

4:

If $y=Tan^{-1}\sqrt{\frac{1-x}{1+x}}$ **, then** $\frac{dy}{d\left(\cos^{-1}x\right)}=$

Sol: Let $x=\cos2\theta\ \Rightarrow\ \theta=\frac{1}{2}\cos^{-1}x\ ;\ (-1<x<1)$

$y=Tan^{-1}\sqrt{\frac{1-\cos2\theta}{1+\cos2\theta}}$

$=Tan^{-1}(\tan\theta)=\theta=\frac{1}{2}\cos^{-1}x$

$\therefore\ y=\frac{1}{2}\cos^{-1}x\Rightarrow\frac{d(y)}{d\left(\cos^{-1}x\right)}=\frac{1}{2}$

5:

$$f(x)=\begin{vmatrix}2\cos x & 1 & 0\\ x-\frac{\pi}{2} & 2\cos x & 1\\ 0 & 1 & 2\cos x\end{vmatrix}\Rightarrow f^1(\pi)=$$

[EAM - 2010]

Sol: $$f^1(x)=\begin{vmatrix}-2\sin x & 0 & 0\\ x-\frac{\pi}{2} & 2\cos x & 1\\ 0 & 1 & 2\cos x\end{vmatrix}+$$

$$\begin{vmatrix}2\cos x & 1 & 0\\ 1 & -2\sin x & 0\\ 0 & 1 & 2\cos x\end{vmatrix}+$$

$$\begin{vmatrix}2\cos x & 1 & 0\\ x-\frac{\pi}{2} & 2\cos x & 1\\ 0 & 0 & -2\sin x\end{vmatrix}$$

put $x=\pi$ we get $f^1(\pi)=2$

6:

If $x=a\cos^3\theta,y=a\sin^3\theta$ **then** $\frac{dy}{dx}=$

Sol: $\frac{dx}{d\theta}=-3a\cos^2\theta.\sin\theta$

and $\frac{dy}{d\theta}=3a\sin^2\theta.\cos\theta$

then $\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=-\tan\theta$

7:

$x^y = y^x$ *then* $\frac{dy}{dx} =$

Sol: Take logarithm , we get, $y\log x = x\log y$ differentiating w.r.t x, we get

$$\log x\frac{dy}{dx} + y.\frac{1}{x} = 1.\log y + x.\frac{1}{y}\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y}{x}.\frac{(x\log y - y)}{(y\log x - x)}$$

8:

If $f(x) = e^x, g(x) = Sin^{-1}x$ **and**

$h(x) = f(g(x))$**, then** $\frac{h^1(x)}{h(x)} =$

Sol: $h^1(x) = f^1(g(x))g^1(x)$

$$\Rightarrow \frac{h^1(x)}{h(x)} = \frac{f^1(g(x))g^1(x)}{f(g(x))} = \frac{1}{\sqrt{1-x^2}}$$

9:

$y = Tan^{-1}\left(\frac{x}{\sqrt{1+x^2}-1}\right)$, **then** $\frac{dy}{dx} =$

Sol: Put $x = \tan\theta$ then

$$y = \tan^{-1}\left(\cot\frac{\theta}{2}\right)$$

$$= \frac{\pi}{2} - \frac{1}{2}\tan^{-1}x$$

$$\therefore \frac{dy}{dx} = -\frac{1}{2(1+x^2)}$$

10:

If $y = \sin(\log_e x)$**, then** $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} =$ **(EAM-2008)**

Sol: $\frac{dy}{dx} = \frac{\cos(\log x)}{x} \Rightarrow x\frac{dy}{dx} = \cos(\log_e x)$

$$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{-\sin(\log_e x)}{x}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y$$

EXERCISE - I

1. $\frac{d}{dx}\left(\sin\sqrt{2x-3}\right) =$

1) $\frac{\cos\sqrt{2x-3}}{\sqrt{2x-3}}$ 2) $\frac{-\cos\sqrt{2x-3}}{2\sqrt{2x-3}}$

3) $\sqrt{2x-3}\cos\sqrt{2x-3}$ 4) $\cos\sqrt{2x-3}$

2. $\frac{d}{dx}\left\{e^{\log\sqrt{1+\cot^2 x}}\right\} =$

1) cosecx cotx 2) -cosecx.cotx

3) cosec²x.cotx 4) 0

3. $\frac{d}{dx}\left(\sqrt{Sin\sqrt{x}}\right) =$

1) $\frac{\cos\sqrt{x}}{4\sqrt{x\sin\sqrt{x}}}$ 2) $\frac{\sin\sqrt{x}}{4\sqrt{x\cos\sqrt{x}}}$

3) $\frac{-\cos\sqrt{x}}{4\sqrt{x\sin\sqrt{x}}}$ 4) $\frac{-\tan\sqrt{x}}{4\sqrt{x\cos\sqrt{x}}}$

4. $\frac{d}{dx}$ **[log{log(logx)}]=**

1) $\frac{1}{x\log x\log(\log x)}$ 2) $\frac{-1}{x\log x\log(\log x)}$

3) $\frac{x}{\log x\log(\log x)}$ 4) $\frac{1}{\log x\log(\log x)}$

5. If $y = 2^{ax}$ **and** $\frac{dy}{dx} = \log 256$ **at x=1, then the value of a is**

1) 0 2) 1 3) 2 4) 3

6. If $y = Tan^{-1}(\sec x + \tan x)$ **then** $\frac{dy}{dx} =$

1) 1 2) $\frac{1}{2}$ 3) -1 4) 0

7. If y=$Tan^{-1}\left(\frac{(3-x)\sqrt{x}}{1-3x}\right)$ **then** $\frac{dy}{dx} =$

1) $\frac{3}{(1+x)\sqrt{x}}$ 2) $\frac{3}{2(1+x)\sqrt{x}}$

3) $\frac{-3}{2(1+x)\sqrt{x}}$ 4) 0

8. If y=$Tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ then $\frac{dy}{dx}$= [EAM - 2020]

1) 1 2) -1 3) $\frac{-1}{2}$ 4) $\frac{1}{3}$

9. If y=$Tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)$ then $\frac{dy}{dx}$=

1) $\frac{3a}{a^2+x^2}$ 2) $\frac{1}{a^2+x^2}$

3) $\frac{-3a^2}{a^2+x^2}$ 4) $\frac{-3a}{a^2+x^2}$

10. If $y=(\sin x)^x$ then $\frac{dy}{dx}$=

1) $y(\log(\sin x)+x\cot x)$

2) $y(\log(\sin x)-x\cot x)$

3) $-y(\log(\sin x)-x\cot x)$

4) $-y(\log(\sin x)+x\cot x)$

11. If $x^y=e^{x-y}$ then $\frac{dy}{dx}$=

1) $\frac{\log x}{(1+\log x)^2}$ 2) $-\frac{\log x}{(1+\log x)^2}$

3) $-\frac{\log x}{(1-\log x)^2}$ 4) $\frac{\log x}{(1-\log x)^2}$

12. If y=2^{2^x} then $\frac{dy}{dx}$=

1) y.(log2)^2)2^x 2) y(log2).2^x

3) y2(log2)^2)2^x 4) -y(log2).2^x

13. $\frac{d}{dx}\left(\sinh^{-1}(3x)\right)$=

1) $\frac{1}{\sqrt{1+9x^2}}$ 2) $\frac{2}{\sqrt{1+9x^2}}$

3) $\frac{3}{\sqrt{1+9x^2}}$ 4) $\frac{9}{\sqrt{1+9x^2}}$

14. $\frac{d}{dx}\left(\cosh^{-1}\frac{x}{2}\right)$=

1) $\frac{1}{\sqrt{x^2+4}}$ 2) $\frac{1}{\sqrt{x^2-4}}$ 3) $\frac{-1}{\sqrt{x^2+4}}$ 4) $\frac{-1}{\sqrt{x^2-4}}$

15. If x=a(t+sint), y=a(1-cost) if $\frac{dx}{dy}$=cot p then p=

1) t 2) 2t 3) $\frac{t}{2}$ 4) $-t^2$

16. If x=a(cost+log(tan $t/2$)), y=asint then $\frac{dy}{dx}$= (EAM-2018)

1) sin t 2) cot t 3) tan t 4) $\tan^2$ t

17. The derivative of $Cos^{-1}\frac{1-x^2}{1+x^2}$ w.r.t. $Tan^{-1}\frac{2x}{1-x^2}$ is

1) 0 2) 1 3) 2 4) $\frac{1}{2}$

18. The derivative of $Tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ w.r.t. $Tan^{-1}x$ is [EAM -2019]

1) 0 2) 1 3) 2 4) $\frac{1}{2}$

19. If $ax^2+2hxy+by^2=0$ then $\frac{dy}{dx}$=

1) $-\left(\frac{ax+hy}{hx+by}\right)$ 2) $\left(\frac{ax+hy}{hx+by}\right)$

3) -(ax+hy)(hx+by) 4) (ax+hy)(hx+by)

20. If e^{x+y} = xy then $\frac{dy}{dx}$= [EAM -2017]

1) $\frac{y(1-x)}{x(y-1)}$ 2) $\frac{-y(1-x)}{x(y-1)}$ 3) $\frac{x(y-1)}{y(1+x)}$ 4) $\frac{-x(y-1)}{y(1+x)}$

21. If $\sin(x+y)=\log(x+y)$, then $\frac{dy}{dx}$=

1) 2 2) -2 3) 1 4) -1

22. If $x^2-y^2=a(x-y)$ and $x\neq y$, then $\frac{dy}{dx}$=

1) 1 2) $\frac{1}{2}$ 3) $\frac{1}{3}$ 4) -1

KEY

01) 1	02) 2	03) 1	04) 1	05) 3	06) 2
07) 2	08) 3	09) 1	10) 1	11) 1	12) 1
13) 3	14) 2	15) 3	16) 3	17) 2	18) 4
19) 1	20) 1	21) 4	22) 4		

SOLUTIONS

1. $\cos\left(\sqrt{2x-3}\right).\frac{1}{2\sqrt{2x-3}}$

2. $\frac{d}{dx}\left(e^{\log_e \operatorname{cosecx}}\right) = \frac{d}{dx}(\operatorname{cosecx})$

3. $\frac{1}{2\sqrt{\sin\sqrt{x}}}\cos\sqrt{x}.\frac{1}{2\sqrt{x}}$

4. $\frac{1}{\log(\log x)}.\frac{1}{\log x}.\frac{1}{x}$

5. Given y = 2^{ax} and $\frac{dy}{dx}$ = log 256 at x = 1

logy = ax log 2 Differentiate w.r.t. x

$\frac{1}{y}\frac{dy}{dx} = a\log 2$ at x - 1m y = 2^9

$\frac{dy}{dx}$ = a 2^9 log 2 $\Rightarrow$ log 2^8 = a2^9 log 2

8 = a2^9 $\Rightarrow$ a = 2

6. y = $\tan^{-1}$(secx+tanx) differentiatte w.r.t x

$$\frac{dy}{dx} = \frac{1}{1+(\sec x+\tan x)^2}.\frac{d}{dx}(\sec x+\tan x) = \frac{\sec x(\sec x+\tan x)}{2\sec^2 x+2\tan x\sec x}$$

$$\frac{dx}{dy} = \frac{1}{2}$$

7. Put $\sqrt{x}$ tan $\theta \Rightarrow \theta = \tan^{-1}\sqrt{x}$

y = $\tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)$ = $\tan^{-1}(\tan 3\theta)$

= 3θ

y = 3 $\tan^{-1}\sqrt{3}$ differentiate w.r.t. x

$$\frac{dy}{dx} = \left(\frac{3}{1+x}\right).\frac{1}{2\sqrt{x}}$$

8. $y = \tan^{-1}\left(\frac{\sin\left(\pi/2 - x\right)}{1+\cos\left(\pi/2 - x\right)}\right)$

9. Put $x = a\tan\theta$

10. y = $(\sin x)^4$ taking log on both sides

logy = x log (sinx) differentate w.r.t. x

$$\frac{1}{y}.\frac{dy}{dx} = x.\frac{1}{\sin x}\cos x + \log(\sin x)$$

$$\frac{dy}{dx} = (\sin x)^x\left(x(\cot x) + \log(\sin x)\right)$$

11. $x^y = e^{x-y}$ taking logrithen on bothsides y

log x = (x - y)

y (1+logx) = x $\Rightarrow$ y = $\frac{x}{1+\log x}$

differentiate w.r.t. x

$$\frac{dy}{dx} = \frac{(1+\log x) - x\left(\frac{1}{x}\right)}{(1+\log x)} = \frac{\log x}{(1+\log x)^2}$$

12. $\frac{d}{dx}\left(a^x\right) = a^x \log a.\frac{d}{dx}(x)$

13. $\frac{d}{dx}\left(\sinh^{-1}x\right) = \frac{1}{\sqrt{1+x^2}}$

15. $\frac{dx}{dy} = \frac{\left(dx/dt\right)}{\left(dy/dt\right)} = \cot p$

16. x = a (cos t + log (tan t/2)), y = a sint

$$\frac{dy}{dx} = a\left(-\sin t + \frac{1}{\tan\frac{t}{2}}.\sec^2\frac{t}{2}.\frac{1}{2}\right)\frac{dy}{dx} = a\cos t$$

= a (-sin t + $\frac{1}{\sin t}$)

$$\frac{dy}{dx} = \frac{\frac{dy}{dx}}{\frac{dx}{dt}} = \frac{a\cos t}{a\cos^2 t}x\sin t = \tan t$$

$$\frac{dy}{dx} = a\left(\frac{\cos^2 t}{\sin t}\right)$$

17. Let y = cos-¹ $\left(\frac{1-x^2}{1+x^2}\right)$ z = tan⁻¹ $\left(\frac{2x}{1-x^2}\right)$

put x = tan θ , θ = $\tan^{-1}$x

y = $\cos^{-1}(\cos 2\theta)$ z = $\tan^{-1}(\tan 2\theta)$

y = 2tan-¹ x z = $2\tan^{-1}$x $\frac{dy}{dx} = 1$

18. Let y = $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ z = $\tan^{-1}$ x put

x = tan $\theta \Rightarrow \theta$= $\tan^{-1}$ x

y = $\tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$

= $\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$ z = $\tan^{-1}$ (tan θ)

= $\tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$ z = θ = $\tan^{-1}$ x

y = $\tan^{-1}(\tan\frac{\theta}{2}) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$ $\frac{dy}{dx} = \frac{1}{2}$

19. $ax^2 + 2hxy + by^2 = 0$

$$\frac{dy}{dx} = \frac{\frac{-\delta f}{\delta x}}{\frac{-\delta f}{\delta y}} = \frac{-(2ax+2hy)}{2hx+2by} = \frac{-(ax+hy)}{hx+by}$$

20. $e^y e^y = xy \Rightarrow e^y e^y - xy = 0$

$$\frac{dy}{dx} = \frac{\frac{-\delta f}{\delta x}}{\frac{\delta f}{\delta y}} = \frac{-(e^x e^y - y)}{e^x e^y - x} = -\frac{(xy-y)}{xy-x} = \frac{-y(x-1)}{x(y-1)}$$

21. common 1 + $\frac{dy}{dx} \Rightarrow \frac{dy}{dx} + 1 = 0 \Rightarrow \frac{dy}{dx} = -1$

22. x + y = a = differentiate w.r.t x

$$1 + \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -1$$

EXERCISE - II

1. If $y=\sqrt{\frac{\sec x-1}{\sec x+1}}$ **then** $\frac{dy}{dx}=$

1) $\frac{1}{2}\sec^2\frac{x}{2}$ 2) $\sec^2\frac{x}{2}$ 3) $\frac{1}{2}\tan\frac{x}{2}$ 4) $\tan\frac{x}{2}$

2. $\frac{d}{dx}\left\{\log(x+\sqrt{a^2+x^2})\right\}=$

1) $\frac{1}{(x+\sqrt{a^2+x^2})}$ 2) $\frac{x}{\sqrt{a^2+x^2}}$

3) $\frac{1}{x(x+\sqrt{a^2+x^2})}$ 4) $\frac{1}{\sqrt{a^2+x^2}}$

3. If $y=e^{\sin^2 x+\sin^4 x+\sin^6 x+....+\infty}$**, then** $\frac{dy}{dx}=$

1) $e^{\tan^2 x}$ 2) $e^{\tan^2 x}\sec^2 x$

3) $2e^{\tan^2 x}\tan x.\sec^2 x$ 4) 1

4. $If\ y=x+e^x$**, then** $\frac{d^2x}{dy^2}$ **is equal to**

1) $\frac{1}{(1+e^x)^2}$ 2) $\frac{-e^x}{(1+e^x)^2}$

3) $\frac{-e^x}{(1+e^x)^3}$ 4) e^x

5. If $x.e^{xy}=y+\sin^2 x$**. then at** $x=0$**,** $\frac{dy}{dx}=$

1) 1 2) 2 3) 3 4) 0

6. If $y=Tan^{-1}\left(\frac{2^x}{1+2^{2x+1}}\right)$ **then** $\frac{dy}{dx}$ **at** $x=0$ **is**

1) $\frac{1}{10}\log 2$ 2) $\frac{1}{5}\log 2$

3) $-\frac{1}{10}\log 2$ 4) $\log 2$

7. $\frac{d}{dx}\left\{Cot^{-1}\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right\}=$

1) $\frac{1}{\sqrt{1-x^2}}$ 2) $\frac{-1}{2\sqrt{1-x^2}}$ 3) $\frac{1}{1+x^2}$ 4) $\frac{-1}{2(1+x^2)}$

8. $\frac{d}{dx}\left\{Tan^{-1}\left(\frac{a\sin x+b\cos x}{a\cos x-b\sin x}\right)\right\}=$

1) 0 2) 1 3) -1 4) $\frac{1}{1+x^2}$

9. $\frac{d}{dx}\left[\sin^2\left(\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)\right]=$ **[EAM -2016]**

1) 1 2) -1 3) $\frac{1}{2}$ 4) $\frac{-1}{2}$

10. $\frac{d}{dx}\left[(\cos x)^{\log x}+(\log x)^x\right]=$

1) $(\log x)^x\left[\frac{1}{\log x}+\log(\log x)\right]$

$+(\cos x)^{\log x}\left[\frac{1}{x}\log(\cos x)-\log x.\tan x\right]$

2) $(\cos x)^{\log x}\left[\log(\cos x)-\cot x.\log x\right]$

$+(\log x)^{\log x}\left[1+\log(\log x)\right]$

3) $\left((\cos x)^{\log x}+(\log x)^x\right)\left[\log x.\cos x+x\log x\right]$

4) $(\log x)^x\left[\log x+\log(\log x)\right](\cos x)^{\log x}$

11. $\frac{d}{dx}\left\{\left(x^x\right)^x\right\}=$ **[EAM -2015]**

1) $\left(x^x\right)^x$ {x(1+logx)} 2) $\left(x^x\right)^x$ .x(1-2logx)

3) $\left(x^x\right)^x$ (1+2logx)x 4) $\left(x^x\right)^x$ .x^2(1-2logx)

12. If $(\cos x)^y=(\sin y)^x$ **then** $\frac{dy}{dx}=$

1) $\frac{\log(\sin y)+y\tan x}{\log(\cos x)-x\cot y}$ 2) $\frac{\log(\sin y)-y\tan x}{\log(\cos x)+\cot y}$

3) $\frac{\log(\sin y)}{\log(\cos x)}$ 4) $\frac{\log(\cos x)}{\log(\sin y)}$

13. If $x=\sin^{-1}t$ **and** $y=\log(1-t^2)$**, then** $\left.\frac{d^2y}{dx^2}\right|_{t=\frac{1}{2}}$ **is**

1) $\frac{-8}{3}$ 2) $\frac{8}{3}$ 3) $\frac{3}{4}$ 4) $\frac{-3}{4}$

14. If $x^2 + y^2 = t + \frac{1}{t}$ and $x^4 + y^4 = t^2 + \frac{1}{t^2}$ then $x^3y\frac{dy}{dx} =$ **[EAM-2019]**

1) 0 2) 1 3) -1 4) 2

15. If $\sqrt{y-\sqrt{y-\sqrt{y.....\infty}}} = \sqrt{x+\sqrt{x+\sqrt{x.....\infty}}}$ then $\frac{dy}{dx}$ is equal to

1) $\frac{y-x+1}{y-x-1}$ 2) $\frac{y-x}{y+x}$

3) $\frac{x+y+1}{y-x}$ 4) $\frac{y-x+1}{y+x}$

16. If $y=\sqrt{x+\sqrt{y+\sqrt{x+\sqrt{y+...\infty}}}}$ then $\frac{dy}{dx}$ is equal to

1) $\frac{y^2+x}{2y^3-2xy-1}$ 2) $\frac{y^2-x}{2y^3-2xy-1}$

3) $\frac{y^2-x}{2y^3-2xy+1}$ 4) $\frac{y^2}{2y^3-2xy+1}$

17. If x siny = 3 siny + 4 cosy , then $\frac{dy}{dx} =$

1) $\frac{-\sin^2 y}{4}$ 2) $\frac{\sin^2 y}{4}$

3) $\frac{-\cos^2 y}{4}$ 4) $\frac{\cos^2 y}{4}$

18. If cosy=x.cos(a+y) then $\frac{dy}{dx} =$

1) $\frac{\cos^2(a+y)}{\sin a}$ 2) $\frac{\cos^2(a+y)}{\cos a}$

3) $\frac{\cos a}{\sin^2(a+y)}$ 4) $\frac{\cos(a+y)}{\sin a}$

19. If $y=\sqrt{a^{3x+1}+\sqrt{a^{3x+1}+\sqrt{a^{3x+1}+......to\infty}}}$ then $\frac{dy}{dx} =$ **[EAM -2020]**

1) $\frac{a^{3x+1}.\log a}{(2y-1)}$ 2) $\frac{3.a^{3x+1}.\log a}{(2y-1)}$

3) $\frac{3.a^{3x+1}.\log a}{(2y+1)}$ 4) $\frac{a^{3x+1}.\log a}{(2y+1)}$

20. If $x\sqrt{1+y}+y\sqrt{1+x}=0$ and $x \neq y$ then $\frac{dy}{dx} =$

1) $\frac{x+1}{(1-x)^2}$ 2) $\frac{1}{(1-x)^2}$

3) $\frac{-1}{(1+x)^2}$ 4) $\frac{1}{(1+x)^2}$

21. If $y=\frac{1}{x}$ then $\frac{dy}{\sqrt{1+y^4}}+\frac{dx}{\sqrt{1+x^4}} =$

1) 0 2) 1 3) $\frac{x}{y}$ 4) $\frac{y}{x}$

22. If g is the inverse of f and $f^1(x)=\frac{1}{2+x^n}$, then $g^1(x)$ is equal to

1) $2+x^n$ 2) $2+(f(x))^n$ 3) $2+(g(x))^n$ 4) 0

KEY

01) 1	02) 4	03) 3	04) 3	05) 1	06) 3
07) 2	08) 2	09) 4	10) 1	11) 3	12) 1
13) 1	14) 3	15) 1	16) 2	17) 1	18) 1
19) 2	20) 3	21) 1	22) 3		

SOLUTIONS

1. $y = \sqrt{\frac{2\sin^2 x/2}{2\cos^2 x/2}} = \tan x/2$

2. $\frac{d}{dx}\left\{\log\left(x+\sqrt{a^2+x^2}\right)\right\}=\frac{d}{dx}\left(\sinh^{-1}\frac{x}{a}\right)=\frac{1}{\sqrt{x^2+a^2}}$

3. y = $e^{\tan^2 x}$ differentiate w.r.t. x

$$\frac{dy}{dx}=e^{\tan^2 x}.\frac{d}{dx}\left(\tan^2 x\right)=e^{\tan^2 x}2\tan x.\sec^2 x$$

$$= e^{\left(\sin^2 x/\cos^2 x\right)} = e^{\tan^2 x}$$

4. $\Rightarrow \frac{dy}{dx}=1+e^x$; $\Rightarrow \frac{dx}{dy}=\frac{1}{1+e^x}$

$$\Rightarrow \frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{1}{1+e^x}\right) \Rightarrow \frac{d^2x}{dy^2} = \frac{-1}{\left(1+e^x\right)^2}\frac{d}{dy}\left(1+e^x\right)$$

$$\Rightarrow \frac{d^2x}{dy^2} = \frac{-1}{\left(1+e^x\right)^2} e^x \frac{dx}{dy} = \frac{-e^x}{\left(1+e^x\right)^3}$$

5. If x = 0 then y = 0

$$e^{xy} + x.e^{xy}\left[x.\frac{dy}{dx} + y\right] = \frac{dy}{dx} + 2\sin x\cos x$$

Put x = 0, y = 0, we get $\left(\frac{dy}{dx}\right)_{x=0} = 1$

6.y = $\tan^{-1}\left(\frac{z^x}{1+2^{2x+1}}\right)$ at x = 0

$$y = \tan^{-1}\left(\frac{2^x(2-1)}{1+2^x.2^{x+1}}\right) = \tan^{-1}\left(\frac{2^{x+1}-2^x}{1+2^x-2^{x+1}}\right)$$

y = $\tan^{-1}(2^{x+1}) - \tan^{-1}(2^x)$ differentiate w.r.t. x

$$\frac{dy}{dx} = \frac{1}{1+\left(2^{x+1}\right)} - 2^{x+1}\log 2 - \frac{1}{1+\left(2^x\right)^2}2^x\log 2$$

$$\left(\frac{dy}{dx}\right)_{x=0} = \frac{1}{5}2\log 2 - \frac{1}{2}\log 2 \quad \log 2\left(\frac{-1}{10}\right)$$

7. Put x = $\cos\theta \Rightarrow \theta = \cos^{-1}x$

$$\frac{d}{dx}\left(\cot^{-1}\left(\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right)\right) = \frac{d}{dx}\left(\cot^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right)\right)$$

$$= \frac{d}{dx}\left(\cot^{-1}\left(\cot\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right)\right)$$

$$= \frac{d}{dx}\left(\cot^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right)\right)$$

$$= \frac{d}{dx}\left(\cot^{-1}\left(\cot\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right)\right) =$$

$$\frac{d}{dx}\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x\right)$$

$$= \frac{-1}{2\sqrt{1-x^2}}$$

8. Put $a = \cos\alpha, b = \sin\alpha$, then

$$\tan^{-1}\left(\frac{\sin(x+\alpha)}{\cos(x+\alpha)}\right) = \tan^{-1}\left(\tan(x+\alpha)\right)$$

9. Put x = $\cos\theta \Rightarrow \theta = \cos^{-1}x$

$$\frac{d}{dx}\left(\sin^2\left(\cot^{-1}\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\right)\right) =$$

$$\frac{d}{dx}\left(\sin^2\left(\cot^{-1}\cot\frac{\theta}{2}\right)\right)$$

$$= \frac{d}{dx}\sin^2\left(\frac{1}{2}\cos^{-1}x\right) = \frac{d}{dx}\left(\frac{1-\cos\left(\cos^{-1}x\right)}{2}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}-\frac{x}{2}\right) = \frac{-1}{2}$$

10. $\frac{d}{dx}\left\{f(x)^{g(x)}\right\} =$

$$f(x)^{g(x)}\left\{g(x).\frac{1}{f(x)}.f^1(x) + \log f(x).g^1(x)\right\}$$

11. $\frac{1}{y}\frac{dy}{dx} = x^2 - \frac{1}{x} + \log x.2x$

$\frac{dy}{dx}$ = y(x+2xlogx) = $(2^x)^x$(1+2logx)x

12. $(\cos x)^y = (\sin y)^x$ faking logar than on both sides

y log(cosx) = x log (siny) differentiate w.r.t x

y $\frac{1}{\cos x}$(-sinx) + log(cosx) $\frac{dy}{dx}$ = x.

$$\frac{1\cos y}{\sin y}\frac{dy}{dx}+\log(\sin y)$$

$\frac{dy}{dx}\left(\log(\cos x)-x\cot y\right)$=(log(siny) + y tanx)

$$\frac{dy}{dx}=\frac{\log(\sin y)+y\tan x}{\log(\cos x)-x\cot y}$$

13. x = $\sin^{-1}$ t y = log(1-t^2) differetiate w.r.t. 't'

$$\frac{dy}{dx}=\frac{1}{\sqrt{1+t^2}}\quad \frac{dy}{dx}=\frac{1}{1-t^2}(-2t)$$

$\frac{dy}{dx}=\frac{-2t}{1-t^2}x\sqrt{1-t^2}=\frac{-2t}{\sqrt{1+t^2}}$ again

differentiate wl.r.t. x

$$=\frac{d^2y}{dx^2}=\frac{-2d}{dt}\left(\frac{t}{\sqrt{1-t^2}}\right)\frac{dt}{dx}$$

$$=-2\left(\frac{\sqrt{1+t^2}-1-\frac{1}{2\sqrt{1-t^2}}^{1-2t}}{1-t^2}\right)\frac{dt}{dx}$$

$$= -2\left(\frac{1-t^2+t^2}{\left(1-t^2\right)^{3/2}}\right).\sqrt{1-t^2}=\frac{-2}{1-t^2}$$

$\left(\frac{d^2y}{dx^2}\right)_{t=\frac{1}{2}}=\frac{-2}{1-\frac{1}{4}}=\frac{-8}{3}$ put t = 1/2

14. $\left(x^2+y^2\right)^2=\left(t+\frac{1}{t}\right)^2$

$$\Rightarrow x^4+y^4+2x^2y^2=t^2+\frac{1}{t^2}+2$$

$$\Rightarrow x^4+y^4+2x^2y^2=x^4+y^4+2$$

$\Rightarrow x^2y^2=1$ and then differentiate

15. $\sqrt{y-\sqrt{y-\sqrt{y.....\infty}}}=\sqrt{x+\sqrt{x+...\infty}}=v$

$$\sqrt{y-v}=\sqrt{x+v}=v$$

$\Rightarrow \left.\begin{matrix} y=v^2+v \\ x=v^2-v\end{matrix}\right\}\Rightarrow \frac{y+x}{2}=v^2$ and $\frac{y-x}{2}=v$

By eliminating v we get the solution.

16.y^2 = x + $\sqrt{2y}$ differentiate w.r.t. x

$$2y\frac{dy}{dx}=1+\frac{1}{\not 2\sqrt{2y}}.\not 2\frac{dy}{dx}$$

$$=\frac{dy}{dx}\left(2y-\frac{1}{\sqrt{2y}}\right)=1$$

=

$\frac{dy}{dx}=\frac{1}{2y-\frac{1}{y^2-x}}=\frac{y^2-x}{2y^3-2xy-1}$ $\quad\therefore \sqrt{2y}=y^2-x$

17. siny (x-3) = 4cosy differentiate w.r.t. x

$$\therefore x-3=\frac{4\cos y}{\sin y}$$

x -3 = 4 cot y

1 = -4 cose^2y $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-\sin^2 y}{4}$$

18. $$1 = \frac{-\cos(a+y)\sin y\frac{dy}{dx} + \cos y\sin(a+y)\frac{dy}{dx}}{\cos^2(a+y)}$$

$$\Rightarrow \cos^2(a+y) = \sin(a+y-y)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

19. $\frac{dy}{dx} = \frac{f^1(x)}{(2y-1)}$

20. $x\sqrt{1+y} = -y\sqrt{1+x} \Rightarrow x^2(1+y) = y^2(1+x)$

$\Rightarrow x^2 - y^2 = xy^2 - x^2y$

$\Rightarrow (x-y)(x+y+xy) = 0$, if $x \neq y$

$\Rightarrow x + y + xy = 0$

$\Rightarrow y(x+1) = -x \Rightarrow y = \frac{-x}{x+1}$, differentiae

21. $\frac{dy}{dx} = -\frac{1}{x^2}$

$$\frac{dy}{dx} + \frac{\sqrt{1+y^4}}{\sqrt{1+x^4}} = \frac{dy}{dx} + \frac{1}{x^2} = -\frac{-1}{x^2} + \frac{1}{x^2} = 0$$

22. $g(x) = f^{-1}(x) \Rightarrow (fog)(x) = x \Rightarrow f'[g(x)]g'(x) = 1$

$\Rightarrow f'[g(x)] = \frac{1}{g'(x)} \Rightarrow \frac{1}{2+[g(x)]^n} = \frac{1}{g'(x)}$

then $g'(x) = 2 + [g(x)]^n$

EXERCISE - III

1. If $(f(x))^{g(y)} = e^{f(x)-g(y)}$ **then** $\frac{dy}{dx} =$

1) $\frac{f^1(x)\log f(x)}{g(y)(1+\log f(x))^2}$ 2) $\frac{f^1(x)\log f(x)}{g^1(y)(1+\log f(x))^2}$

3) $\frac{f(x).\log f(x)}{g^1(y)(1-\log f(x))^2}$ 4) $2\frac{f^1(x)\log f(x)}{g(y)(1+\log f(x))^2}$

2. If $\cos^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right) = \log a$ *then* $\frac{dy}{dx} =$

1) $-\frac{x}{y}$ 2) $-\frac{y}{x}$ 3) $\frac{y}{x}$ 4) $\frac{x}{y}$

3. $\phi(x) = f(x)g(x)$ *and* $f'(x)g'(x) = k$ **, then** $\frac{2k}{f(x)g(x)} =$

1) $\frac{\phi''(x)}{\phi(x)} - \frac{f''(x)}{f(x)} - \frac{g''(x)}{g(x)}$ 2) $\frac{\phi''(x)}{\phi(x)} + \frac{f''(x)}{f(x)} + \frac{g''(x)}{g(x)}$

3) $\frac{\phi''(x)}{\phi(x)} + \frac{f''(x)}{f(x)} - \frac{g''(x)}{g(x)}$ 4) $\frac{\phi''(x)}{\phi(x)} - \frac{f''(x)}{f(x)} + \frac{g''(x)}{g(x)}$

4. Let f : R $\to$ R such that for all 'x' and y in R, $|f(x) - f(y)| \leq |x - y|^3$ then f(x)

1) e^x 2) e^{-x} 3) x 4) 'c'(constant)

5. If $\sqrt{\tan y} = e^{\cos 2x}.Sinx$, **then** $\frac{dy}{dx} =$

1) $Sin2y.(Cotx - 2Sin2x)$

2) $Sin2x(\cot y - \sin y)$

3) Sin2y.Sin2x 4) cos2y.cos2x

6. If x < 1, then

$$\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + ...\infty =$$

1) x 2) $\frac{1}{1+x}$ 3) $\frac{1}{1-x}$ 4) $\frac{1}{x}$

7. If f(x) = $\frac{g(x)+g(-x)}{2} + \frac{2}{[h(x)+h(-x)]^{-1}}$ **where g and h are differentiable functions then $f^1(0)$=**

1) 1 2) 0 3) $\frac{1}{2}$ 4) $\frac{3}{2}$

8. If $x=\sqrt{\frac{1-t^2}{1+t^2}}$, $y=\frac{\sqrt{1+t^2}-\sqrt{1-t^2}}{\sqrt{1+t^2}+\sqrt{1-t^2}}$ **then** $\frac{dy}{dx}=$

1) $\frac{1-x}{1+x}$ 2) $\frac{-2}{(1+x)^2}$

3) $\frac{2}{(1+x)^2}$ 4) $\frac{1+x}{1-x}$

9. $\frac{d}{dx}\left\{e^{\frac{1}{2}\log\left(1-Tanh^2x\right)}+3^{\frac{1}{2}\log_3\left(\cot h^2x-1\right)}\right\}=$

1) $\frac{-\left(\sin h^3x+\cos h^3x\right)}{\cos h^2x\sin h^2x}$ 2) $\frac{Sin\,h^3x+\cos h\,3x}{\cos h^2x\sin h^2x}$

3) $\frac{\sin h^2x+\cos h^2x}{\sin h^2x\cos h^2x}$ 4) $\sec h^2x\cos h^2x$

10. If $x=\cos\theta, y=\sin 5\theta$ **then**

$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=$

1) -5y 2) 5y 3) 25y 4) -25y

11. If $y=\log\left\{\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}\right\}-\frac{1}{2}\tan^{-1}(x)$, **then** $\frac{dy}{dx}=$

1) $\frac{x}{1-x^2}$ 2) $\frac{x^2}{1-x^4}$ 3) $\frac{x}{1+x^4}$ 4) $\frac{x}{1-x^4}$

12. If f(x)= $\begin{vmatrix} Sin\ x & 1 & 0 \\ x-\frac{\pi}{2} & Sin\ x & 1 \\ 0 & 1 & Sin\ x \end{vmatrix}$ **then** $\frac{df}{dx}$ at $x=\frac{\pi}{2}$

is

1) 2 2) $\frac{\pi}{2}$ 3) 1 4) 8

13. If y = $\log\cos\left(\tan^{-1}\left(\frac{e^x-e^{-x}}{2}\right)\right)$, **then** $\frac{dy}{dx}=$

1) -tanhx 2) sinhx 3) coshx 4) cothx

14. Let f(x) = x^n , n being a positive integer. The value of n for which $f^1(a+b) = f^1(a) + f^1(b)$, when a, b > 0 is

1) 1 2) 2 3) 3 4) 4

15. If P (x) = ax^3+bx^2+cx+d **and** $p(0)=4$, $P^1(0)=3,\ P^{11}(0)=4, P^{111}(0)=6$ **then arrange the values of a, b, c, d in the descending order of their values**

1) a, b, c, d 2) a, b, d, c
3) d, c, b, a 4) b, a, c, d

16. If $f^1(x)=g(x)$ and $g^1(x)=-f(x)$ for all x and f(2) = 4 = f^1 (2) then $f^2(24)+g^2(24)$ is

1) 32 2) 24 3) 64 4) 48

17. If f(x)=(cos x+i sin x)(cos 3x+i sin 3x)..... (cos(2n-1)x + i sin(2n-1)x), then $f^{11}(x)$ is

1) $n^2f(x)$ 2) $-n^4f(x)$ 3) $-n^2f(x)$ 4) $n^4f(x)$

18. If y^2=p(x), a polynomial of degree 3, then $2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$ **is equal to**

1) $p^{111}(x)+p^1(x)$ 2) $p^{11}(x)+p^{111}(x)$
3) $p(x)p^{111}(x)$ 4) constant

19. If y= $\cos^{-1}\frac{9-x^2}{9+x^2}$ **then $y^1(-1)$ is equal to**

1) -3/5 2) 3/5 3) 2/7 4) 3/8

20. If f(x) = sgn (cos x), then $f^1\left(\frac{\pi}{2}\right)$ **is**

1) 0 2) -1 3) 1 4) 2

21. If (sinx)(cosy)=1/2 then $\frac{d^2y}{dx^2}$ **at** $\left(\frac{\pi}{4},\frac{\pi}{4}\right)$ **is equal to**

1) -4 2) -2 3) -6 4) 0

22. Let y=e^{2x}. Then $\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right)=$

1) 1 2) e^{-2x} 3) $2e^{-2x}$ 4) $-2e^{-2x}$

23. If $\sqrt{x+y}+\sqrt{y-x}=c$ **, then** $\frac{d^2y}{dx^2}=$

1) $\frac{2}{c}$ 2) $\frac{-2}{c^2}$ 3) $\frac{2}{c^2}$ 4) $\frac{-2}{c}$

24. If f(x)=cosx cos2x cos4x cos8x cos16x then $f^1\left(\frac{\pi}{4}\right)$ **equals**

1) $-\sqrt{2}$ 2) 0 3) $\frac{1}{\sqrt{2}}$ 4) $Cosec\frac{\pi}{4}$

25. **If** $y^{y^{y^{\cdot^{\cdot^{\infty}}}}} = \log_e(x + \log_e(x + ...))$ **then** $\frac{dy}{dx}$ **at**

(x=e²-2,y= $\sqrt{2}$ **) is**

1) $\frac{\log\left(e/2\right)}{2\sqrt{2}(e^2-1)}$ 2) $\frac{\log 2}{2\sqrt{2}(e^2-1)}$

3) $\frac{\sqrt{2}\log\left(e/2\right)}{(e^2-1)}$ 4) $\frac{\log\left(e/2\right)}{(e^2-1)}$

26. **If** $y = \frac{\sin^2 x}{1+\cot x} + \frac{\cos^2 x}{1+\tan x}$ **then** $y^1(x)$ **is equal to**

1) cos2x 2) -cos2x 3) $2\cos^2 x$ 4) $\cos^3 x$

27. **If** $t = \sin^{-1} 2^s$ **then** $\frac{ds}{dt}$ **is equal to**

1) $\frac{\log 2}{\sqrt{1-t^2}}$ 2) $\frac{\sin t}{\log 2}$ 3) $\frac{\cot t}{\log 2}$ 4) $\frac{\tan t}{\log 2}$

28. **If** $y = \cfrac{x}{a + \cfrac{x}{b + \cfrac{x}{a + \cfrac{x}{b + ...+ to\ \infty}}}}$ **then** $\frac{dy}{dx} =$

1) $\frac{1}{a(2y+b)}$ 2) $\frac{b}{a(2y+b)}$

3) $\frac{1}{ab(2y+b)}$ 4) $\frac{ab}{(2y+b)}$

29. **If** $\sqrt{1-x^2} + \sqrt{1-y^2}$ **=a(x-y) then** $\frac{dy}{dx} =$

1) $\sqrt{\frac{1-x^2}{1-y^2}}$ 2) $\sqrt{(1-x^2)(1-y^2)}$

3) $\sqrt{\frac{1-y^2}{1-x^2}}$ 4) $\frac{1}{\sqrt{(1-x^2)(1-y^2)}}$

30. **If** $Y = \tan^{-1}\left(\frac{1}{1+x+x^2}\right) + \tan^{-1}\left(\frac{1}{x^2+3x+3}\right) + \tan^{-1}\left(\frac{1}{x^2+5x+7}\right) + --- +$ **upto n terms**

then $y^1(0) =$

1) $\frac{-1}{1+n^2}$ 2) $\frac{-n^2}{1+n^2}$ 3) $\frac{n}{1+n^2}$ 4) n

31. **If** $y = Cot^{-1}(1+x^2-x)$**, then** $\frac{dy}{dx} =$

1) $\frac{1}{1+x^2} + \frac{1}{1+(x-1)^2}$ 2) $\frac{1}{1+(x-1)^2} - \frac{1}{1+x^2}$

3) $\frac{1}{1+x^2} - \frac{1}{1+(x-1)^2}$ 4) $-\frac{1}{1+x^2} - \frac{1}{1+(x-1)^2}$

32. **If f(x)=|x|**$^{|\sin x|}$**, then** $f^1\left(-\frac{\pi}{4}\right)$ **is equals**

1) $\left(\frac{\pi}{4}\right)^{1/\sqrt{2}}\left(\frac{\sqrt{2}}{2}\ln\frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}\right)$

2) $\left(\frac{\pi}{4}\right)^{1/\sqrt{2}}\left(\frac{\sqrt{2}}{2}\ln\frac{4}{\pi} + \frac{2\sqrt{2}}{\pi}\right)$

3) $\left(\frac{\pi}{4}\right)^{1/\sqrt{2}}\left(\frac{\sqrt{2}}{2}\ln\frac{\pi}{4} - \frac{2\sqrt{2}}{\pi}\right)$

4) $\left(\frac{\pi}{4}\right)^{1/\sqrt{2}}\left(\frac{\sqrt{2}}{2}\ln\frac{\pi}{4} + \frac{2\sqrt{2}}{\pi}\right)$

33. **If** $f^1(x)=\sin(\log x)$ **and** $y = f\left(\frac{2x+3}{3-2x}\right)$**, then** $\frac{dy}{dx}$ **is**

1) $\sin(\log x).\frac{1}{x\log x}$ 2) $\frac{12}{(3-2x)^2}\sin\left[\log\left(\frac{2x+3}{3-2x}\right)\right]$

3) $\sin\left[\log\left(\frac{2x+3}{3-2x}\right)\right]$ 4) $\sin(\log x)$

34. $y = \log^n x$ **where** $\log^n$ **means log. log.log........(repeated n times)**

then $x.\log x.\log^2 x.\log^3 x --- \log^{n-1} x.\frac{dy}{dx} =$

1) logx 2) $\log^n x$ 3) $\frac{1}{\log x}$ 4) 1

35. $\frac{d^2x}{dy^2} =$ **[AIEEE - 2011]**

1) $\dfrac{1}{\left(\dfrac{dy}{dx}\right)^2}$ 2) $\dfrac{\left(\dfrac{d^2y}{dx^2}\right)}{\left(\dfrac{dy}{dx}\right)^2}$

3) $\left(\dfrac{d^2y}{dx^2}\right)$ 4) $\dfrac{-\left(\dfrac{d^2y}{dx^2}\right)}{\left(\dfrac{dy}{dx}\right)^3}$

36. $\dfrac{d}{dx}\left[\dfrac{\tan x-\cot x}{\tan x+\cot x}\right]=$

1) 2 sin2x 2) -2 sin2x
3) 2 cos2x 4) -2 cos2x

37. If |x| < 1, then

$\dfrac{d}{dx}\left[1+\dfrac{p}{q}x+\dfrac{P(p+q)}{\angle 2}\left(\dfrac{x}{q}\right)^2+\dfrac{p(p+q)(p+2q)}{\angle 3}\left(\dfrac{x}{q}\right)^3.....\infty\right]=$

1) $\dfrac{p}{q(1-x)^{\frac{p}{q}+1}}$ 2) $\dfrac{p}{q(1-x)^{\frac{p}{q}}}$

3) $(1-x)^{-pq-1}$ 4) $(1-x)^{pq+1}$

38. If y=$\left(x+\sqrt{x^2-a^2}\right)^n$ **then** $\left(x^2-a^2\right)\left(\dfrac{dy}{dx}\right)^2=$

1) n^2y 2) $-n^2y$ 3) ny^2 4) n^2y^2

39. $\dfrac{d}{dx}\left(\sqrt{Sec^{-1}x^2}\right)=$

1) $\dfrac{1}{x\sqrt{Sec^{-1}x^2}.\sqrt{x^4-1}}$ 2) $\dfrac{x}{\sqrt{Sec^{-1}x^2}.\sqrt{x^4-1}}$

3) $\dfrac{-1}{x\sqrt{Sec^{-1}x^2}.\sqrt{x^4-1}}$ 4) $\dfrac{-1}{\sqrt{Sec^{-1}x^2}.\sqrt{x^4-1}}$

40. The value of $3\,f(x)-2\,f\left(\dfrac{1}{x}\right)=x$ then $f^1(2)=$

1) $\dfrac{2}{7}$ 2) $\dfrac{1}{2}$ 3) 2 4) $\dfrac{7}{2}$

41. If $x=exp\left\{\tan^{-1}\left(\dfrac{y-x^2}{x^2}\right)\right\}$ **then** $\dfrac{dy}{dx}=$

1) $2x\left[1+\tan(\log x)\right]+x.\sec^2(\log x)$

2) $x\left[1+\tan(\log x)\right]+\sec^2(\log x)$

3) $2x\left[1+\tan(\log x)\right]+x^2\sec^2(\log x)$

4) $2x\left[1+\tan(\log x)\right]+\sec^2(\log x)$

42. If $af(Tanx)+bf(\cot x)=x$ **then** $f^1(\cot x)=$

1) $\dfrac{1}{a-b}$ 2) $\dfrac{\sin^2 x}{a+b}$ 3) $\dfrac{\sin^2 x}{a-b}$ 4) $\dfrac{\sin^2 x}{b-a}$

43. If x=secθ **-cos**θ **and y=sec**$^n\theta$ **-cos**$^n\theta$ **then** $\left(\dfrac{dy}{dx}\right)^2$ **is equal to**

1) $\dfrac{n^2(y^2+4)}{x^2+4}$ 2) $\dfrac{n^2(y^2-4)}{x^2}$

3) $n\dfrac{(y^2-4)}{x^2-4}$ 4) $\left(\dfrac{ny}{x}\right)^2-4$

44. If $y=\sin x\left[\dfrac{1}{\sin x.\sin 2x}+\dfrac{1}{\sin 2x.\sin 3x}+...+\dfrac{1}{\sin nx\,\sin(n+1)x}\right]$ **then** $\dfrac{dy}{dx}=$

1) cotx - cot(n+1)x

2) $(n+1)\cos ec^2(n+1)x-\cos ec^2x$

3) $\cos ec^2x-(n+1)\cos ec^2(n+1)x$

4) cotx + cot(n+1)x

45. Let $f(x)=x^5+2x^3+3x+4$ **then the value of** $28\dfrac{d}{dx}\left(f^{-1}(x)\right)$ **at** $x=-2$ **is**

1) 1 2) 2 3) 1/14 4) -2

46. If f(x) = $\begin{vmatrix} sec\theta & tan^2\theta & 1 \\ \theta secx & tanx & x \\ 1 & tanx-tan\theta & 0 \end{vmatrix}$ **then** $f^1(\theta)$ **is**

1) 0 2) - 1
3) independent of θ 4) none

47. If $\sin y+e^{-x\cos y}=e$ **then** $\dfrac{dy}{dx}$ **at (1,** π **) is equal to**

1) sin y 2) - x cosy 3) e 4) siny - x cosy

48. **If the function** $f(x)=x^3+e^{\frac{x}{2}}$ **and** $g(x)=f^{-1}(x)$**, then the value of** $g^1(1)$ **is**

1) $\frac{1}{2}$ 2) 2 3) 1 4) $-\frac{1}{2}$

49. **If** $f(x)=\cos x\cos 2x\cos 2^2 x\cos 2^3 x.....\cos 2^{n-1}x$ **and** $n>1$**, then** $f^1\left(\frac{\pi}{2}\right)$ **is**

1) 1 2) 0 3) -1 4) $(-1)^{n-1}$

50. **If u = f(x²), v = g(x³), f¹ (x) = sin x, g¹(x) = cos x then find** $\frac{du}{dv}$

1) 1 2) $\frac{2}{3}$ 3) $\frac{2\sin x^2}{3x.\cos x^3}$ 4) $\frac{2x^2}{3x^3}$

51. **If f(0) = 0, f¹(0) = 2, then the derivative of y = f(f(f(f(x)))) at x = 0 in**

1) 2 2) 8 3) 16 4) 4

52. **If x =** $\phi(t)$**, y =** $\psi(t)$ **then** $\frac{d^2y}{dx^2}$ **is**

1) $\frac{\phi^1\psi^{11}-\psi^1\phi^{11}}{(\phi^1)^2}$ 2) $\frac{\phi^1\psi^{11}-\psi^1\phi^{11}}{(\phi^1)^3}$

3) $\frac{\phi^{11}}{\psi^{11}}$ 4) $\frac{\psi^{11}}{\phi^{11}}$

53. $\frac{d}{dx}\left[\cos^2\left(Tan^{-1}\left(\sin\left(Cot^{-1}x\right)\right)\right)\right]=$

1) $\frac{2}{(x^2+2)^2}$ 2) $\frac{2x}{(x^2+2)^2}$

3) $\frac{x^2+1}{x^2+2}$ 4) $\frac{-2x}{(x^2-1)^2}$

54. **If** $x\sin y=\sin(y+a)$ **and** $\frac{dy}{dx}=\frac{A}{1+x^2-2x\cos a}$ **then the value of A is**

1) 2 2) $\cos a$ 3) $\sin a$ 4) -2

55. **If y = Tan⁻¹** $\left(\frac{1}{\cos^2 x+\cos x+1}\right)+Tan^{-1}\left(\frac{1}{\cos^2 x+3\cos x+3}\right)$ **+ .... upto n terms then f¹ (0) =**

1) 0 2) 1 3) $\frac{\pi}{2}$ 4) π

56. **f(x) = Cot⁻¹** $\left(\frac{x^x-x^{-x}}{2}\right)$ **then f¹ (1) =**

1) - log 2 2) log 2 3) 1 4) -1

57. **If y =** $\left(1+\frac{1}{x}\right)^x$ **then** $\frac{2\sqrt{y_2(2)+\frac{1}{8}}}{\log\frac{3}{2}-\frac{1}{3}}$ **is**

1) 1 2) 3 3) 0 4) 1/3

58. $If\ \ y=\cfrac{\sin x}{1+\cfrac{\cos x}{1+\cfrac{\sin x}{1+\cos x....\infty}}}$ **then** $\frac{dy}{dx}=$

1) $\frac{(1+y)\cos x+y\sin x}{1+2y+\cos x-\sin x}$ 2) $\frac{(1+y)\sin x+y\cos x}{1+2y+\cos x-\sin x}$

3) $\frac{(1+y)\cos x-y\sin x}{1+2y-\cos x+\sin x}$ 4) $\frac{(1+y)\cos x+y\sin x}{1+2y-\cos x-\sin x}$

59. **If** $p(x)$ **is a polynomial such that** $p(x^2+1)=\{p(x)\}^2+1$ **and** $p(0)=0$ **then** $p^1(0)$ **is equal to**

1) -1 2) 0 3) 1 4) 2

60. **If x² + y² = 2 and y¹¹ + A.** y^{-3} **= 0 then A =**

1) 0 2) 2 3) 1 4) 3

61. $y=Tan^{-1}\left(\frac{\log\left(\frac{e}{x^2}\right)}{\log(ex^2)}\right)+Tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$

then $\frac{d^2y}{dx^2}=$

1) 2 2) 1 3) 0 4) - 1

62. **If** $y=Tan^{-1}\left(\frac{4\sin 2x}{\cos 2x-6\sin^2 x}\right)$ **then** $\frac{dy}{dx}$ **at** $x=0$ **is**

1) 10 2) 12 3) 6 4) 8

63. **If** $\sqrt{\frac{\upsilon}{\mu}}+\sqrt{\frac{\mu}{\upsilon}}=6,\ then\frac{d\upsilon}{d\mu}=$

1) $\frac{17\mu-\upsilon}{\mu-17\upsilon}$ 2) $\frac{\mu-17\upsilon}{17\mu-\upsilon}$

3) $\frac{17\mu+\upsilon}{\mu-17\upsilon}$ 4) $\frac{\mu+17\upsilon}{17\mu-\upsilon}$

64. $\frac{d}{dx}\left\{e^{x^e}+x^{e^x}+e^{x^x}\right\}=$

1) $e^{x^e}+x^{e^x}+e^{x^x}$ 2) $x^2.e^{x^e}+e^x.x^{e^x}+x^x.e^{x^x}$

3) $e.e^{x^e}.x^{e-1}+x^{e^x}.e^x\left(\frac{1}{x}+\log x\right)+e^{x^x}.x^x(1+\log x)$

4) $e.e^{x^e}.x^{e-1}+x^{e^x}.e^x\left(\frac{1}{x}-\log x\right)+e^{x^x}.x^x(1-\log x)$

65. The derivative of y=(1-x) (2-x) (3-x)...(n-x) at $x=1$ is

1) $n!$ 2) $(n-1)!$ 3) $-(n-1)!$ 4) 0

66. If $y=\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}$ then $\frac{dy}{d\theta}$ at $\theta=\frac{3\pi}{4}$ is

1) -2 2) 2 3) 1 4) -1

67. If $\sqrt{1-x^{2n}}+\sqrt{1-y^{2n}}=a\left(x^n-y^n\right)$, then $\sqrt{\frac{1-x^{2n}}{1-y^{2n}}}\frac{dy}{dx}$ is equal to

1) $\frac{x^{n-1}}{y^{n-1}}$ 2) $\frac{y^{n-1}}{x^{n-1}}$ 3) $\frac{x}{y}$ 4) 1

68. If $y=|\cos x|+|\sin x|$, then $\frac{dy}{dx}$ at $x=\frac{2\pi}{3}$ is

1) $\frac{1-\sqrt{3}}{2}$ 2) 0 3) $\frac{\sqrt{3}-1}{2}$ 4) $\frac{\sqrt{3}+1}{2}$

69. If $f(x)=|\cos x-\sin x|$, then $f^1\left(\frac{\pi}{4}\right)=$

1) $\sqrt{2}$ 2) $-\sqrt{2}$ 3) 0 4) does not exist

70. Let $u(x)$ and $v(x)$ are differentiable function such that $\frac{u(x)}{v(x)}=7$. If $\frac{u'(x)}{v'(x)}=p$ and $\left(\frac{u(x)}{v(x)}\right)'=q$, then $\frac{p+q}{p-q}=$

1) 1 2) 0 3) 7 4) -7

71. Differential coefficient of

$\left(x^{\frac{l+m}{m-n}}\right)^{\frac{1}{n-l}}.\left(x^{\frac{m+n}{n-l}}\right)^{\frac{1}{l-m}}.\left(x^{\frac{n+l}{l-m}}\right)^{\frac{1}{m-n}}$ **w.r.t. x, is**

1) 1 2) 0 3) -1 4) x^{lmn}

72. If $x=\operatorname{cosec}\theta-\sin\theta,\ y=\operatorname{cosec}^n\theta-\sin^n\theta$ then

$\left(x^2+4\right)\left(\frac{dy}{dx}\right)^2-n^2y^2=$

1) n^2 2) $2n^2$ 3) $3n^2$ 4) $4n^2$

73. If $y=f\left(\frac{3x+4}{5x+6}\right)$ and $f^1(x)=\tan x^2$ then $\frac{dy}{dx}$ is equal to

1) $-2\tan\left(\frac{3x+4}{5x+6}\right)^2\times\frac{1}{(5x+6)^2}$

2) $f\left(\frac{3\tan x^2+3}{5\tan x^2+6}\right)\tan x^2$ 3) $\tan x^2$

4) $2\tan\left(\frac{3x+4}{5x+6}\right)^2\times\frac{1}{(5x+6)^2}$

74. If $y=\cos^{-1}(\cos x)$, then $\frac{dy}{dx}$ at $x=\frac{5\pi}{4}$ is

1) 1 2) -1 3) $\frac{1}{\sqrt{2}}$ 4) $\frac{5\pi}{4}$

75. If $f(\theta)=\sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4}<\theta<\frac{\pi}{4}$, then the value of $\frac{d}{d(\tan\theta)}f(\theta)$ is

1) 1 2) -1 3) 0 4) $\sqrt{2}$

76. The first derivative of the function $\cos^{-1}\left(\sin\sqrt{\frac{1+x}{2}}\right)+x^x$ with respect to x at x=1 is

1) $\frac{3}{4}$ 2) 0 3) $\frac{1}{2}$ 4) $-\frac{1}{2}$

77. If $\sin\theta\sin(2\alpha+\theta)\sin(4\alpha+\theta)$.......

$\sin(2(n-1)\alpha+\theta)=\frac{\sin n\theta}{2^{n-1}}$ **where** $2n\alpha=\pi$

then

$\cot(\theta)+\cot(2\alpha+\theta)+\cot(4\alpha+\theta)+.....+\cot(2(n-1)\alpha+\theta)=$

1) - n cot $n\theta$ 2) n cot $n\theta$
3) n tan $n\theta$ 4) - n tan $n\theta$

78. A function is represented parametrically by the equations $x=\frac{1+t}{t^3},\quad y=\frac{3}{2t^2}+\frac{2}{t}$ **then** $\frac{dy}{dx}-x\left(\frac{dy}{dx}\right)^2$ **has the absolute value equal to**

1) -1 2) 1 3) 0 4) 2

79. If $f(x)=\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$, **then** $f^1(x)=$

1) 0 2) $6x$ 3) $12x^3$ 4) $6x^2$

80. Let y **be an implict function of** x **defined by** $x^{2x}-2x^x\cot y-1=0$. **Then** $y^1(1)$ **equals**

[AIEEE-2009]

1) -1 2) 1 3) $\log 2$ 4) $-\log 2$

81. Let $f:(-1,1)\to R$ **be a differentiable function with** $f(0)=-1$ **and** $f^1(0)=1$. **Let** $g(x)=\left[f(2f(x)+2)\right]^2$, **then** $g'(0)$ =

[AIEEE 2010]

1) -4 2) 0 3) -2 4) 4

82. Let f(x+y)=f(x)f(y) and f(x)=1+(sin2x)g(x) where g(x) is continuous. Then $f^1(x)$ **equals**

1) f(x)g(0) 2) 2f(x)g(0) 3) 2g(0) 4) 2f(0)

83. Given that f(x) is a differentiable function of x and that f(x).f(y)=f(x)+f(y)+f(xy)-2 and that f(2)=5. Then f(3) is equal to

1) 10 2) 24 3) 15 4) 19

84. Let f be twice differentiable function such that $g^1(x)=-f(x)$ **and** $f^1(x)=g(x)$, $h(x)=\left(f(x)\right)^2+\left(g(x)\right)^2$. **If h(5)=11, then h(10) is**

1) 22 2) 11 3) 0 4) 8

KEY

01) 2	02) 3	03) 1	04) 4	05) 1	06) 3
07) 2	08) 2	09) 1	10) 4	11) 2	12) 3
13) 1	14) 2	15) 3	16) 1	17) 2	18) 3
19) 1	20) 1	21) 1	22) 4	23) 3	24) 4
25) 1	26) 2	27) 3	28) 2	29) 3	30) 2
31) 3	32) 1	33) 2	34) 4	35) 4	36) 1
37) 1	38) 4	39) 1	40) 2	41) 1	42) 3
43) 1	44) 2	45) 2	46) 2	47) 3	48) 2
49) 1	50) 3	51) 3	52) 2	53) 2	54) 3
55) 1	56) 4	57) 2	58) 1	59) 3	60) 2
61) 3	62) 4	63) 2	64) 3	65) 3	66) 2
67) 1	68) 3	69) 4	70) 1	71) 2	72) 4
73) 1	74) 2	75) 1	76) 3	77) 2	78) 2
79) 4	80) 1	81) 1	82) 2	83) 1	84) 2

SOLUTIONS

1. g(y) log f(x) = f(x) - g(y)

$$\Rightarrow\frac{dy}{dx}=\frac{f^1(x).\log f(x)}{g^1(y)\left(1+\log f(x)\right)^2}$$

2. $\left(\frac{x^2-y^2}{x^2+y^2}\right)=\text{Cos}(\log a)$

Applying Componendo and dividendo

3. Differentiate $\phi(x)=f(x)g(x)$ two times and simplify

4. $\underset{x\to y}{\text{Lt}}\frac{|f(x)-f(y)|}{|x-y|}\le|x-y|^2$

$f'(x)=0\Rightarrow f(x)$ is constant

5. Taking Logarithm on both sides, then

$$\frac{dy}{dx}=\frac{-\left(\partial f/\partial x\right)}{\left(\partial f/\partial y\right)}$$

6. $(1+x)(1+x^2)...(1+x^n)=\frac{1-x^{2n}}{1-x}$ and taking logarithms and differentiate

7. Since $f(x)$ is an even function $f'(0)=0$

8. $\frac{1}{x}=\frac{\sqrt{1+t^2}}{\sqrt{1-t^2}}\Rightarrow\frac{1-x}{1+x}=\frac{\sqrt{1+t^2}-\sqrt{1-t^2}}{\sqrt{1+t^2}+\sqrt{1-t^2}}$

$$\Rightarrow\frac{1-x}{1+x}=y$$

9. $\frac{d}{dx}\{\sec hx + \cos echx\}$

$= -\sec hx \tanh x - \cos echx \coth x$

$= -\left[\frac{\sinh x}{\cosh^2 x} + \frac{\cosh x}{\sinh^2 x}\right] = -\left[\frac{\sinh^3 x + \cosh^3 x}{\cosh^2 x \sinh^2 x}\right]$

10. $\frac{dy}{dx} = \frac{\left(dy/d\theta\right)}{\left(dx/d\theta\right)}$

11. $\log\left(a^m\right) = m\log a$,

$\frac{d}{dx}(\log x) = 1/x, \frac{d}{dx}\left(\tan^{-1} x\right) = 1/\left(1+x^2\right)$

12. Find its determinant and put $x = \frac{\pi}{2}$.

13. Express 'y' as $\log\cos\tan^{-1}(\sinh x) = \log(\sec hx)$ and simplify

14. Verify the options

15. $P(0) = 4 \Rightarrow d = 4$, $P^1(x) = 3ax^2 + 2bx + c$
at x = 0, $P^1(0) = 3$ then c = 3
similarly to find a and b calculate $P^{11}(0)$ and $P^{111}(0)$

16. $\frac{d}{dx}\left(f^2(x)+g^2(x)\right)=2\left[f(x)f^1(x)+g(x)g^1(x)\right]$

$= 2\left[f(x)g(x) - g(x)f(x)\right] = 0$

hence $f^2(x) + g^2(x)$ is constant. Thus

$f^2(24) + g^2(24) = f^2(2) + g^2(2) = 16 + 16 = 32$

17. f(x)=cisx.cis3x.....cis(2n-1)x
=cis[x+3x+...+(2n-1)x]=cis[(1+3+...+(2n-1))x]
$=cis(n^2x) \Rightarrow f(x)=\cos n^2x + i\sin n^2x$
differentiate w.r.t 'x' on both sides

18. $y^2=p(x) \Rightarrow 2yy_1=p^1(x)$ then $2yy_2+2y_1y_1=p^{11}(x)$

$\Rightarrow 2\left[y^3y_2 + y^2y_1^2\right] = y^2p^{11}(x)$

$\Rightarrow 2\left[y^3y_2 + \frac{\left(p^1(x)\right)^2}{4}\right] = y^2p^{11}(x)$

$\Rightarrow 2y^3y_2 = y^2p^{11}(x) - \frac{\left(p^1(x)\right)^2}{2}$

$\Rightarrow y^3y_2 = \frac{1}{2}\left[y^2p^{11}(x) - \frac{\left(p^1(x)\right)^2}{2}\right]$

differentiate w.r.t 'x' on both sides

19. $\cos^{-1}\frac{9-x^2}{9+x^2} = \cos^{-1}\frac{1-(x/3)^2}{1+(x/3)^2} = 2\tan^{-1}(x/3)$

If $0 \le x < \infty$ and is equal to $-2\tan^{-1}(x/3)$

If $-\infty < x \le 0$ .Hence $y'(x) = \begin{cases} \frac{2}{1+(x/3)^2}.\frac{1}{3} & \text{if } 0 < x < \infty \\ -\frac{2}{1+(x/3)^2}.\frac{1}{3} & \text{if } -\infty < x < 0 \end{cases}$

Hence $y'(-1) = -\frac{3}{5}$

20. $f(x) = 0,\ x = \frac{\pi}{2};\ = 1,\ x < \frac{\pi}{2};\ = -1,\ x > \frac{\pi}{2}$

21. Differentiating, we have,

$\cos x\cos y - \sin x\sin y\frac{dy}{dx} = 0$.

Putting $x = y = \pi/4$, we have $\left.\frac{dy}{dx}\right|_{(\pi/4,\pi/4)} = 1$.

differentiating again, we have

$-\sin x\cos y - \cos x\sin y\frac{dy}{dx} - \cos x\sin y\frac{dy}{dx}$

$-\sin x\sin y\left(\frac{dy}{dx}\right)^2 - \sin x\sin y\frac{d^2y}{dx^2} = 0$,

Putting $x=y=\frac{\pi}{4}$, we have $\left.\frac{d^2y}{dx^2}\right|_{(\pi/4,\pi/4)} = -4$

22. $\frac{dy}{dx} = 2e^{2x}$ and $\frac{d^2y}{dx^2} = 4e^{2x}$. Also $x = \frac{1}{2}\log y$.

so $\frac{dx}{dy} = \frac{1}{2y}$ and $\frac{d^2x}{dy^2} = -\frac{1}{2y^2} = -\frac{1}{2}e^{-4x}$

Hence $\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right) = -2e^{-2x}$

23. consider $\left(\sqrt{x+y}\right)^2 - \left(\sqrt{y-x}\right)^2$

$= \left(\sqrt{x+y} - \sqrt{y-x}\right)\left(\sqrt{x+y} + \sqrt{y+x}\right)$

$2x = \left(\sqrt{x+y} - \sqrt{y-x}\right).c$; $\Rightarrow \sqrt{x+y} - \sqrt{y-x} = \frac{2x}{c} \rightarrow (1)$

given $\sqrt{x+y} + \sqrt{y-x} = c \rightarrow (2)$

$(1)+(2):\ 2\sqrt{x+y} = \frac{2x}{c} + c$

$4(x+y) = \frac{4x^2}{c^2} + c^2 + 4x$, then $4y = \frac{4x^2}{c^2} + c^2$

differentiate w.r.t 'x' two times

24. f(x)=cosx cos2x cos4x cos8x cos16x

$= \cos A \cos 2A \cos 4A \ldots \cos 2^n A = \frac{\sin 2^{n+1}x}{2^{n+1}\sin x}$

$$= \frac{\sin 2^5 x}{2^5 \sin x} = \frac{\sin 32x}{32 \sin x}$$

$$f^1(x) = \frac{32\sin x \cos 32x - \cos x \sin 32x}{32\sin^2 x}$$

then $f'(\pi/4) = \frac{32\sin \pi/4 \cos 8\pi - 0}{32(1/2)} = 2\sin \pi/4 = \cos ec\, \pi/4$

25. Let $y^{y^{y^{\infty}}} = \log_e(x + \log_e(x + ...)) = v$

$\therefore y^v = \log_e(x + v) = v \qquad \Rightarrow y = v^{1/v}$ and $x = e^v - v$

$\Rightarrow \frac{dy}{dv} = v^{1/v}\left(\frac{d}{dx}\frac{1}{v}\log v\right)$ and $\frac{dx}{dv} = (e^v - 1)$

i.e $\frac{dy}{dv} = v^{1/v}\left(\frac{1}{v}\cdot\frac{1}{v} - \frac{1}{v^2}\log v\right) = v^{1/v - 2}(1 - \log v)$

and $\frac{dx}{dv} = (e^v - 1) \therefore \frac{dy}{dx} = \frac{v^{1/v-2}(1-\log v)}{e^v - 1}$

at $x = e^2 - 2, y = \sqrt{2}$, $v^{1/v} = \sqrt{2}, e^v - v = e^2 - 2$

$\Rightarrow$ v=2 or 4 which is not valid

$\therefore$ v=2, e^v-v=e^2-2=x given so true

$$\left(\frac{dy}{dx}\right)_{e^2-2,\sqrt{2}} = \left(\frac{dy}{dx}\right)_{v=2} = \frac{1-\log 2}{2\sqrt{2}(e^2-1)}$$

26. $y = \frac{\sin^3 x}{\sin x + \cos x} + \frac{\cos^3 x}{\cos x + \sin x} = \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x}$

$= \sin^2 x + \cos^2 x - \sin x \cos x = 1 - \frac{1}{2}\sin 2x$ then

$y'(x) = -\cos 2x$

27. $t = Sin^{-1}(2^s) \Rightarrow Sin\, t = 2^s$ differentiate w.r.t.x 't' on both sides $\cos t = 2^s \log 2.\frac{ds}{dt}$

$\Rightarrow \cos t = \sin t \log 2.\frac{ds}{dt} \Rightarrow \frac{ds}{dt} = \frac{\cot t}{\log 2}$

28. Express 'y' as $\frac{x}{x + \frac{x}{b+y}}$ and then differentiate

29. Put $x = \sin \alpha, y = \sin \beta$

30. $Tan^{-1}(x+1) - Tan^{-1}x + Tan^{-1}(x+2)$

$-Tan^{-1}x + 1 - - - - - = Tan^{-1}x + n - Tan^{-1}x$

31. $y = Tan^{-1}\left(\frac{1}{1+x(x-1)}\right) = Tan^{-1}\left(\frac{x-(x-1)}{1+x(x-1)}\right)$

32. In the neighbourhood of $-\pi/4$, we have

$f(x) = (-x)^{-\sin x} = e^{-\sin x \log(-x)}$

$\Rightarrow f'(x) = e^{-\sin x \log(-x)}\left(-\cos x.\log(-x) - \frac{\sin x}{x}\right)$

$\Rightarrow f'(x) = (-x)^{-\sin x}\left(-\cos x.\log(-x) - \frac{\sin x}{x}\right)$

$\Rightarrow f'(-\pi/4) = (\pi/4)^{(1/\sqrt{2})}\left(\frac{-1}{\sqrt{2}}\log(\pi/4) + \frac{4}{\pi}\times\frac{-1}{\sqrt{2}}\right)$

$= (\pi/4)^{(1/\sqrt{2})}\left(\frac{\sqrt{2}}{2}\log(\pi/4) - \frac{2\sqrt{2}}{\pi}\right)$

33. $\frac{dy}{dx} = f^1\left(\frac{2x+3}{3-2x}\right)\cdot\frac{d}{dx}\left(\frac{2x+3}{-2x+3}\right); \; = \sin\left[\log\left(\frac{2x+3}{3-2x}\right)\right]\cdot\frac{12}{(3-2x)^2}$

34. n=2, verification

35. $\frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dy}\left(\frac{1}{\left(\frac{dy}{dx}\right)}\right)$

36. $\frac{d}{dx}\left(\frac{\tan x - 1/\tan x}{\tan x + 1/\tan x}\right) = \frac{d}{dx}\left(\frac{-(1-\tan^2 x)}{1+\tan^2 x}\right)$

$= \frac{d}{dx}(-\cos 2x)$

37. $(1-x)^{-p/q} = 1 + p\left(x/q\right) + \frac{p(p+q)}{2!}\left(x/q\right)^2 + ...$

$\Rightarrow \frac{d}{dx}(1-x)^{-p/q}$

38. $\frac{dy}{dx} = n\left(x + \sqrt{x^2 - a^2}\right)^{n-1}\cdot\left(1 + \frac{1}{2\sqrt{x^2-a^2}}\cdot 2x\right)$

$$= n\frac{\left(x+\sqrt{x^2-a^2}\right)^n}{\left(x+\sqrt{x^2-a^2}\right)}\left(\frac{\left(x+\sqrt{x^2-a^2}\right)}{\sqrt{x^2-a^2}}\right)$$

39. $\frac{d}{dx}(Sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$

40. $3f(x) - 2f\left(\frac{1}{x}\right) = x$ -----(1)

Put $x = \frac{1}{x}$ we get $3f\left(\frac{1}{x}\right) - 2f(x) = \frac{1}{x}$ ----- (2) Solve (1) and (2).

41. $\log x = Tan^{-1}\left(\frac{y-x^2}{x^2}\right)$ $y = x^2 + x^2\tan(\log x)$

42. Eliminate $f(\tan x)$

$$f(\cot x)=\frac{x}{b-a}-\frac{a\pi}{2(b^2-a^2)}$$

43. $\frac{dy}{d\theta}=n\sec^{n-1}\theta.\sec\theta\tan\theta-n\cos^{n-1}\theta(-\sin\theta)$

$=n\tan\theta(\sec^n\theta+\cos^n\theta)$

$\frac{dx}{d\theta}=\sec\theta\tan\theta+\sin\theta$

$$\left(\frac{dy}{dx}\right)^2=\left[\frac{n\tan\theta(\sec^n\theta+\cos^n\theta)}{\tan\theta(\sec\theta+\cos\theta)}\right]^2$$

$$=\frac{n^2(\sec^n\theta+\cos^n\theta)^2}{(\sec\theta+\cos\theta)^2}$$

$$=\frac{\left[n^2(\sec^n\theta-\cos^n\theta)^2+4\sec^n\theta\cos^n\theta\right]}{(\sec\theta-\cos\theta)^2+4\sec\theta\cos\theta}$$

$$=\frac{n^2(y^2+4)}{x^2+4}$$

44. Express 'y' as $\cot x-\cot(n+1)x$ and differentiate

45. Let $g(x)$ be the inverse of $f(x)$

$\Rightarrow g^1(f(x))f^1(x)=1$

$g^1(f(-1))f^1(-1)=1\Rightarrow g^1(-2)=\frac{1}{14}$

46. $$f^1(x)=\begin{vmatrix}0&0&0\\ \theta\sec x&\tan x&x\\ 1&\tan x-\tan\theta&\theta\end{vmatrix}$$

$$+\begin{vmatrix}\sec\theta&\tan^2\theta&1\\ \theta\sec x\tan x&\sec^2 x&1\\ 1&\tan x-\tan\theta&0\end{vmatrix}+\begin{vmatrix}\sec\theta&\tan^2\theta&1\\ \theta\sec x&\tan x&x\\ 0&\sec^2 x&0\end{vmatrix}$$

$$f^1(\theta)=0+\begin{vmatrix}\sec\theta&\tan^2\theta&1\\ \theta\sec\theta\tan\theta&\sec^2\theta&1\\ 1&0&0\end{vmatrix}+\begin{vmatrix}\sec\theta&\tan^2\theta&1\\ \theta\sec\theta&\tan\theta&\theta\\ \theta&\sec^2\theta&0\end{vmatrix}=-1$$

47. $\frac{dy}{dx}=-\frac{f_x}{f_y}$

48. $(fog)(x)=x$ *for all* x

$\Rightarrow f^1(g(x))g^1(x)=1$ *for all* x

$\Rightarrow f^1(g(1))g^1(1)=1$; $\Rightarrow g^1(1)=\frac{1}{f^1(g(1))}$

But $(gof)(x)=x\Rightarrow g(f(0))=0$

$\Rightarrow g(1)=0$ and $f^1(0)=\frac{1}{2}$

Hence, $g^1(1)=\frac{1}{\frac{1}{2}}=2$

49. $f(x)=\cos x\cos 2x\cos 2^2x\cos 2^3x....\cos 2^{n-1}x$

$\Rightarrow f(x)=\frac{\sin 2^n x}{2^n\sin x}$

$\Rightarrow f^1(x)=\frac{2^n\cos 2^n x\sin x-\sin 2^n x\cos x}{2^n\sin^2 x}$

$\Rightarrow f^1\left(\frac{\pi}{2}\right)=\frac{2^n\cos 2^{n-1}\pi}{2^n}=\cos 2^{n-1}\pi=(-1)^{2^{n-1}}=1$

50. $\frac{du}{dx}=f^1(x^2)(2x)=(\sin x^2)(2x)$

$\frac{du}{dx}=g^1(x^3)(3x^2)=\cos(x^3)(3x^2)$

$\frac{du}{dv}=\frac{du/dx}{dv/dx}=\frac{2x\sin x^2}{3x^2\cos x^3}=\frac{2\sin x^2}{3x\cos x^3}$

51. $\frac{dy}{dx}=f^1[f(f(f(x)))].f^1[f(f(x))].f^1(f(x))f^1(x)$

$\left(\frac{dy}{dx}\right)_{x=0}=f^1[f(f(f(0)))].f^1[f(f(0))].f^1[f(0)].f^1(0)$

$=f^1[f(f(0))].f^1[f(0)].f^1(0).(2)$

$=f^1[f(0)].f^1(0)(2)(2)$

$=f^1(0).(2)(2)(2)=(2)(2)(2)(2)=16$

52. $\frac{dy}{dx}=\frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\psi^1}{\phi^1}$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{\psi^1}{\phi^1}\right)=\frac{d}{dt}\left(\frac{\psi^1}{\phi^1}\right)\frac{dt}{dx}=\frac{\phi^1\psi^{11}-\psi^1\phi^{11}}{(\phi^1)^2}.\frac{1}{\phi^1}$$

$$=\frac{\phi^1\psi^{11}-\psi^1\phi^{11}}{(\phi^1)^3}$$

53. $\frac{d}{dx}\left[\cos^2\left(Tan^{-1}\left(Sin\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)\right)\right]$

$$\frac{d}{dx}\left[Cos^2\left[Tan^{-1}\frac{1}{\sqrt{1+x^2}}\right]\right]=\frac{d}{dx}\left[\frac{1+\cos 2\left(Tan^{-1}\frac{1}{\sqrt{1+x^2}}\right)}{2}\right]$$

$$=\frac{d}{dx}\left[1+\frac{1-\frac{1}{1+x^2}}{1+\frac{1}{1+x^2}}\right]=\frac{2x}{\left(2+x^2\right)^2}$$

54. $x\sin y=\sin y\cos a+\cos y\sin a$

$\Rightarrow \sin y\left(x-\cos a\right)=\cos y\sin a$

$\frac{\sin y}{\cos y}=\frac{\sin a}{x-\cos a}\Rightarrow y=Tan^{-1}\left(\frac{\sin a}{x-\cos a}\right)$

55. $f(x)=Tan^{-1}(\cos x+n)-Tan^{-1}(\cos x)$

$f^1(x)=\frac{1}{1+(n+\cos x)^2}(-\sin x)-\frac{1}{1+\cos^2 x}(-\sin x)$

$f^1(0)=0$

56. $f^1(x)=\frac{-1}{1+\left(\frac{x^x-x^{-x}}{2}\right)^2}\frac{d}{dx}\left(\frac{x^x-x^{-x}}{2}\right)$

$f^1(1)=\frac{-4}{(1+1)^2}.\frac{1(1+\log 1)+1(1+\log 1)}{2}=(-1)\left(\frac{2}{2}\right)$

$=-1$

57. $\log y=x[\log(1+1/x)]$

$\frac{1}{y}y_1=\frac{x^2}{x+1}\left(\frac{-1}{x^2}\right)+\log\left(1+\frac{1}{x}\right)\Rightarrow y_1=\frac{-y}{x+1}+y\log\left(1+\frac{1}{x}\right)$

$y(2)=\frac{9}{4}$; $y_1(2)=\frac{9}{4}\left(\frac{-1}{3}+\log\frac{3}{2}\right)$

$y_2(2)=y_1(2)\left(\frac{-1}{3}+\log\frac{3}{2}\right)+y(2)\left(\frac{1}{9}-\frac{1}{6}\right)$

$y_2(2)=\frac{9}{4}\left(\frac{-1}{3}+\log\frac{3}{2}\right)^2-\frac{1}{8}$

58. $y=\frac{f(x)}{1+\frac{g(x)}{1+\frac{f(x)}{1+...\infty}}}\Rightarrow\frac{dy}{dx}=\frac{(1+y)f^1(x)-yg^1(x)}{1+2y+g(x)-f(x)}$

59. $p\left(x^2+1\right)=\left(p(x)\right)^2+1\Rightarrow p(x)=x$,

$p^1(x)=1\Rightarrow p^1(0)=1$

60. $y^1=-\frac{x}{y}\Rightarrow y^{11}=\frac{-y+xy^1}{y^2}=-\frac{2}{y^3}$

$-\frac{2}{y^3}+\frac{A}{y^3}=0\Rightarrow A=2$

61. $y=Tan^{-1}\left(\frac{1-\log x^2}{1+\log x^2}\right)+Tan^{-1}\left(\frac{3+2\log x}{1-3(2\log x)}\right)$

$=Tan^{-1}(1)-Tan^{-1}(\log x^2)+Tan^{-1}(3)+Tan^{-1}(3\log x)$

$=Tan^{-1}(1)+Tan^{-1}(3)$; $\frac{d^2y}{dx^2}=0$

62. $y=Tan^{-1}\left(\frac{8\sin x\cos x}{\cos^2 x-7\sin^2 x}\right)$

$=Tan^{-1}\left(\frac{8\tan x}{1-7\tan^2 x}\right)=Tan^{-1}\left(\frac{7\tan x+\tan x}{1-7\tan x.\tan x}\right)$

$y=Tan^{-1}\left(7\tan x\right)+x$

63. $v+u=6\sqrt{uv}\Rightarrow v^2+u^2=34uv$ then differentiate

64. $e^{x^e}.e.x^{e-1}+x^{e^x}\left[e^x.\frac{1}{x}+e^x.\log x\right]$

$+e^{x^x}\left[x^x\left(1+\log x\right)\right]$

65. Taking logarithms on both sides & dfferentiating

66. $y=|\cot\theta|=-\cot\theta,\left(\theta=\frac{3\pi}{4}\right)$

67. Put $x^n=\sin\alpha, y^n=\sin\beta$

68. $y=-\cos x+\sin x$

69. $f(x)=\cos x-\sin x, 0<x\le\frac{\pi}{4}$

$\Rightarrow \sin x-\cos x,\frac{\pi}{4}<x<\frac{\pi}{2}$

70. $u(x)=7.v(x)\Rightarrow u^1(x)=7v^1(x)\Rightarrow p=7$

$\left(\frac{u(x)}{v(x)}\right)^1=0\Rightarrow q=0$

71. Exponent of $x=\frac{l^2-m^2+m^2-n^2+n^2-l^2}{(l-m)(m-n)(n-l)}=0$

$y=x^0=1$

72. $\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

73. $\frac{dy}{dx} = f^1\left(\frac{3x+4}{5x+6}\right)\frac{d}{dx}\left(\frac{3x+4}{5x+6}\right)$

74. [$\because 0 \le \cos^{-1} x \le \pi \Rightarrow$ in the neighbourhood of $x = \frac{5\pi}{4}$ we have $0 < 2\pi - x < \pi$]

75. Differentiate

76. $f(x) = \frac{\pi}{2} - \sqrt{\frac{1+x}{2}} + x^x$

77. Taking logarithms on both sides & dfferentiating

78. $\frac{dx}{dt} = -\left(\frac{3}{t^4} + \frac{2}{t^3}\right) = -\left(\frac{3+2t}{t^4}\right)$

$\frac{dy}{dt} = -\left(\frac{3}{t^3} + \frac{2}{t^2}\right) = -\left(\frac{3+2t}{t^3}\right)$; $\frac{dy}{dx} = t$

79. $f^1(x) = 0 + 0 + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix}$

80. Put $x = 1 \Rightarrow y = \frac{\pi}{2}$ and then differentiate.

81. $g^1(x) = 2\left(f\left(2f(x)+2\right)\right)\left(\frac{d}{dx}\left(f\left(2f(x)+2\right)\right)\right)$

$= 2f\left(2f(x)+2\right)f^1\left(2f(x)+2\right).\left(2f^1(x)\right)$

$= g^1(0) = 2f\left(2f(0)+2\right).f^1\left(2f(0)+2\right).2\left(f^1(0)\right)$

$= 4f(0)f^1(0)$

$= 4(-1)(1) = -4$

82. $f^1(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h}$

$= f(x)\lim_{h\to 0}\frac{f(h)-1}{h} = f(x)\lim_{h\to 0}\frac{1+(\sin 2h)g(h)-1}{h}$

$= f(x)\lim_{h\to 0}\frac{\sin 2h}{h}.\lim_{h\to 0} g(h) = 2f(x)g(0)$

83. we have, $f(x).f(y) = f(x)+f(y)+f(xy)-2$

replace y by $\frac{1}{x}$

$\Rightarrow f(x).f\left(1/x\right) = f(x) + f\left(1/x\right) + f(1) - 2$

$\Rightarrow f(x).f\left(1/x\right) = f(x) + f\left(1/x\right)$

(since $f(1) = 2$ putting x=y=1)

$\Rightarrow f(x) = x^n + 1 \Rightarrow f(2) = x^2 + 1$ (since f(2)=5)

$\Rightarrow n = 2 \quad \therefore f(x) = x^2 + 1 \Rightarrow f(3) = 10$

84. $h^1(x) = 2f(x)f^1(x) + 2g(x)g^1(x)$

as $f^1(x) = g(x), \quad g^1(x) = -f(x)$

$\therefore h^1(x) = 0 \quad \therefore h(x)$ is constant function.

JEE MAINS QUESTIONS

1. If x^2+y^2 + siny = 4 then the value of $\frac{d^2y}{dx^2}$ at the point (-2, 0) is [2018]

1) -34 2) -32 3) 4 4) -2

2. If f(x) = $\sin^{-1}\left(\frac{2x3^x}{1+9^x}\right)$ then $f^1\left(\frac{-1}{2}\right)$ equals [2018]

1) - $\sqrt{3}$ log e$\sqrt{3}$ 2) $\sqrt{3}$ log e$\sqrt{3}$

3) -$\sqrt{3}$ log e^3 4) $\sqrt{3}$ log e^3

3. If x = 3 tan and y = 3 sec t then the value of $\frac{d^2y}{dx^2}$ at t = $\frac{\pi}{4}$ is [2019]

1) $\frac{3}{2\sqrt{2}}$ 2) $\frac{1}{6}$ 3) $\frac{1}{6\sqrt{2}}$ 4) $\frac{1}{3\sqrt{2}}$

4. For x > 1, if $(2x)^{2y}$ = 4 $e^{2x - 2y}$ then $(1+\log_e{}^{2x})^2 \frac{dy}{dx}$ is equal to [2019]

1) $\frac{x\log_e^{2x}-\log_e^2}{x}$ 2) $\log_e{}^{2x}$

3) x $\log_e{}^{2x}$ 4) $\frac{x\log_e^{3x}-\log_e^2}{x}$

5. Let $x^K + y^K = a^K$ where a, K>0 and $\frac{dy}{dx}+\left(\frac{y}{x}\right)^{\frac{1}{3}}$=0 tjem fomd K [2020]

1) $\frac{1}{3}$ 2) $\frac{2}{3}$ 3) $\frac{4}{3}$ 4) 2

6. Let y - y(x) be a function of x satisfying y = $\sqrt{1-x^2} = K = x\sqrt{1-y^2}$. Where K is a constant and y $\left(\frac{1}{2}\right)$ = $\frac{-1}{4}$. Then $\frac{dy}{dx}$ at x = $\frac{1}{2}$ is =

1) $\frac{5}{\sqrt{7}}$ 2) - $\frac{5}{\sqrt{7}}$ 3) - $\frac{\sqrt{5}}{2}$ 4) $\frac{\sqrt{5}}{2}$

7. If (a + 2b cos x) (a-2b cos y) = a^2 - b^2 where a > b> 0, then $\frac{dx}{dy}$ at $\left(\frac{-\pi}{4},\frac{\pi}{4}\right)$ is

1) $\frac{2a+b}{2a-b}$ 2) $\frac{a+b}{a-b}$ 3) $\frac{a+b}{a+b}$ 4) $\frac{a-2b}{a+2b}$

8. Let f " R be defined as (f (x) =

$$\begin{cases} x^5\sin\left(\frac{1}{x}\right)+5x^2 & x<0 \\ 0 & x=0 \\ x^5\cos\left(\frac{1}{x}\right)+\lambda x^2 & x>0 \end{cases}$$

Then the value of λ for which $f^{11}(0)$ exist. is

KEY

1) 1 2) 2 3) 3 4) 1

5) 2 6) 3 7) 3 8) 5

SOLUTIONS

1. $x^2+y^2+\sin y = 4$ differentiate w.r.t. x

$2x + 2y\frac{dy}{dx} + \cos\frac{dy}{dx} = 0$

again differentiate w.r.t. x

$\frac{dy}{dx} = \frac{-2x}{2y+coxy}, \left(\frac{dy}{dx}\right)_{(-2,0)} = 4$

$$\frac{d^2y}{dx^2} = \frac{(2x+\cos y)(-2)+2x\left(2\frac{dy}{dx}-\sin y\frac{dy}{dx}\right)}{(2y+\cos y)^2}$$

$$\left(\frac{d^2y}{dx^2}\right)_{(-2,0)} = \frac{(0+1)(-2)-4(2(4))-0}{(0+1)^2} = -34$$

2. Let $3^x = \tan\theta$ then $f(\theta) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$f(0) = \sin^{-1}(\sin 2\theta) = 2\theta \Rightarrow f(x) = 2\tan^{-1}(3^x)$

$f^1(x) = \frac{2}{1+9^x}$. 3x log 3 $\Rightarrow f^1\left(\frac{-1}{2}\right) = \sqrt{3}\log_e\sqrt{3}$

3.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \sin t \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{3}\cos^3 t \Rightarrow \left(\frac{d^2y}{dx^2}\right)_{t-\frac{\pi}{4}} = \frac{1}{6\sqrt{2}}$$

4. Taking lofg on both sides

$2y\log 2x = \log 4 + 2x - 2y \Rightarrow y = \frac{x+\log 2}{1+\log 2x}$

$$\frac{dy}{dx} = \frac{(1+\log 2x)1-(x+\log 2).\frac{1}{2x}}{(1+\log 2x)^2} \Rightarrow (1+\log 2x)^2$$

$$\frac{dy}{dx} = \frac{x\log 2x - \log 2}{x}$$

5. Differentiate w.r.t. x

K . x^{k-1} + K . $yd^{k-1} = 0$

$$\frac{dy}{dx} = -\left(\frac{x}{y}\right)^{k-1} \Rightarrow \frac{dy}{dx} + \left(\frac{x}{y}\right)^{k-1} = 0$$

$K - 1 = \frac{-1}{3} \Rightarrow K = 1 - \frac{1}{3} = \frac{2}{3}$

6. $x = \frac{1}{2}$, $y = \frac{1}{4} \Rightarrow xy = -\frac{1}{8}$ diffrentiate w.r.t. x

$$y\frac{1(-2x)}{2\sqrt{1-x^2}} + y^1\sqrt{1-x^2}$$

$$= -\left(1.\sqrt{1-y^2} + x\frac{-2y}{2\sqrt{1-y^2}}y^1\right)$$

$$\Rightarrow \frac{-xy}{\sqrt{1-x^2}} + y^1\sqrt{1-x^2} = -\sqrt{1-y^2} + \frac{xy\ y^1}{\sqrt{1-y^2}}$$

$$y^1\left(\sqrt{1-x^2}-\frac{xy}{\sqrt{1-y^2}}\right)=\frac{xy}{\sqrt{1-x^2}}-\sqrt{1-y^2}$$

$$y^1\left(\frac{\sqrt{45}+1}{2\sqrt{15}}\right)=-\left(\frac{1+\sqrt{45}}{4\sqrt{3}}\right)\therefore y^1=-\frac{\sqrt{5}}{2}$$

7. (a+$\sqrt{2}$b cosx) (a-$\sqrt{2}$b cosy) = a^2 - b^2

Differentiate w.r.t. x

(-$\sqrt{2}$b sinx) (a-$\sqrt{2}$b cosy) + (a+$\sqrt{2}$b cosx)($\sqrt{2}$bsiny) y^1 = 0 at $\left(\frac{-\pi}{4},\frac{\pi}{4}\right)$

- b(a-b) +(a+b) b y^1 = 0

$$\frac{dx}{dy}=\frac{a+b}{a-b}$$

8. If g(x) = x^5 sin $\left(\frac{1}{x}\right)$ and h(x) - x^5 cos$\left(\frac{1}{x}\right)$

then $g^{11}(0)$ = 0 and $h^{11}(0)$ = 0

So $f^{11}(0^+) = g^{11}(0^+) + 10 = 10$

and $f^{11}(0^-) = h^{11}(0^-) + 2\lambda = f^{11}(10^+)$

$\Rightarrow 2\lambda = 10$

λ = 5

TANGENT & NORMAL

SYNOPSIS

Slope of tangent & normal :

- If the tangent drawn to the curve y = f(x) at $P(x_1, y_1)$ on it makes an angle $\theta(\neq 90^0)$ with $\overrightarrow{OX}$ then $\tan\theta$ is defined as the slope of the tangent and it is also called the gradient of the curve

  $i.e., m = \tan\theta = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

  i) For the curve f(x, y) = 0, slope of the tangent at $P(x_1, y_1) = -\left(\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}\right)_{(x_1, y_1)}$

  ii) For the curve , y = f(x) here x = f(t); y = g(t), Then the slope of the tangent at P(t) is $\frac{g^1(t)}{f^1(t)}$.

  iii) If $\frac{dy}{dx} = 0$ then the tangent is horizontal

  iv) If $\frac{dx}{dy} = 0$ then the tangent is vertical

- A straight line which is perpendicular to the tangent to the curve at the point of contact is called the normal to the curve.

  i) Slope of normal at any point $P(x_1, y_1)$ on a curve is given by $\left(\frac{-1}{dy/dx}\right)_{(x_1, y_1)}$

  ii) For the curve f(x, y) = 0 the slope of the normal at $P(x_1, y_1)$ is $\left(\frac{\partial f}{\partial y} \Big/ \frac{\partial f}{\partial x}\right)_{(x_1, y_1)}$

  iii) For the curve x = f(t); y = g(t) the slope of the normal at P(t) is $-\left(\frac{f^1(t)}{g^1(t)}\right)$

Equation of tangent and normal :

- Equation of the tangent to the curve y = f(x) at (x_1, y_1) is $y - y_1 = m(x - x_1)$

- Equation of the normal to the curve y = f(x) at (x_1, y_1) is $(y - y_1) = \left(\frac{-1}{m}\right)(x - x_1)$.

  where $m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

- If a curve passes through the origin then the equation of tangent(s) at the origin can be directly written by equating the lowest degree terms appearing in the equation of the curve to zero.

- Equation of the tangent to the curve

  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ at

  (x_1, y_1) is

  $axx_1 + h(xy_1 + yx_1) + byy_1 +$
  $g(x + x_1) + f(y + y_1) + c = 0$

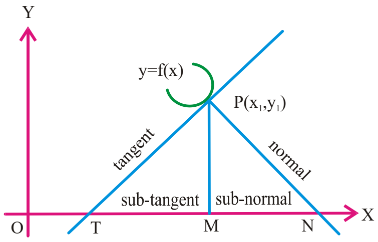
- The condition for the line $y = mx + c$ to be a tangent to

  i) The parabola $y^2 = 4ax$ is $c = \frac{a}{m}$

  ii) An ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $c^2 = a^2m^2 + b^2$

  iii) The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $c^2 = a^2m^2 - b^2$

Length of tangent , normal, sub-tangent and sub-normal :



- Let the tangent and normal drawn to the curve $y = f(x)$ at $P(x_1, y_1)$ meet the x-axis at T & N. Draw the line PM perpendicular to x-axis. If m is the slope of the tangent then

  i) PT = Length of the tangent $= \left|\dfrac{y_1 \cdot \sqrt{1+m^2}}{m}\right|$

  ii) PN = Length of the normal $= \left|y_1 \cdot \sqrt{1+m^2}\right|$

  iii) TM = Length of the sub-tangent $= \left|\dfrac{y_1}{m}\right|$

  iv) MN = Length of the sub-normal $= |y_1 m|$

  where $m = \left(\dfrac{dy}{dx}\right)_{p(x_1,y_1)}$

- Length of sub-tangent, ordinate of the point, Length of sub-normal at any point on the curve y=f(x) are in G.P

  i.e., $(\text{ordinate})^2 = (\text{L.S.T})(\text{L.S.N})$

Leibnitz Rule :

- $$\frac{d}{dx}\int_{\varphi(x)}^{\phi(x)} f(t).dt = f(\phi(x)).\phi'(x) - f(\varphi(x)).\varphi'(x)$$

Angle between two curves :

- The angle between any two curves at the point of intersection is defined as the angle between the tangents to the curves at that point of intersection.
- Let P be a point of intersection of the two curves $y = f(x)$, $y = g(x)$ and m_1, m_2 be the slopes of the tangents to the curves at P. If θ is the angle between the curves then

  $$\tan\theta = \left(\frac{m_1 - m_2}{1+m_1m_2}\right) \text{ where } m_1m_2 \neq -1$$

- The curves $y = f(x)$ and $y = g(x)$

  i) Touch each other if $m_1 = m_2$

  ii) Cut each other orthogonally if $m_1\, m_2 = -1$.
- The curves $f(x,y) = 0$, $g(x, y) = 0$

  i) Touch each other if $\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial g}{\partial y} = \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial g}{\partial x}$

  ii) cut each other orthogonally

  if $\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial g}{\partial x} + \dfrac{\partial f}{\partial y}\cdot\dfrac{\partial g}{\partial y} = 0$
- Angle between two curves $y^2 = 4ax$ and $x^2 = 4by$ not at origin is

  $$\theta = \tan^{-1}\left(\frac{3a^{1/3}b^{1/3}}{2\left(a^{2/3}+b^{2/3}\right)}\right)$$

- The family of curves $\dfrac{x^2}{a^2+\lambda} + \dfrac{y^2}{b^2+\lambda} = 1$ is self orthogonal (λ is a parameter)
- The family of curves $y^2 = 4a(x+a)$ is self orthogonal (a is a parameter)
- If the curves $a_1x^2 + b_1y^2 = 1$ and $a_2x^2 + b_2y^2 = 1$ cut each other orthogonally then $\dfrac{1}{a_1} - \dfrac{1}{a_2} = \dfrac{1}{b_1} - \dfrac{1}{b_2}$
- The area of the triangle formed by any tangent on the curve $xy = c^2$ and the coordinate axes is $2c^2$ sq.units.
- If the area of the triangle formed by any tangent to the curve $x.y^n = a^{n+1}$ and the co-ordinate axes is constant then $n = 1$.
- If the area of the triangle formed by any tangent to the curve $x^m y^n = k$, $(m \neq 0, n \neq 0)$ and the coordinate axes is a constant then $m = n$
- . The area of the triangle formed by the tangent,normal at a point $P(x_1, y_1)$ on the curve $y = f(x)$ and the line

i. $x=k$ is $\dfrac{(x_1-k)^2(m^2+1)}{2|m|}$ sq.units

ii. $y=k$ is $\dfrac{(y_1-k)^2(m^2+1)}{2|m|}$ sq.units

iii. x-axis is $\dfrac{y_1^2(m^2+1)}{2|m|}$ sq.units

iv. y - axis is $\dfrac{x_1^2(m^2+1)}{2|m|}$ sq.units

- The tangent and normal at a point (x_1, y_1) on the curve meets the x-axis in T and G then $TG=\left|y_1\left(m+\frac{1}{m}\right)\right|$
- At any point on the curve $y^2=4ax$, the length of subnormal is constant
- If the normal at (x_1, y_1) on the curve $y=f(x)$ makes equal intercepts on the coordinate axes then $\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=1$.

Some standard results :

- At any point on the curve $by^2=(x+a)^3, \dfrac{(L.S.T)^2}{L.S.N}=\dfrac{8b}{27}$
- The equation of the tangent at (a, b) to the curve $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ is $\frac{x}{a}+\frac{y}{b}=2$, $(n\neq 0)$
- Point on the curve $ay^2=x^3$ at which the normal makes equal intercepts on the axes is $\left(\frac{4a}{9},\frac{8a}{27}\right)$.
- If p, q are the lengths of perpendiculars from the origin to tangent and normal at a point on the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ respectively then $4p^2+q^2=a^2$.
- If p and q are the lengths of perpendiculars from the origin to the tangent and normal to the curve. $x=ae^{\theta}(\sin\theta-\cos\theta)$ and $y=ae^{\theta}(\sin\theta+\cos\theta)$ then $p=q$.
- If the curves $xy=c^2$ and $y^2=4ax$ cut each other orthogonally then $c^4=32a^4$
- A tangent to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ (or) $x=a\cos^4\theta; y=a\sin^4\theta$ cuts the axes in A and B then OA + OB = a.
- A tangent to the curve $x^{2/3}+y^{2/3}=a^{2/3}$ (or) $x=a\cos^3\theta; y=a\sin^3\theta$ cuts the co-ordinate axes in A and B then AB = $|a|$.
- The tangent at any point 't' on the curve $x=at^3$ and $y=at^4$ divides the abscissa of the point of contact in the ratio 1 : 3.

EXERCISE -I

1. The slope of the tangent to the curve $y=\dfrac{8}{4+x^2}$ at x = 2 on it is

1) -2 2) $\frac{-1}{2}$ 3) $\frac{1}{2}$ 4) 2

2. The slope of the tangent at (-2, 0) on the curve $y=6+x-x^2$

1) 3 2) 5 3) -1 4) -3

3. The slope of the normal to the curve $y^2=4x$ at (1,2)

1) -1 2) 1 3) 2 4) -2

4. A point on the curve $y=x^4-4x^3+4x^2+1$, the tangent at which is parallel to x-axis is

1) (1, 1) 2) (2, 1) 3) (3, 1) 4) (1, 3)

5. If V is the set of points on the curve $y^3-3xy+2=0$ where the tangent is vertical then V =

1) ϕ 2) $\{(1,0)\}$

3) $\{(1,1)\}$ 4) $\{(0,0),(1,1)\}$

6. The gradient of the curve $x^m \cdot y^n=(x+y)^{m+n}$ is

1) $\frac{x}{y}$ 2) $\frac{y}{x}$ 3) $-\frac{x}{y}$ 4) $-\frac{y}{x}$

7. **The inclination of the tangent at $\theta=\frac{\pi}{3}$ on the curve $x=a(\theta+\sin\theta)$, $y=a(1+\cos\theta)$ is**
1) $\frac{\pi}{3}$ 2) $\frac{\pi}{6}$ 3) $\frac{2\pi}{3}$ 4) $\frac{5\pi}{6}$

8. **The point on the curve $x^2+y^2-2x-3=0$ at which the tangent is parallel to x-axis is**
1) (1,0), (-1, -4) 2) (0, -1), (-2, 3)
3) (2, 13),(-2,-3) 4) (1,2), (1,-2)

9. **For the curve $x=t^2-1, y=t^2-t$, the tangent is perpendicular to x-axis then**
1) t = 0 2) $t=\frac{1}{2}$ 3) t = 1 4) $t=\frac{1}{\sqrt{3}}$

10. **The slope of the normal to the curve given by $x=a(\theta-\sin\theta), y=a(1-\cos\theta)$ at $\theta=\frac{\pi}{2}$**
1) $\frac{-1}{2}$ 2) $\frac{1}{2}$ 3) -1 4) 2

11. **The point at which the tangent line to the curve $x^3+y^3=a^3$ is parallel to y-axis is**
1) (0, a) 2) (a, 0) 3) (–a, 0) 4) (0, –a)

12. **The equation of the tangent to the curve $6y=7-x^3$ at (1, 1) is**
1) $2x+y=3$ 2) $x+2y=3$
3) $x+y=-1$ 4) $x+y+2=0$

13. **The equation of the normal to the curve $y=x+\sin x.\cos x$ at $x=\frac{\pi}{2}$ on it is**
1) $x-\pi=0$ 2) $x+\pi=0$
3) $2x-\pi=0$ 4) $2x+\pi=0$

14. **The point on the curve $y=5x-x^2$ at which the normal is perpendicular to the line $x+y=0$ is**
1) (3, -6) 2) (3,6) 3) (-3,-6) 4) (6,3)

15. **The equation of the tangent to the curve $y=2\sin x+\sin 2x$ at $x=\frac{\pi}{3}$ on it is**
1) y - 3 = 0 2) $y+\sqrt{3}=0$
3) $2y-3=0$ 4) $2y-3\sqrt{3}=0$

16. **If the curve $y=ax^2+bx$ passes through (-1,0) and y = x is the tangent line at x = 1 then (a,b)**
1) (1,1) 2) (1/2,1/2)
3) (1/3, 1/3) 4) (3, 3)

17. **The equation of the normal at $t=\frac{\pi}{2}$ to the curve x=2sint, y = 2 cost is**
1) x = 0 2) y = 0 3) y = 2x+3 4) y = 3

18. **Equation of the tangent to the curve $y=1-e^{\frac{x}{2}}$ at the point where the curve cuts y-axis is**
1) x + y = 0 2) x + 2y = 0
3) 2x + y = 0 4) 2x - y = 0

19. **The equation of the normal to the curve $y^2=4ax$ at the origin is**
1) x = 0 2) x = 2 3) y = 0 4) y = 2

20. **The equation of the normal to the curve given by $x=at^2$, y = 2at at the point 't' is**
1) $xt+y=2at+at^3$ 2) $x+yt=2at+at^3$
3) $xt-y=at+at^3$ 4) $x=0$

21. **The equation of the tangent to the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point θ on it is**
1) $bx\cos\theta-ay\sin\theta=ab$
2) $bx\sin\theta+ay\cos\theta=ab$
3) $bx\cos\theta+ay\sin\theta=ab$
4) y = 0

22. **The length of the sub-normal at any point on the curve $y^2=2px$ is [EAM -2019]**
1) Constant 2) Varies as abscissa
3) Varies as ordinate 4) Varies as p

23. **The length of the subtangent to the curve $x^2+xy+y^2=7$ at (1, -3) is**
1) 15 2) 7 3) 12 4) 10

24. **The tangent at A(2,4) on the curve $y=x^3-2x^2+4$ cuts the x-axis at T then the length of AT =**
1) $\sqrt{10}$ 2) $\sqrt{12}$ 3) $\sqrt{15}$ 4) $\sqrt{17}$

25. **The length of normal at (2,1) on the curve $xy+2x-y=5$ is**
1) $\sqrt{5}$ 2) $\sqrt{\frac{5}{3}}$ 3) $\sqrt{\frac{10}{3}}$ 4) $\sqrt{10}$

26. **The length of sub-normal to the curve $xy = a^2$ at (x,y) on it varies as**

1) x^2 2) y^2 3) x^3 4) y^3

27. **If the length of the subtangent is 9 and the length of the subnormal is 4 at (x,y) on y = f(x) then y =**

1) 36 2) ± 9 3) ± 4 4) ± 6

28. **For the parabola $y^2 = 4ax$ the ratio of the subtangent to the abscissae is**

1) 1 : 1 2) 2 : 1 3) x : y 4) $x^2 : y$

29. **The length of sub-tangent to the curve $y^n = a^{n-1}.x$ at (x, y) on it is**

1) $\left|\frac{n}{x}\right|$ 2) $|nx|$ 3) $n^2|x|$ 4) $\frac{n^2}{|x|}$

30. **If at any point on a curve the subtangent and subnormal are equal, then the length of the tangent is equal to**

1) ordinate 2) $\sqrt{2}\,|\text{ordinate}|$
3) $\sqrt{2\text{ ordinate}}$ 4) $2\sqrt{\text{ordinate}}$

31. **The length of subnormal to the curve $y = be^{x/a}$ at any point (x, y) is proportional to**

1) x 2) y 3) x^2 4) y^2

32. **If the subnormal to the curve $x^2.y^n = a^2$ is a constant then n = [EAM -2020]**

1) -4 2) -3 3) -2 4) -1

33. **At any point on the curve y = f(x), the sub-tangent, the ordinate of the point and the sub-normal are in**

1) A.P. 2) G.P. 3) H.P. 4) A.G.P.

34. **The curve $x^4 - 2xy^2 + y^2 + 3x - 3y = 0$ cuts the x-axis at (0,0) at an angle**

1) $\frac{\pi}{4}$ 2) $\frac{\pi}{2}$ 3) $\frac{\pi}{6}$ 4) $\frac{\pi}{3}$

35. **If θ is an angle between the curves $y^2 = x^3, y = 2x^2 - 1$ at (1,1) then $|\tan\theta|$**

1) 5/14 2) 5/12 3) 25/12 4) 14/5

36. **The angle between the curves $x^3 - 3xy^2 = 2$ and $3x^2y - y^3 = 2$ is [EAM -2018]**

1) $\frac{\pi}{6}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{3}$ 4) $\frac{\pi}{2}$

37. **The curves $y = x^2$ and $6y = 7 - x^3$ intersect at (1,1) at an angle is**

1) $\frac{\pi}{4}$ 2) $\frac{\pi}{3}$ 3) $\frac{\pi}{2}$ 4) π

38. **If the curves $ay + x^2 = 7$ and $x^3 = y$ cut orthogonally at (1,1) then a =**

1) 1 2) -6 3) 6 4) $\frac{1}{6}$

39. **The angle between the curves y = sinx and $y = \cos x$ is**

1) $\frac{\pi}{3}$ 2) $\frac{\pi}{2}$ 3) $\mathrm{Tan}^{-1}(2)$ 4) $\mathrm{Tan}^{-1}(2\sqrt{2})$

40. **If the curves $x = y^2$ and xy = k cut each other orthogonally then k^2 =**

1) $\frac{1}{2}$ 2) $\frac{1}{4}$ 3) $\frac{1}{8}$ 4) $\frac{1}{16}$

41. **The tangent at the point P(x, y) on the curve $x^m.y^n = a^{m+n}$ meets the axes at A and B. The ratio in which P divides $\overline{AB}$ is**

1) m : 1 2) 1 : n 3) n : m 4) m : n

42. **If the tangent at $\theta = \frac{\pi}{4}$ to the curve $x = a\cos^3\theta, y = a\sin^3\theta$ meets the x and y axes in A and B then the area of the triangle OAB is**

1) $\frac{a^2}{4}$sq.units 2) $\frac{a^2}{2}$sq.units
3) $\frac{3a^2}{4}$sq.units 4) $\frac{5a^2}{4}$sq.units

43. **If the area of the triangle formed by a tangent to the curve $x^n y = a^{n+1}$ and the coordinate axes is constant, then n =**

1) 2 2) -2 3) –1 4) 1

44. **If the line ax + by + c = 0 is normal to the curve xy = 1 then**

1) a > 0, b > 0 2) a > 0, b < 0
3) a < 0, b < 0 4) a = 0, b = 0

45. **The sum of the squares of the intercepts on the axes of the tangent at any point on the**

curve $x^{2/3}+y^{2/3}=a^{2/3}$ **is**

1) $\frac{a^2}{2}$ 2) a^2 3) 2a 4) $\frac{3a}{2}$

KEY

01) 2	02) 2	03) 1	04) 2	05) 3	06) 2
07) 4	08) 4	09) 1	10) 3	11) 4	12) 2
13) 3	14) 2	15) 4	16) 3	17) 2	18) 2
19) 3	20) 1	21) 3	22) 1	23) 1	24) 4
25) 4	26) 4	27) 4	28) 2	29) 2	30) 2
31) 4	32) 1	33) 2	34) 1	35) 3	36) 4
37) 3	38) 3	39) 4	48) 3	49) 3	50) 1
51) 4	52) 2	53) 2			

SOLUTIONS

1. Find $\frac{dy}{dx}$ and sustitute $x=2$

2. Find $\frac{dy}{dx}$ at (–2,0)

3. Slope of normal $=\frac{-1}{\left(\frac{dy}{dx}\right)_p}$

4. y = x^4 - $4x^3+4x^2+1$

$\frac{dy}{dx}$ = $4x^3-12x^2+8x$ slope is parallel to x-axis

$4x(4x^2-3x+2)=0$

x=0, x=1, 2

y=1, 2, 11 (1,2) or (2, 1)

5. $y^3-3xy+2=0$

$\frac{dy}{dx}=\frac{3y}{3y^2-3x}$. If the tangnet is vertical

$m=\infty$

$\therefore\ \therefore 3y^2-3x=0, y^2=x$

By verification {(1,1)} is on the given curve

6. $x^m y^n=(x+y)^{m+n}$

$m\log x+n\log y=(m+n)\log(x+y)$

$\frac{m}{x}+\frac{n}{y}\frac{dy}{dx}=\frac{m+n}{x+y}\left[1+\frac{dy}{dx}\right]$

$\frac{dy}{dx}\left[\frac{n}{y}-\frac{m+n}{x+y}\right]=\frac{m+n}{x+y}-\frac{m}{x}$

$\frac{dy}{dx}\left[\frac{nx-ny-my-ny}{y(x+y)}\right]=\frac{mx+nx-mx-my}{(x+y)x}$

$\frac{dy}{dx}=\frac{y}{x}$

7. Find $\frac{dy}{dx}$ and $\theta=\frac{\pi}{3}$

$m=\tan\alpha=\tan\frac{5\pi}{6}$, $\alpha=\frac{5\pi}{6}$

8. $\frac{dy}{dx}=\frac{-(2x-2)}{2y}=0$

$\Rightarrow x=1, 1+y^2-2-3=0\Rightarrow y=\pm 2$

points= $(1,2),(1,-2)$

9. $\frac{dy}{dx}=\frac{2t-1}{2t}$

tangent is $\perp^{lar}$ to x–axis $m=\infty$

$\frac{2t-1}{2t}=\frac{1}{0}\Rightarrow t=0$

10. $\left(\frac{dx}{dt}\right)_{at\theta}=\left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right)_{\theta=\frac{\pi}{2}}$

11. $x^3+y^3=a^3\Rightarrow 3x^2+3y^2\frac{dy}{dx}=0$

m = $\frac{dy}{dx}$ = $-\frac{x^2}{y^2}$ parullel to y-axis

m = ∞ = 1/0

1/0 = $-\frac{x^2}{y^2}\Rightarrow$ y^2 = 0 ,, y = 0, x = a

(a, 0)

12. $m=\left(\frac{dy}{dx}\right)_{(1,1)}$ and apply $y-y_1=m(x-x_1)$

13. $x=\frac{\pi}{2}\Rightarrow y=\frac{\pi}{2}$, $m=\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{2}}$ and apply

$y-y_1=\frac{-1}{m}(x-x_1)$

14. y = 5x -x^2 differentiate w.r.t x

$\frac{dy}{dx}=5-2x$

slope of normal 5 - 2x = -1, 2 = 3

they y = 6 (3, 6)

15. y = 2sinx+sin2x at x = $\frac{\pi}{3}$ then y =

$\sqrt{3}+\frac{\sqrt{3}}{2}=\frac{3\sqrt{3}}{2}$

$\left(\frac{\pi}{3},\frac{3\sqrt{3}}{2}\right)$

$\frac{dy}{dx}$ = 2 cosx + 2cos2x

$\Rightarrow$m = $\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{3}}$ =1 - 1 = 0

y - $\frac{3\sqrt{3}}{2}$ = 0

2y - 3$\sqrt{3}$ = 0

16. At $x=1$, $\frac{dy}{dx}=1$ and substitute (-1,0) in the curve and solve the two equations for a and b

17. x = 2 sint, y = 2cot (x, y) =)2, 0)

$$\frac{dy}{dx}=\frac{\frac{dy}{dx}}{\frac{dx}{dy}}=\frac{-2\sin t}{2\cos t}=-\tan t$$

$$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{2}}=-\tan\frac{\pi}{2}=\infty$$

slope or normal = 0

equation of normal y - 0 = 0(x-2)

18. y = 1 - $e^{x/2}$ put x = 0 then y = 1 point

(0, 0)

$\frac{dy}{dx}=-\frac{1}{2}e^{\frac{x}{2}}$, m = $\left(\frac{dy}{dx}\right)_{(0,0)}=-\frac{1}{2}$

equation to tangent y -0 = $-\frac{1}{2}$

2y = -x

x+2y = 0

19. Slope of the normal = 0

20. $\frac{dy}{dx}=\left(\frac{dy}{dt}\right)\Big/\left(\frac{dx}{dt}\right)$

21. Equation of the tangent at 'θ' to

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$

22. y^2 = 2px dofferentiate w.r.t. s

2y $\frac{dy}{dx}$ = 2p

m $\left(\frac{dy}{dx}\right)_{(x,y)} = \frac{p}{y_1}$

Length of sub normal = y_1 m = p is a costant

23. Length of sub–tangent$=\left|\frac{y_1}{m}\right|$

24. $\frac{dy}{dx}$ at (2, 4) is m = 4

Length of tangent

$$AT = \left|\frac{y_1\sqrt{1+m^2}}{m}\right| = \left|\frac{4.\sqrt{17}}{4}\right| = \sqrt{17}$$

25. xy + 2x - y = 5 differentiate w.r.t. x P (2, 1)

$$x\frac{dy}{dx} + y + 2 - \frac{dy}{dx} = 0$$

$$\left(\frac{dy}{dx}\right)_{(2,1)} = \frac{-3}{1}$$

Length or normal at P(x, y) = (2, 1) = y $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ = 1 $\sqrt{1+9} = \sqrt{10}$

26. Length of sub-normal $= |y_1 m|$

27. $L.S.T = \left|\frac{y}{m}\right| = 9$, $L.S.T = |ym| = 4$

$\frac{y}{m}.ym = 36$, $\Rightarrow y^2 = 36$, $y = \pm\ 6$

28. $\frac{S.T}{x} = \frac{y}{y^1 x}$, Now 2y y^1=4a

$y^1 = \frac{2a}{y}$, $\frac{y^2}{2ax} = \frac{4ax}{2ax} = \frac{2}{1}$

29. $y^n = a^{n-1}x^1$, $x^{-1}y^n = a^{n-1}$, m=-1, n=n

Length of the sub tangent $= \left|\frac{nx}{m}\right| = \left|\frac{nx}{-1}\right| = |nx|$

30. Let P (x_1, y_1) be and point

$\frac{y_1}{m} = y_1 m$ $m^2 = 1 \Rightarrow m = \pm\ 1$

length of tangent = $\left|y_1\frac{\sqrt{1+m^2}}{m}\right|$ = $y_1\ \sqrt{2}$

31. Length of sub-normal $= |y_1 m|$

32. $x^2y^n = a^2$, $m = 2, n = n$, $2m + n = 0$

$4 + n = 0$, $n = -4$

33. Length of sub-tangent$=\left|\frac{y_1}{m}\right|$

Length of sub normal $= |y_1 m|$

34. $\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$, $m = \tan\theta \therefore \tan\theta = 1$,

$\Rightarrow \theta = \pi/4$

35. $|\tan\theta| = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \frac{5}{14}$

36. $m_1 m_2 = -1$

37. Given circues y = x^2 and 6y = 7 - x^3 point (1, 1)

$\frac{dy}{dx} = 2x$ $\frac{6dy}{dx} = -3x^2$

$m_1 = \left(\frac{dy}{dx}\right)_{(1,1)} = 2$ $m_2 = \left(\frac{dy}{dx}\right)_{(1,1)} = \frac{-1}{2}$

now $m_1 m_2$ =-1 angle is = $\frac{\pi}{2}$s

38. Slope of the first curve at (1,1) is $m_1 = \frac{-2}{a}$

Slope of the second curve at (1,1) is $m_2 = 3$

$m_1 m_2 = -1 \Rightarrow \frac{-2}{a}.3 = -1 \Rightarrow a = 6$

39. $x = \frac{\pi}{4}$, $m_1 = \frac{1}{\sqrt{2}}, m_2 = \frac{-1}{\sqrt{2}}$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

40. Given circues $x = y^2$ and $xy = K$ cuts orthogonally

$1 = 2y\frac{dy}{dx}$ $\quad x\frac{dy}{dx} + y = 0$

$m_1 m_2 = -1$

$\frac{1}{2y} x \frac{-y}{x} = -1$ $\quad x = 1/2 \quad y = 1/\sqrt{2}$

now $x^2 y^2 = k^2$

$k^2 = \frac{1}{8}$

41. $\frac{PA}{PB} = \frac{n}{m}$, $PA : PB = n : m$,

42. $(x_1, y_1) = (a\cos^3\theta, a\sin^3\theta), \theta = \frac{\pi}{4}$

$(x_1, y_1) = \left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}\right)$

Equation of the tangent $\frac{x}{x_1^{\frac{1}{3}}} + \frac{y}{y_1^{\frac{1}{3}}} = a^{\frac{2}{3}}$

43. Find equation of the tangent and then use $\frac{c^2}{2|ab|}$

44. $y = \frac{1}{x}, \frac{dy}{dx} = \frac{-1}{x^2} = \frac{b}{a}, x = \sqrt{-\frac{a}{b}}$

45. Equation of the tangent at $P(\theta)$ to $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is $\frac{x}{a\cos\theta} + \frac{y}{a\sin\theta} = 1$

EXERCISE -II

1. **The area of the triangle formed by the positive x–axis, the normal and the tangent to the curve $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ in sq. units is (EAM-2016)**
 1) $2\sqrt{3}$ 2) $\sqrt{3}$ 3) $4\sqrt{3}$ 4) 6

2. **Equation of the tangent line to $y = be^{\frac{-x}{a}}$ where it crosses y-axis is**
 1) ax+by=1 2) $\frac{x}{a} + \frac{y}{b} = 1$
 3) $\frac{x}{b} + \frac{y}{a} = 1$ 4) ax–by=1

3. **The number of tangents to the curve $x^{3/2} + y^{3/2} = a^{3/2}$, where the tangents are equally inclined to the axes, is**
 1) 2 2) 1 3) 0 4) 4

4. **The equation of the tangent to the curve $y = e^{-|x|}$ at the point where the curve cuts the line x = 1 is**
 1) x+y = e 2) e(x+y)=1
 3) y+ex=1 4) x+ey=2

5. **If the slope of the tangent to the curve $y=x^3$ at a point on it is equal to the ordinate of the point then the point is**
 1) (27, 3) 2) (3, 27) 3) (3, 3) 4) (1,1)

6. **If the slope of the tangent to the curve xy+ax+by=0 at the point (1, 1) on it is 2 then values of a and b are**
 1) 1, 2 2) 1, –2 3) –1, 2 4) –1, –2

7. **The point of intersection of the tangents drawn to the curve $x^2y=1-y$ at the points where it is met by the curve xy=1–y is given by**
 1) (0, –1) 2) (1, 1) 3) (0, 1) 4) $(0, \infty)$

8. The tangent to the curve $y = 2 + bx + 3x^2$ at the point where the curve meets y-axis has the equation $4x - y + 2 = 0$ then b is

1) 7 2) 27 3) 3 4) 4

9. If the normal line at (1, -2) on the curve $y^2 = 5x - 1$ is ax - 5y + b = 0 then the values of a and b are

1) -14,4 2) 4, -14 3) 4, 6 4) 4, 10

10. The slope of the tangent to the curve at a point (x, y) on it is proportional to (x–2). If the slope of the tangent to the curve at (10, –9) on it is –3. The equation of the curve is

1) $y = k(x-2)^2$ 2) $y = \frac{-3}{16}(x-2)^2 + 1$

3) $y = \frac{-3}{16}(x-2)^2 + 3$ 4) $y = K(x+2)^2$

11. Area of the triangle formed by the normal to the curve $x = e^{\sin y}$ at (1,0) with the coordinate axes is [EAM -2017

1) $\frac{1}{4}$ 2) $\frac{1}{2}$ 3) $\frac{3}{4}$ 4) 1

12. If tangent at any point on the curve $e^y = 1 + x^2$ makes an angle θ with positive direction of the x–axis then

1) $|Tan\theta| > 1$ 2) $|Tan\theta| < 1$

3) $Tan\theta > 1$ 4) $|Tan\theta| \le 1$

13. If the length of the subnormal is equal to the length of the subtangent at any point (3,4) on the curve y=f(x) and the tangent at (3,4) to y = f(x) meets the coordinate axes A and B the maximum area of the ΔOAB is

1) $\frac{45}{2}$ 2) $\frac{49}{2}$ 3) $\frac{25}{2}$ 4) $\frac{81}{2}$

14. At any point on the curve y = f(x), the length of the sub normal is constant, then the curve is

1) circle 2) ellipse 3) parabola 4) straight line

15. Sub normal to $xy = c^2$ at any point on it varies directly as

1) cube of ordinate 2) square of ordinate
3) ordinate 4) cube of abscissa

16. The value of k for which the length of the sub tangent to the curve $xy^k = c^2$ is constant is

1) 0 2) 1 3) 2 4) –2

17. The angle between the curves $x^2=4y$ and $y^2 = 4x$ at (4,4) is

1) $\frac{\pi}{2}$ 2) $Tan^{-1}(3)$ 3) $\tan^{-1}\left(\frac{3}{4}\right)$ 4) $\tan^{-1}\left(\frac{4}{3}\right)$

18. The angle between the curves $2x^2+y^2=20$ and $4y^2-x^2=8$ at the point $\left(2\sqrt{2}, 2\right)$ is

1) $\frac{\pi}{2}$ 2) $\tan^{-1}\left(\frac{1}{2}\right)$ 3) $Tan^{-1}(2)$ 4) $\tan^{-1}\left(\frac{2}{3}\right)$

19. The angle between the curves $x^3+y^3+x+2y=0$ and $xy+2x=y$ at the origin is

1) $\frac{\pi}{4}$ 2) $\frac{\pi}{6}$ 3) $\frac{\pi}{3}$ 4) $\frac{\pi}{2}$

20. If α is the angle between the curves $y^2 = 2x$ and $x^2 + y^2 = 8$, then tan α is

1) 1 2) 2 3) 3 4) 4

21. The curves $x^2 - y^2 = 5$ and $\frac{x^2}{18} + \frac{y^2}{8} = 1$ cut each other at the common point at an angle

1) $\frac{\pi}{4}$ 2) $\frac{\pi}{3}$ 3) $\frac{\pi}{2}$ 4) π

22. If the curves $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{l^2} - \frac{y^2}{m^2} = 1$ cut each other orthogonally then...

1) $a^2 + b^2 = l^2 + m^2$ 2) $a^2 - b^2 = l^2 - m^2$
3) $a^2 - b^2 = l^2 + m^2$ 4) $a^2 + b^2 = l^2 - m^2$

23. The circle $x^2+y^2=a^2$ and the hyperbola $x^2-y^2=a^2$

1) Touch each other at (a,0)
2) Intersect at (a,0)
3) Touch each other at $(\pm a, 0)$
4) touch each other at $\left(a\sqrt{2}, 0\right)$

24. The curves $x^2 + py^2 = 1$ and $qx^2 + y^2 = 1$ are orthogonal to each other then (EAM-2014)

1) $p - q = 2$ 2) $\frac{1}{p} - \frac{1}{q} = 2$

3) $\frac{1}{p}+\frac{1}{q}=-2$ 4) $\frac{1}{p}+\frac{1}{q}=2$

25. If the curves $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{25}+\frac{y^2}{16}=1$ cut each other orthoganally then $a^2-b^2=$

(EAM-2015)

1) 9 2) 400 3) 75 4) 41

KEY

01) 1	02) 2	03) 2	04) 4	05) 2	06) 2
07) 3	08) 4	09) 2	10) 3	11) 2	12) 4
13) 2	14) 3	15) 1	16) 1	17) 3	18) 1
19) 4	20) 3	21) 3	22) 3	23) 3	24) 4
25) 1					

SOLUTIONS

1. Given curcue $x^2+y^2 = 4$ at p(1, $\sqrt{3}$) differentiate w.r.t. x

$$2x + 2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-x}{y}$$

$$m=\left(\frac{dy}{dx}\right)_{(1,\sqrt{3})}=\frac{-1}{\sqrt{3}}$$

Required area $=\frac{y^2\left(1+m^2\right)}{2|m|}=\frac{3\left(1+\frac{1}{3}\right)}{\frac{2}{3}}=2\sqrt{3}$

2.Given $y = be^{-x/a}$ cuts y - axis put x = 0 , y = b

p(0, b)

$$\frac{dy}{dx}=\frac{-b}{a}e^{-x/a} \qquad m=\left(\frac{dy}{dx}\right)_{(0,b)}=\frac{-b}{a}$$

equation of tangent y - b = -b/a(x-0)

ay-ab = -bx

bx +ay -ab = 0 $\Rightarrow \frac{x}{a}+\frac{y}{b}=1$

3. $\frac{dy}{dx}=1\Rightarrow-\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}=1$, $\Rightarrow x=y$

$\therefore 2x^{\frac{3}{2}}=a^{\frac{3}{2}}\Rightarrow x=\frac{a}{2^{\frac{2}{3}}}$, $(x,y)=\left(\frac{a}{2^{\frac{2}{3}}},\frac{a}{2^{\frac{2}{3}}}\right)$

$\therefore$ Number of tangents = 1

4. $y=e^{-(x)},(x,y)=\left(1,\frac{1}{e}\right)$

$\frac{dy}{dx}$ at $\left(1,\frac{1}{e}\right)$ is $m=\frac{-1}{e}$

equation of tangent at $\left(1,\frac{1}{e}\right)$ is

$$y-\frac{1}{e}=\frac{-1}{e}(x-1)\Rightarrow x+ey=2$$

5. $\frac{dy}{dx}=3x^2$, by verification $(3,27)$ is statisfied

6. $\frac{dy}{dx}$ at $(1, 1)$ is 2 and obtain equation in a and b $(1, 1)$ also lies on curve and obtain another equation and solve

7. Solving the two equations, we get

$x^2y=xy\Rightarrow xy(x-1)=0\Rightarrow x=0,y=0,x=1$

$\Rightarrow\frac{dy}{dx}=\frac{2xy}{x^2+1}$

$\Rightarrow\left(\frac{dy}{dx}\right)_{(0,1)}=0$ and $\left(\frac{dy}{dx}\right)_{(1,3/2)}=\frac{1}{2}$

The equation of the required tangents are

$\Rightarrow y=1$ and $x+2y-2=0$

These two tangents interest at (0, 1)

8. Put x=0 in the curve find the point at that point find tangetnt and compare

9. Given $y^2 = 5x - 1$ A(1, -2) differentiatew.r.t x

$2y\frac{dy}{dx} = 5$

tangent at A(1, -2) = $\frac{dy}{dx} = \frac{5}{-4}$

slope of normal m = $\frac{-1}{\frac{-5}{4}} = \frac{4}{5}$

$\sqrt{2}$ radion of normal y + 2 = $\frac{4}{5}$(x-1

5y + 10 = 4x -4

4x-5y -14 = 0

Comparing with ax - 5y + b = 0
a=4, b = -14

10. Given $\frac{dy}{dx}\alpha(x-2) \Rightarrow \frac{dy}{dx} = K(x-2)$

given $\left(\frac{dy}{dx}\right)_{(10,-9)} = -3$

-3 = k(8)

k = -3/8

$\frac{dy}{dx} = \frac{-3}{8}(x-2)$ integrate on both sides

y = $\frac{-3}{8}\left(\frac{x-2}{2}\right)^2 + C$ -----1 passes through (10, -9)

- 9 = $\frac{-3}{16} x 64^4 + c^2 + C$, c = 3

y = $\frac{-3}{16}(x-2)^2 + 3$

11. Find normal at (1, 0) and apply $\frac{c^2}{2|ab|}$

12. $y = \log\left(1+x^2\right)$,,

13. Given $\frac{y_1}{m} = y_1 m \Rightarrow m = \pm 1$

Equation of tangent $y - 4 = \pm(x-3)$

tangents are x+y–7=0, x+y+1=0

area with coordinate axis is $\frac{49}{2}$

14. Given longthe of subformal is cons tant

y m = k(say)

y du = l dx integrating

$\frac{y^2}{2}$ = kx + c

y^2 = 2kx + c is a parabola

15. $xy = c^2 \Rightarrow y = \frac{c^2}{x} \Rightarrow \frac{dy}{dx} = \frac{-c^2}{x^2}$

length of subnormal = |y m|

= $\frac{yc^2}{x^2} = \frac{c^4}{x^3}\left(\because xy = c^2\right)$

cube of abscissa

16. $xy^k = c^2$ P(x, y)

xK $y^{k-1}\frac{dy}{dx} + y^k = 0$

$m = \frac{dy}{dx} = \frac{-y}{xk} = 0$

Length of subtangent = $\left(\frac{y}{m}\right)$ is a constant

= xk

= $\frac{Kc^2}{y^k}$ is a constant if k = 0

17. Angle between the curves $x^2 = 4ay$ and $y^2 = 4ax$ at $(4a, 4a)$ *is* $\tan^{-1}\left(\frac{3}{4}\right)$

18. Find $\frac{dy}{dx}$ for the two curves m_1 *and* m_2 at $\left(2\sqrt{2}, 2\right)$ *then* $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

19. Slope of the first curve at (0, 0), $m_1 = -\frac{1}{2}$

Slope of the second curve at (0, 0),

20. Given $y^2 = 2x$ ......(1) and $x^2+y^2=8$ 2

$2y\frac{dy}{dx} = 2$ soluing 1 and 2

$m^1 = \left(\frac{dy}{dx}\right)_{(2,2)} = \frac{1}{2}$ $x^2 + 2x - 8 = 0$, x = 2, y = 2

differentiate w.r.t.x $2x + 2y\frac{dy}{dx} = 0$

21. Point of intersection for two curves is (3, 2) and

$m_1 = \frac{3}{2}, m_2 = \frac{-2}{3}$

22. Apply $\frac{1}{a} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{b_1}$

23. $x^2 + y^2 = a^2$ is a circle with centre (0, 0)and radius = a

$x^2 - y^2 = a^2$ *ie.,* $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$ is hyperbola with transverse axis as the x–axis and ends of transverse axis as $A^1(-a, 0)\ A(a, 0)$

24. Given circus $x^2 + py^2 = 1$ and $9x^2+y^2 = 1$ cuts or thogonally

condition $\frac{1}{a_1} - \frac{1}{a_2} = \frac{1}{b_1} - \frac{1}{b_2}$

$1 - \frac{1}{q} = \frac{1}{p-1} \Rightarrow \frac{1}{p} + \frac{1}{q} = 2$

25. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{25} + \frac{y^2}{16} = 1$ cuts or thogonally

$a^2 - 25 = b^2 - 16$

$a^2 - b^2 = 9$

EXERCISE -III

1. **The angle between the curves** $x^2 + y^2 = \sqrt{2}a^2$ **and** $x^2 - y^2 = a^2$ **is**

1) $\pi/4$ 2) $\pi/6$ 3) $\pi/3$ 4) $\pi/2$

2. **A : Angle between the curves** $y^2 = x, y^2 = -x$ **at (0, 0). B : Angle between the curves** $y = 3x^2; y^2 = 2x$ **at (0, 0) C : Angle between the curves** $y^2 = 4x, x^2 = 4y$ **at (4, 4) Then the descending order of the above values are**

1) A,B,C 2) C,B,A 3) B,C,A 4) A,C,B

3. **The angle between the curves**

$\frac{x^2}{a^2+k_1} + \frac{y^2}{b^2+k_1} = 1$ **and** $\frac{x^2}{a^2+k_2} + \frac{y^2}{b^2+k_2} = 1$ **is**

1) $\frac{\pi}{6}$ 2) $\frac{\pi}{3}$ 3) $\frac{\pi}{2}$ 4) $Tan^{-1}\left(\frac{k_1}{k_2}\right)$

4. **The equations of the tangents at the origin to the curve** $y^2 = x^2(1+x+x^2)$ **are**

1) $y = \pm x$ 2) $y = \pm 2x$ 3) $y = \pm 3x$ 4) $x = \pm 2y$

5. **The equation of the common normal at the point of contact of the curves** $x^2 = y$ **and** $x^2 + y^2 - 8y = 0$

1) $x = y$ 2) $x = 0$ 3) $y = 0$ 4) $x + y = 0$

6. **If the parametric equations of a curve given by** $x = e^t \cos t, y = e^t \sin t$**, then the tangent to the curve at the point** $t = \pi/4$ **makes an angle with positive x-axis is**

1) 0 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{3}$ 4) $\frac{\pi}{2}$

7. **The portion of the tangent to the curve**

$$x = \sqrt{a^2 - y^2} + \frac{a}{2}\log\frac{a - \sqrt{a^2 - y^2}}{a + \sqrt{a^2 - y^2}}$$

intercepted between the curve and x - axis, is of length.

1) $\frac{|a|}{2}$ 2) $|a|$ 3) $2|a|$ 4) $\frac{|a|}{4}$

8. **If the normal at the point** $P(\theta)$ **of the curve** $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ **passes through the origin then**

1) $\theta = \pi/3$ 2) $\theta = \pi/6$
3) $\theta = \pi/4$ 4) $\theta = \pi/2$

9. **In the curve** $y = be^{\frac{x}{a}}$ **the**

a) Subtangent is constant
b) Subnormal varies as the square of the ordinate
1) Both a, b are correct 2) Only 'b' is correct
3) Only 'a' is correct 4) Both a, b are wrong

10. **If the tangent at** $(1,1)$ **on** $y^2 = x(2-x)^2$ **meets the curve again at P, then P is**

1) $(4,4)$ 2) $(-1,2)$ 3) $\left(\frac{9}{4},\frac{3}{8}\right)$ 4) $(0,0)$

11. **The curve** $y = ax^3 + bx^2 + cx + 8$ **touches x-axis at** $P(-2,0)$ **and cuts the y-axis at a point** Q **where its gradient is 3. The values of** a, b, c **are respectively.**

1) $-\frac{5}{4},-3,3$ 2) $0,\frac{1}{4},3$ 3) $\frac{1}{4},0,3$ 4) $\frac{1}{4},-\frac{1}{4},3$

12. **The point P on the curve** $x = a(\theta + \sin\theta)$**,** $y = a(1-\cos\theta)$**, where the tangent is inclined at an angle** $\frac{\pi}{4}$ **to the x-axis is (EAM-2012)**

1) $\left(a\left(\frac{\pi}{2}-1\right), a\right)$ 2) $\left(a\left(\frac{\pi}{2}+1\right), a\right)$

3) $\left(a\frac{\pi}{2}, a\right)$ 4) (a, a)

13. **The normal to a curve at** $P(x,y)$ **meets the x-axis at G . If the distance of G from the origin is twice the abscissa of P then the curve is a (an) (AIEEE-2007)**

1) Ellipse 2) Parabola
3) Circle 4) Straight line

14. **The equation of tangent to the curve** $y = x + \frac{4}{x^2}$ **that is parallel to x-axis is**

(AIEEE-2010)

1) $y=1$ 2) $y=2$ 3) $y=3$ 4) $y=4$

15. The normal to the curve $x=a(\cos\theta+\theta\sin\theta)$, $y=a(\sin\theta-\theta\cos\theta)$ at any point θ is such that

1) it passes through origin
2) it passes through the point (1, 1)
3) it passes through $\left(\frac{a\pi}{2},-a\right)$
4) it is at a constant distance from the origin

16. A function $y=f(x)$ has a second derivative $f''(x)=6(x-1)$. If its graph passes through the point $(2,1)$ and at that point the tangent to the graph is $y=3x-5$ then the function is

1) $(x-1)^2$ 2) $(x+1)^2$
3) $(x+1)^3$ 4) $(x-1)^3$

17. The intercepts on $x-$ axis made by tangents to the curve, $y=\int_0^x |t|\,dt, x\in R$, which are parallel to the line $y=2x$, are equal to

(MAINS - 2013)

1) ± 1 2) ± 2 3) ± 4 4) ± 3

18. If the curves $y^2=4ax$ and $xy=c^2$ cut orthogonally then $\frac{c^4}{a^4}=$

1) 4 2) 8 3) 16 4) 32

19. If the chord joining the points where x=p, x=q on the curve $y = ax^2 + bx + c$ is parallel to the tangent drawn to the curve at(α,β) then $\alpha =$

1) 2pq 2) $\sqrt{pq}$ 3) $\frac{p+q}{2}$ 4) $\frac{p-q}{2}$

20. If the tangent to the curve $2y^3 = ax^2 + x^3$ at the point (a,a) cuts off intercepts α and β on the coordinate axes such that $\alpha^2+\beta^2 = 61$ then a =

1) ± 30 2) ± 5 3) ± 6 4) ± 61

21. Area of the triangle formed by the tangent, normal at (1,1) on the curve $\sqrt{x}+\sqrt{y}=2$ and the y–axis is (in sq.units)

1) 1 2) 2 3) $\frac{1}{2}$ 4) 4

22. The sum of the length of the sub-tangent and tangent drawn at the point (x, y) on the curve $y=a\log(x^2-a^2)$ varies as

1) x^2 2) y^2 3) xy 4) y/x

23. A curve is given by the equations $x=\sec^2\theta$, $y=\cot\theta$. If the tangent at P where $\theta=\frac{\pi}{4}$ meets the curve again at Q, then length of PQ is

1) $\frac{\sqrt{5}}{2}$ 2) $\frac{3\sqrt{5}}{2}$ 3) 10 4) 20

24. I. If the subnormal to the curve $x.y^n=a^{n+1}$ is constant then the value of n is -2.
II. The length of the subtangent, ordinate of a point, (not the origin) length ofthe subnormal on $y^2=4ax$ are in G.P.
Which of the above statements is correct.

1) Only I 2) Only II
3) Both I and II 4) neither I nor II

25. I. If $x+y=k$ is normal to $y^2=12x$ then k is 6
II. If m is the slope of the tangent to the curve $e^y=1+x^2$ then $|m|\le 1$

1) only I 2) only II
3) both I and II 4) neither I nor II

KEY

01) 1	02) 3	03) 3	04) 1	05) 2	06) 4
07) 2	08) 3	09) 1	10) 3	11) 1	12) 2
13) 1	14) 3	15) 4	16) 4	17) 1	18) 4
19) 3	20) 1	21) 1	22) 3	23) 2	24) 3
25) 2					

SOLUTIONS

1. $\tan\theta = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right|$

2. $\tan\theta = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right|$

3. Apply $\frac{1}{a}-\frac{1}{b}=\frac{1}{a_1}-\frac{1}{b_1}$

$a^2+k_1-b^2-k_1=a^2+k_2-b^2-k_2$

$a^2-b^2=a^2-b^2 \therefore \theta=\frac{\pi}{2}$

or use synopsis

4. Since the curve passing through origin therefore tangents at origin is obtaines by equating the lowest degree terms of the equation is zero i.e. $y^2-x^2=0$, $y=\pm x$

5. Common tangent is y=0 $\Rightarrow$ common normal is x=0

6. $\frac{dx}{dt}=e^t(\cos t-\sin t)\, and\, \frac{dy}{dt}=e^t(\sin t+\cos t)$

$\therefore \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\sin t+\cos t}{\cos t-\sin t}\Rightarrow\left(\frac{dy}{dx}\right)_{t=\pi/4}=\infty$

So, tangent at $t=\frac{\pi}{4}$ makes with axis of x the angle $\frac{\pi}{2}$.

7. $\frac{dx}{dy}=\frac{\sqrt{a^2-y^2}}{y}\Rightarrow\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{y_1}{\sqrt{a^2-y_1^2}}$

$LT=\left|\frac{y_1}{m}\sqrt{1+m^2}\right|$, $=\left|\sqrt{a^2-y_1^2}.\sqrt{1+\frac{y_1^2}{a^2-y_1^2}}\right|=|a|$

8. $\frac{dy}{dx}=\left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right)$

9. The given curve is $y=be^{\frac{x}{a}}, y_1=be^{\frac{x_1}{a}}$

$m=\left(\frac{dy}{dx}\right)_{(x_1,y_1)}\frac{b}{a}e^{\frac{x_1}{a}}$, L.S T= a

$L.S.N=|y_1.m|=\left|y_1\cdot\frac{b}{a}.e^{\frac{x}{a}}\right|=\left|\frac{y_1^2}{a}\right|\propto y_1^2$

10. Equation of tangent is $y-y_1=m(x-x_1)$

11. Put $(-2,0)$ in the curve

$\left(\frac{dy}{dx}\right)_{(0,8)}=3, \left(\frac{dy}{dx}\right)_{(P)}=0$ solve these

12. $m=\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=1$

13. Equation of normal is $y-y_1=-\frac{1}{m}(x-x_1)$

$G=(x_1+my_1,0)$, $|x_1+my_1|=2|x_1|$

$x+y\frac{dy}{dx}=\pm 2x$ solve this

14. $m=0$

15. Equation of normal is $y-y_1=-\frac{1}{m}(x-x_1)$

16. $y''=6(x-1)$

$f'(x)=3(x-1)^2+c_1$, $\left(\frac{dy}{dx}\right)_{(2,1)}=3\Rightarrow c_1=0$

$f'(x)=3(x-1)^2$, $f(x)=(x-1)^3+c_2$

$c_2=0$, $f(x)=(x-1)^3$

17. $\frac{dy}{dx}=|x|=2$, $y=\int_0^2 t\,dt=2$

18. $m_1.m_2=-1$, then eliminate x, y by using given equations

19. $A=(p, ap^2+bp+c), B=(q, aq^2+bq+c)$

Slope $\left(\frac{dy}{dx}\right)_{(\alpha,\beta)} = 2a\alpha + b$

Slope of $\overline{AB} = a(p+q)+b$

20. Find the equation of the tangent

21. $\Delta = \frac{1}{2}x_1^2\left|\frac{m^2+1}{m}\right|$

22. $L.S.T = \left|\frac{y_1}{m}\right|$, $\text{L.T} = \left|\frac{y_1}{m}\sqrt{1+m^2}\right|$

23. $\Rightarrow \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = -\frac{1}{2}$

The coordinates of P are (2, 1)
the equation of the tangent at P (2, 1) is

$\Rightarrow x+2y-4=0, \therefore \frac{x}{x-1} = 1+y^2 \Rightarrow y^2 = \frac{1}{x-1}$

$\Rightarrow 2y^3-3y^2+1=0$, $\Rightarrow y=1, y=-\frac{1}{2}$

The coordinates of $\text{Q} = \left(5,-\frac{1}{2}\right)$, $PQ = \frac{3\sqrt{5}}{2}$

24. (i) $2\text{m}+\text{n}=0$
$\text{m}=1, \text{n}=\text{n}$

(ii) $L.S.T = \left|\frac{y_1}{m}\right|$

$L.S.N = |y_1 m|$

25. I) $m=1$, $a=3$,

point of contact $= \left(\frac{a}{m^2}, \frac{2a}{m}\right) = (3,6)$

$3+6=k \Rightarrow k=9$

II) $e^y y' = 2x$, $y' = \frac{2x}{1+x^2} = m$

$m \in [-1,1]$

JEE MAINS QUESTIONS

1.If the tangent to the curve, y = f (x) = $\log_e x$ x, (x > 0) at a\point (c, f(c)) is parallel to the line segement joining thepoints (1, 0) and (e, e), then c is equal to [2020]

1) $\frac{e-1}{e}$ 2) $e^{\left(\frac{1}{e-1}\right)}$

3) $e^{\left(\frac{1}{1-e}\right)}$ 4) $\frac{e}{1-e}$

2.Which of the following points lies on the tangent to thecurve $x^4 e^y + 2\sqrt{y+1} = 3$ at the point (1, 0)? [2020]

(1) (2, 2) (2) (2, 6)
(3) (– 2, 6) (4) (– 2, 4)

3.If the lines x + y = a and x – y = b touch the curve $y = x^2 - 3x + 2$ at the points where the curve intersects thex-axis, then then $\frac{a}{b}$ s equal to _____. [2020]

4.If the tangent to the curve, $y = e^x$ at a point (c, e^c)and thenormal to the parabola $y^2 = 4x$ at the point (1, 2) intersectat the same point on the x-axis, then the value of c is [2020]

5.Let the normal at a point P on the curve $y^2 - 3x^2 + y + 10$ intersect the y-axis at $\left(0, \frac{3}{2}\right)$ If m is the slope of thetangent at P to the curve, then |m| is equal to -- [2020]

6.The length of the perpendicular from the origin, on thenormal to the curve $x^2 + 2xy - 3y^2$ at the point (2, 2) is [2020]

1) $\sqrt{2}$ 2) $4\sqrt{2}$
3) 2 4) $2\sqrt{2}$

7. If the tangent to the curve, y = x^3 + ax – b at the point(1, –5) is perpendicular to the line, – x + y + 4 = 0, thenwhich one of the following points lies on the curve? [2019]

(1) (–2, 1) (2) (–2, 2)
(3) (2, –1) (4) (2, –2)

8.The tangent and the normal lines at the point ($\sqrt{3}, 1$) tothe circle $x^2 + y^2 = 4$ and the x-axis form a triangle. The areaof this triangle (in square units) is

1) $\frac{4}{\sqrt{3}}$ 2) $\frac{1}{3}$ [2019]

3) $\frac{2}{\sqrt{3}}$ 4) $\frac{1}{\sqrt{3}}$

9.The maximum area (in sq. units) of a rectangle having itsbase on the x-axis and its other two vertices on the parabola,y = 12 – x^2 such that the rectangle lies inside the parabola, [2019]

(1) 36 (2) $20\sqrt{2}$
(3) 32 (4) $18\sqrt{3}$

10.If θ denotes the acute angle between the curves $y = 10 - x^2$ and $y = 2 + x^2$ at a point of their intersection, then |tan θ | is equal to:

{1} $\frac{4}{9}$ {2} $\frac{8}{15}$

{3} $\frac{7}{17}$ {4} $\frac{8}{17}$

11.If the curves $y^2 = 6x, 9x^2 + by^2 = 16$ intersect each otherat right angles, then the value of b is [2018]

1) $\frac{7}{2}$ 2) 4

3) $\frac{9}{2}$ 4) 6

12. The eccentricity of an ellipse whose centre is at the origin is $\frac{1}{2}$ If one of its directices is x = –4, then the equation of the normal to it at $\left(1, \frac{3}{2}\right)$ is : [2018]

(1) x + 2y = 4 (2) 2y – x = 2
(3) 4x – 2y = 1 (4) 4x + 2y = 7

KEY

1) 2 2) 3 3)0.50 4) 4 5) 4 6)4

7) 4 8) 3 9}3 10} 2 11}3 12}3

SOLUTIONS

1.

The given tangent to the curve is,

$y = x\log_e x \qquad (x > 0)$

$\Rightarrow \frac{dy}{dx} = 1 + \log_e x$

$\Rightarrow \left.\frac{dy}{dx}\right]_{x=c} = 1 + \log_e c \qquad \text{(slope)}$

$\because$ The tangent is parallel to line joining (1, 0), (e, e)

$\therefore 1 + \log_e c = \frac{e-0}{e-1}$

$\Rightarrow \log_e c = \frac{e}{e-1} - 1 \Rightarrow \log_e c = \frac{1}{e-1}$

$\Rightarrow c = e^{\frac{1}{e-1}}$

2.

The given curve is, $x^4 \cdot e^y + 2\sqrt{y+1} = 3$

Differentiating w.r.t. x, we get

$$(4x^3 + x^4 \cdot y')e^y + \frac{y'}{\sqrt{1+y}} = 0$$

$$\Rightarrow \left(\frac{dy}{dx}\right) = \frac{-4x^3 e^y}{\left(\frac{1}{\sqrt{y+1}} + e^y x^4\right)}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)} = -2$$

$\therefore$ Equation of tangent;

$$y - 0 = -2(x-1) \Rightarrow 2x + y = 2$$

Only point (–2, 6) lies on the tangent.

3.

The given curve $y = (x-1)(x-2)$, intersects the x-axis at $A(1, 0)$ and $B(2, 0)$.

$$\therefore \frac{dy}{dx} = 2x - 3; \left(\frac{dy}{dx}\right)_{(x=1)} = -1 \text{ and } \left(\frac{dy}{dx}\right)_{(x=2)} = 1$$

Equation of tangent at $A(1, 0)$,

$$y = -1(x-1) \Rightarrow x + y = 1$$

Equation of tangent at $B(2, 0)$,

$$y = 1(x-2) \Rightarrow x - y = 2$$

So $a = 1$ and $b = 2$

$$\Rightarrow \frac{a}{b} = \frac{1}{2} = 0.5.$$

4.

For (1, 2) of $y^2 = 4x \Rightarrow t = 1, a = 1$

Equation of normal to the parabola

$$\Rightarrow tx + y = 2at + at^3$$

$\Rightarrow x + y = 3$ intersect x-axis at (3, 0)

$$y = e^x \Rightarrow \frac{dy}{dx} = e^x$$

Equation of tangent to the curve

$$\Rightarrow y - e^c = e^c(x - c)$$

$\because$ Tangent to the curve and normal to the parabola intersect at same point.

$$\therefore 0 - e^c = e^c(3 - c) \Rightarrow c = 4.$$

5.

$$P \equiv (x_1, y_1)$$

$$2yy' - 6x + y' = 0$$

$$\Rightarrow \quad y' = \left(\frac{6x_1}{1 + 2y_1}\right)$$

$$\left(\frac{\frac{3}{2} - y_1}{-x_1}\right) = -\left(\frac{1 + 2y_1}{6x_1}\right)$$

$\Rightarrow \quad 9 - 6y_1 = 1 + 2y_1$

$\Rightarrow \quad y_1 = 1$

$\therefore \quad x_1 = \pm 2$

$\therefore \quad$ Slope of tangent $(m) = \left(\frac{\pm 12}{3}\right) = \pm 4$

$\therefore \quad |m| = 4$

6

Given equation of curve is

$x^2 + 2xy - 3y^2 = 0$

$\Rightarrow \quad 2x + 2y + 2xy' - 6yy' = 0$

$\Rightarrow \quad x + y + xy' - 3yy' = 0$

$\Rightarrow \quad y'(x - 3y) = -(x + y)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x + y}{3y - x}$$

Slope of normal = $\frac{-dx}{dy} = \frac{x-3y}{x-3y}$

Normal at point (2, 2) = $\frac{2-6}{2+2} = -1$

Equation of normal to curve = $y - 2 = -1\,(x-2)$

$\Rightarrow\ x + y = 4$

$\therefore$ Perpendicular distance from origin

$= \left|\frac{0+0-4}{\sqrt{2}}\right| = 2\sqrt{2}$

7.

$y = x^3 + ax - b$

Since, the point (1, –5) lies on the curve.

$\Rightarrow 1 + a - b = -5$

$\Rightarrow a - b = -6$

$\frac{dy}{dx} = 3x^2 + a$

$\left(\frac{dy}{dx}\right)_{at\ x=1} = 3 + a$

Since, required line is perpendicular to y =

slope of tangent at the point P (1, –5) = –1

$3 + a = -1$

$a = -4$

$b = 2$

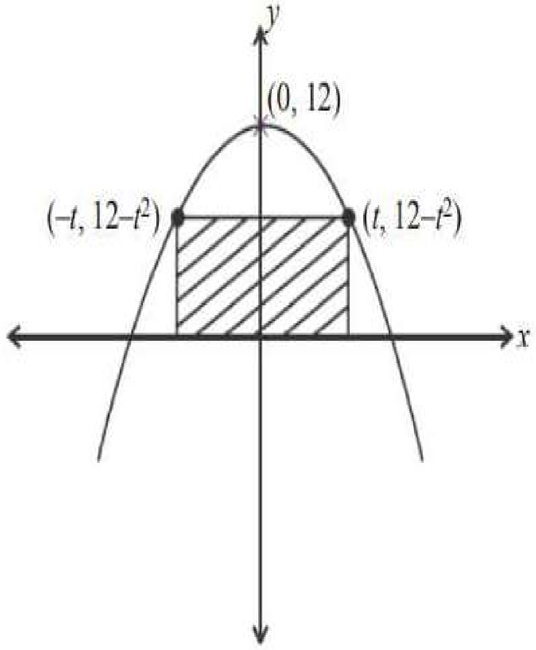
the equation of the curve is $y = x^3 - 4x - 2$

(2, –2) lies on the curve

9.

Given, the equation of parabola is,

$x^2 = 12 - y$



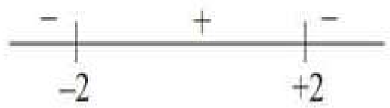
Area of the rectangle = $(2t)\,(12 - t^2)$

$A = 24t - 2t^3$

$\frac{dA}{dt} = 24 - 6t^2$

Put $\frac{dA}{dt} = 0 \Rightarrow 24 - 6t^2 = 0$

$\Rightarrow\ t = \pm 2$



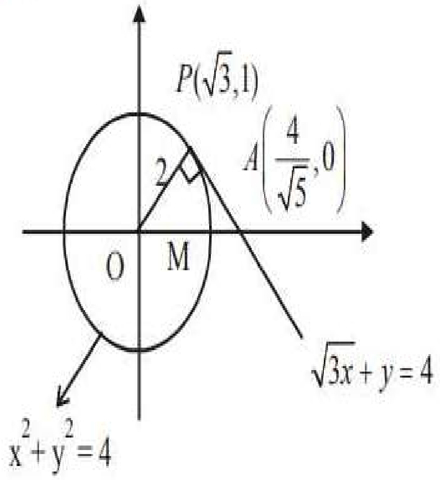
At $t = 2$, area is maximum $= 24(2) - 2(2)^3$

$= 48 - 16 = 32$ sq. units

8.

Equation of tangent to circle at point $(\sqrt{3},1)$ is

$\sqrt{3}x+y=4$



coordinates of the point $A=\left(\frac{4}{\sqrt{3}},0\right)$

$$\text{Area}=\frac{1}{2}\times OA\times PM=\frac{1}{2}\times\frac{4}{\sqrt{3}}\times 1=\frac{2}{\sqrt{3}} \text{ sq. units}$$

10.

Since, the equation of curves are

$y=10-x^2$ ...(i)

$y=2+x^2$ ...(ii)

Adding eqn (i) and (ii), we get

$2y=12\Rightarrow y=6$

Then, from eqn (i)

$x=\pm 2$

Differentiate equation (i) with respect to x

$$\frac{dy}{dx}=-2x\Rightarrow\left(\frac{dy}{dx}\right)_{(2,6)}=-4 \text{ and } \left(\frac{dy}{dx}\right)_{(-2,6)}=4$$

Differentiate equation (ii) with respect to x

$$\frac{dy}{dx}=2x\Rightarrow\left(\frac{dy}{dx}\right)_{(2,6)}=4 \text{ and } \left(\frac{dy}{dx}\right)_{(-2,6)}=-4$$

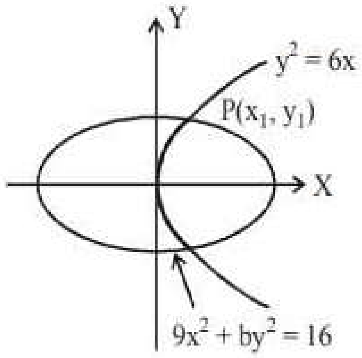
$$\text{At } (2, 6)\ \tan\theta=\left(\frac{(-4)-(4)}{1+(-4)\times(4)}\right)=\frac{8}{15}$$

$$\text{At } (-2, 6),\ \tan\theta=\frac{(4)-(-4)}{1+(4)(-4)}=\frac{8}{-15}\Rightarrow|\tan\theta|=\frac{8}{15}$$

$$\therefore|\tan\theta|=\frac{8}{15}$$

11.

Let curve intersect each other at point $P(x_1, y_1)$



Since, point of intersection is on both the curves, then

$y_1^2=6x_1$ ...(i)

and $9x_1^2+by_1^2=16$ ...(ii)

Now, find the slope of tangent to both the curves at the point of intersection $P(x_1, y_1)$

For slope of curves:

Curve (i):

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=m_1=\frac{3}{y_1}$$

Curve (ii):

and $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = m_2 = -\frac{9x_1}{by_1}$

Since, both the curves intersect each other at right angle then,

$m_1m_2 = -1 \Rightarrow \frac{27x_1}{by_1^2} = 1 \Rightarrow b = 27\frac{x_1}{y_1^2}$

$\therefore$ from equation (i), $b = 27 \times \frac{1}{6} = \frac{9}{2}$

12.

Eccentricity of ellipse = $\frac{1}{2}$

Now, $-\frac{a}{e} = -4 \Rightarrow a = 4 \times \frac{1}{2} = 2 \Rightarrow a = 2$

We have $b^2 = a^2(1 - e^2) = a^2\left(1 - \frac{1}{4}\right)$

$= 4 \times \frac{3}{4} = 3$

$\therefore$ Equation of ellipse is

$\frac{x^2}{4} + \frac{y^2}{3} = 1$

Now differentiating, we get

$\Rightarrow \frac{x}{2} + \frac{2y}{3} \times y' = 0 \Rightarrow y' = -\frac{3x}{4y}$

$y'\big|_{(1,3/2)}\big| = -\frac{3}{4} \times \frac{2}{3} = -\frac{1}{2}$

Slope of normal = 2

$\therefore$ Equation of normal at $\left(1, \frac{3}{2}\right)$ is

$y - \frac{3}{2} = 2(x - 1) \Rightarrow 2y - 3 = 4x - 4$

$\therefore \quad 4x - 2y = 1$

RATE MEASURE

SYNOPSIS

✈ **Derivative as the Rate of Change:**

If a variable quantity y is a function of time t i.e., $y = f(t)$, then small change in time Δt have a corresponding change in Δy in y.

Thus, the average rate of change $= \frac{\Delta y}{\Delta t}$

When limit $\Delta t \to 0$ is applied, the rate of change becomes instantaneous and we get the rate of change with respect to 't' at the instant x.

i.e., $\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}$

Hence, it is clear that the rate of change of any variable with respect to some other variable is derivative of first variable with respect to other variable.

i) If x is any variable, $\frac{dx}{dt}$ represents the rate of change of x at time 't'.

ii) If y = f(x), then $\frac{dy}{dx}$ is the rate of change of y w.r.t. x.

iii) If 's' is the distance travelled by a particle in time t. The relation between s and t can be expressed as s = f(t).

iv) $v = \frac{ds}{dt}$ is the rate of change of displacement is called velocity.It is a vector, measured in unit per second.

a) $v = 0 \Rightarrow$ the particle moving on a straight line comes to rest and the distances becomes maximum where it changes its direction after $v = 0$

b) $v > 0 \Rightarrow$ s increases

c) $v < 0 \Rightarrow$ s decreases

$t = -\frac{2\sqrt{3}}{2}$ is rejected $\Rightarrow t > \frac{2\sqrt{2}}{3}$

v) The rate of change in velocity is called the acceleration of the particle at 't' and is denoted by a

$$a = \frac{dv}{dt} = \frac{d}{dt}\left[\frac{ds}{dt}\right] = \frac{d^2s}{dt^2} = \frac{dv}{ds}.\frac{ds}{dt} = v.\frac{dv}{ds}$$

It is a vector . It is measured in units /Sec^2

a) a=0 $\Rightarrow$ velocity v becomes maximum

b) a>0 $\Rightarrow$ v increases. S Minimum

c) a<0 $\Rightarrow$ v decreases. S Maximum

d) A particle moving on a straight line comes to rest *if* $\frac{ds}{dt} = 0$ & $\frac{d^2s}{dt^2} = 0$

e) A particle moving on a straight line is at rest *if* $\frac{ds}{dt} = 0$ & $\frac{d^2s}{dt^2} \neq 0$

f) A particle, projected vertically upwards, attains the maximum height when $\frac{ds}{dt} = 0$.

✈ **Retardation :** If the acceleration of a particle is negative, it is called Retardation.

✈ **Angular velocity and angular acceleration:**

If P is any point which moves on a curve and θ is the angle made by OP with the positive direction of the initial line $\overline{OX}$, the angular velocity of P at O=$\frac{d\theta}{dt}$.It is denoted by ω.

i) The angular acceleration of P at O is

$$\frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}$$

ii) The equations of motion of a particle p(x,y) on a plane curve are given by x = f(t), y = g(t) then the velocity of the particle is given

by $\frac{ds}{dt} = \sqrt{[f'(t)]^2 + [g'(t)]^2}$

iii) The equations of motion of a particle p(x,y) on a plane curve are given by x = f(t), y = g(t) then the acceleration of the particle is given by $\frac{d^2s}{dt^2} = \sqrt{(f''(t))^2 + (g''(t))^2}$

EXERCISE - I

1. A particle moves along a straight line according to the equation $s = 8\cos 2t + 4\sin t$. The initial velocity is

1) -5 units/sec 2) -4units/sec
3) 4 units/sec 4) 5 units/sec

2. The motion of a particle along a straight line is given by $v^2 = u^2 + 90s$. If the particle starts from rest, then the acceleration is

1) 15 units/sec^2 2) 30 units/sec^2
3) 45 units/sec^2 4) 75 units/sec^2

3. If the distance s travelled by a particle in time t is given by $s = t^2 - 2t + 5$ then its acceleration is [EAM-2011]

1) 0 2) 1 3) 2 4) 3

4. The distance moved by the particle in time 't' is given by $S = t^3 - 12t^2 + 6t + 8$. At the instant, when its acceleration is zero. The velocity is

1) 42 2) -42 3) 48 4) -48

5. A particle moves along a line by $s = \frac{1}{3}t^3 - 3t^2 + 8t + 5$, it changes its direction when

1) t = 1, t = 2 2) t = 2, t = 4
3) t = 0, t = 4 4) t = 2, t = 3

6. The displacement 's' of a particle measured from a fixed point 'O' on a line is given by $s = 16 + 48t - t^3$. After 4sec, the direction of motion of the particle.

1) is towards 'O' 2) is away from 'O'
3) is at rest 4) is at 'O'

7. A stone is thrown vertically upwards and the height reached by it in time t is given by $S = 80t - 16t^2$ then the stone reaches the maximum height in time t =

1) 2 sec 2) 2.5 sec 3) 3 sec 4) 3.5 sec

8. A particle moves along a line by $s = t^3 - 9t^2 + 24t$, then S is decreasing when $t \in$

1) $(2, 4)$ 2) $(-\infty, 2) \cup (4, \infty)$
3) $(-\infty, 2)$ 4) $(4, \infty)$

9. The displacement of a particle in time 't' is given by $S = t^3 - t^2 - 8t - 18$. The acceleration of the particle when its velocity vanishes is

1) 15 units/sec^2 2) 10 units/sec^2
3) 5 units/sec^2 4) 20 units/sec^2

10. If k is the diameter of a circle and A is the area of a sector of the circle whose vertical angle is θ then $\frac{dA}{dt}$ =

1) $\frac{k^2}{8}\left(\frac{d\theta}{dt}\right)$ 2) $\left(\frac{k^2}{4}\right)\left(\frac{d\theta}{dt}\right)$ 3) $\frac{d\theta}{dt}$ 4) $k\left(\frac{d\theta}{dt}\right)$

11. The rate of change of area of a square plate is equal to that of the rate of change of its perimeter. Then length of the side is

1) 2 units 2) 3 units 3) 4 units 4) 6 units

12. The relation between P and V is given by $PV^{\frac{1}{4}} =$ constant. If the percentage decrease in V is $\frac{1}{2}$ then percentage increase in 'P' is

1) -1/8 2) 1/16 3) 1/8 4) 1/2

13. An angle θ through which a pulley turns with time 't' is completed by $\theta = t^2 + 3t - 5$ sq.cms /min Then the angular velocity for t = 5sec.

1) 5^c/sec 2) 13^c/sec 3) 23^c/sec 4) 35^c/sec

KEY

01) 3 02) 3 03) 3 04) 2 05) 2 06) 1
07) 2 08) 1 09) 2 10) 1 11) 1 12) 3
13) 2

SOLUTIONS

1. $V = -16\sin 2t + 4\cos t$, $t = 0 \Rightarrow V = 4$
2. $V^2 = u^2 + 90s$, $2V.a = 90\dfrac{ds}{dt}(\because u = 0) \Rightarrow a = 45$
3. $a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$
4. a = 0, $\Rightarrow t = 4$, $v = 3t^2 - 24t + 6 = -42$
5. $V = t^2 - 6t + 8$, $v = 0 \Rightarrow t = 2, t = 4$
6. $V = 0 \Rightarrow t = \pm 4$, If t=4 $\Rightarrow$ V=0
   After t = 4 means we should put t = 5.
7. $S = 80t - 16t^2$, $V = \dfrac{ds}{dt} = 80 - 32t$
   maximum height $\Rightarrow V = 0$, $t = 5/2 = 2.5\sec$
8. Solve $V < 0$
9. $V = 0 \Rightarrow t = 2$ or $-\dfrac{4}{3}$, $a = 6t - 2$; at t= 2
   $\Rightarrow a = 10$
10. $k = 2r$, $A = \dfrac{1}{2}r^2\theta = \dfrac{k^2\theta}{8}$
11. $\dfrac{d}{dt}$ (area) = $\dfrac{d}{dt}$ (perimeter)
12. Given $PV^{1/4}$ = constant & $\dfrac{\Delta V}{V}\times 100 = -\dfrac{1}{2}$
    Take log on both sides and diff.
    $\Rightarrow \dfrac{\Delta P}{P}\times 100 = -\dfrac{1}{4}\times\dfrac{\Delta V}{V}\times 100 = \dfrac{1}{8}$
13. $\theta = t^2 + 3t - 5$, $\dfrac{d\theta}{dt} = 2t + 3$

EXERCISE - II

1. If the distance travelled by a particle is $x = \sqrt{pt^2 + 2qt + r}$ then the acceleration is proportional to

1) $\dfrac{1}{x}$ 2) $\dfrac{1}{x^2}$ 3) $\dfrac{1}{\sqrt{x}}$ 4) $\dfrac{1}{x^3}$

2. The position of a point in time "t" is given by $x = a + bt - ct^2$, $y = at + bt^2$. Its acceleration at time "t" is

1) $b - c$ 2) $b + c$
3) $2b - 2c$ 4) $2\sqrt{b^2 + c^2}$

3. A particle 'p' moves along a straight line away from a fixed point 'O' obeying the relation $S = 16 + 48t - t^3$. The direction of 'P' after $t = 4$ is

1) $\overrightarrow{OP}$ 2) $\overrightarrow{PO}$
3) Rest at the instant 4) Perpendicular to $\overrightarrow{OP}$

4. The velocity v of a particle is given by $v^2 = s^2 + 4s + 4$. The acceleration of the particle when it is 30 cms away from the starting point is

1) 30 cms/sec^2 2) 32 cms/sec^2
3) 34 cms/sec^2 4) 35 cms/sec^2

5. If a particle moving along a line following the law $t = ps^2 + qs + r$ then the retardation of the particle is proportional to

1) Square of displacement
2) Square of velocity
3) Cube of displacement
4) Cube of velocity

6. The equation of motion of a particle p(x,y) on a plane are given by $x = 4 + b\cos t, y = 5 + b\sin t$. Its velocity at time 't' is

1) 4 2) 5 3) b 4) tan t

7. A stone projected vertically upwards raises 's' feets in 't' seconds where $s = 112t - 16t^2$. Then maximum height it reached is

1) 195 ft 2) 194 ft 3) 196 ft 4) 216 ft

8. A particle moves along a line OA which is at a distance 5 cm from O where $s = 6t^2 - \dfrac{t^3}{2}$, then the greatest velocity along OA is

1) 32cm/s 2) 24 cm/s 3) 18cm/s 4) 19 cm/s

9. If the velocity v of a particle varies as the square of its displacement x then the acceleration varies as

1) x^2 2) x^3 3) v^2 4) v^3

10. A particle moves along the curve $y = x^2 + 2x$ then the point on the curve such that x and y coordinates of the particle change with the same rate is [EAM-2009]

1)(1,3) 2) (1/2,3/4) 3) (-1/2,-3/4) 4) (-1,-1)

11. The point on the ellipse $16x^2 + 9y^2 = 400$, at which the ordinate decreases at the same rate at which the abscissa increases is

1) $\left(3,\frac{16}{3}\right)$ 2) $\left(-3,\frac{16}{3}\right)$ 3) $\left(3,\frac{-16}{3}\right)$ 4) $\left(-4,\frac{-16}{3}\right)$

12. The area of an equilateral triangle of side 'a' feet is increasing at the rate of 4 sq.ft./sec. The rate at which the perimeter is increasing is

1) $\frac{3\sqrt{8}}{2}$ 2) $\frac{8\sqrt{3}}{a}$ 3) $\frac{\sqrt{3}}{a}$ 4) $\frac{2\sqrt{3}}{a}$

13. A car starts from rest and attains the speed of 1 km/hr and 2 k.ms/hr at the end of 1st and 2nd minutes. If the car moves on a straight road, the distance travelled in 2 minutes is

1) $\frac{1}{4}km$ 2) $\frac{1}{30}km$ 3) $15\,km$ 4) $20\,km$

14. A point is moving along $y^3 = 27x$. The interval in which the abscissa changes at slower rate than ordinate is

1) $(-2,2)$ 2) $(-\infty,\infty)$

3) $(-1,1)$ 4) $(-\infty,-3)\cup(3,\infty)$

15. A point 'P' is moving with constant velocity V along a line AB. O is a point on the line perpendicular to AB at A and at a distance "l" from A. The Angular velocity of P about O is

1) $\frac{lv}{op}$ 2) $\frac{lv}{op^2}$ 3) $\frac{lv^2}{op}$ 4) $\frac{op^2}{lv}$

16. An angle is increasing at a constant rate. The rate of increase of tan when the angle is $\pi/3$ is

1) 4 times the increase of sine

2) 8 times the increase of cosine

3) 8 times the increase of sine

4) 4 times the increase of cosine

17. The volume of metallic hallow sphere is constant. If the outer radius is increasing at the rate of V cm/sec. Then the rate at which the inner radius increasing when the radii are $a+d, a$ is

1) $\frac{V(a+d)^2}{a^2}$ 2) $\frac{V(a+d)}{a}$

3) $V(a+d)$ 4) $a+d$

18. In a simple pendulum, if the rate of change in the time period is equal to the rate of change in the length then the length of the pendulum is

1) $\frac{\pi}{g}$ 2) $\frac{\pi^2}{g}$ 3) $\pi^2 g$ 4) πg^2

19. The side of an equilateral triangle expands at the rate of 2 cms/sec. The rate of increase of its area when each side is 10cms. is

1) $10\sqrt{2}$ sq.cms/sec 2) $10\sqrt{3}$ sq.cms/sec

3) 10 sq.cms/sec 4) 5 sq.cms/sec

20. Two cars started from a place one moving due east and the other due north with equal speed V. Then the rate at which they were being seperated from each other is

1) $\frac{\sqrt{2}}{V}$ 2) $\frac{V}{\sqrt{2}}$ 3) $\frac{1}{\sqrt{2}V}$ 4) $\sqrt{2}V$

21. A point p moves with an angular velocity 2 radians/sec on the circumference of a circle with centre O and radius 2 cms. PM is perpendicular to the diameter of the circle such that $\angle POM = \theta$ .If the velocity of the point M is zero, then values of θ are

1) $0,\pi$ 2) $\frac{\pi}{2},\pi$ 3) $\frac{\pi}{3},\frac{\pi}{6}$ 4) $\frac{\pi}{4},\frac{3\pi}{4}$

22. A variable triangle is inscribed in a circle of radius R. If the rate of change of a side is R times the rate of change of the opposite angle, then the opposite angle is

1) $\frac{\pi}{6}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{3}$ 4) $\frac{\pi}{2}$

23. At a given instant, the sides OA and OB of a right angled triangle AOB are 8 cm and 6 cms respectively. If OA increases at the rate of 2 cm/sec and OB decreases at the rate of 1 cm/sec, the rate of decrease of the area of ΔAOB after 2 seconds is

1) 2 sq cm/sec 2) 1 sq cm/sec
3) 3 sq cm/sec 4) 4 sq cm/sec

KEY

01) 4	02) 4	03) 2	04) 2	05) 4	06) 3
07) 3	08) 2	09) 2	10) 3	11) 1	12) 2
13) 2	14) 3	15) 2	16) 3	17) 1	18) 2
19) 2	20) 4	21) 1	22) 3	23) 1	

SOLUTIONS

1. Squaring and then differentiate two times
2. Hint: Resultant Acceleration $= \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$
3. $s = 16 + 48t - t^3$, $t = 4 \Rightarrow s = 144$
   $t = 5, s = 131$, So it move to words O
4. $2v\frac{dv}{dt} = 2s\frac{ds}{dt} + 4\frac{ds}{dt}$, $\frac{dv}{dt} = (s+2)$
   s =30, a = 30+2 = 32
5. diff. two times
6. $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$
7. $v = 112 - 32t = 0$, $t = \frac{7}{2}$
   max. height $= 112.\frac{7}{2} - 16\left(\frac{7}{2}\right)^2$ =196
8. $V = \frac{ds}{dt} = 12t - \frac{3}{2}t^2$, $a = 12 - 3t = 0$, t=4
   $v = 12.4 - \frac{3}{2}(16)$ =24
9. $V \propto x^2$, $V = kx^2$ ..... (1)
   $a = k.2x.kx^2$ from (1), $= (2k^2)x^3 \Rightarrow a \propto x^3$
10. We have $\frac{dy}{dt} = (2x+2)\frac{dx}{dt} \Rightarrow x = -\frac{1}{2}; y = -\frac{3}{4}$
11. $\frac{dy}{dt} = \frac{-dx}{dt}$
12. $x = a\ ft, \frac{dA}{dt} = 4sq.ft/\sec, \frac{dc}{dt} = ?$
    use $12\sqrt{3}A = C^2$
13. $s = ut + \frac{1}{2}at^2$, $v = u + at \Rightarrow v - \mu = at$
    $a = 1km/h = \frac{1}{60}km/\min$,
    $v = 2kmph = \frac{1}{30}km/\min$
    $v^2 - \mu^2 = 2as \Rightarrow s = \frac{1}{30}$
14. $\frac{dx}{dt} < \frac{dy}{dt} \Rightarrow y \in (-3,3) \Rightarrow x \in (-1,1)$
15. $\tan\theta = \frac{x}{l} \Rightarrow \sec^2\theta.\frac{d\theta}{dt} = \frac{1}{l} \times \frac{dx}{dt} = \frac{V}{l}$
    $\Rightarrow \frac{d\theta}{dt} = \cos^2\theta.\frac{V}{l} = \frac{V}{l} \times \frac{l^2}{op^2} = \frac{Vl}{op^2}$
16. $\frac{d\theta}{dt} = k$, If $\theta = \frac{\pi}{3}$,
    $\frac{d}{dt}(\tan\theta) = \sec^2\theta \times \frac{d\theta}{dt}$
    $= 4 \times \frac{d\theta}{dt} = 8\left[\frac{1}{2} \times \frac{d\theta}{dt}\right] = 8\left[\frac{d}{dt}(\sin\theta)\right]$
17. Outer radius = R_1, Inner radius = R_2
    $V = \frac{4}{3}\pi(R_1^3 - R_2^3)$, $0 = 3R_1^2.\frac{dR_1}{dt} - 3R_2^2.\frac{dR_2}{dt}$

$$\therefore \frac{dR_2}{dt} = \frac{R_1^2 \cdot \left(\frac{dR_1}{dt}\right)}{R_2^2},$$

$$= \frac{V(a+d)^2}{a^2}$$

18. $T = 2\pi\sqrt{\frac{l}{g}}, \frac{dT}{dt} = \frac{dl}{dt}$

19. $\frac{dx}{dt} = 2$ Cm/sec, x=10, $A = \frac{\sqrt{3}}{4}x^2$

$\frac{dA}{dt} = 10\sqrt{3}$ sqcm/sec

20. $S = \sqrt{x^2 + x^2}$ $S = \sqrt{2}\,x \Rightarrow \frac{dS}{dt} = \sqrt{2}\,V$

21. $OM = \cos\theta, \frac{d(OM)}{dt} = 0$

22. $\frac{4}{3}\pi\left(r_1^3 - r_2^3\right) = C \Rightarrow \frac{dr_1}{dt} = \frac{r_2^2}{r_1^2}.\frac{dr_2}{dt}$

23. Given $OA = 8$, $OB = 6$

after 2 seconds $OA = 12$, $OB = 4$

$\Delta = Area = \frac{1}{2}xy$, $\frac{d\Delta}{dt} = \frac{1}{2}\left[x.\frac{dy}{dt} + y.\frac{dx}{dt}\right]$

$= \frac{1}{2}\left[12(-1) + 4(2)\right] = -2$

EXERCISE - III

1. The volume of a ball increases at $2\pi c.c/\sec$. The rate of increase of radius when the volume is $288\pi\, c.cms$ is [E-2012]

1) 1/36 cm/sec 2) 1/72 cm/sec
3) 1/18 cm/sec 4) 1/9 cm/sec

2. A particle moving on a straight line so that its distnace 's' from a fixed point at any time 't' is proportional to 't^n' if 'v' be the velocity and 'a' the acceleration at any time then $\frac{nas}{(n-1)} =$

1) v 2) v^2 3) v^3 4) 2v

3. A ladder AB of 10 mts long moves with its ends on the axes. When the end A is 6 mts from the origin, it moves away from it at 2mts/minute. The rate of increase of the area of the ΔOAB is... $sq.mts/\min$

1) $\frac{4}{3}$ 2) $\frac{8}{3}$ 3) $\frac{14}{3}$ 4) $\frac{7}{2}$

4. A is a fixed point on the circumference of a circle with centre '0' and radius 'r', A particle starts at A and moves on the circumference with an angular velocity 4 radians/sec. If PM is perpendicular to OA and $\angle POM = \pi/3$, then the rate at which area of ΔPOM decreases is

1) $\frac{r^2}{2}\, sq.cms/\sec$ 2) $r^2\ sq.cms/\sec$

3) $\frac{3r^2}{2}\, sq.cms/\sec$ 4) $2r^2\ sq.cms/\sec$

5. A source of light is hung h mts., directly above a straight horizontal path on which a boy 'a' mts., in height is walking. If a boy walks at a rate of b mts/sec. from the light then the rate at which his shadow increases.

1) $\frac{ab}{h-a}\, mt/\sec$ 2) $\frac{ab}{h+a}\, mt/\sec$

3) $\frac{ab}{2(h-a)}\, mt/\sec$ 4) $\frac{ab}{2(h+a)}\, mt/\sec$

6. The slant height of a cone is fixed at 7cm. The rate of increase in the volume of the cone corresponding to the rate of increase of 0.3 cm/s in the height when h = 4cm is

1) $\frac{\pi}{10}\, cc/s$ 2) $\frac{3\pi}{10}\, cc/s$ 3) $\frac{\pi}{5}\, cc/s$ 4) $\frac{7\pi}{10}\, cc/s$

7. A kite flying at a height 'h' mts has “x” meters of string paid out at a time t seconds. If the kite moves horizontally with constant velocity v mts/sec. Then the rate at which the string is paid out is

1) $\frac{\sqrt{x^2-h^2}}{v}\, mt/\sec$ 2) $\sqrt{x^2-h^2}\ mt/\sec$

3) $\frac{v\sqrt{x^2-h^2}}{x}\, mt/\sec$ 4) $\frac{\sqrt{x^2-h^2}}{h}\, mt/\sec$

8. **A wheel rotates so that the angle of rotation is proportional to the square of the time. The first revolution was performed by the wheel for 8 seconds the angular velocity at this time is**

1) $\pi \, rad/\sec$ 2) $2\pi \, rad/\sec$

3) $\frac{\pi}{2} rad/\sec$ 4) $\frac{\pi}{3} rad/\sec$

9. **A is an end of diameter of a cirlce with centre O and radius 2 units. If a particle 'p' starting from A moves on a circle with angular velocity 4 radians/sec and M is the foot of the perpendicular of 'p' on the diameter then the rate at which M moving on the diameter when it is at a distance of 1 unit from O is**

1) $4\sqrt{3}$ units/sec 2) - $4\sqrt{3}$ units/sec

3) 4 units/sec 4) -4 units/sec

10. **Two cars are travelling along two roads which cross each other at right angles at A. One car is travelling towards A at 21 kmph and the other is travelling towards A at 28 kmph. If initially their distances from A are 1500 km and 2100 km respectively, then the nearest distance them is**

1) 30 2) 45 3) 60 4) 75

11. **A dynamite blast blows a heavy rock straight up with a launch velocity of 160m/sec. It reaches a height of $s = 160t - 16t^2$ after *t* sec. The velocity of the rock when it is 256 m above the ground on the way up is**

1) 98 m/s 2) 96 m/s 3) 104 m/s 4) 48 m/s

12. **A body falling from rest under gravity passes a certain point P. It was a distance of400 m from P, 4sec prior to passing through P. If $g = 10 m/\sec^2$, then the height above the point "P" from where the body began to fall is (AIE-2006)**

1) 900 m 2) 320 m 3) 680 m 4) 720 m

13. **A spherical balloon is filled with 4500 π cu.m of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cu.m/ min, then the rate (in m/min) at which the radius of the balloon decreases 49** min after the leakage began is **(AIE-2012)**

1) $\frac{9}{7}$ 2) $\frac{7}{9}$ 3) $\frac{2}{9}$ 4) $\frac{9}{2}$

14. **A lamp of negligible height is placed on the ground l_1 away from a wall. A man l_2 m tall is walking at a speed of $\frac{l_1}{10} m/s$ from the lamp to the nearest point on the wall. When he is midway between the lamp and the wall. the rate of change in the length of this shadow on the wall is**

1) $-\frac{5l_2}{2} m/s$ 2) $-\frac{2l_2}{5} m/s$

3) $-\frac{l_2}{2} m/s$ 4) $-\frac{l_2}{5} m/s$

KEY

01) 2	02) 2	03) 4	04) 2	05) 1	06) 1
07) 3	08) 3	09) 2	10) 3	11) 2	12) 4
13)3	14) 2				

SOLUTIONS

1. $\frac{dv}{dt} = 2\pi \, c.c/\sec$, $V = 288\pi$, $\frac{4}{3}\pi r^3 = 288\pi$,

$\Rightarrow r = 6 \therefore \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{4}{3}\pi.3r^2 \frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{1}{72}$

2. $S \propto t^n \Rightarrow s = k\left(t^n\right)$, differentiate

3. $x^2 + y^2 = 100 \Rightarrow y = 8mts$, $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$

$\frac{dy}{dt} = \frac{-3}{2} mts/\min$, $A = \frac{1}{2}xy$

$\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right) = \frac{7}{2} sq.mts/\min$

4. OM= $r\cos\theta$, PM=r $\sin\theta$, $A = \frac{1}{2}(OM)(PM)$

$\frac{dA}{dt} = \frac{r^2}{4}(2\cos 2\theta)\frac{d\theta}{dt} = -r^2$

5. $\frac{a}{h}=\frac{y}{x+y} \Rightarrow ax+ay=hy$

$\Rightarrow (h-a)\frac{dy}{dt}=ab \Rightarrow \frac{dy}{dt}=\frac{ab}{(h-a)}$

6. $V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\left(l^2-h^2\right)h$

$=\frac{1}{3}\pi\left(l^2h-h^3\right) \Rightarrow \frac{dv}{dt}=\frac{1}{3}\pi\left(l^2-3h^2\right)\frac{dh}{dt}$

7. $x^2=y^2+h^2 \Rightarrow x\frac{dx}{dt}=v.y$

$\Rightarrow \frac{dx}{dt}=\frac{vy}{x}=\frac{v\sqrt{x^2-h^2}}{x}$

8. $\theta \propto t^2 \Rightarrow \theta = kt^2$ (k constant), $k=\frac{2\pi}{64} \Rightarrow k=\frac{\pi}{32}$

$\frac{d\theta}{dt}=k.2t=\frac{\pi}{32}\times 2\times 8=\frac{\pi}{2}\,R/\sec$

9. $V=\frac{ds}{dt}=0$

10. If t is the time.

$f(t)=(1500-21t)^2+(2100-28t)^2$

$f'(t)=0 \Rightarrow -42(1500-21t)$

$-56(2100-28t)=0 \Rightarrow t=\frac{516}{7}$

$f(t)=(1500-1548)^2+(2100-2064)^2=3600$

$\therefore$ The minimum distance is 60.

11. $v=\frac{ds}{dt}=160-32t$. We now find values of t for which $s(t)=256$. So, $160t-16t^2=256$

$t=2, t=8$, $v(2)=96, v(8)=-96$

So the velocity on the way up in $96\,m/s$

12. Let the body is at a height h_1 at a time 't' and is at a height "h" at a time $(t-4)$ from above.

$h_1-h=400 \Rightarrow \frac{1}{2}gt^2-\frac{1}{2}g(t-4)^2=400$

$\Rightarrow t^2-(t-4)^2=80 \Rightarrow t=12\sec$

$\therefore h=\frac{1}{2}g(t-4)^2=320\,m$ 7

Hence, total distance $=320+400=720\,m$

13. Volume of the balloon $v=\frac{4}{3}\pi r^3$

$\Rightarrow \frac{dr}{dt}=\frac{\frac{dv}{dt}}{4\pi r^2}----(1)$ Now, to find $\frac{dr}{dt}$ at $t=49$ min, we require $\frac{dv}{dt}$ the radius(r) at that stage $\frac{dv}{dt}=-72\pi m^3/\min$.

Also, amount of volume lost in 49 min

$=72\pi\times 49m^3=3528\pi m^3$

$\therefore$ Final volume at the end of 49 min

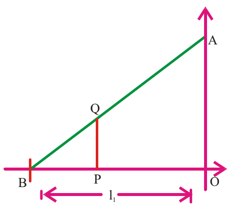
$=4500\pi-3528\pi=972\pi m^3$

If r is the radius at the end of 49min, then

$\frac{4}{3}\pi r^3=972\pi \Rightarrow r=9$

But $\frac{dr}{dt}=\frac{dv/dt}{4\pi r^2}$

$\Rightarrow \left(\frac{dr}{dt}\right)_{t=49}=\frac{72\pi}{4\pi(9)^2}=\frac{2}{9}m/\min$

14. 

Let $BP=x$. from similar $\Delta's$ property. we get

$\frac{AO}{l_1}=\frac{l_2}{x} \Rightarrow AO=\frac{l_1l_2}{x} \Rightarrow \frac{d(AO)}{dt}=\frac{-l_1l_2}{x^2}\frac{dx}{dt}$,

when $x=\frac{l_1}{2}, \frac{d(AO)}{dt}=-\frac{2l_2}{5}m/s$

ERRORS & APPROXIMATIONS

SYNOPSIS

- If $y = f(x)$, δx is any change in x then the corresponding change in y is δy. It is given by $\delta y = f(x+\delta x) - f(x)$
- $\left(\frac{dy}{dx}\right)\delta x$ is called differential of y It is denoted by dy or df.
  $\therefore dy = f^1(x)\delta x$
- The approximate value of the function is
  $f(x+\delta x) \cong f(x) + f^1(x)\delta x$
- $\delta y \cong dy$

Error, Relative Error, Percentage Error:

- Let y=f(x) be a function defined on an interval A and $x \in A$. Let δx be any change in x and δy be the corresponding change in y.Then
  i) δy is called error in y.
  ii) $\frac{\delta y}{y}$ is called relative error in y.
  iii) $\frac{\delta y}{y}$ X 100 is called percentage error in y.
- If $y = f(x) = K.x^n$ then the approximate relative error (or percentage error) in y is 'n' times the relative error (or percentage error) in x where n and k are constants.
- **Circle** If r is the radius , x is the diameter , p is perimeter (circumference) and A is the area of a circle then
  i) x = 2r
  ii) $p = 2\pi r$ or $p = \pi x$
  iii) $A = \pi r^2$ or A= $\frac{\pi x^2}{4}$
- **Sector :** If r is the radius, l is the length of the arc and θ is the angle , p is the perimeter and A is the area of a sector, then
  i) $l = r\theta$
  ii) $p = l + 2r$ or $p = r\theta + 2r = r(\theta + 2)$
  iii) $A = \frac{1}{2}lr$ or $A = \frac{1}{2}r^2\theta$
- **Cube:** If x is the side , S is the surface area and V is the volume of a cube then
  $S = 6x^2$; $V = x^3$
- **Sphere:** If r is the radius , S is the surface area V is the volume of a sphere then
  $S = 4\pi r^2$; $V = \frac{4}{3}\pi r^3$
- **Cylinder :** If r is the radius (of cross section) h is the height , L is the lateral surface area, S is the total surface area, V is the volume of a cylinder (right circular) then
  $L = 2\pi rh$, $S = 2\pi rh + 2\pi r^2$, $V = \pi r^2 h$
- **Cone :** If r is the base radius , h is the height, l is the slant height , θ is the semivertical angle α is the vertical angle , L is the lateral surface area , S is the total surface area and V is the volume of a (right circular) cone then
  i) $l^2 = r^2 + h^2$ ii) $Tan\theta = \frac{r}{h}$
  iii) $\alpha = 2\theta$
  iv) $L = \pi rl$ (or) $L = \pi r\sqrt{r^2 + h^2}$
  v) $S = \pi rl + \pi r^2$ vi) $V = \frac{1}{3}\pi r^2 h$
- **Simple pendulum :** If l is the length, T is the period of oscillation of a simple pendulum and g is the acceleration due to gravity then, $T = 2\pi\sqrt{l/g}$
- An electric current 'C' is measured by tangent galvonometer. If θ is the deflection of the galvonometer then $C \,\alpha\, Tan\theta$

EXERCISE - I

1. **If f(x)=** $3x^2 - x$ **where x =1 and** $\delta x = 0.02$ **then** $\delta f =$
   1) 0.1012 2) 1.012 3) 0.101 4) 0.1
2. **The approximate value of** $\sqrt{50}$ **is**
   1) 7.0704 2) 7.0741 3) 7.0714 4) 7.0785
3. **The approximate value is** $\cos 61^0$ **is**
   1) 0.4848 2) 0.4849 3) 0.4948 4) 0.5059
4. **If** 1^0 **=0.01745 radians .Then the approximate value of** $\tan 46^0$ **is**
   1) 1.0259 2) 1.0394 3) 1.0349 4) 1.0493
5. ΔABC **is not a right angled and is inscribed in a fixed circle . If a, A,b,B be slightly varied keeping c, C fixed then** $\frac{\delta a}{\cos A} + \frac{\delta b}{cosB} =$
   1) 2 2)1 3)0 4)5
6. **If the sides of** ΔABC **are changed slightly but its circum radius remains constant then** $\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} =$
   1) 0 2) a+b+c 3) A+B+C 4) 2R
7. **The diameter of a circle found by measurement 5.2cms with a maximum error 0.05cms. The maximum error in its area is**
   1) 4.1 sq cms 2) 0.041 sq.cms
   3) 0.41 sq.cms 4) 0.5 sq.cms
8. **A circular plate expands when heated from a radius of 5cms to 5.06 cm then the percentage increase in its area is**
   1) 0.6 2) 1.2 3) 2.4 4) 0.12
9. **When the radius of a sphere decreases from 3 cm to 2.98 cm then the approximate decrease in volume of sphere is**
   1) $0.002\pi cm^3$ 2) $0.072\pi cm^3$
   3) $0.72\pi cm^3$ 4) $0.008\pi cm^3$
10. **If an error of** $\left(\frac{1}{10}\right)\%$ **is made in measuring the radius of a sphere then percentage error in its volume is**
    1) 0.3 2) 0.03 3) 0.003 4) 0.0003
11. **The area of square is 9sq cms and the error in its is 0.02 sq.cm The percentage error in the measurement of the length of the diagonal of the square is**
    1) $\frac{2}{9}$ 2) $\frac{1}{9}$ 3) $\frac{4}{9}$ 4) $\frac{1}{3}$
12. **The height of a cylinder is equal to its radius. If an error of 1% is made in its height. Then the percentage error in its volume is**
    1) 1 2) 2 3) 3 4) 4
13. **Pressure P and Volume V of a gas are connected by the relation** $PV^{\frac{1}{4}} = C$ **(constant). The percentage increase in p corresponding to a diminition of** $\frac{1}{2}\%$ **in the volume is**
    1) $\frac{1}{2}$ 2) $\frac{1}{4}$ 3) $\frac{1}{8}$ 4) $\frac{1}{16}$
14. **The voltage E of a thermo couple as a function of temperature T is given by** $E = 6.2T + 0.0002T^3$ **when T changes from** 100^0 **to** 101^0 **the approximate change in E is**
    1) 12 2) 12.1 3) 12.12 4)12.2
15. **If there is an error of** $\pm 0.04cm$ **in the measurement of the diameter of sphere then the percentage error in its volume, when radius is 10 cm (EAM-2014)**
    1) ± 1.2 2) ± 0.06 3) ± 0.006 4) ± 0.6
16. **The circumference of a circle is measured as 28cm with an error of 0.01 cms. Then the percentage error in the area of the circle is**
    1) $\frac{2}{21}$ 2) $\frac{1}{7}$ 3) $\frac{2}{7}$ 4) $\frac{1}{14}$
17. **If there is an error of 0.01% in the radius of a sphere then the percentage error in its volume**
    1) 0.005 cu.cms 2) 0.05 cu.cms
    3) 0.03 cu.cms 4) 0.2 cu.cms
18. **If the length of simplependulum decreases by 3% then the percentage error in the period T is decreased by**
    1) 2 2) 2.5 3) 1.8 4) 1.5
19. **The pressure p and volume v of a gas are connected by the relation PV=C (constant). If** δp **and** δv **are the errors respectively in p and v.Then the approximate value of** $\frac{C.\delta v}{v^2}$ **is**
    1) $-\delta p$ 2) δp 3) $\frac{1}{\delta p}$ 4) $\frac{-1}{\delta p}$

KEY

01) 1	02) 3	03) 2	04) 3	05) 3	06) 1
07) 3	08) 3	09) 3	10) 1	11) 2	12) 3
13) 3	14) 4	15) 4	16) 4	17) 3	18) 4
19) 1					

SOLUTIONS

1. $\delta f = f(x+\delta x) - f(x)$
2. $f(x)=\sqrt{x}, x=49, \delta x=1$
   $f(x+\delta x) \cong f(x)+f^1(x)\delta x$
3. $f(x)=\cos x, x=60, \delta x=1^0$
   $f(x+\delta x) \cong f(x)+f^1(x)\delta x$
4. $f(x)=\tan x, x=45^0, \delta x=1^0$
5. $A+B+C=180$
   $\delta a = 2R\cos A\delta A, \delta b = 2R\cos B\delta B$
   $a=2R\sin A, b=2R\sin B$, $\delta A+\delta B=0$
6. $A+B+C=180^0$, $\delta A+\delta B+\delta C=0$
   $a=2R\sin A$, $b=2R\sin B$, $c=2r\sin C$
   $\delta a=2R\cos A\delta A$, $\delta b=2R\cos B\delta B$
   $\delta c=2R\cos C\delta C$, $\frac{\delta a}{\cos A}=2R\delta A,$
   $\frac{\delta b}{\cos B}=2R\delta B,$ $\frac{\delta c}{\cos C}=2R\delta C,$
   $\frac{\delta a}{\cos A}+\frac{\delta b}{\cos B}+\frac{\delta c}{\cos C}=2R(\delta A+\delta B+\delta C)=2R(0)=0$
7. $\delta A \cong dA, A=\frac{\pi}{4}x^2, x=$ dia meter
8. $A=\pi r^2, r=5, \delta r=0.06$
9. $V=\frac{4}{3}\pi r^3, r=3, \delta r=-0.02,$ $\delta v \cong dv$
10. $V\%=3(S\%)$
11. A=9, $\ell=\sqrt{2}x$, $\delta A=0.02$, $A=x^2$
    $A=\frac{\ell^2}{2}$, $1\%=\frac{1}{2}\frac{\delta A}{A}\times 100$
12. $h=r$ and $v=\pi h^3$, $V\%=3(h\%)$
13. $P\%=\left(\frac{-1}{2}\right)\left(\frac{-1}{4}\right)$
14. $T=100^0, \delta T=1$, $\delta E=6.2\delta T+0.0006T^2.\delta T$
15. Given $\Delta r=\pm\frac{0.04}{2}=\pm 0.02; r=10$
    Volume , $V=\frac{4}{3}\pi r^3$
    Take log on both sides & diff.
    $\Rightarrow \frac{\Delta V}{V}\times 100=3.\frac{\Delta r}{r}\times 100=\pm 0.6$
16. $4\pi A=c^2$, $A\%=2\frac{\delta c\times 100}{c}$
17. $V=\frac{4}{3}\pi r^3 \Rightarrow V\%=3r\%$
18. $T=2\pi\sqrt{\frac{\ell}{g}}$
19. $pv=c \quad \Rightarrow p\delta v+v\delta p=0$
    $c\frac{\delta v}{v^2}=p\frac{\delta v}{v}=-\delta p$

EXERCISE - II

1. **The radius and height of a cone are measured as 6cms each by scale in which there is an error of 0.01 cm in each cm. Then the approximate error in its volume is**
   1) $216\pi c.c$ 2) $2.16\pi c.c$
   3) $21.6\pi c.c$ 4) $0.216\pi c.c$
2. **The height and slant height of a cone are measured as 15cms and 25cms. Errors 2% are to allowed in both of these lengths. The possible error in its volume is**
   1) 30π c.c 2) 60π c.c
   3) 100π c.c 4) 120π c.c
3. **If there is an error 0.04 sq.cms in the surface area of a sphere then the error in its volume when the radius is 30cms is**
   1) 0.06.c.c 2) 0.006c.c
   3) 0.6 c.c 4) 0.0006 c.c
4. **The area of triangle is measured in terms of b,c, A. If A=63^0 and there is an error of 15^1 in A; the percentage error in the area is**
   1) $\frac{5\pi}{36}\cot 63^0$ 2) $\frac{\pi}{36}\cot 63^0$
   3) $\frac{2\pi}{36}\cot 63^0$ 4) $\frac{4\pi}{36}\cot 63^0$
5. **In a triangle ABC, the sides b,c are given . If there is an error δA in measuring angle A. Then error δa in the side a is**
   1) $\frac{\Delta.\delta A}{2a}$ 2) $\frac{2.\Delta\delta A}{a}$ 3) $bc\sin A\delta A$ 4) $\frac{3.\Delta\delta A}{a}$
6. **If there are 1%, 2%, 3%, 4% errors in r, r_1, r_2, r_3 then find the % error in area of triangle**
   1) 10 2) 5 3) 6 4) 8

7. The focal length of a mirror is given by $\frac{1}{v}-\frac{1}{u}=\frac{2}{f}$. If equal errors α are made in measuring u and v. Then relative error in f is (EAM-2013)

1) $\frac{2}{\alpha}$ 2) $\alpha\left(\frac{1}{u}+\frac{1}{v}\right)$ 3) $\alpha\left(\frac{1}{u}-\frac{1}{v}\right)$ 4) $\frac{3}{\alpha}$

8. A balloon is in the form of right circular cylinder of radius 1.5 m and length 4m and is surmounted by hemispherical ends. If the radius is increased by 0.01 m and the length by 0.05m, the percentage change in the volume of the balloon is

1) 2.389 2) 2.489 3) 2.0389 4) 2.589

9. The radius of a cylinder is half of its height. Error in the measurement of the raidus is 0.5% then percentage error in its surface area is

1) 5 2) 1 3) 1.5 4) 2

10. The distance S travelled by a particle is calculated using the formula $S=ut-\frac{1}{2}at^2$. If there is 1% error in t, the approximate percentage error in S is

1) $\left(\frac{u-at}{2u-at}\right)$ 2) $2\left(\frac{u-at}{2u-at}\right)$

3) $\frac{1}{2}\left(\frac{u-at}{2u-at}\right)$ 4) $\left(\frac{u-at}{3u-at}\right)$

11. The maximum error in T due to possible errors upto 1% in l and 2.5% in g where period T of a simple pendulum is

$$T=2\pi\sqrt{l/g}$$

1) 1.75% 2) 1.57% 3) 1.68% 4) 1.73%

12. The approximate value of $(0.007)^{1/3}$

1) 0.1919 2) 0.1619 3) 0.1816 4) 0.1716

13. The approximate value of

$\sqrt{(1.97)^2+(4.02)^2+(3.98)^2}$

1) 5.99 2) 5.099 3) 5.009 4) 5.734

14. The approximate value of

$\left\{(3.92)^2+3(2.1)^4\right\}^{1/6}$

1) 2.0466 2) 2.755 3) 2.345 4) 2.732

15. In an acute angled triangle ABC, if sides a, b be constants and the base angles A and B vary then

1) $\frac{\delta A}{\sqrt{a^2-b^2\sin^2 A}}=\frac{\delta B}{\sqrt{b^2-a^2\sin^2 B}}$

2) $\frac{\delta A}{\sqrt{b^2-a^2\sin^2 A}}=\frac{\delta B}{\sqrt{a^2-b^2\sin^2 B}}$

3) $\frac{\delta A}{\sqrt{a^2\sin^2 A-b^2}}=\frac{\delta B}{\sqrt{a^2\sin^2 B-b^2}}$

4) $\frac{\delta A}{\sqrt{a^2+b^2\sin^2 A}}=\frac{\delta B}{\sqrt{b^2+a^2\sin^2 B}}$

16. With the usual meaning for a, b, c and s if Δ be the area of a triangle then the error in Δ resulting from a small error in the measurement of c, is

1) $\frac{\Delta}{4}\left(\frac{1}{s}+\frac{1}{s-a}+\frac{1}{s-b}-\frac{1}{s-c}\right)\delta c$

2) $\frac{1}{4}\left(\frac{1}{s}+\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right)\delta c$

3) $\frac{\Delta}{4}\left(\frac{1}{s}+\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right)$

4) $\left(\frac{1}{s}+\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right)\delta c$

17. Which of the following statements are true

I: In ΔABC ,b,c are fixed and error in A is δA then error is $a=\frac{2\Delta.\partial A}{a}$

II: If semi vertical angle of a cone is 45^0 then error in volume is base area times of error in radius

1) only I 2) only II

3) both I and II 4) neither I nor II

KEY

01) 4 02) 1 03) 2 04) 3 05) 2 06) 2
07) 2 08) 1 09) 2 10) 2 11) 1 12) 1
13) 1 14) 1 15) 1 16) 1 17) 3

SOLUTIONS

1. $h = 15\text{cm}, \ \delta h = \frac{2h}{100}, \quad l = 25cm \Rightarrow \delta l = \frac{2l}{100},$

   $v = \frac{1}{3}\pi r^2 h, \ \delta v = \frac{\pi}{3}\left(r^2 \delta h + h.2r\delta r\right)$

2. Area $S = \frac{1}{2}bc\sin A, A = 63^0, \delta A = 15^1$

   $= \frac{15}{60} \times \frac{\pi}{180}, \ \frac{\delta s}{s} \times 100 = \cot A \ \delta A \times 100$

   $= \cot 63 \times \frac{15}{60} \times \frac{\pi}{180} \times 100 \ = \frac{5\pi}{36}\cot 63^0$

3. $r = h = 6cm \Rightarrow \delta r = \delta h = 6(0.01) = 0.06cm$

   $V = \frac{1}{3}\pi r^3 \Rightarrow \Delta V = \pi r^2 \Delta r$

4. $v = \frac{1}{6\sqrt{\pi}} s^{\frac{3}{2}}$

5. $a^2 = b^2 + c^2 - 2bc\cos A, 2a\delta a = 2bc\sin A\delta A$

6. $\Delta^2 = rr_1r_2r_3, \ 2\Delta\% = r\% + r_1\% + r_2\% + r_3\%$

7. $\delta x = \delta v = \alpha \ \Rightarrow \frac{-1}{v^2}\delta v + \frac{1}{u^2}\delta u = \frac{2}{f^2}\delta f$

8. Volume $V = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3$

   $= \pi r^2 h + \frac{4}{3}\pi r^3$, Find $\frac{\delta v}{v} \times 100$

9. $r = \frac{h}{2}, s = 2\pi rh + 2\pi r^2$

   $r\% = 0.5\% \ s = 6\pi r^2, \quad s\% = 2r\%$

10. taking logarithms and differentiate

11. $T = 2\pi\sqrt{l/g}$

    $\log T = \log 2\pi + 1/2\log l - 1/2\log g$

12. $f = x^{1/3}$, taking $x = 0.008, \Delta x = -0.001$

13. $f = \sqrt{x^2 + y^2 + z^2}$, taking

    $x = 2, \ y = 4, \ z = 4$

    $\Delta x = -0.03, \ \Delta y = 0.02, \ \Delta z = -0.02$

14. $x = 4, \ y = 2, \ \Delta x = -0.08, \ \Delta y = 0.1$

    $f = \left(x^2 + 3y^4\right)^{1/6}$

15. $\frac{a}{\sin A} = \frac{b}{\sin B}$ differentiate

16. $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

    $\log\Delta = \frac{1}{2}[(\log s + \log(s-a)$

    $+\log(s-b) + \log(s-c)]$, $S = \frac{a+b+c}{2}$

17. i. use $a^2 = b^2 + c^2 - 2bc\cos A$

    ii. $\theta = 45^0, \ r = h, \ v = \frac{1}{3}\pi r^2 h$

    $v = \frac{1}{3}\pi r^3 \Rightarrow \delta v = \pi r^2 \delta r$

MEAN VALUE THEOREMS

SYNOPSIS

Rolle's Theorem :

✈ If a function $f:[a,b]\to R$ is such that

i) f is continuous on $[a,b]$

ii) f is derivable on (a, b) and

iii) f(a) = f(b) then there exists atleast one value 'c' of x in the interval (a,b) such that f '(c) = 0.

Geometrical Interpretation of Rolle's Theorem :

✈ If $f:[a,b]\to R$ be a function satisfying the three conditions of Rolle's theorem. Then the graph of y = f(x) is such that

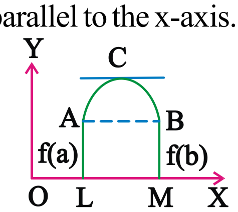
i) it is continuous curve from the point A(a, f(a)) to the point B(b, f(b)).

ii) It is a curve having unique tangent line at every intermediate point between A and B and

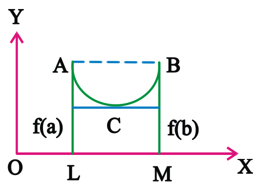
iii) The ordinates f(a), f(b) at the end points A, B are equal.

By Rolle's theorem there is atleast one $c \in (a, b)$ such that $f'(c) = 0$.

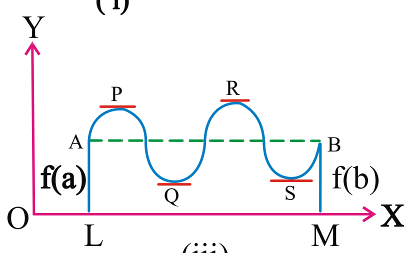
$\therefore$ There is atleast one point C(c, f(c)) between A and B on the curve at which the tangent line is parallel to the x-axis.



(i)



(ii)



(iii)

Note :

✈ The conditions of the Rolle's theorem for $f(x)$ on $[a,b]$ are only sufficient but not neccessary for $f'(x)$ to vanish at some point in (a, b).That is

i) If $f(x)$ satisfies the conditions of the Rolle's theorem in $[a,b]$ then the theorem guarantees the existance of at least one point $c\in(a,b) \ni f'(c)=0$.

ii) Even if function f does not satisfy the conditions of Rolle's theorem in $[a,b]$ there may exist points $x\in(a,b)$ at which $f'(x)$ vanishes

Ex: Let $f(x)=x-\sin x,\ x\in[\pi,5\pi]$. Clearly $f(\pi)\neq f(5\pi)$

But $f'(x)=1-\cos x=0$

at $x=2\pi,4\pi\in(\pi,5\pi)$.

Another form of Rolle's theorem :

✈ If $f:[a,a+h]\to R$ is such that

i) f is continuous on $[a,a+h]$

ii) f is derivable on (a, a+h) and

iii) f(a) = f(a+h) then there exists at least one number θ $(0<\theta<1)$ such that $f'(a+\theta h)=0$.

Lagrange's Mean Value Theorem (or) First Mean Value Theorem :

✈ If a function f(x) is such that $f:[a,b]\to R$

i) It is continuous on $[a,b]$

ii) It is derivable in (a, b), then there exists at least one value 'c' of x in (a, b) such that

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

Geometrical Interpretation of Lagrange's Theorem :

✈ Let $f:[a,b]\to R$ be a function satisfying the two conditions of lagrange's theorem. Then the graph of $y=f(x)$ is such that

i) it is continuous curve from the point A(a, f(a)) to the point B(b, f(b)) and

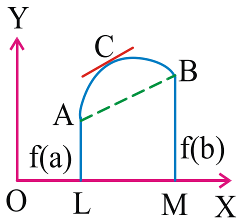
ii) It is a curve having unique tangent line at every intermediate point between A and B.

$\dfrac{f(b)-f(a)}{b-a}$ = slope of the chord $\overline{AB}$,

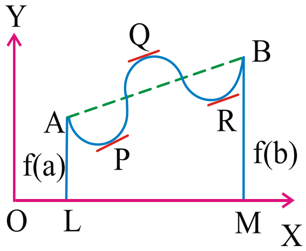
$f'(c)$ = slope of the tangent line at C(c, f(c)).

$\dfrac{f(b)-f(a)}{b-a}=f'(c)\Rightarrow$ chord $\overline{AB}$ is parallel to the tangent line at 'C'.

$\therefore\ \exists$ at least one point C(c, f(c)) on the curve between A and B such that the tangent line is parallel to the chord



(i)



(ii)

Note :

✈ The two condtions of LMVT are only sufficient conditions but not neccessary for the conclusion.

Ex: Let $f(x)=x^{\frac{1}{3}}, x\in[-1,1]$

$f'(x)=\dfrac{1}{3x^{\frac{2}{3}}}$ Which does not exist finitely at $x=0\in(-1,1)\Rightarrow f(x)$ is not differentiable in $(-1,1)$

$\therefore$ Lagrange's mean value thoerem is not applicable.

However, $\dfrac{f(1)-f(-1)}{1-(-1)}=f'(c)$

$$\Rightarrow\frac{1-(-1)}{2}=\frac{1}{3c^{\frac{2}{3}}}$$

$$\Rightarrow c^{\frac{2}{3}}=\frac{1}{3}\Rightarrow c=\frac{1}{3\sqrt{3}}\in(-1,1)$$

Another form of Lagrange's Mean Value Theorem :

✈ If a function $f:[a,a+h]\to R$ is such that

i) f is continuous on $[a,a+h]$ and

ii) f is derivable on (a, a+h) then there exists at least one number θ $(0<\theta<1)$ such that

$f(a+h)=f(a)+h\,f'(a+\theta h)$.

Intermediate Mean value Theorem :

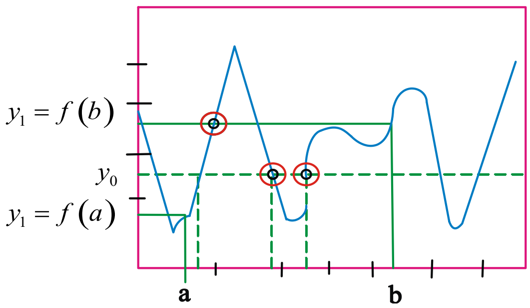
✈ Let $f(x)$ be a function which is continuous on the closed interval $[a,b]$ and let y_0 be a real number lying between $f(a)$ and $f(b)$, i.e., with

$f(a)\le y_0\le f(b)$

or $f(b)\le y_0\le f(a)$.

Then there is at least one c with $a\le c\le b$ such that

$$y_0=f(c)=\frac{f(a)+f(b)}{2}$$



Cauchy's Mean Value Theorem :

✈ If two functions $f(x)$ & $\phi(x)$ are such that

i) both are continuous in the closed interval $[a,b]$

ii) both are derivable in the open interval (a,b)

iii) $\phi'(x)\neq 0$ for any value of x in the open interval (a,b) then there exists at least one value c of x in the open interval (a,b) such that

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)}=\frac{f'(c)}{\phi'(c)}$$

Another Form of Cauchy's Mean Value theorem :

✈ If two functions $f(x)$ and $\phi(x)$ are such that

i) both are continuous in the closed interval $[a, a+h]$

ii) both are derivable in the open interval $(a, a+h)$

iii) $\phi'(x) \neq 0$ for any value of x in the open interval $(a, a+h)$ then there exists at least one number θ such that

$$\frac{f(a+h)-f(a)}{\phi(a+h)-\phi(a)} = \frac{f'(a+\theta h)}{\phi'(a+\theta h)}$$

where $0 < \theta < 1$.

EXERCISE - I

1. **For the function** $f(x) = x^3 - 6x^2 + ax + b$, **if Rolle's theorem holds in [1, 3] with** $c = 2 + \frac{1}{\sqrt{3}}$ **then (a, b) =**
   1) (11, 12) 2) (11, 11)
   3) (11, any value) 4) (any value, 0)
2. **Rolle's theorem cannot be applicable for**
   1) $f(x) = \sqrt{4 - x^2}$ in [-2, 2]
   2) f(x) = [x] in [-1, 1]
   3) $f(x) = x^2 + 3x - 4$ in [-4, 1]
   4) f(x) = cos2x in [0, π]
3. **Rolle's theorem cannot be applicable for**
   1) $f(x) = \cos x - 1$ in $[0, 2\pi]$
   2) $f(x) = x(x-2)^2$ in [0, 2]
   3) $f(x) = 3 + (x-1)^{2/3}$ in [0, 3]
   4) $f(x) = \sin^2 x$ in $[0, \pi]$
4. **Value of 'c' of Rolle's theorem for** $f(x) = \log(x^2 + 2) - \log 3$ **on [-1, 1] is**
   1) 0 2) 1 3) –1 4) does not exists
5. **If** $2a + 3b + 6c = 0$, **then at least one root of the equation** $ax^2 + bx + c = 0$ **lies in the interval**
   1) (0, 1) 2) (1, 2) 3) (2, 3) 4) (0, 4)
6. **If** $27a + 9b + 3c + d = 0$, **then the equation** $4ax^3 + 3bx^2 + 2cx + d = 0$ **has atleast one real root lying between**
   1) 0 and 1 2) 1 and 3
   3) 0 and 3 4) 0 and 2
7. **The quadratic equation** $3ax^2 + 2bx + c = 0$ **has at least one root between 0 and 1 if**
   1) a+b+c=02) c=0
   3) 3a+2b+c=0 4) a+b=c
8. **The value of 'c' in Lagrange's mean value theorem for f(x) = logx on [1, e] is**
   1) e/2 2) e-1 3) e-2 4) 1-e
9. **The value of 'c' in Lagrange's mean value theorem for** $f(x) = x(x-2)^2$ **in [0, 2] is**
   1) 0 2) 2 3) 2/3 4) 3/2
10. **The value of '*c*' in Lagrange's mean value theorem for** $f(x) = x^3 - 2x^2 - x + 4$ **in [0, 1] is**
    1) 1/3 2) 1/2 3) 2/3 4) 1
11. **The value of** θ **of mean value theorem for the function** $f(x) = ax^2 + bx + c$ **in [1,2] is**
    1) $\frac{1}{3}$ 2) $\frac{1}{2}$ 3) $\frac{1}{4}$ 4) $\frac{1}{5}$
12. **The value of 'c' in Lagrange's mean value theorem for f(x) = log (sinx) in** $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ **is**
    1) $\frac{\pi}{4}$ 2) $\frac{\pi}{2}$ 3) $\frac{2\pi}{3}$ 4) $\frac{3\pi}{4}$
13. **Lagrange's mean value theorem cannot be applied for [EAM -2019]**
    1) $f(x) = \log x$ in [1, e]
    2) $f(x) = x - \frac{1}{x}$ in [1, 3]
    3) $f(x) = \sqrt{x^2 - 4}$ in [2, 4]
    4) $f(x) = |x|$ in [-1, 2]

14. **The chord joining the points where x = p and x = q on the curve $y = ax^2 + bx + c$ is parallel to the tangent at the point on the curve whose abscissa is**

1) $\frac{p+q}{2}$ 2) $\frac{p-q}{2}$ 3) $\frac{pq}{2}$ 4) $\frac{p}{q}$

15. **If $f(x)$ is differentiable in the interval [2, 5], where $f(2)=\frac{1}{5}$ and $f(5)=\frac{1}{2}$, then there exists a number c, $2 < c < 5$ for which $f'(c)$ is equal to**

1) $\frac{1}{2}$ 2) $\frac{1}{5}$ 3) $\frac{1}{10}$ 4) $-\frac{1}{2}$

16. **The value of 'c' in Lagrange's mean value theorem for $f(x) = lx^2 + mx + n, (l \neq 0)$ on [a, b] is [EAM -2020]**

1) $\frac{a}{2}$ 2) $\frac{b}{2}$ 3) $\frac{(a-b)}{2}$ 4) $\frac{(a+b)}{2}$

17. **In [0,1], Lagrange's mean value theorem is not applicable to**

i) $f(x)=\begin{cases} \frac{1}{2}-x, & x<\frac{1}{2} \\ \left(\frac{1}{2}-x\right)^2, & x \geq \frac{1}{2} \end{cases}$

ii) $f(x)=\begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x=0 \end{cases}$

iii) $f(x) = x|x|$ iv) $f(x) = |x|$

1) i 2) ii 3) iii 4) iv

18. **The value of 'a' for which $x^3 - 3x + a = 0$ has two distinct roots in [0, 1] is given by**

1) –1 2) 1

3) 3 4) does not exists

KEY

01) 3	02) 2	03) 3	04) 1	05) 1	06) 3
07) 1	08) 2	09) 3	10) 1	11) 2	12) 2
13) 4	14) 1	15) 3	16) 4	17) 1	18) 4

SOLUTIONS

1. $f(1) = f(3) \Rightarrow a = 11$

   $f(1) = f(3)$ is independent of b $\therefore a = 11, b \in R$
2. f(x) = [x] is discontinuous function in [–1, 1]
3. $f(x) = 3 + (x-1)^{2/3}$

   $f'(x) = \frac{2}{3}(x-1)^{-1/3}$ is not defined at x= 1
4. $f'(c) = 0 \Rightarrow \frac{2c}{c^2+2} = 0 \Rightarrow c = 0$
5. Let $f'(x) = 6ax^2 + 6bx + 6c$

   $\Rightarrow f(x) = 2ax^3 + 3bx^2 + 6cx + d$

   f(0) = d, f(1) = 2a + 3b + 6c + d

   $\Rightarrow f(1) = d,\ f(0) = f(1)$

   $\therefore \exists$ at least one root of the equation $f'(x) = 0$ lies in (0, 1).
6. Let $f(x) = \frac{4ax^4}{4} + \frac{3bx^3}{3} + \frac{2cx^2}{2} + dx$

   $f(0) = 0 = f(3) \Rightarrow \exists\, c \in (0,3) \ni f'(c) = 0$
7. Let $f(x) = ax^3 + bx^2 + cx$

   $f'(x) = 3ax^2 + 2bx + c$

   $f'(c) = 0$, f(0) = f(1) $\Rightarrow a + b + c = 0$
8. Using formula $f'(c) = \frac{f(b) - f(a)}{b-a}$
9. $f'(c) = 0$, $2c(c-2) + (c-2)^2 = 0$

   c = 2, 3/2 $\therefore c = 3/2\,(c \neq 2)$
10. $f'(c) = \frac{f(1) - f(0)}{1-0}$
11. $f(a+h) = f(a) + h\,f'(a + \theta h)$
12. $f'(c) = \frac{\log\left(\sin\left(\frac{5\pi}{6}\right)\right) - \log\left[\sin(\pi/6)\right]}{\frac{5\pi}{6} - \frac{\pi}{6}}$

    $f'(c) = 0$, $\cot c = 0 \Rightarrow c = \pi/2$
13. f(x) = | x | is not differentiable at x = 0
14. Apply Lagrange's theorem

    $f'(c) = \frac{f(q) - f(p)}{q-p}$
15. $f'(c) = \frac{f(5) - f(2)}{5-2}$

16. $f'(c)=\dfrac{lb^2+mb+n-la^2-ma-n}{b-a}$

$2lc+m=l(a+b)+m$, $c=\dfrac{a+b}{2}$

17. $f(x)$ is not differentiable at $x=\dfrac{1}{2}\in(0,1)$

18. Let $\alpha,\beta\in[0,1]$

f(x) is continuous on $[\alpha,\beta]$ and differentiable on (α,β) and $f(\alpha)=f(\beta)=0$

$\therefore c\in(\alpha,\beta)$ such that

$f'(c)=0\Rightarrow c=\pm1\notin(0,1)$

EXERCISE - II

1. Value of 'c' of Rolle's theorem for f(x) = sinx – sin2x on [0, π] is

1) $\cos^{-1}\left(\dfrac{1+\sqrt{33}}{8}\right)$ 3) $\cos^{-1}\left(\dfrac{1+\sqrt{35}}{8}\right)$

3) $\cos^{-1}\left(\dfrac{1-\sqrt{38}}{5}\right)$ 4) does not exists

2. If *a, b, c* are non-zero real numbers such that $\int_0^1 (1+\cos^8 x)(ax^2+bx+c)\,dx =$ $\int_0^2 (1+\cos^8 x)(ax^2+bx+c)\,dx = 0$, then the equation $ax^2+bx+c=0$ will have

1) one root between 0 and 1 and other root between 1 and 2

2) both the roots between 0 and 1

3) both the roots between 1 and 2

4) both the roots between (3, ∞)

3. If *f*(*x*) and *g*(*x*) are differentiable functions in [0,1] such that *f*(0) = 2, *g*(0) = 0, *f*(1) = 6, *g*(1) = 2, then there exists *c*, 0 < *c* < 1 such that *f*′(*c*) = (JEE MAINS 2014)

1) $g'(c)$ 2) $-g'(c)$ 3) $2g'(c)$ 4) $3g'(c)$

4. If $f(x)=\begin{cases} x^{\alpha}\log x, & x>0 \\ 0, & x=0\end{cases}$ and Rolle's theorem is applicable to *f*(*x*) for $x\in[0, 1]$ then α may be equal to

1) –2 2) –1 3) 0 4) 1/2

5. For which interval, the function $\dfrac{x^2-3x}{x-1}$ satisfies all the conditions of Rolle's theorem

1) $[0,3]$ 2) $[-3,0]$

3) $[1.5,3]$ 4) For no interval

6. Let f be differentiable for all x. If f(1) = -2 and $f'(x)\geq 2$ for all $x\in[1,6]$, then

1) f(6) < 8 2) f(6) $\geq$ 8 3) f(6) $\geq$ 5 4) f(6) $\leq$ 5

7. Value of 'c' of Lagrange's mean theorem for $f(x)=\begin{cases} 2+x^3, & if\ x<1 \\ 3x, & if\ x>1\end{cases}$ on [-1, 2] is

1) $\pm\dfrac{\sqrt{5}}{3}$ 2) $\pm\dfrac{\sqrt{3}}{2}$ 3) $\pm\dfrac{\sqrt{2}}{5}$ 4) $\pm\dfrac{3}{\sqrt{5}}$

8. If $0<\alpha<\beta<\dfrac{\pi}{2}$, and if $\dfrac{\tan\beta}{\tan\alpha}>k$, then k is

1. $\dfrac{\alpha}{\beta}$ 2) $\dfrac{\beta}{\alpha}$ 3) $\dfrac{2\alpha}{\beta}$ 4) $\dfrac{2\beta}{\alpha}$

9. If $a_1<(28)^{\frac{1}{3}}-3<b_1$, then (a_1,b_1) is

1) $\left(\dfrac{1}{28},\dfrac{1}{27}\right)$ 2) $\left(\dfrac{1}{27},\dfrac{1}{28}\right)$

3) $(27,28)$ 4) $\left(\dfrac{1}{27},\dfrac{1}{26}\right)$

10. If $f(x)=\cos x, 0\leq x\leq\dfrac{\pi}{2}$, then the real number 'c' of the mean value theorem is

1) $\dfrac{\pi}{6}$ 2) $\dfrac{\pi}{4}$ 3) $\sin^{-1}\left(\dfrac{2}{\pi}\right)$ 4) $\cos^{-1}\left(\dfrac{2}{\pi}\right)$

11. If $\dfrac{a_0}{n+1}+\dfrac{a_1}{n}+\dfrac{a_2}{n-1}+......+\dfrac{a_{n-1}}{2}+a_n=0$

then [EAM -2017]

$a_0x^n+a_1x^{n-1}+.......+.....+a_{n-1}x+a_n=0$ **has**

1) no solution in (0, 1)

2) at least one solution in (0, 1)

3) exactly one solution in (0, 1)

4) at least one solution in (2, 3)

12. Let $f(x)=(x-4)(x-5)(x-6)(x-7)$ then

1) $f'(x)=0$ has four roots

2) three roots of $f'(x)=0$ lie in (4,5)U(5,6)U(6,7)

3) the equation $f'(x)=0$ has only one root.

4) three roots of $f'(x)=0$ lie in (3,4)U(4,5)U(5,6)

13. Let f(x) and g(x) be differentiable for $0 \le x \le 1$, such that $f(0)=2, g(0)=0, f(1)=6$. Let there exist a real number c in [0, 1] such that $f'(c)=2g'(c)$, then the value of g(1) must be

1) 1 2) 2 3) -2 4) -1

14. If $a<c<b$, and if $1-k_1<ln\left(\frac{b}{a}\right)<k_2-1$, then (k_1,k_2) is [EAM - 2018]

1) $\left(\frac{a}{b},\frac{b}{a}\right)$ 2) $\left(\frac{b}{a},\frac{a}{b}\right)$ 3) $(2a,2b)$ 4) (a,b)

15. The value of 'c' of Lagrange's mean value theorem for $f(x)=x^3-5x^2-3x$ in $[1,3]$ is

1) 2 2) 5/4 3) 3 4) 7/3

16. In the mean value theorem, $f(b)-f(a)=(b-a)f'(c)$, if a=4, b=9 and $f(x)=\sqrt{x}$ then the value of c is

1) 8 2) 5.25 3) 6.25 4) 4

KEY

01) 1 02) 1 03) 3 04) 4 05) 4 06) 2
07) 1 08) 1 09) 1 10) 3 11) 2 12) 2
13) 2 14) 1 15) 4 16) 3

SOLUTIONS

1. $f'(c)=0 \Rightarrow \cos c-2\cos 2c=0$

$4\cos^2 c-\cos c-2=0$

$\cos c=\frac{1\pm\sqrt{33}}{8}$, $c=\cos^{-1}\left(\frac{1\pm\sqrt{33}}{8}\right)$

2. $f(x)=\int_0^x\left(1+\cos^8 x\right)\left(ax^2+bx+c\right)dx$

$f(0)=0, f(1)=0, f(2)=0$

$f'(x)=0 \Rightarrow \left(1+\cos^8 x\right)\left(ax^2+bx+c\right)=0$

$\Rightarrow ax^2+bx+c=0$

It gives two roots

$f(0)=0=f(1)$ and $f'(x)=0 \Rightarrow$ At least one x between 0 and 1

$f(1)=0=f(2)$ and $f'(x)=0 \Rightarrow$ At least one x between 1and 2

3. Let $\phi(x)=f(x)-2g(x)$

$\phi(0)=2=\phi(1)$, $\phi'(c)=0 \Rightarrow f'(c)=2g'(c)$

4. By Rolle's theorem, f is continuous at $x=0$

$\underset{x\to 0}{Lt} f(x)=f(0)$, $\underset{x\to 0}{Lt} x^{\alpha}\log x=0 \Rightarrow \alpha$ is positive

5. $f'(x)$ is not defined at x = 1 i.e, in $(0,3)$

Also $f(a)=f(b)$ does not hold for $[-3,0]$ and $[1.5,3]$

6. By Lagrange's theorem $c \in (1,6)$ such that

$f'(c)=\frac{f(6)-f(1)}{6-1}=\frac{f(6)+2}{5}\ge 2$

$\Rightarrow f(6)+2\ge 5(2)$

7. $f^1(c)=\frac{f(b)-f(a)}{b-a}$

8. $f(x)=x\tan x, f^1(x)=\tan x+x\sec^2 x>0$

$\frac{f(\beta)-f(\alpha)}{\beta-\alpha}=f^1(c)>0 \Rightarrow f(\beta)>f(\alpha)$

$\beta\tan\beta>\alpha\tan\alpha \Rightarrow \frac{\tan\beta}{\tan\alpha}>\frac{\alpha}{\beta}.$

9. Let $f(x)=x^{1/3}$ in [27,28] $\Rightarrow f^1(x)=\frac{1}{3x^{2/3}}$

By Lagrange's theorem, $(28)^{1/3}-3=\frac{1}{3c^{2/3}}$.....(1)

$27<c<28 \Rightarrow 9<c^{2/3}<(28)^{2/3}$

$\Rightarrow 27<3.c^{2/3}<3.(28)^{2/3}$

$\Rightarrow \frac{1}{27}>\frac{1}{3.c^{\frac{2}{3}}}>\frac{1}{3(28)^{\frac{2}{3}}}=\frac{(28)^{\frac{1}{3}}}{3}.\frac{1}{28}>\frac{1}{28}$

$\therefore \frac{1}{28}<\frac{1}{3(c)^{2/3}}<\frac{1}{27}$ and by (1)

$\frac{1}{28}<(28)^{1/3}-3<\frac{1}{27}.$

10. $f'(c)=\frac{f(b)-f(a)}{b-a}$

11. $\phi(x)=a_0\frac{x^{n+1}}{n+1}+a_1\frac{x^n}{n}+......+a_nx$

$\phi(0)=0;\phi(1)=\frac{a_0}{n+1}+......+a_n=0$

using Rolle's theorem.

12. f(4) = f(5) = f(6) = f(7) = 0

By Rolle's theorem

$\exists\ \alpha_1\in(4,5),\ \alpha_2\in(5,6),\ \alpha_3\in(6,7)$

such that $f'(\alpha_i)=0,\ i=1,2,3$

13. Let $\phi(x)=f(x)-2g(x)$

$\phi(0)=f(0)-2g(0) = 2 - 0 = 2$

$\phi(1)=f(1)-2g(1)=6-2g(1)$

$\phi'(x)=f'(x)-2g'(x)$

$\phi'(c)=f'(c)-2g'(c)=0 \qquad \therefore\ g(1)=2$

14. $f(x)=l\text{n}\,x, f^1(x)=\frac{1}{x},\frac{f(b)-f(a)}{b-a}=\frac{1}{c}$

$\Rightarrow \ln\frac{b}{a}=\frac{b-a}{c}$

$a<c<b\Rightarrow\frac{1}{b}<\frac{1}{c}<\frac{1}{a}\Rightarrow\frac{b-a}{b}<\frac{b-a}{c}<\frac{b-a}{a}$

$\Rightarrow 1-\frac{a}{b}<\ln\frac{b}{a}<\frac{b}{a}-1$

15. $f'(c)=\frac{f(3)-f(1)}{2}$

16. $f'(c)=\frac{f(b)-f(a)}{b-a}$

EXERCISE - III

1. In $[0,\pi]$ Rolle's theorem is not applicable to

1) $f(x)=\sin x$

2) $f(x)=\begin{cases}\frac{\sin x}{x}, & x\neq 0\\ 1, & x=0\end{cases}$

3) $f(x)=\cos 2x$ 4) $\sin^2 x+\sin x$

2. Let $f(x)$ be continuous on $[a,b]$, differentiable in (a,b) and $f(x)\neq 0$ for all $x\in[a,b]$. Then, there exists $\theta\in(a,b)$ such that $\frac{f'(\theta)}{f(\theta)}$ is equal to

1) $\frac{1}{a+\theta}-\frac{1}{b-\theta}$ 2) $\frac{1}{a-\theta}+\frac{1}{b-\theta}$

3) $\frac{1}{a+\theta}+\frac{1}{b-\theta}$ 4) $(a+b)\theta$

3. Let $f(x)$ be non-constant differentiable function for all real x and $f(x)=f(1-x)$ Then Rolle's theorem is not applicable for $f(x)$ on

1) $[0,1]$ 2) $[-1,2]$ 3) $[-2,3]$ 4) $\left[0,\frac{2}{3}\right]$

4. The real number k for which the equation, $2x^3+3x+k=0$ has two distinct real roots in $[0,1]$ (JEE MAINS 2013)

1) lies between 1 and 2

2) lies between 2 and 3

3) lies between -1 and 0

4) does not exist

5. **If f be a continuous function on $[0,1]$, differentiable in $(0,1)$ such that $f(1)=0$, then there exists some $c\in(0,1)$ such that**

1. $cf'(c)-f(c)=0$ 2. $f'(c)+cf(c)=0$

3. $f'(c)-cf(c)=0$ 4. $cf'(c)+f(c)=0$

6. **If a, b, c are real numbers such that $\frac{3a+2b}{c+d}+\frac{3}{2}=0$ then the equation $ax^3+bx^2+cx+d=0$ has**

1) at least one root in $[-2,0]$
2) at least one root in $[0,2]$
3) at least two roots in $[-2,2]$
4) No root in $[-2,2]$

7. **Let** $f(x)=\begin{vmatrix}1 & 1 & 1\\ 3-x & 5-3x^2 & 3x^3-1\\ 2x^2-1 & 3x^5-1 & 7x^8-1\end{vmatrix}$ **then the equation $f(x)=0$ has**

1) no real root 2) atmost one real root
3) atleast 2 real roots
4) exactly one real root in $(0,1)$ and no other real root.

8. **Consider the function $f(x)=8x^2-7x+5$ on the interval $[-6,6]$. Then the value of c that satisfies the conclusion of Lagrange's mean value theorem is**

1) 0 2) 1 3) 2 4) 4

9. **If f is continuous on $[a,b]$ and differentiable in (a,b) $(ab>0)$, then there exists $c\in(a,b)$ such that** $\dfrac{f(b)-f(a)}{\frac{1}{b}-\frac{1}{a}}=$

1) $-c^2f'(c)$ 2) $c^2f'(c)$

3) $-cf'\left(\frac{1}{c}\right)$ 4) $cf'\left(\frac{1}{c^2}\right)$

10. **Using Lagrange's mean value theorem for $f(x)=\cos x$, we get that $|\cos a-\cos b|\le$**

1) $|a+b|$ 2) $|a-b|$ 3) $|2a+b|$ 4) $|2a-b|$

11. **If f is continuous function in $[1,2]$ such that $|f(1)+3|<|f(1)|+3$ and $|f(2)+10|=|f(2)|+10,(f(2)\neq 0)$ then the function f in $(1,2)$ has**

1) Atleast one root 2) No root
3) Exactly one root 4) None of these

12. **In $[-1,1]$, Lagrange's Mean Value theorem is applicable to**

1) $f(x)=|x|$ 2) $f(x)=\begin{cases}\cot x, & x\neq 0\\ 0, & x=0\end{cases}$

3) $f(x)=\begin{cases}\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$ 4) $f(x)=x^2$

13. **If the functions $f(x)$ and $\phi(x)$ are continuous in $[a,b]$ and differentiable in (a,b), then the value of 'c' for the pair of functions $f(x)=\sqrt{x}$, $\phi(x)=\frac{1}{\sqrt{x}}$ is**

1) $\sqrt{a}$ 2) $\sqrt{b}$ 3) $\sqrt{ab}$ 4) $-\sqrt{ab}$

14. **If the functions $f(x)$ and $\phi(x)$ are continuous in $[a,b]$ and differentiable in (a,b), then the value of 'c' for the pair of functions $f(x)=e^x$, $\phi(x)=e^{-x}$ is**

1) $\frac{a}{2}$ 2) $\frac{a-b}{2}$ 3) $\frac{a+b}{2}$ 4) $\frac{-a+b}{2}$

15. **There is a point P between $(1,0)\,\&\,(3,0)$ on $y=x^2-4x+3$ such that tangent at P is parallel to x-axis. Then the ordinate of the point of contact is**

1) 2 2) -1 3) 1 4) 3

KEY

01) 2   02) 2   03) 4   04) 4   05) 4   06) 2

07) 3   08) 1   09) 1   10) 2   11) 1   12) 4

13) 3   14) 3   15) 2

SOLUTIONS

1. (1) $f(x)=\sin x,\ x\in[0,\pi]$ f is continuous and differentiable and $f(0)=f(\pi)=0$. Hence Rolle's theorem is applicable

(2) $f(x)=\begin{cases}\frac{\sin x}{x}, & x\neq 0\\ 1 \ for & x=0\end{cases}$, $\underset{x\to 0}{Lt}\frac{\sin x}{x}=1=f(0)$

f is continuous in $[0,\pi]$ and also differentiable in $(0,\pi)$. $f(0)=1$ and $f(\pi)=0$. Rolle's theorem is not applicable.

(3) $f(x)=\cos 2x, x\in[0,\pi]$ is continuous and differentiable. $f(0)=1=f(\pi)$.
Hence Rolle's theorem is applicable.

(4) $f(0)=f(\pi)$.
$\therefore$ Rolle's theorem is applicable.

2. Let $\phi(x)=(a-x)(b-x)f(x)$ on $[a,b]$
Using the Rolle's theorem, there exists $\theta\in(a,b)$ such that $\phi'(\theta)=0$.
therefore,
$-(b-\theta)f(\theta)-(a-\theta)f(\theta)+(a-\theta)(b-\theta)f'(\theta)=0$
$\therefore \frac{f'(\theta)}{f(\theta)}=\frac{1}{a-\theta}+\frac{1}{b-\theta}$

3. Clearly $f(0)=f(1), f(-1)=f(2)$
$f(-2)=f(3)$ and $f\left(\frac{1}{3}\right)=f\left(\frac{2}{3}\right)$

4. Clearly $f'(x)=(6x^2+3)>0$
$f(x)$ is increasing function
$f(x)=0$ will have no real roots in $[0,1]$

5. Let $g(x)=xf(x)$, As $f(1)=0$,
$g(0)=0=g(1)$ then use Rolle's theorem

6. $f'(x)=ax^3+bx^2+cx+d$

$f(x)=\frac{ax^4}{4}+\frac{bx^3}{3}+\frac{cx^2}{2}+dx$
given $6a+4b+3c+3d=0, f(+2)=0=f(0)$
use Rolle's theorem

7. f(0)=f(1)=0 (obviously) and f(x) is a polynomial of degree 10.Therefore by Rolle's theorem we must have at least one root in (0, 1).Since the degree of f(x) is even, hence atleast two real roots

8. $f'(c)=16c-7=\frac{f(6)-f(-6)}{12}$
$=\frac{(8\times 36-7\times 6+5)-(8\times 36+7\times 6+5)}{12}=-7\Rightarrow c=0$

9. Let $F(x)=f\left(\frac{1}{x}\right),\ x\in\left[\frac{1}{b},\frac{1}{a}\right]$ use LMVT for $F(x)$. Then there exists $d\in\left(\frac{1}{b},\frac{1}{a}\right)$ such that

$\frac{F\left(\frac{1}{a}\right)-F\left(\frac{1}{b}\right)}{\frac{1}{a}-\frac{1}{b}}=F'(d)=-\frac{1}{d^2}f'\left(\frac{1}{d}\right)$

Put $c=\frac{1}{d}$

10. Use Lagrange's theorem

11. $|f(1)+3|<|f(1)|+3\Rightarrow f(1)<0$
$|f(2)+10|=|f(2)|+10\Rightarrow f(2)>0$

12. **(1)** $f(x)=|x|$ is not differentiable in $(-1,1)$
$\therefore$ LMVT is not applicable.

(2) $f(x)$ is not differentiable at $x=0$
$\therefore$ Lagrange's Theorem is not applicable

(3) $f(x)=\begin{cases}\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

$\underset{x\to 0-}{Lt}f(x)=-\infty$ and $\underset{x\to 0-}{Lt}f(x)=+\infty$

$\underset{x\to 0}{Lt}f(x)$ does not exist
$\therefore$ Lagrange's Theorem is not applicable

(4) $f(x) = x^2$ is continuous and differentiable in

$[-1,1]$

$\therefore$ Lagrange's Theorem is applicable

13. $f(x) = \sqrt{x},\ \phi(x) = \dfrac{1}{\sqrt{x}}$

(Assuming $0 < a < b$)

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{f'(c)}{\phi'(c)}$$

$$\Rightarrow \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}} = \frac{\frac{1}{2\sqrt{c}}}{-\frac{1}{2c\sqrt{c}}} \quad \Rightarrow c = \sqrt{ab}$$

14. By Cauchy's mean value theorem we have

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{f'(c)}{\phi'(c)}$$

$$\Rightarrow \frac{e^b - e^a}{\frac{1}{e^b}-\frac{1}{e^a}} = -e^{2c} \Rightarrow -e^{a+b} = -e^{2c}$$

$$\Rightarrow c = \frac{a+b}{2}$$

15. Use LMVT

MAXIMA & MINIMA

SYNOPSIS

Increasing and decreasing functions on an interval :

- f be a real valued function with domain D and $(a,b) \subseteq D$. f is said to be

  i) increasing function on (a,b) if

  $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) \forall x_1, x_2 \in (a,b)$

  ii) strictly increasing function on (a,b) if

  $x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \forall x_1, x_2 \in (a,b)$

  iii) decreasing function on (a,b) if

  $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2) \forall x_1, x_2 \in (a,b)$

  iv) strictly deacreasing on (a,b) if

  $x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \forall x_1, x_2 \in (a,b)$;

  Ex: i) sin$x$ is increasing in $\left[0, \frac{\pi}{2}\right]$

  ii) cos$x$ is decreasing in $\left[0, \frac{\pi}{2}\right]$

Monotonic Function :

- A function which is either increasing (or) decreasing in its domain is called a monotonic function.

Test for Monotonicity :

- If f is strictly increasing on (a, b) then

  $x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \forall x_1, x_2 \in (a,b)$

  If h is very small positive real number and

  $x \in (a,b)$ then $x < x+h \Rightarrow f(x) < f(x+h)$

  $$f^1(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

  $$= \lim_{h \to 0} \frac{+ve}{+ve} \Rightarrow f^1(x) > 0$$

  thus we have the following conditions for monotonicity

  i) If f is increasing on (a, b) then

  $f^1(x) \geq 0 \ \forall x \in (a,b)$

  ii) If f is strictly increasing on (a, b) then

  $f^1(x) > 0 \ \forall x \in (a,b)$

  iii) If f is decreasing on (a, b) then

  $f^1(x) \leq 0 \ \forall x \in (a,b)$

  iv) If f is strictly decreasing on(a, b) then

  $f^1(x) < 0 \ \forall x \in (a,b)$

Increasing and decreasing functions at a point :

- f is a real valued function defined in the neighbour hood of 'a' then

  i) f is said to be increasing at 'a' if $f(a-h) < f(a) < f(a+h)$ for a small positive real number 'h'

  ii) f is said to be decreasing at 'a' if

  $f(a-h) > f(a) > f(a+h)$

  iii) f is neither increasing nor decreasing at 'a'

  if $f(a-h) > f(a) < f(a+h)$

  or $f(a-h) < f(a) > f(a+h)$

Critical Point :

- f is a real valued function with domain D and $a \in D$ then $f(x)$ is said to have a critical point at x = a, if f^1(a)=0 (or) f^1(a) does not exist .

Stationary Point :

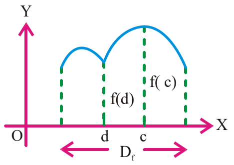
- If $f^1(a)=0$ then y = $f(x)$ is said to be stationary at $x = a$. $f(a)$ is called the stationary value of f at $x = a$ then $(a, f(a))$ is called a stationary point of f.

Maxima or Absolute maxima or global maxima or greatest value :

- Let $f(x)$ be a function with domain D then $f(x)$ is said to have maximum value at a point $a \in D$ if $f(x) \leq f(a)$ for all $x \in D$. In such a case, the point 'a' is called the point of maximum and $f(a)$ is called maximum value or the absolute maximum or global maximum or the greatest value of $f(x)$

Minimum or Absolute minimum or global minimum or least value :

✈ Let $f(x)$ be a function with domain D then $f(x)$ is said to have minimum value at a point $a \in D$ if $f(x) \geq f(a)$ for all $x \in D$. In such a case, the point 'a' is called the point of minimum and $f(a)$ is called minimum value or the absolute minimum or global minimum or the least value of $f(x)$

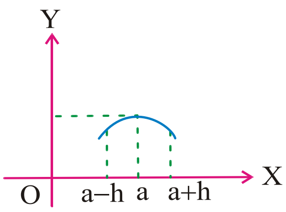


Note: absolute maxima and absolute minima values of a function, if they exits, are unique.

Local (Relative) maxima & Local (Relative) minima :

✈ **Local Maximum :**

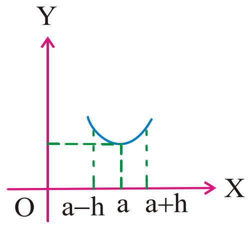
A function $y = f(x)$ is said to have a local maximum at a point x = a if $f(x) \leq f(a)$ for all $x \in (a-h,\ a+h)$ where h is a small positive quantity



The point $x = a$ is called a point of local maximum of the function $f(x)$ and $f(a)$ is local maximum.

Local minimum :

✈ The function $y = f(x)$ is said to have a local minimum at a point $x = a$ if $f(x) \geq f(a)$ for all $x \in (a-h,\ a+h)$ where h is a small positive quantity



The point $x = a$ is called a point of local minimum of the function $f(x)$ and $f(a)$ is local minimum.

Extreme points and Extreme values :

✈ If a function $f(x)$ has local maximum $f(a)$ at $x = a$ and local minimum $f(b)$ at $x = b$ then the points $x = a, x = b$ are called extreme points, and the values $f(a), f(b)$ are called extreme values.

Properties of local maximum and local minimum :

✈ i) Local maximum, local minimum of a continuous function occur alternately and between two consecutive maximum values there is a minimum value and vice versa

ii) Even a local minimum value may be greater than a local maximum value

Note : Local maximum, minimum values of a continuous function are also called turning values .

✈ Let $f(x)$ be a differentiable function on an interval I. Let $a \in I$

1). $f(x)$ has local maximum at $x = a$ if $f^1(a) = 0,\ f^{11}(a) < 0$

2) $f(x)$ has local minimum at $x = a$ if $f^1(a) = 0,\ f^{11}(a) > 0$

Note: If $f^1(a) = 0,\ f^{11}(a) = 0$ then second derivative test fails, then we use the first derivative test or the following n^{th} derivative test

n^{th} Derivative Test :

- Let $f(x)$ be a differentiable function on an interval I. Let $a \in I$ such that

  $f^{1}(a) = f^{11}(a) = f^{111}(a) = ... = f^{(n-1)}(a) = 0$ and

  $f^{(n)}(a) \neq 0$ then $f(x)$ has

  i) Local maximum at $x = a$ if n is even and $f^{(n)}(a) < 0$

  ii) Local minimum at $x = a$ if n is even and $f^{(n)}(a) > 0$

  iii) If n is odd and $f^{(n)}(a) > 0$ then f is increasing at a

  iv) If n is odd and $f^{(n)}(a) < 0$ then f is decreasing at a

  v) $f(x)$ has neither local maximum nor local minimum if n is odd

Methods to find global maximum/ minimum of continuous functions :

- Global maximum / minimum in $[a,b]$ would occur at the critical points of $f(x)$ within $[a,b]$ or at the end points of the interval.

Global maximum/minimum in $[a,b]$

- To find the global maximum / minimum of $f(x)$ in $[a,b]$. Find out all critical points of $f(x)$ in (a,b). Let $c_1, c_2, ...c_n$ be the critical points in (a,b)

  Let $M_1 = \max\left\{\begin{matrix} f(a), f(c_1), f(c_2) \\ .....f(c_n), f(b) \end{matrix}\right\}$

  Let $M_2 = \min\left\{\begin{matrix} f(a), f(c_1), f(c_2) \\ .....f(c_n), f(b) \end{matrix}\right\}$

  Now global maximum of $f(x)$ in $[a,b]$ is M_1

  Global minimum of $f(x)$ in $[a,b]$ is M_2

Global maximum/minimum in (a,b)

- To find the global maximum / minimum of $f(x)$ in (a,b). Find out all critical points of $f(x)$ in (a,b).

  Let $c_1, c_2, ...c_n$ be the critical points in (a,b)

  Let $M_1 = \max\{f(c_1), f(c_2).....f(c_n)\}$

  Let $M_2 = \min\{f(c_1), f(c_2).....f(c_n)\}$

  i)Global maximum of $f(x)$ in (a,b) is M_1 and $f(x)$ does not have global maximum in (a,b) if the limiting values at the end points are greater than M_1

  ii)Global minmum of $f(x)$ in (a,b) is M_2 and $f(x)$ does not have global minimum in (a,b) if the limiting values at the end points are less than M_2

Maximum and minimum of non-differentiable function :

- f is continous real valued function on interval I and $a \in I$ and $f^{1}(a)$ does not exist then $f(x)$ has

  i) Local maximum if $f^{1}(x)$ changes its sign at $x = a$ from +ve to -ve while moving from left to right

  ii) Local minimum if $f^{1}(x)$ changes its sign at $x = a$ from -ve to +ve while moving from left to right

Standard Results :

- The minimum value of (x-a) (x-b) is $\frac{-(a-b)^2}{4}$
- The maximum value of a $a\cos^2 x + b\sin^2 x$ is a and minimum value is b $(If\ a > b)$
- The least value of $a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x$ is $(a+b)^2$ when $x = \tan^{-1}\sqrt{\frac{b}{a}}$.
- $\sin^p \theta \cos^q \theta$ attains a maximum value at $\theta = \tan^{-1}\sqrt{\frac{p}{q}}$ and that max. value is $\left(\frac{p^p . q^q}{(p+q)^{p+q}}\right)^{1/2}$

- The minimum value of $a\sec x + b\operatorname{cosec} x$ is $(a^{2/3}+b^{2/3})^{3/2}$ at $x = \tan^{-1}\left(\frac{b}{a}\right)^{1/3}$.
- The minimum value of $\left(1+\frac{1}{\sin^n\alpha}\right)\left(1+\frac{1}{\cos^n\alpha}\right)$ is $\left(1+2^{n/2}\right)^2$
- If the sum of two positive numbers is k, then their product will be maximum when the two numbers are $\frac{k}{2},\frac{k}{2}$
- If the sum of two positive numbers is k, then sum of their squares is minimum then the numbers are $\frac{k}{2},\frac{k}{2}$
- If the product of two positive numbers is k, then their sum of the squares will be least when the two numbers are $\sqrt{k},\sqrt{k}$.
- The least value of each of $a^2\sin^2x+b^2\operatorname{cosec}^2x$, $a^2\sec^2x+b^2\cos^2x$, $a^2\tan^2x+b^2\cot^2x$ is 2ab.
- The minimum value of $a\cot x + b\tan x$ is $2\sqrt{ab}$ at $x = \tan^{-1}\sqrt{\frac{a}{b}}$.
- The maximum rectangle inscribed in a circle of radius r is a square of side $\sqrt{2}r$
- The maximum triangle inscribed in a circle of radius r is an equilateral triangle of side $\sqrt{3}r$
- The perimeter of a sector is 'K' cms. Then maximum area of sector is $\frac{K^2}{16}$ sq.cm
- The area of sector is 'k' sq.cm. Then the least perimeter of sector is $4\sqrt{k}$ cm
- When perimeter is given, the area of sector is maximum then $\theta = 2^c$.
- In a right angled triangle, the sum of a side and hypotenuse is given. If the area of the triangle is maximum, then the angle between them is 60^0.
- The least area of the triangle formed by any line through (p,q) and the co-ordinate axes is 2pq sq units
- The least value of the portion of tangent to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intercepted between the co-ordinate axes is a+b.
- A normal is drawn at a variable point P of the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ then the maximum distance of the normal from the centre of the curve is a-b.
- The minimum distance from the origin to a point on the curve $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$ is (a+b).
- The area of greatest isosceles triangle that can be inscribed in a given ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ having its vertex coincident with one extremity of major axis is $\frac{3\sqrt{3}}{4}$ ab sq units.
- The area of greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is 2ab sq units.
- From the four corners of rectangular sheet of metal of sides a,b, four equal squares are cut off and the remaining edges are folded up to form an open box. If the volume of the box is to be maximum the side of a square removed is $\frac{a+b-\sqrt{a^2+b^2-ab}}{6}$.
- From the four corners of a square sheet of metal of side 'a', four equal squares are cut off and the remaining edges are folded up to form a rectangular open box. If the volume of the box formed is to be maximum, the side of the square removed is $\frac{a}{6}$.
- A cone is drawn circumscribing a sphere of radius 'R'. If the volume of the cone is maximum, its height is $\frac{4R}{3}$ and its semivertical angle is $\sin^{-1}\frac{1}{3}$ (If surface area is constant).

Some useful formulae :

- ✈ Volume of sphere (radius r) = $\frac{4}{3}\pi r^3$

  Surface area of sphere (radius r) = $4\pi r^2$
- ✈ Volume of right circular cylinder (Base radius r and height h) = $\pi r^2 h$

  Surface area of right circular cylinder(open top) = $2\pi rh + \pi r^2$ (Base radius r and height h)

  Curved Surface area of right circular cylinder = $2\pi rh$
- ✈ Volume of right circular cone = $\frac{1}{3}\pi r^2 h$

  (Base radius r, height h and slant height l)

  Curved surface area of cone = πrl

  Total surface area of cone = $\pi r^2 + \pi rl$

  Cuboid: Volume = xyz. x, y, z are length edges

  Surface area = 2 ($xy + yz + zx$)

  Cube : Volume = x^3, surface area $= 6x^2$

EXAMPLES

1. **The least value of k for which the function $f(x) = x^2 + kx + 1$ is a increasing function in the interval $1 \le x \le 2$**

Sol: f is increasing $\Rightarrow f^1(x) \ge 0$

$$\Rightarrow 2x + k \ge 0 \Rightarrow x \ge \frac{-k}{2}$$

Since $1 \le x \le 2 \Rightarrow 1 \ge -\frac{k}{2} \Rightarrow k \ge -2$

$\therefore$ least value of k is -2

2:

The interval in which $f(x) = x^3 - 3x^2 - 9x + 20$ is strictly increasing or strictly decreasing

Sol: Given $f(x) = x^3 - 3x^2 - 9x + 20$

$\Rightarrow f^1(x) = 3x^2 - 6x - 9$

$\Rightarrow f^1(x) = 3(x-3)(x+1)$

$f^1(x) > 0 \Rightarrow x \in (-\propto, -1) \cup (3, \propto)$

$f^1(x) < 0 \Rightarrow x \in (-1, 3)$

Thus, $f(x)$ is strictly increasing for $x \in (-\propto, -1) \cup (3, \propto)$ and strictly decreasing for $x \in (-1, 3)$

3:

The complete set of values of λ for which the function $f(x) = \begin{cases} x+1, x<1 \\ \lambda, x=1 \\ x^2 - x + 3, x > 1 \end{cases}$

is strictly increasing at x = 1

Sol: $f(x)$ is strictly increasing at $x = 1$

$\Rightarrow f(1-h) < f(1) < f(1+h)$

$\Rightarrow \lim_{x \to 1^-}(x+1) < \lambda < \lim_{x \to 1^+}(x^2 - x + 3)$

$\Rightarrow 2 < \lambda < 3$

4:

The critical points of

$f(x) = (x-2)^{\frac{2}{3}}(2x+1)$ **are**

Sol : $f(x) = (x-2)^{\frac{2}{3}}(2x+1)$

$$f^1(x) = 2\left[\frac{5x-5}{(x-2)^{\frac{1}{3}}}\right]$$

$f^1(x) = 0 \Rightarrow x = 1$

$f^1(x)$ does not exist at $x = 2$

$\therefore x = 1$ *and* $x = 2$ are two critical points

5:

The number of stationary points of $f(x) = \sin x$ in $[0, 2\pi]$ are

Sol: $f(x) = \sin x \Rightarrow f^1(x) = \cos x$

$$\Rightarrow f^1(x) = 0 \Rightarrow x = (2n+1)\frac{\pi}{2}$$

Therefore number of stationary points of $f(x)$ in $[0, 2\pi]$ is 2

. 6:

If the function $f(x)=\frac{a}{x}+x^2$. has maximum at x = - 3, then the value a

Sol: $f^1(x)=-\frac{a}{x^2}+2x$ since $f(x)$ has local maximum at x = -3

$\Rightarrow f^1(-3)=0$ and $f^{11}(-3)<0$

For $f^1(-3)=0 \Rightarrow a=-54$

For $x=-3, a=-54$

Now, $f^{11}(-3)<0$, Hence, a = -54

7:

The point at which $f(x)=(x-1)^4$ assumes local maximum or local minimum values are

Sol: $f^1(x)=4(x-1)^3$

$f^{11}(x)=12(x-1)^2$

$f^{111}(x)=24(x-1)$; $f^{iv}(x)=24$

$f^1(1)=0$, $f^{11}(1)=0, f^{111}(1)=0, f^{iv}(1)\neq 0$

therfore n = iv is even and $f^{iv}(1)=24>0$

therefore $f(x)$ has local minimum at x = 1

8:

The global maximum and global minimum of $f(x)=2x^3-9x^2+12x+6$ in $[0,2]$

Sol: $f^1(x)=6(x-1)(x-2)$

$f^1(x)=0 \Rightarrow x=1, x=2$

$\Rightarrow f(0)=6, f(1)=11, f(2)=10$

therefore global maximum

$M_1=\max\{f(0), f(1), f(2)\}=11$

global minimum

$M_2=\min\{f(0), f(1), f(2)\}=6$

9:

Let $f(x)=2x^3-9x^2+12x+6$ discuss the global maximum and global minimum of $f(x)$ in $(1,3)$

Sol: $f^1(x)=6(x-1)(x-2)$

$f^1(x)=0 \Rightarrow x=1, x=2$ and $f(2)=10$

$M_1=\max\{f(2)\}$

$M_2=\max\{f(2)\}$

Now $\lim_{x\to 1^+} f(x)=11>M_1$ and

$\lim_{x\to 3^-} f(x)=15>M_1$ therefore global maximum in (1,3) does not exist and global minimum in (1,3) is 10

10:

Discuss the extremum of $f(x)=2x+3x^{2/3}$

Sol: $f(x)=2x+3x^{2/3}$

$f^1(x)=2+3\times\frac{2}{3}x^{-1/3}=2\left(1+x^{-1/3}\right)$

Let $f^1(x)=0$

$\Rightarrow x^{1/3}+1=0 \Rightarrow x=-1$

$\Rightarrow f^{11}(x)=-\frac{2}{3}x^{-\frac{4}{3}}$

and $\Rightarrow f^{11}(-1)=-\frac{2}{3}(-1)^{-4/3}=-\frac{2}{3}<0$

$\Rightarrow x=-1$ is the point of maxima

Also, $f(x)$ is non-differentiable at x = 0 but $f^1(x)$ changes its sign -ve to +ve therefore at x = 0, $f(x)$ has local minima

EXERCISE - I

1. **If $y = 8x^3 - 60x^2 + 144x + 27$ is a decreasing function in the interval (a,b), then (a,b) is**
1) (-3,2) 2) (2,3) 3) (5,6) 4) (3,2)

2. $f(x) = \frac{x}{5} + \frac{5}{x} (x \neq 0)$ **is increasing in**
1) (-5, 0) 2) (0, 5)
3) $(-\infty, -5) \cup (5, \infty)$ 4) (-5, 5)

3. **The condition that f(x) = $x^3 + ax^2 + bx + c$ is an increasing function for all real values of 'x' is**
1) $a^2 < 12b$ 2) $a^2 < 3b$ 3) $a^2 < 4b$ 4) $a^2 < 16b$

4. **The set of values 'x' for which f(x) = $x^3 - 6x^2 + 27x + 10$ is increasisng in**
1) (1, 2) 2) $(-\infty, 1) \cup (2, \infty)$
3) $(-\infty, \infty)$ 4) $(-\infty, 1)$

5. **The set of values of 'a' for which f(x) = $x^3 - ax^2 + 48x + 1$ is increasing for all real values of 'x' is**
1) (-12, 12) 2) $(-\infty, -12)$
3) $(12, \infty)$ 4) $(-\infty, \infty)$

6. $f(x) = \frac{x}{\log x} - \frac{\log 5}{5}$ **is decreasing in**
1) (e, ∞) 2) (0, 1) U (1, e)
3) (0, 1) 4) (1, e)

7. **f(x) = sin x - ax is decreasing in R if**
1) $a > 1$ 2) $a < 1$ 3) $a > \frac{1}{2}$ 4) $a < \frac{1}{2}$

8. **f(x) = Tan^{-1} (sin x) is decreasing in**
1) $\left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right)$ 2) $\left(2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right)$
3) $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ 4) $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

9. **If f(x) = sin x - cos x - ax + b decreases for a where x $\in$ R then**
1) $a < 1$ 2) $a > 1$ 3) $a < \sqrt{2}$ 4) $a > \sqrt{2}$

10. **A stationary point of** $f(x) = \sqrt{16 - x^2}$ **is**
1) (4, 0) 2) (-4, 0) 3) (0, 4) 4) (-4, 4)

11. **f(x) = $(Sin^{-1}x)^2 + (Cos^{-1}x)^2$ is stationary at**
1) $x = \frac{1}{\sqrt{2}}$ 2) $x = \frac{\pi}{4}$ 3) x = 1 4) x = 0

12. **The number of stationary points of f(x) = cos x in [0, 2 π] are**
1) 1 2) 2 3) 3 4) 4

13. The function $f(x) = x^{1/x}$ has stationary point at
1) $x = e$ 2) $x = 1$ 3) $x = \sqrt{e}$ 4) $x = 1/2$

14. **If** $-4 \le x \le 4$ **then critical points of** $f(x) = x^2 - 6|x| + 4$ **are**
1) 3,-2 2) 6,-6 3) 3,-3,0 4) 0,1,3

15. **The critical point of** $f(x) = |2x + 7|$ **at x =**
1) 0 2) 7 3) $\frac{-7}{2}$ 4) -7

16. **The maximum of** $f(x) = \frac{\log x}{x^2} (x > 0)$ **occurs at x=**
1) e 2) $\sqrt{e}$ 3) $\frac{1}{e}$ 4) $\frac{1}{\sqrt{e}}$

17. **f(x) = sin x (1 + cos x) is maximum at x =**
1) $\frac{\pi}{4}$ 2) $\frac{\pi}{6}$ 3) $\frac{\pi}{3}$ 4) $\frac{\pi}{2}$

18. **The maximum and minimum values of** $f(x) = 4x^3 + 3x^2 - 6x + 5$ **are**
1) 8, 7/2 2) 10, 13/4
3) 3, 5/7 4) 2, 8/7

19. **The minimum value of** $f(x) = x^2 + \frac{250}{x}$ **is**
1) 15 2) 25 3) 45 4) 75

20. **Minimum value of** $\frac{(6+x)(11+x)}{2+x}$ **is**
1) 5 2) 15 3) 45 4) 25

21. **The function** $f(x) = \sin^2 x \cos^3 x$ **attains a maximum when** $x =$ **[EAM -2019]**
1) $Tan^{-1}\frac{2}{3}$ 2) $Tan^{-1}\sqrt{\frac{2}{3}}$
3) $Tan^{-1}\frac{3}{2}$ 4) $Tan^{-1}\sqrt{\frac{3}{2}}$

22. $f(x) = \frac{x}{1 + x\tan x}$ **is maximum when x =**
1) sin x 2) cos x 3) tan x 4) cot x

23. **The least value of $f(x)=\frac{x^3}{3}-abx$ occurs at x =**

1)G.M of a,b 2) A.M of a,b
3)H.M of a,b 4) A.G.M of a,b

24. **The maximum value of f(x) =100-$|45-x|$ is**

1)100 2)145 3) 55 4) 45

25. **For a particle moving on a straight line it is observed that the distance 'S' at time 't' is given by S = 6t - $\frac{t^3}{2}$, the maximum velocity during the motion is**

1) 3 2) 6 3) 9 4) 12

26. **The minimum value of 64 sec θ +27cosec θ when θ lies in $\left(0,\frac{\pi}{2}\right)$ is**

1) 125 2) 625 3) 25 4) 1025

27. **The minimum value of $\frac{7}{4\sin x+3\cos x+2}$ is**

1) 1 2) $\frac{7}{9}$ 3) $\frac{7}{5}$ 4) $\frac{7}{3}$

28. **Let f(x) = $a_0 + a_1x^2 + a_2x^4 + ... + a_nx^{2n}$ when $0 < a_0 < a_1 < ...... < a_n$ then f(x) has**

1) No extremum 2) Only one maximum
3) Only one minimum 4) Two maximums

29. **The condition for $f(x)=x^3+px^2+qx+r\left(x\in R\right)$ to have no extreme value is (EAMCET 2017)**

1) $p^2<3q$ 2) $2p^2<q$

3) $p^2<\frac{1}{4}q$ 4) $p^2>3q$

30. **The smallest value of x^2 - 3x + 3 in the interval $\left[-3,\frac{3}{2}\right]$ is**

1) $\frac{3}{4}$ 2) 5 3) -15 4) -20

31. **The greatest value of $\sin^3x + \cos^3x$ in $\left[0,\frac{\pi}{2}\right]$ is**

1) 1 2) 2 3) 3 4) 4

32. **If m and M respectively denote the minimum and maximum of $f(x)=(x-1)^2+3$ for $x\in[-3,1]$, then the ordered pair (m ,M) is equal to**

1) $(-3,19)$ 2) $(3,19)$

3) $(-19,3)$ 4) $(-19,-3)$

33. **The least and the greatest values of f(x) = x^2 log x in [1, e] are [EAM -2018]**

1) log 2, log 4 2) 0, e^2

3) e^2, e^4 4) $\frac{1}{e}$, e

34. **The sum of two numbers is 6. The minimum value of the sum of their reciprocals is**

1) $\frac{3}{4}$ 2) $\frac{6}{5}$ 3) $\frac{2}{3}$ 4) $\frac{2}{5}$

35. **The sum of two +ve numbers is 20. If the sum of their squares is minimum then one of the number is**

1) 6 2) 4 3) 10 4) 8

36. **If the product of two + ve numbers is 256 then the least value of their sum is**

1) 32 2)16 3) 48 4) 40

37. **x and y are two +ve numbers suchs that xy = 1. Then the minimum value of x + y is**

1) 4 2) $\frac{1}{4}$ 3) $\frac{1}{2}$ 4) 2

38. **If A > 0, B > 0, and A + B = $\frac{\pi}{3}$ then the maximum value of tan A tan B is**

1) $\frac{1}{\sqrt{3}}$ 2) $\frac{1}{3}$ 3) 3 4) $\sqrt{3}$

39. **The minimum value of 16 cot x + 9 tan x is**

1) 12 2) 6 3) 24 4) 25

40. **The sides of a rectangle are (6 - x) cm and (x - 3) cm. If its area is maximum then x =**

1) 4 2) 4.5 3) 4.8 4) 4. 6

41. **The length of diagonal of the rectangle of maximum area having perimeter 100 cm is**

1) $10\sqrt{2}$ 2) 10 3) $25\sqrt{2}$ 4) 15°

42. **A triangle of maximum area is inscribed in a circle. If a side of the trinagle is 20$\sqrt{3}$ then the radius of the circle is**

1) 20 2)30 3) 40 4) 60

43. **The maximum height of the curvey = 6 cos x - 8 sin x above the X - axis**

1) 6 2) 8 3) 14 4) 10

KEY

01) 2	02) 3	03) 2	04) 3	05) 1	06) 2
07) 1	08) 2	09) 1	10) 3	11) 1	12) 3
13) 1	14) 3	15) 3	16) 2	17) 3	18) 2
19) 4	20) 4	21) 2	22) 2	23) 1	24) 1
25) 2	26) 1	27) 3	28) 3	29) 1	30) 1
31) 1	32) 2	33) 2	34) 3	35) 3	36) 1
37) 4	38) 2	39) 3	40) 2	41) 3	42) 1
43) 4					

SOLUTIONS

1. $f^1(x) < 0$ if $2 < x < 3$
2. On verification $f^1(x) > 0 \;\forall x \in (-\infty, -5) \cup (5, \infty)$
3. $f^1(x) = 3x^2 + 2ax + b > 0,$

   $D = 4\left[a^2 - 3b\right] < 0 \quad \Rightarrow a^2 < 3b$
4. On verification $f^1(x) > 0$, for all $x \in (-\infty, \infty)$
5. f(x) = x^3-ax^2+48x+1 is increasing in for all real x

   $f^1(x) = 3x^2 - 2ax + 48 > 0$

   $\Delta = b^2 - 4ac < 0$

   $4a^2 - 4.3.48 <$

   $a^2 - 144 < 0$

   $a \in (-12, 12)$
6. On verification $f^1(x) < 0, \;\forall x \in (0,1) \cup (1,e)$
7. For decreasing $f^1(x) < 0, \cos x < a \Rightarrow a > 1$
8. $f^1(x) = \dfrac{\cos x}{1 + \sin^2 x}$, f(x) is decreasing $f^1(x) < 0$

   $\cos x < 0 \; x \in \left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$
9. $f^1(x) < 0 \Rightarrow \cos x + \sin x < a$ But cosx + sinx $< \sqrt{2}$ from (1) and (2) $a > \sqrt{2}$
10. $f^1(x) = \dfrac{-x}{\sqrt{16 - x^2}}$ for maxima and minima

    $f^1(x) = 0 \Rightarrow x = 0$
11. $f^1(x) = 0 \Rightarrow \sin^{-1} x = \cos^{-1} x \Rightarrow x = \dfrac{1}{\sqrt{2}}$
12. $f^1(x) = 0 \sin x = 0, x = 0, \pi, 2\pi$
13. $y = x^{\frac{1}{x}}$; $\log y = \dfrac{\log x}{x}$

    $y^1 = 0$; $\dfrac{1 - \log x}{x^2} = 0$; $\log x = 1$; $x = e$
14. $f^1(x) = 2x - 6\dfrac{|x|}{x}$; $f^1(x) = 0$ at $x = 3, -3$

    $f^1(x)$ does not exist at $x = 0$
15. f(x) = |2x+7| then $f^1(x) = \dfrac{|2x+7|}{2x+7}.2 = 0$

    critical point at x = $\dfrac{-7}{2}$
16. A function $f(x)$ is said to be maximum or minimum

    $f^1(x) = 0 \Rightarrow 1 - 2\log x = 0 \Rightarrow x = e^{\frac{1}{2}}$
17. $f^1(x) = 0 \Rightarrow \cos 2x + \cos x = 0, x = \dfrac{\pi}{3}, \pi$, At $x = \dfrac{\pi}{3}, f^{11}\left(\dfrac{\pi}{3}\right) < 0 \Rightarrow f(x)$ is max. at $x = \dfrac{\pi}{3}$
18. $f^1(x) = 12x^2 + 6x - 6 = 0$

    $6(2x^2 + x - 1) = 0$; $x = \dfrac{1}{2}, x = -1$

    $f^{11}(x) = 24x + 6$

    $f^{11}\left(\dfrac{1}{2}\right) > 0, \quad f^{11}(-1) < 0$

    Maximum value is $f(-1) = 10$

    Minimum value is $f\left(\dfrac{1}{2}\right) = \dfrac{13}{4}$

19. f(x) = $x^2 + \frac{250}{x} \Rightarrow f^1(x) = 2x - \frac{250}{x^2}$

for maxima or minima $f^1(x) = 0$

$2x = \frac{250}{x^2}$

$x^3 = 125 = 5^3$, x = 5, $f^{11}(x) = 2 + \frac{500}{x^3} > 0$

minimum at x = 5

minimum value is f(5) = $5^2 \frac{250}{5} = 25 + 50 = 75$

20. $f^{1}(x) = 0$ Then put x = 4

21. Maximum at $x = Tan^{-1}\sqrt{\frac{m}{n}} = Tan^{-1}\sqrt{\frac{2}{3}}$

22. For maxima and minima

$f^1(x) = 0 \Rightarrow 1 - x^2 \sec^2 x = 0$

23. $f^1(x)$. For maxima or minima

$f^1(x) = 0 \Rightarrow x = \sqrt{ab}$, $f^{11}(x) > 0$ for $x = \sqrt{ab}$

24. f(x) = 100-|45-x| then $f^1(x) = \frac{|45-x|}{45-x}$

for maimum $f^1(x) = 0 \Rightarrow x = 45$

maximum value f(45) = 100

25. $v = 6 - \frac{3t^2}{2} = f(t)$

$v = 0 \Rightarrow t = \pm 2$

$f^1(t) = 0 \Rightarrow t = 0$, $f^{11}(0) = -3 < 0$

26. The minimum value of $a\sec\theta + b\cos ec\theta$ is

$\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

27. The minimum value of $\frac{7}{4\sin x + 3\cos x + 2} =$

$\frac{7}{\max of\ (4\sin x + 3\cos x + 2)} = \frac{7}{\sqrt{16+9}+2}$

28. $f^1(x) = 0 \Rightarrow x_1 = 0$

$f^{11}(0) = 2a_1 > 0$

29. $f^1(x) = 3x^2 + 2xp + q \neq 0\ \forall x \in R$

$f^1(x) > 0 \forall x \in R \Rightarrow b^2 - 4ac < 0 \Rightarrow p^2 < 3q$

30. $f(x) = x^2 = 3x + 3 \Rightarrow f^1(x) = 2x - 3 \Rightarrow f(x)$ is said to be maximum or minimum

$f^1(x) = 0 \Rightarrow x = \frac{3}{2}$. Minimum

$\left\{f(-3), f\left(\frac{3}{2}\right)\right\} = \left\{21, \frac{3}{4}\right\}$

31. $f(x) = \sin^3 x + \cos^3 x$

$f^1(x) = 3\sin^2 x\cos x - 3\cos^2 x \sin x = 0$

$\Rightarrow \sin x = \cos x$ $\sin x = 0$, $\cos x = 0$

$x = 0, x = \frac{\pi}{2}, x = \frac{\pi}{4}$

$Max = \max\left\{f(0), f\left(\frac{\pi}{2}\right), f\left(\frac{\pi}{4}\right)\right\} = 1$

32. $f^1(x) = 2(x-1) = 0 \Rightarrow x = 1$

$f(-3) = 19, f(1) = 3$

$\therefore m = 3$ and $M = 19$

33. $f^1(x) = 2x\log x + x = 0 \Rightarrow x = 0 (or) x = e^{\frac{-1}{2}}$

Max.=$\max\left\{f(1), f\left(e^{\frac{-1}{2}}\right), f(e)\right\} = e^2$

Min.=$\min\left\{f(1), f\left(e^{\frac{-1}{2}}\right), f(e)\right\} = 0$

34. x = y = 6/2 = 3, 1/x+1/y = 2/3

35. $x + y = 20$, let $f(x) = x^2 + (20-x)^2$, for maxima and minima $f^1(x) = 0 \Rightarrow x = 10$

36. $x = y = \sqrt{256} = 16$ Then sum = 2(16) = 32.

37. x = y = 1 Then x+y=2

38. $A = B = \frac{\pi}{6}$. Then TanA .TanB = 1/3

39. Minimum value of a cot x + b tan x is $2\sqrt{ab}$

40. $A = (6-x)(x-3) = -x^2 + 9x - 18$

Area is maximum at $x = \frac{9}{2} = 4.5$

41. $2x + 2y = 100 \Rightarrow x + y = 50$
area xy maximum if x=25 ,y=25 therefore diagonal= $25\sqrt{2}$

42. The triangle is equilateral , $h = \frac{\sqrt{3}}{2} 20\sqrt{3} = 30$

4\3. Maximum of $y = \sqrt{6^2 + 8^2} = 10$

EXERCISE - II

1. $f(x) = \sqrt{x^2 - 4}$ **is decreasing in**
1) (-2, 2) 2) (2, ∞)
3) (-∞, -2) 4) (-∞, ∞)

2. In the interval $(7, \infty), f(x) = |x-5| + 2|x-7|$ **is**
1) Increasing 2) Decreasing
3) Constant 4) Cannot be estimated

3. f(x) = 2x-Tan⁻¹x-log $\left(x + \sqrt{1+x^2}\right)(x > 0)$ **is increasing in**
1) (1, 2) 2) (0, 1) U (2, ∞)
3) (0, ∞) 4) (-∞, ∞)

4. In $\left(0, \frac{\pi}{2}\right)$**, f(x) = x sin x + cos x +** $\frac{1}{2}\cos^2 x$ **is**
1) Increasing 2) Decreasing
3) Constant
4) Nothing can be determined

5. If log (1+x) - $\frac{2x}{2+x}$ **is increasing then**
1) $-1<x<\infty$ 2) $-\infty<x<0$
3) $-\infty<x<\infty$ 4) $1<x<2$

6. If $0 < x < \frac{\pi}{2}$ **then**
1) $\frac{2}{\pi} > \frac{\sin x}{x}$ 2) $\frac{2}{\pi} < \frac{\sin x}{x}$
3) $\frac{\sin x}{x} > 1$ 4) $2 < \frac{\sin x}{x}$

7. The greatest of the numbers
$1, 2^{1/2}, 3^{1/3}, 4^{1/4}, 5^{1/5}, 6^{1/6}$, **and** $7^{1/7}$ **is**
1) $2^{1/2}$ 2) $3^{1/3}$ 3) $7^{1/7}$ 4) $4^{1/4}$

8. The function $f(x) = \cos x - 2\lambda x$ **is monotonically decreasing when**
1) $\lambda > \frac{1}{2}$ 2) $\lambda < \frac{1}{2}$ 3) $\lambda < 2$ 4) $\lambda > 2$

9. The number of ciritical point of f(x) = $\frac{|x-1|}{x^2}$ **is**
1)1 2)2 3)3 4)0

10. $f(x) = x(\log x)^2$ **then** f **is stationary at**
1) $-1, \frac{4}{e}$ 2) $1, e^{-2}$ 3) $1, 4e^2$ 4) $1, e^2$

11. The Stationary points of $8x^2 - x^4 - 4$ **are**
1) (0,-4), (2,12), (-2, 12) 2) (1,2)
3) (1,12) 4) (1,1)

12. If $f(x) = x^5 - 5x^4 + 5x^3 - 10$ **has local max. and min. at x=a and x=b respectively, then (a,b) is**
1) (0,1) 2) (1,3) 3) (1,0) 4) (3,0)

13. The minimum of $f(x) = \frac{1+x+x^2}{1-x+x^2}$ **occurs at** $x =$
1) -1 2) 1 3) 2 4) -2

14. If f(x) = a log x + bx² + x has extreme values at x = -1, x = 2 then a = ..., b = ...
1) $2, \frac{-1}{2}$ 2) $\frac{-1}{2}, 2$ 3) $\frac{1}{2}, 2$ 4) $2, \frac{1}{2}$

15. The value of "*a*" for which the sum of the squares of the roots of the equation $x^2 - (a-2)x - a - 1 = 0$ **assume the least value is**
1) 1 2) 0 3) 3 4) 2

16. The least value of "a" for which the equation $\frac{4}{\sin x} + \frac{1}{1-\sin x} = a$ **for atleast one solution of the interval** $(0, \pi/2)$ **is**
1)1 2) 4 3) 8 4) 9

17. If $2 \le x \le 4$ **then the max value of** $f(x) = (x-2)^6 (4-x)^5$ **is**
1) $\left(\frac{11}{12}\right)^5 \left(\frac{11}{10}\right)^5$ 2) $\left(\frac{2}{11}\right)^6 \left(\frac{10}{9}\right)^5$
3) $\left(\frac{12}{11}\right)^6 \left(\frac{10}{11}\right)^5$ 4) $\left(\frac{2}{11}\right)^6 \left(\frac{10}{11}\right)^5$

18. If $xy(x-y) = 2a^3 (a > 0)$ **then** y **has minimum when** $x =$

1) $\frac{1}{a}$ 2) $-a$ 3) $\frac{a}{2}$ 4) a

19. The maximum value of $\sin^2 x \cos^3 x$ is

1) $\frac{6\sqrt{3}}{25\sqrt{5}}$ 2) $\frac{9\sqrt{3}}{25\sqrt{5}}$ 3) $\frac{9\sqrt{2}}{6\sqrt{5}}$ 4) $\frac{\sqrt{2}}{\sqrt{5}}$

20. If 2x+y=5 then the maximum value of $x^2+3xy+y^2$ is

1) $\frac{125}{4}$ 2) $\frac{4}{125}$ 3) $\frac{625}{4}$ 4) $\frac{4}{625}$

21. A cubic function of x has maximum value 10 and minimum $\frac{-5}{2}$ when x = -3, x= 2 respectively then the function is [EAM -2020]

1) $\frac{1}{5}x^3+\frac{3}{10}x^2-\frac{18}{5}x+\frac{19}{10}$

2) $x^3 + 3x^2 - 18x + 19$

3) $2x^3 + 3x^2 - 36x + 10$

4) $x^3 + x^2 + x + 1$

22. f(x) = (x -1)(x - 2)(x - 3) is minimum at x =

1) $3+\frac{1}{\sqrt{2}}$ 2) $3-\frac{1}{\sqrt{2}}$

3) $2+\frac{1}{\sqrt{3}}$ 4) $2-\frac{1}{\sqrt{3}}$

23. Maximum value of $(x+5)^4 (13-x)^5$ is

1) $7^4 11^5$ 2) $6^4 14^5$ 3) $8^4 10^5$ 4) $7^5 10^5$

24. Minimum value of f(x) = $(x - 1)^2 + (x -2)^2 + .. + (x - 10)^2$ occurs at x=

1) 7 2) 6 3) 4 4) 5.5

25. A particle is moving in a straight line such that its distance at any time 't' is given by S = $\frac{t^4}{4}-2t^3+4t^2+7$ then its acceleration is minimum at t = [EAM -2015]

1) 1 2) 2 3) 1 / 2 4) 3 / 2

26. If x=-1 and x=2 are extreme points of $f(x)=\alpha\log|x|+\beta x^2+x$, then

(JEE MAIN 2014)

1) $\alpha=-6, \beta=\frac{1}{2}$ 2) $\alpha=-6, \beta=-\frac{1}{2}$

3) $\alpha=2, \beta=-\frac{1}{2}$ 4) $\alpha=2, \beta=\frac{1}{2}$

27. The largest value of $f(x)=2x^3-3x^2-12x+5$ for $-2\le x\le 4$ occurs at x =

1) -2 2) -1 3) 2 4) 4

28. The image of the interval [-1, 3] under the mapping f(x) = $4x^3$ - 12x is

1) [-2, 0] 2) [-8, 72] 3) [-8,0] 4) [-8, -2]

29. The sum of two +ve numbers is 100. If the product of the square of one number and the cube of the other is maximum then the numbers are

1) 60, 40 2) 20, 80

3) 80, 20 4) 40, 60

30. If **$x+y$ = 6543298 and $x^{11}y^5$ is maximum then the ratio of the numbers is**

1) 12 : 4 2) 3 : 13 3) 14 : 2 4) 11 : 5

31. The minimum value of

$$\frac{(A^2+A+1)(B^2+B+1)(C^2+C+1)(D^2+D+1)}{BACD},$$

where A, B, C, D are positive

1) 3^4 2) 3^{-4} 3) 2^4 4) 2^{-4}

32. The difference of two positive numbers is 10. If the square of the greater exceeds twice the square of the smaller by maximum value then they are

1) 15, 5 2) 20, 10 3) 30, 20 4) 25, 35

33. Let a,b,c,d,e,f,g,h be distinct elements in the set{-7,-5,-3,-2,2,4,6,13}. The minimum value of $(a+b+c+d)^2+(e+f+g+h)^2$ is

1) 30 2) 32 3) 34 4) 40

34. The minimum value of $(px+qy)$ when $xy=n^2$ is equal to

1) $2n\sqrt{pq}$ 2) $2pq\sqrt{n}$

3) $2\sqrt{npq}$ 4) $2pqn$

35. Maximum area of the rectangle inscribed in a circle of radius 10 cms is

1) 100 2) 200 3) 400 4) 1600

36. The perimeter of a sector is given. The area is maximum when the angle of the sector is

1) 1 radian 2) 2 radians

3) 3 radians 4) 4 radians

37. ABCD is a rectangle in which AB = 10 cms, BC = 8 cms. A point P is taken on AB such

that PA = x. Then the minimum value of PC^2+PD^2 is obtained when x =

1)10 2) 5 3) 8 4) 4

38. The maximum possible area that can be enclosed by a wire of length 20 cm by bending it into the form of a sector in sq. cms. is

1) 20 2) 25 3) 30 4) 15

39. A straight line segment through the point (3, 4) in the first quadrant meets the coordinate axes in A and B. The minimum area of ΔAOB is

1) 42 2) 64 3) 48 4) 24

40. P(3, 4), Q(-7, 6). The point A on x-axis for which PA + AQ is least is

1) (-2, 0) 2) (-1, 0) 3) (3, 0) 4)(2, 0)

41. The point on the curve $x^2 = 2y$ which is closest to the point (0, 5) is [EAM -2017]

1) $(2\sqrt{2}, 4)$ 2) (4, 8)

3) $(\sqrt{2}, 1)$ 4) (2, 2)

42. The area of the rectangle of maximum area inscribed in the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ is

1) 48 2) 41 3) 40 4) 50

43. A rod AB of length 10 cms slides between two perpendicular lines OX, OY. The maximum area of the ΔOAB

1) 50 2) 20 3) 25 4) 60

44. The least intercept made by the coordinate axes on a tangent to the ellipse $\frac{x^2}{64}+\frac{y^2}{49}=1$ is

1) 40 2) 30 3)15 4) 100

45. The longest distance of the point (*a*, 0) from the curve $2x^2 + y^2 = 2x$ is **(EAMCET-2010)**

1) $1 + a$ 2) $|1 - a|$

3) $\sqrt{1-2a+2a^2}$ 4) $\sqrt{1-2a+3a^2}$

46. The point on the parabola $y=x^2+7x+2$ which is closest to the straight line $y = 3x - 3$ is

1) (–1, –4) 2) (–2, –8)

3) (1, 10) 4) (0, 2)

47. If l, m, n are the direction cosines of a half line OP then the maximum value of l.m.n is

1) $\frac{1}{\sqrt{3}}$ 2) $\frac{1}{3\sqrt{3}}$ 3) $\frac{1}{3}$ 4) 1/4

48. The fraction exceeds its P^{th} power by the greatest number possible, where $p \geq 2$ is

1) P^P 2) $\left(\frac{1}{P}\right)^{P-1}$ 3) $P^{\frac{1}{1-P}}$ 4) $P^{\frac{1}{P}}$

49. The total cost of producing x pocket radio sets per day is Rs. $\left(\frac{1}{4}x^2+35x+25\right)$ and the price per set at which they may be sold is Rs. $\left(50-\frac{x}{2}\right)$ to obtain maximum profit the daily out put should be .... radio sets

1)10 2) 5 3)15 4) 20

50. The point on the curve $y = \frac{x}{1+x^2}$ where the tangent to the curve has the greatest slope is

1) $\left(1,\frac{1}{2}\right)$ 2) $\left(-1,\frac{-1}{2}\right)$ 3) $\left(2,\frac{2}{5}\right)$ 4) (0, 0)

KEY

01) 3	02) 1	03) 3	04) 1	05) 1	06) 2
07) 2	08) 1	09) 3	10) 2	11) 1	12) 2
13) 1	14) 1	15) 1	16) 4	17) 3	18) 2
19) 1	20) 1	21) 1	22) 3	23) 3	24) 4
25) 2	26) 3	27) 4	28) 2	29) 4	30) 4
31) 1	32) 2	33) 2	34) 1	35) 2	36) 2
37) 2	38) 2	39) 4	40) 2	41) 1	42) 3
43) 3	44) 3	45) 3	46) 2	47) 2	48) 3
49) 1	50) 4				

SOLUTIONS

1. $f^1(x) = \frac{x}{\sqrt{x^2-4}} < 0$

   If $x < 0, x \in (-\infty, -2)$

2. If $x > 7$ then $f^1(x) > 0$

3. $f^1(x) > 0 \Rightarrow (2x^2+1) - \sqrt{1+x^2} > 0$

   $\Rightarrow (2x^2+1)^2 \geq (1+x^2)$

   $\Rightarrow 4x^4 + 3x^2 > 0 \Rightarrow x^2(4x^2+3) > 0 \Rightarrow x > 0$

4. f(x) = x sin x + cosx + $\frac{1}{2}\cos^2 x$

$\Rightarrow f^1(x) = x\cos x + \sin x - \sin x - \frac{\sin 2x}{2}$

$f^1(x) = x\cos x - \frac{\sin 2x}{2} > 0$

f(x) = is in creasing in $\left(0, \frac{\pi}{2}\right)$

5. f(x) = log(1+x)-f¹(x) = $\frac{1}{1+x} - \frac{4}{(2+x)^2}$

$= \frac{4+x^2+4x-4}{(1+x)(2+x)^2} = \frac{x^2+4x}{(1+x)(2+x)^2} > 0$

$\Rightarrow x(x+4) > 0$

x < - 4 (or) x>0

$(-\infty, -4) \cup (0, \infty)$ and $(-1, \infty)$

6. f(x) = $\frac{\sin x}{x} \Rightarrow f^1(x) = \frac{x\cos x - \sin x}{x^2} < 0$

$\Rightarrow x < \frac{\pi}{2} \Rightarrow f(x) > f(\frac{\pi}{2}), \frac{\sin x}{x} > \frac{2}{\pi}$

$m^2 = \left(\frac{dy}{dx}\right)_{(2_1-2)} = \frac{-x}{y} = -1$

$\tan\alpha = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right| = \left|\frac{\frac{1}{2}+1}{1-\frac{1}{2}}\right| = \frac{\frac{3}{2}}{\frac{-1}{2}} = 3$

7. Let $f(x) = x^{1/x}$: f is decreasing if $x > e$

f is increasing if $x < e$

$e < 3 < 4 < 5 < 6 < 7$

$f(e) > f(3) > f(4) > f(5) > f(6) > f(7)$

$\therefore f(3)$ is maximum $\Rightarrow$ greatest number $=3^{1/3}$

8. On verfication $f'(x) < 0$

9. $f(x) = \frac{|x-1|}{x^2}$. f is not defined for x = 0,

$f^1(x) = 0 \Rightarrow x = 2$

f is not differentiable for x = 1.

critical points are 0, 1, 2.

10. $f^1(x) = (\log x)[\log x + 2] = 0$

$\log x = 0 \quad \log x = -2$

$x = 1 \quad x = e^{-2}$

11. $f^1(x) = 0$

x = 0, 2, -2

(0, f(0)), (2, f(2)), (-2, f(-2))

12. $f^1(x) = 0 \Rightarrow x = 0, 1, 3$

$f^{11}(x) < 0$ at x=1 ; $f^{11}(x) > 0$ at x=3

13. $f^1(x) = 0 \Rightarrow (2x+1)(1-x+x^2) - (2x-1)(1+x+x^2) = 0$

$\Rightarrow x = \pm 1 \;\&\; f^{11}(-1) > 0$

14. $f^1(-1) = 0 \Rightarrow -a - 2b + 1 = 0$

$f^1(2) = 0 \Rightarrow \frac{a}{2} + 4b + 1 = 0$

15. $\alpha + \beta = a - 2$

$\alpha\beta = -(a+1)$

$f(a) = \alpha^2 + \beta^2 = (a-2)^2 + 2(a+1)$

$f^1(a) = 0$

$a = 1$

16. $a = \frac{4}{\sin x} + \frac{1}{1-\sin x}$ is least

$\frac{da}{dx} = \left(\frac{-4}{\sin^2 x} + \frac{1}{(1-\sin x)^2}\right)\cos x = 0$

$\cos x \neq 0 \Rightarrow \sin x = 2/3$

$\frac{d^2 a}{dx^2} = 45 > 0$ for $\sin x = 2/3$

$\therefore a = \frac{4}{2/3} + \frac{1}{1-2/3} = 6 + 3 = 9$

17. $f^1(x) = (x-2)^5 (4-x)^4 (34 - 11x) = 0$

$$\Rightarrow x = \frac{34}{11}$$

$$f\left(\frac{34}{11}\right) = \left(\frac{12}{11}\right)^6 \left(\frac{10}{11}\right)^5$$

18. $\frac{dy}{dx} = \frac{-\left(2xy - y^2\right)}{\left(x^2 - 2xy\right)} = 0$

$\Rightarrow y = 2x$

$x(2x)(x-2x) = 2a$

$-x^3 = a^3$; $x = -a$

19. For $\sin^p x \cos^q x$.

The max value is $\left(\frac{p^p.q^q}{(p+q)^{p+q}}\right)^{1/2}$

20. y = 5-2x and substitute and derivative is zero and substitute

21. $f^1(-3) = 0$, $f^1(2) = 0$

f(-3) = 10 f(2) = $-\frac{5}{2}$ verify

22. $f^1(x) = 0 \Rightarrow x = 2 \pm \frac{1}{\sqrt{3}}$; $f^{11}(x) = 6x - 12$

23. $\frac{4}{x+5} = \frac{5}{13-x} = \frac{9}{18}$

$f(x)$ is max at $x = 3$

24. $10x = 55$

25. Let $S = \frac{t^4}{4} - 2t^3 + 4t^2 + 7$, Acceleration $a = \frac{d^2s}{dt^2} = 3t^2 - 12t + 8$ for a is maximum or minimum $\frac{da}{dt} = 0 \Rightarrow t = 2$.

At t=2, $\frac{d^2a}{dt^2} > 0$,

$\therefore$ a is maximum at t=2

26. $f^1(x) = \frac{\alpha}{x} + 2\beta x + 1$

$f^1(-1) = -\alpha - 2\beta + 1 = 0$

$f^1(2) = \frac{\alpha}{2} + 4\beta + 1 = 0 \Rightarrow \alpha = 2, \beta = \frac{-1}{2}$

27. $f^1(x) = 0$

x = -1, 2

Largest value at x = 4

28. $f^1(x) = 0 \Rightarrow x = \pm 1$

$f(-1) = 8, f(1) = -8, f(3) = (1)8 - 36 = 72$

29. $x + y = 100$

$x^2 y^3$ is maximum when $\frac{x}{2} = \frac{y}{3}$

30. $x^m y^n$

$x = \frac{mk}{m+n}$ and $y = \frac{nk}{m+n}$

= m : n = 11 : 51

31. Consider $\frac{A^2 + A + 1}{A} = A + 1 + \frac{1}{A}$

$\therefore \frac{A + 1 + \frac{1}{A}}{3} \geq 1 \Rightarrow \frac{A^2 + A + 1}{A} \geq 3$

Hence $\left(\frac{A^2 + A + 1}{A}\right)\left(\frac{B^2 + B + 1}{B}\right)$

$\left(\frac{C^2 + C + 1}{C}\right)\left(\frac{D^2 + D + 1}{D}\right) \geq 3^4$

32. x - y = 10

$f(x) = x^2 - 2y^2 \quad = x^2 - 2(x-10)^2$

$f^1(x) = 0$; $x = 20$

33. Sum of the elements is 8

$\therefore (a+b+c+d) + (e+f+g+h) = 8$

and $(a+b+c+d)^2 + (e+f+g+h)^2$ is minimum

$\therefore a+b+c+d = e+f+g+h = 4$

$\therefore (a+b+c+d)^2 + (e+f+g+h)^2 = 32$

34. $f^1(x) = P - \frac{qn^2}{x^2}$

$f^1(x) = 0 \Rightarrow x = \pm n\sqrt{\frac{q}{p}}$

$f^{11}(x) > 0$ for $x = n\sqrt{\frac{q}{p}}$

$\therefore$ Minimum value= $2n\sqrt{pq}$

35. $2r^2$

36. $2r + r\theta = k$; area $= \frac{1}{2}r^2\left(\frac{k-2r}{r}\right) = f(r)$

$f^1(r) = 0 \Rightarrow k - 4r = 0$

37. $PC^2 + PD^2 = 64 + (10-x)^2 + 64 + x^2 = f(x)$

38. The perimeter of a sector is C cm. The maximum area of the sector is $\frac{c^2}{16}$ square meter.

39. area = $2x_1.y_1$

40. y= 0, verify the distance

41. $\frac{dy}{dx} = x$, slope of (x,y) (0,5) is $\frac{y-5}{x}$

$x \times \frac{y-5}{x} = -1$

42. 2ab

43. $a^2 + b^2 = 100 \Rightarrow a^2 = b^2 = 50 \Rightarrow$
$a = b = 5\sqrt{2} \Rightarrow$ area = 25 Sq.units

44. (a+b)

45. $f(x) = (PA)^2 = (x-a)^2 + 2x - 2x^2$

$f^1(x) = 0$; $x = (1-a)$

$PA = \sqrt{1 - 2a + 2a^2}$

46. $f(x) = \frac{|x^2 + 4x + 5|}{\sqrt{10}}$

$f^1(x) = 0 \Rightarrow 2x + 4 = 0 \Rightarrow x = -2$

47. $f(x) = \frac{|x^2 + 4x + 5|}{\sqrt{10}}$

$f^1(x) = 0 \Rightarrow 2x + 4 = 0 \Rightarrow x = -2$

48. Let $y = x - x^p$ where x is the fraction for maxima or minima $\frac{dy}{dx} = 0 \Rightarrow x = \left(\frac{1}{p}\right)^{\frac{1}{p-1}}$. At $x = \left(\frac{1}{p}\right)^{\frac{1}{p-1}}, \frac{d^2y}{dx} < 0$ $\therefore$ y is maximum at $x = \left(\frac{1}{p}\right)^{\frac{1}{p-1}}$

49. If daily out put is x sets and p be the total point then

$p = x\left(50 - \frac{1}{2}x\right) - \left(\frac{1}{4}x^2 + 35x - 25\right),$

for maxima or minima $\frac{dp}{dx} = 0 \Rightarrow x = 10$.

At $x = 10 \frac{d^2p}{dx^2} = \frac{-3}{2} < 0$

50. $m = \frac{dy}{dx} = \frac{1-x^2}{(1+x^2)^2} \Rightarrow \frac{dm}{dx} = 0 \Rightarrow x = 0$

EXERCISE - III

1. **The function** $f(x)=\frac{ln(\pi+x)}{ln(e+x)}$ **is**

   1) Increasing on (o,∞)

   2) Decreasing on (o,∞)

   3) Increasing on $\left(o,\frac{\pi}{e}\right)$, decreasing on $\left(\frac{\pi}{e},\infty\right)$

   4) Decreasing on $\left(o,\frac{\pi}{e}\right)$, increasing on $\left(\frac{\pi}{e},\infty\right)$

2. **The least value of $(x + 100)^2 + (x + 99)^2 + ...+ (x + 1)^2 + x^2 + (x-1)^2 + (x - 2)^2 + ... +(x-100)^2$ is**

   1) 6767 2) 67670 3) 676700 4) 767600

3. **The minimum value of**

   $f(x)=2^{\left(\log_8 3\right)\cos^2 x}+3^{\left(\log_8 2\right)\sin^2 x}$ **is**

   1) $2^{1-\log_8\sqrt{3}}$ 2) $2^{\log_8\sqrt{3}}$ 3) $3^{\log_8\sqrt{2}}$ 4) $2^{1+\log_8\sqrt{3}}$

4. **The minimum value of** $\left(1+\frac{1}{\sin^n x}\right)\left(1+\frac{1}{\cos^n x}\right)$

   1) $\left(1+2^n\right)^2$ 2) $\left(1+2^{\frac{n}{2}}\right)^2$ 3) 2 4) 1

5. **Let** $f(x)=(x-3)^5(x+1)^4$ **then**

   1) x = 7/9 is a point of maxima
   2) x = 3 is a point of minimum
   3) x = –1 is a point of maxima
   4) *f* has no point of maximum or minimum

6. **If the function**

   $f(x)=x^3+3(a-7)x^2+3(a^2-9)x-1$ **has a positive point of maximum, then**

   1) $a\in(3,\infty)\cup(-\infty,-3)$

   2) $a\in(-\infty,-3)\cup\left(3,\frac{29}{7}\right)$

   3) $(-\infty,7)$ 4) $\left(-\infty,\frac{29}{7}\right)$

7. **If** $f(x)=\begin{cases}|x|, & \text{if } 0<|x|\le 2\\ 1, & \text{if } x=0\end{cases}$ **then at x=0 f has**

   1) local maximum 2) local minimum
   3) no extreme value 4) not determined

8. **A window is in the shape of a rectangle surmounted by a semi circle. If the perimeter of the window is of fixed length '*l*' then the maximum area of the window is**

   1) $\frac{l^2}{2\pi+4}$ 2) $\frac{l^2}{\pi+8}$ 3) $\frac{l^2}{2\pi+8}$ 4) $\frac{l^2}{8\pi+4}$

9. **A running track 440 ft. is to be laid out enclosing foot ball field the shape of which a rectangle with a semi circle at each end. If the area of the rectangular position is to be maximum then the dimensions of the rectangle are**

   1) 100, 70 2) 110, 70 3) 100, 80 4) 110, 60

10. **A wire of length 'a' is cut into two parts which are bent in the form of a square and a circle. The least value of the sum of the areas thus formed is**

    1) $\frac{a^2}{\pi+4}$ 2) $\frac{a^2}{2(\pi+4)}$ 3) $\frac{a^2}{3(\pi+4)}$ 4) $\frac{a^2}{4(\pi+4)}$

11. **The least perimeter of an isosceles traingle in which a circle of radius r can be inscribed is**

    1) $4\sqrt{3}r$ 2) $2\sqrt{3}r$ 3) $6\sqrt{3}r$ 4) $8\sqrt{3}r$

12. **The sum of the hypotenuse and a side of a right angled triangle is constant. If the area of the triangle is maximum then the angle between the hypotenuse and the given side is**

    1) $\frac{\pi}{2}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{3}$ 4) $\frac{\pi}{6}$

13. **The maximum distance from origin to any point on** $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ **is**

    1) a 2) a/2 3) 2a 4) $a^{2/3}$

14. **A(0, a), B(0, b) be fixed points. P(x, 0) a variable point. The angle $\angle APB$ is maximum if**

    1) $x^2 = ab$ 2) x = ba
    3) $x^2 = 2ab$ 4) $2x^2 = ab$

15. The radius of a right circular cylinder of maximum volume which can be inscribed in a sphere of radius R, is

1) R 2) $\frac{R}{2}$ 3) $\sqrt{\frac{2}{3}}R$ 4) $\sqrt{\frac{3}{2}}R$

16. The height of the cylinder of maximum curved surface area that can be inscribed in a sphere of radius 'R' is

1) $\frac{R}{3}$ 2) $\sqrt{2}R$ 3) $\sqrt{\frac{2}{3}}R$ 4) $\frac{3R}{4}$

17. The volume of the greatest cylinder which can be inscribed in a cone of height h and semi vertical angle α is

1) $\frac{4\pi h^3}{27}\tan^2\alpha$ 2) $4\pi h^2\tan^2\alpha$

3) $\frac{4\pi h^3}{9}\tan^2\alpha$ 4) $\frac{4\pi h^3}{27}\tan^3\alpha$

18. Height of the cylinder of maximum volume that can be inscribed in a sphere of radius 12 cm is

1) $8\sqrt{3}$ cm 2) 8 cm

3) $12\sqrt{3}$ cm 4) 24 cm

19. The height of the cone of maximum volume inscribed in a sphere of radius R is

1) $\frac{R}{3}$ 2) $\frac{2R}{3}$ 3) $\frac{4R}{3}$ 4) $\frac{4R}{\sqrt{3}}$

20. An open rectangular tank with a square base and 32c.c. of capacity has least surface area in sq. cms. is

1) 48 2) 16 3) 32 4) 12

21. A box is made from a piece of metal sheet 24 cms square by cutting equal small squares from each corner and turning up the edges. If the volume of the box is maximum then the dimensions of the box are

1) 16,16,4 2) 9, 9, 6 3) 8, 8, 8 4) 9, 9, 8

22. A box without lid having maximum volume is made out of square metal sheet of edge 60 cms by cutting equal square pieces from the four corners and turning up the projecting pieces to make the sides of the box. The height of the box is

1) 60 2) 10 3) 15 4) 12

23. A box is made with square base and open top. The area of the material used is 192 sq.cms. If the volume of the box is maximum, the dimensions of the box are

1) 4,4,8 2) 2, 2,4 3) 8, 8, 4 4) 2, 2, 2

24. Given: $f(x)=x^{\frac{1}{x}},(x>0)$ **has the maximum value at x=e, then**

1) $e^{\pi}>\pi^{e}$ 2) $e^{\pi}>\pi^{\pi}$ 3) $e^{\pi}=\pi^{e}$ 4) $e^{\pi}\le\pi^{e}$

25. The point on the curve $4x^2+a^2y^2=4a^2, 4<a^2<8$ **that is farthest from the point** $(0,-2)$

1) $(0,2)$ 2) $(2,0)$ 3) $(0,3)$ 4) $(0,4)$

26. For the points on the circle $x^2+y^2-2x-2y+1=0$ **the sum of maximum and minimum value of** $4x+3y$ **is**

1) 26/3 2) 10 3) 12 4) 14

KEY

01) 2	02) 3	03) 4	04) 2	05) 3	06) 2
07) 1	08) 3	09) 2	10) 4	11) 3	12) 3
13) 1	14) 1	15) 3	16) 2	17) 1	18) 1
19) 3	20) 1	21) 1	22) 2	23) 3	24) 1
25) 1	26) 4	27) 2	28) 3		

SOLUTIONS

1. $$f^1(x)=\frac{(e+x)\ln(e+x)-(\pi+x)\ln(\pi+x)}{(\pi+x)(e+x)(\ln(e+x))^2}<0$$
$on\ (0,\infty)\ (\because 1<e<\pi)$

2. first derivative zero.

3. $f(x)=2^{\log_8 3\cos^2 x}+3^{\log_8 2\sin^2 x}$

4. $f(x)=1+\frac{1}{\sin^n x}+\frac{1}{\cos^n x}+\frac{1}{\cos^n x\sin^n x}$

$f^1(x)=\frac{-n\cos x}{\sin^{n+1}x}+\frac{n\sin x}{\cos^{n+1}x}$
$-n(\sin x\cos x)^{-n-1}\left[\cos^2 x-\sin^2 x\right]$

$f^1(x)=0\Rightarrow$
$n\left[\sin^2 x-\cos^2 x+\sin^{n+2}x-\cos^{n+2}x\right]=0$

$\Rightarrow \sin x-\cos x=0$

$$\therefore f(x)_{\frac{\pi}{4}} = 1 + \frac{1}{\left(\frac{1}{2}\right)^{\frac{n}{2}}} + \frac{1}{\left(\frac{1}{2}\right)^{\frac{n}{2}}} + \frac{1}{\left(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\right)^{n}}$$

$$= \left(1 + 2^{\frac{n}{2}}\right)^2$$

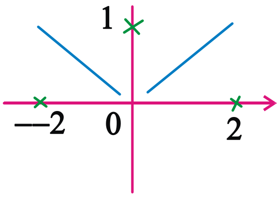
5. $f^1(x) = (x-3)^4 (x+1)^3 (7x-9)$ by first derivative test $x = -1$ is a point of maxima

6. $f^1(x) = 3x^2 + 6x(a-7) + 3(a^2-9)$

$\Rightarrow$ (Discriminant of $f^1(x) = 0) > 0, f^1(0) > 0$ and sum of the roots >0

$$\Rightarrow a < \frac{29}{7}, a < -3 \text{ or } a > 3 \text{ and } a < 7$$

$$\Rightarrow a \in (-\infty, -3) \cup \left(3, \frac{29}{7}\right)$$

7. 

Clearly at x=0 , f has local maximum

8. $2x + 2r + \pi r = l$

$$A = 2rx + \frac{1}{2}\pi r^2$$

A is max or min $\frac{dA}{dr} = 0 \Rightarrow r = \frac{l}{4+\pi}$

$\therefore$ At $r = \frac{l}{\pi + 4}$ $\quad \frac{d^2A}{dr^2} < 0$

9. $2x + 2\pi y = 440 \Rightarrow x + \pi y = 220$

Area $= x(2y) = (220 - \pi y)(2y)$

$$f(y) = 440y - 2\pi y^2$$

$$f^1(y) = 0 \Rightarrow 440 = 4\pi y \Rightarrow 110 = \frac{22}{7}y$$

$$\Rightarrow y = 35$$

10. Let x be the side of a square and r be the radius of the circle

$4x + 2\pi r = a \Rightarrow x = \frac{a - 2\pi r}{4}$ sum of areas

$$A = x^2 + \pi r^2, \frac{dA}{dr} = 0 \Rightarrow r = \frac{a}{2(\pi+4)}$$

11. $S = AF + 2BD$

$$= r(\cot\alpha + 2\tan\alpha + 2\sec\alpha)$$

$$\frac{ds}{d\alpha} = 0 \Rightarrow \alpha = \frac{\pi}{6}$$

12. $z + x = k \;;\; z + z\sin\theta = k$

$$z = \frac{k}{1+\sin\theta}$$

Area $= \frac{1}{2}xy$

$$f(\theta) = \frac{z^2}{4}\sin 2\theta \;;\quad f(\theta) = \frac{x^2}{4}\frac{\sin 2\theta}{(1+\sin\theta)^2}$$

$$f^1(\theta) = 0 \Rightarrow \theta = \frac{\pi}{6}$$

13. $f(\theta) = \sqrt{a^2(\cos^6\theta + \sin^6\theta)}$

$$= \sqrt{a^2\left(1 - \frac{3}{4}\sin^2 2\theta\right)}$$

$$f(\theta)_{\max} = a$$

14. $\cos\theta = \frac{PA^2 + PB^2 - AB^2}{2PAPB} = \frac{x^2+ab}{\sqrt{(x^2+a^2)(x^2+b^2)}}$

Applying $\frac{d\theta}{dx} = 0 \Rightarrow x^2 = ab$

15. $R^2 = r^2 + \frac{h^2}{4} \;;\quad v = \pi\left(R^2h - \frac{h^3}{4}\right)$

For maxima or minima $\frac{dv}{dh} = 0 \Rightarrow h = \frac{2R}{\sqrt{3}}$

At $h = \frac{2R}{\sqrt{3}}$, $\frac{d^2v}{dh^2} < 0$, $\therefore r = \sqrt{\frac{2}{3}}R$

16. Let r be the radius and h be the height of the culinder

$R^2 = r^2 + \frac{h^2}{4} \Rightarrow r^2 = R^2 = \frac{h^2}{4}$. Let $S = 2\pi rh$, S is maximum or minimum

$\frac{ds}{dh} = 0 \Rightarrow h = \sqrt{2}R, \frac{d^2s}{dh^2} < 0$

17. $\tan\alpha = \frac{x}{H} = \frac{r}{h}$; $v = \pi r^2 H$

$f(x) = \frac{\pi}{\tan\alpha}\left(h\tan\alpha - r^2\right)x$

$f^1(x) = 0 \Rightarrow (h\tan\alpha - r)^2 + x.2(h\tan\alpha - r) = \theta$

$x = \frac{h\tan\alpha}{3}$

18. Let r be base radius and h be the height of the cylinder

$V = \pi\left(144h - \frac{h^3}{4}\right), \frac{dv}{dh} = 0 \Rightarrow h = 8\sqrt{3}$.

At $h = 8\sqrt{3}, \frac{d^2v}{dh^2} < 0$

19. Let r be the radius and h be the height of the cone

$R^2 = (h-R)^2 + r^2 \Rightarrow r^2 = R^2 - (h-R)^2$

$v = \frac{1}{3}\pi r^2 h$

$v = \frac{1}{3}.\pi\left(R^2 - (h-R)^2\right)h \Rightarrow \frac{dv}{dh} = 0 \Rightarrow h = \frac{4R}{3}$

20. $x^2 h = 32$

$f(x) = x^2 + \frac{128}{x}$, $f^1(x) = 0 \Rightarrow x = 4$

21. $x = \frac{a}{6}$

22. $x = \frac{a}{6} = 10$

23. Verify the formula x^2+4xy

24. $f^1(x) = x^{\frac{1}{x}}$ Since $x = e$ is a point of maxima

$f(e) > f(x)$

25. first derivative zero and Verify second derivative

26. Centre and radius = (1,1) and '1'

any point on the circle is $(1+\cos\theta, 1+\sin\theta)$

maxi. value +mini. value of$(4x+3y) =$

maxi. of $(7+4\cos\theta+3\sin\theta)$

$+$ min. of $(7+4\cos\theta+3\sin\theta) = 2(7) = 14$

JEE MAINS QUESTIONS

1.Let m and M be respectively the minimum and maximumvalues of

$$\begin{vmatrix} \cos^2 x & 1+\sin^2 x & \sin 2x \\ 1+\cos^2 x & \sin^2 x & \sin 2x \\ \cos^2 x & \sin^2 x & 1+\sin 2x \end{vmatrix}$$

Then the ordered pair (m, M) is equal to :

(1) (– 3, 3) (2) (– 3, – 1)
(3) (– 4, – 1) (4) (1, 3)

2.

Let AD and BC be two vertical poles at A and B respectively on a horizontal ground. If AD = 8 m, BC = 11 m and AB = 10 m; then the distance (in meters) of a point M on AB from the point A such that $MD^2 + MC^2$ is minimum is _____.

3.

The set of all real values of λ for which the function $f(x) = (1-\cos^2 x)\cdot(\lambda + \sin x)$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, has exactly one maxima and exactly minima, is:

1) $\left(-\frac{1}{2}, \frac{1}{2}\right) - \{0\}$ 2) $\left(-\frac{3}{2}, \frac{3}{2}\right)$

3) $\left(-\frac{1}{2}, \frac{1}{2}\right)$ 4) $\left(-\frac{3}{2}, \frac{3}{2}\right) - \{0\}$

4.The area (in sq. units) of the largest rectangle ABCD whosevertices A and B lie on the x-axis and vertices C and D lieon the parabola $y = x^2 - 1$ below the x-axis, is :

1) $\frac{2}{3\sqrt{3}}$ 2) $\frac{1}{3\sqrt{3}}$

3) $\frac{4}{3}$ 4) $\frac{4}{3\sqrt{3}}$

5.Let f(x) be a polynomial of degree 3 such that f(–1) = 10,f(1)=–6, f(x) has a critical point at x = –1 and f ¢(x) has acritical point at x = 1. Then f(x) has a local minima at x = ________.

6.Let f(x) be a polynomial of degree 5 such that x = ±1 are itscritical points. If = 4, then which one of the following isnot true ?

1) f is an odd function.
2) f(l) – 4f(–l) = 4.
3) x = 1 is a point of maxima and x = –1 is a point ofminima of f.
4) x = 1 is a point of minima and x = –1 is a point of maxima of f

KEY

1) 2 2) 5 3) 4

4) 4 5) 3 6) 4

SOLUTIONS

1.

$C_1 \to C_1 + C_2$

Let $f(x) = \begin{vmatrix} 2 & 1+\sin^2 x & \sin 2x \\ 2 & \sin^2 x & \sin 2x \\ 1 & \sin^2 x & 1+\sin 2x \end{vmatrix}$

$R_1 \to R_1 - 2R_3;\ R_2 \to R_2 - 2R_3$

$= \begin{vmatrix} 0 & \cos^2\theta & -(2+\sin 2x) \\ 0 & -\sin^2 x & -(2+\sin 2x) \\ 1 & \sin^2 x & 1+\sin 2x \end{vmatrix} = -2 - 2\sin 2x$

$f'(x) = -2\cos 2x = 0$

$\Rightarrow \cos 2x = 0 \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}$

$f''(x) = 4\sin 2x$

So, $f''\left(\frac{\pi}{4}\right) = 4 > 0$ (minima)

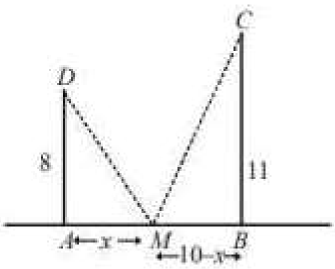
$m = f\left(\frac{\pi}{4}\right) = -2 - 1 = -3$

$f''\left(\frac{3\pi}{4}\right) = -4 < 0$ (maxima)

$M = f\left(\frac{3\pi}{4}\right) = -2 + 1 = -1$

So, $(m, M) = (-3, -1)$

2.



Let $AM = x$ m

$\therefore (MD)^2 + (MC)^2 = 64 + x^2 + 121 + (10-x)^2 = f(x)$ (say)

$f'(x) = 2x - 2(10-x) = 0$

$\Rightarrow 4x = 20 \Rightarrow x = 5$

$f''(x) = 2 - 2(-1) > 0$

$\therefore f(x)$ is minimum at $x = 5$ m.

3.

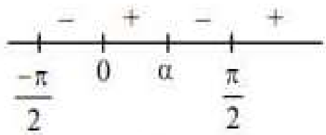
$f(x) = (1-\cos^2 x)(\lambda + \sin x) = \sin^2 x(\lambda + \sin x)$

$\Rightarrow f(x) = \lambda\sin^2 x + \sin^3 x$ ...(i)

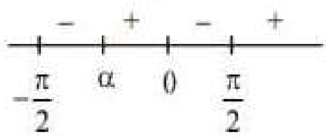
$\Rightarrow f'(x) = \sin x\cos x[2\lambda + 3\sin x] = 0$

$\Rightarrow \sin x = 0$ and $\sin x = -\frac{2\lambda}{3} \Rightarrow x = \alpha$ (let)

So, $f(x)$ will change its sign at $x = 0, \alpha$ because there is exactly one maxima and one minima in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$



OR



Now, $\sin x = -\frac{2\lambda}{3}$

$\Rightarrow -1 \le -\frac{2\lambda}{3} \le 1 \Rightarrow -\frac{3}{2} \le \lambda \le \frac{3}{2} - \{0\}$

$\because$ If $\lambda = 0 \Rightarrow f(x) = \sin^3 x$ (from (i))

Which is monotonic, then no maxima/minima

So, $\lambda \in \left(-\frac{3}{2}, \frac{3}{2}\right) - \{0\}$

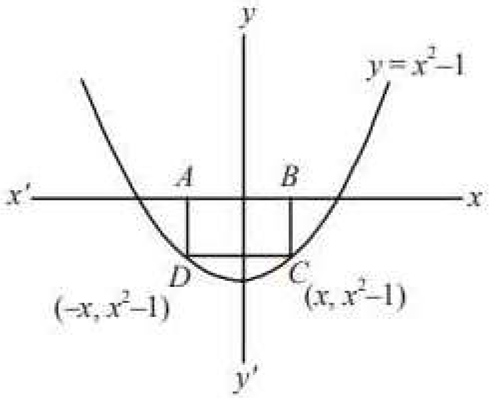
4. Area of rectangle $ABCD$

$$A = 2x \cdot (x^2 - 1) = 2x^3 - 2x$$

$$\therefore \quad \frac{dA}{dx} = 6x^2 - 2$$

For maximum area $\frac{dA}{dx} = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$

$$\frac{d^2A}{dx^2} = (12x) \Rightarrow \left(\frac{d^2A}{dx^2}\right)_{x=\frac{-1}{\sqrt{3}}} = \frac{-12}{\sqrt{3}} < 0$$



$$\therefore \text{ Maximum area } = \left|\frac{2}{3\sqrt{3}} - \frac{2}{\sqrt{3}}\right| = \frac{4}{3\sqrt{3}}$$

5. Let $f(x) = ax^3 + bx^2 + cx + d$

$f(-1) = 10$ and $f(1) = -6$

$-a + b - c + d = 10$ ...(i)

$a + b + c + d = -6$ ...(ii)

Solving equations (i) and (ii), we get

$$a = \frac{1}{4}, d = \frac{35}{4}$$

$$b = \frac{-3}{4}, c = -\frac{9}{4}$$

$\Rightarrow \quad f(x) = a(x^3 - 3x^2 - 9x) + d$

$$f'(x) = \frac{3}{4}(x^2 - 2x - 3) = 0$$

$\Rightarrow \quad x = 3, -1$

Local minima exist at $x = 3$

6. $f(x) = ax^5 + bx^4 + cx^3$

$$\lim_{x \to 0}\left(2 + \frac{ax^5 + bx^4 + cx^3}{x^3}\right) = 4$$

$\Rightarrow \quad 2 + c = 4 \quad \Rightarrow \quad c = 2$

$f'(x) = 5ax^4 + 4bx^3 + 6x^2$

$= x^2(5ax^2 + 4bx + 6)$

Since, $x = \pm 1$ are the critical points,

$\therefore \quad f'(1) = 0 \quad \Rightarrow \quad 5a + 4b + 6 = 0$

$f'(-1) = 0 \quad \Rightarrow \quad 5a - 4b + 6 = 0$

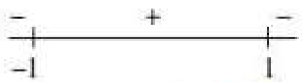
From eqns. (i) and (ii),

$$b = 0 \text{ and } a = -\frac{6}{5}$$

$$f(x) = \frac{-6}{5}x^5 + 2x^3$$

$f'(x) = -6x^4 + 6x^2 = 6x^2(-x^2 + 1)$

$= -6x^2(x + 1)(x - 1)$



$\therefore \quad f(x)$ has minima at $x = -1$ and maxima at $x = 1$

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APPLICATION OF DERIVATIVES

ADVANCED LEVEL QUESTIONS

SINGLE ANSWER TYPE QEUESTIONS

1. **The number of values of k for which the equation $x^3 - 3x + k = 0$ has two distinct roots lyingin the interval (0 , 1) is**

A) three B) two

C) infinitely many D) zero

2.Let the function $g:(-\infty,\infty) \to \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ be given by g(u) = 2 $\tan^{-1}(e^u) - \frac{\pi}{2}$. Then, g is

[IIT 2008]

A) even and is strictly increasing in (0, ¥)

B) odd and is strictly decreasing in (-¥, ¥)

C) odd and is strictly increasing in (-¥, ¥)

D) neither even nor odd, but is strictly increasing in $(-\infty,\infty)$

3.Let f, g and h be real valued functions defined on the interval $[0,1]$ by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = x.e^{x^2} + e^{-x^2}$, $h(x) = x^2.e^{x^2} + e^{-x^2}$. If a, b and c denotes respectively, the absolute maximum of f, g and h on $[0,1]$ then

[IIT 2010]

A) $a = b \, and \, c \neq b$ B) $a = c \, and \, a \neq b$

C) $a \neq b \, and \, c \neq b$ D) $a = b = c$

4. **Let** $f(x) = \begin{cases} |x|, & \text{for } 0 < |x| \leq 2 \\ 1, & \text{for } x = 0 \end{cases}$ **then at x = 0, f has** **[ADV 2020]**

A) a local maximum B) no local maximum

C) a local minimum D) no extremum

5 . **If a, b, c, d are real numbers such that $\frac{3a+2b}{c+d} + \frac{3}{2} = 0$, Then the equation $ax^3 + bx^2 + cx + d = 0$ has.**

A) at least one root in [–2, 0]

B) at least one root in [0, 2]

C) at least two roots in [–2, 2]

D) No root in [–2, 2]

6 . **The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point (1, 1) and the coordinate axes, lies in the first quadrant. If its area is 2, then the value of b is[IIT 2001]**

A) –1 B) 3 C) –3 D) 1

7. **If a variable tangent to the curve $x^2y = c^3$ makes intercepts a, b on x and y axis respectively, then the value of a^2b is**

A) 27 c^3 B) $\frac{4}{27}c^3$ C) $\frac{27}{4}c^3$ D) $\frac{4}{9}c^3$

KEY

1. D	2.C	3. D	4.A
5. B	6.C	7.D	

SOLUTIONS

1. Let there be a value of k for which $x^3 - 3x + k = 0$ has two distinct roots between 0 and 1 .

Let a , be two distinct roots of $x^3 - 3x + k = 0$ lying between 0 and 1 such that a < b . Let $f(x) = x^3 - 3x + k$. Then f (a) = f (b) =0. Since between any two roots of a polynomial f (x) , there exists at least one root of its drvative f '(x). Therefore , $f'(x) = 3x^2 - 3$ has at least one root between a and b . But $f'(x) = 0$ has two roots equal to ± 1 which do not lie between a, b.

Hence : f (x) = 0 has no real roots lying between 0 and 1 for any value of k.

2. $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$

$g'(u) = \frac{2e^u}{1+e^{2u}} > 0$

Hence g(u) is increasing function

$g(u) = \tan^{-1}(e^u) - \cot^{-1}(e^u)$

$g(-u) = \tan^{-1}(e^{-u}) - \cot^{-1}(e^{-u})$

$= \tan^{-1}\left(\frac{1}{e^u}\right) - \cot\left(\frac{1}{e^u}\right) = \cot^{-1}\left(e^u\right) - \tan\left(e^u\right)$

= -g(u)

Hence g(u) is odd strictly increasing in (-¥,¥)

3. $f^1(x) = 2xe^{x^2} - 2x.e^{-x^2} = 2x\left[e^{x^2} - e^{-x^2}\right] \geq 0 \ \forall x \in [0,1]$

$g^1(x) = e^{x^2} + 2x^2.e^{x^2} - 2x.e^{-x^2} > 0 \forall x \in [0,1]$

$h^1(x) > 0 \ \forall x \in [0,1]$ hence f, g, h are increasing functions in $[0,1]$

Maximum of $f = f(1)$,

and that of $g(x)$ and $h(x)$ and $g(1)$ and $h(1)$

Hence $f(1) = g(1) = h(1) = e + \frac{1}{e} \Rightarrow a = b = c$

4. $f(0) > f(0 + h)$

$f(0) > f(0 - h)$

hence it is local maximum.

5. $f'(x) = ax^3 + bx^2 + cx + d$

$f(x) = \frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx + e$

given $6a + 4b + 3c + 3d = 0$,

$f(+2) = e = f(0)$ use rolle's theorem.

6. tangent to $y = x^2 + bx - b$ at (1, 1) is

$x - \text{int ercept} = \frac{b+1}{b+2}$

and y-intercept = $-(b + 1)$

ATQ $Ar(\Delta) = 2$

$\Rightarrow \frac{1}{2}\left(\frac{b+1}{b+2}\right)[-(b+1)] = 2$

= b –3

7. $.x^2y = c^3$

$x^2\frac{dy}{dx} + 2xy = 0 \Rightarrow \frac{dy}{dx} = -\frac{2y}{x}$

equation of tangent at (x,y)

$Y - y = -\frac{2y}{x}(X - x)$

Y = 0, gives , $X = \frac{3x}{2} = a$

and X = 0 , gives , Y = 3y = b

Now $a^2b = \frac{9x^2}{4}.3y$

$= \frac{27}{4}x^2y = \frac{27}{4}c^3 \Rightarrow$ (C)

MULTIPULE ANSWER TYPE QUESTIONS

1. **If the line ax + by + c = 0 is a normal to the rectangular hyperbola xy = 1, then**
A) a > 0, b > 0 B) a > 0, b < 0
C) a < 0, b > 0 D) a < 0, b < 0

2. **Let**
$f(x)=2\sin^3 x-3\sin^2 x+12\sin x+5,\ 0\le x\le \pi/2$
Then f(x) is
A) decreasing in $[0, \pi/2]$
B) increasing in $[0, \pi/2]$
C) increasing in $[0, \pi/4]$ and decreasing in $[\pi/4, \pi/2]$
D) none of these

3. **Let** $f(x)=\dfrac{x^2+1}{[x]}, 1\le x\le 3.9.$ $[.]$ **denotes the greatest integer function. Then**
A) f(x) is monotonically decreasing in [1, 3.9]
B) f(x) is monotonically increasing in [1, 3.9]
C) the greatest value of f(x) is $\frac{1}{3}\times 16.21$
D) the least value of f(x) is 2.

4. **Let h(x) = f(x) – $(f(x))^2$ + $(f(x))^3$ for every real number x. Then [IIT 1998]**
A) h is increasing whenever f is increasing
B) h is increasing whenever f is decreasing
C) h is decreasing whenever f is decreasing
D) nothing can be said in general

5. **Let the parabolas y = x^2 + ax + b and y = x(c - x) touch each other at a point (1, 0). Then**
A) a = – 3 B) b = 1
C) c = 2 D) b + c = 3

6. **Let f(x) = x^3 + ax^2 + bx + 5 $\sin^2 x$ be an** increasing function in the set of real numbers R. Then a and b satisfy the condition
A) $a^2 - 3b - 15 \le 0$ B) $a^2 - 3b + 15 \ge 0$
C) $a^2 - 3b + 15 \le 0$ D) a > 0 and b > 0

7. **If** $f(x)=\sin x, -\pi/2\le x\le \pi/2$ **, then**
A) f(x) is increasing in the interval $[-\pi/2, \pi/2]$
B) f{f(x)} is increasing in the interval $[-\pi/2, \pi/2]$
C) f{f(x)} is decreasing in $[-\pi/2, 0]$ and increasing in $[0, \pi/2]$
D) f{f(x)} is invertible in $[-\pi/2, \pi/2]$

8. **The function**
$f(x)=\int_{-1}^{x} t(e^t-1)(t-1)(t-2)^3(t-3)^5\,dt$ **has a local minimum at x =**
A) 0 B) 1 C) 2 D) 3

9. **The critical point(s) of** $f(x)=\dfrac{|2-x|}{x^2}$ **is/are**
A) x = 0 B) x = 2 C) x = 4 D) x=1

10. **The value of x for which the function** $f(x)=\int_0^x (1-t^2)e^{-t^2/2}dt$ **has an extremum is**
A) 0 B) 1 C) -1 (D) 2

11. **A tangent to the curve** $y=\int_0^x |t|dt$**, which is parallel to the line y = x, cuts off an intercept from the y-axis equals to**
A) 1 B) –1/2 C) 1/2 D) –1

12. **The number of values of x where the function f(x) = cos x + cos $(\sqrt{2}x)$ attains its maximum is [AD 2019]**
A) 0 B) 1 C) 2 D) infinite

13. **f(x) is cubic polynomial with f(2) = 18 and f(1) = –1. Also f(x) has local maxima at x = –1 and f '(x) has local minima at x = 0, then [IIT - 2018]**
A) the distance between (–1, 2), and (a,f(a)), where x = a is the point of local minima is $2\sqrt{5}$
B) f(x) is increasing for $x\in[1, 2\sqrt{5}]$
C) f(x) has local minima at x = 1
D) the value of f(0) = 15

14. **If** $f(x)=\int_0^x e^{t^2}(t-2)(t-3)dt$ **for all** $x\in(0,\infty)$**, then**

A) f has a local maximum at x=2

B) f is decreasing on $(2,3)$

C) there exists some $c \in (0,\infty)$ such that $f''(c)=0$

D) f has a local minimum at x=3

KEY

01) B,C	02) B	03) C ,D
04) A,C	05) A ,D	06) A,C
07) A , B, D	08) B,D	09) B , C
10) B , C	11) B ,C	12) B
13) B,C	14) A,B,C,D	

SOLUTIONS

1. Differentiating w.r.t. x, $y + x\frac{dy}{dx} = 0$

$\therefore$ the equation of the normal at (α,β) is

$y-\beta = \frac{\alpha}{\beta}(x-\alpha)$ or $\alpha x - \beta y = \alpha^2 - \beta^2$

The given line is a normal at (α,β) if

$\frac{\alpha}{a} = -\frac{\beta}{b} = \frac{\alpha^2-\beta}{-c}$

$\Rightarrow \frac{\alpha}{a} = \frac{\beta}{-b} = \frac{\sqrt{\alpha\beta}}{\sqrt{-ab}} = \frac{1}{\sqrt{-ab}}$ $(\because \alpha\beta = 1)$

$\therefore$ a, b are real if ab < 0 i.e., a > 0, b < 0 or a < 0, b > 0.

2. $f'(x)$ = 6 sin² x cosx - 6sinx cosx + 12 cosx

= 6cosx {sin²x - sinx + 2}

$= 6\cos x\left\{\left(\sin x - \frac{1}{2}\right)^2 + \frac{7}{4}\right\}$

$\therefore$ in $\left[0,\frac{\pi}{2}\right]$, $f'(x) \geq 0$. So, f(X) is increasing in $\left[0,\frac{\pi}{2}\right]$

3. Here, $f(x) = x^2 + 1$, $1 \leq x < 2$

$\frac{x^2+1}{2}, 2 \leq x < 3$

$\frac{x^2+1}{3}, 3 \leq x \leq 3.9$

$f'(x) > 0$ in each of the intervals and so f(x) is increasing in each of the intervals.

$\therefore$ $2 \leq f(x) < 5$ in $1 \leq x < 2$; $\frac{5}{2} \leq f(x) < 5$ in $2 \leq x < 3$

$\frac{10}{3} \leq f(x) \leq \frac{1}{3} \times 16.21$ in $3 \leq x \leq 3.9$

Hence the least value is 2 and the greatest value is $\frac{1}{3} \times 16.21$

4. $h'(x) = 3f'(x)[\{f(x) - 1/3\}^2 + 2/9]$

Note that

$h'(x) < 0$ whenever $f'(x) < 0$ and $h'(x) > 0$ whenever $f'(x) > 0$, thus, h(x) increases (decreases) whenever f(x) increases (decreases).

5. (1, 0) is on both the curves.

So, 0 = 1 + a + b and 0 = c - 1

For the first parabola, $\frac{dy}{dx} = 2x + a$

$\therefore \frac{dy}{dx})_{1,0} = 2 + a$

For the second parabola, $\frac{dy}{dx} = c - 2x$

$\therefore \frac{dy}{dx})_{1,0} = c - 2$

$\therefore$ 2 + a = c - 2 and 0 = c - 1

$\Rightarrow$ c = 1, a = - 3 $\therefore$ 0 = 1 + (- 3) + b or b = 2

6. $f'(x)$ = 3x² + 2ax + b + 5sin 2x

f(x) increases always, so $f'(x) > 0 \ \forall \ x \in R$

$\Rightarrow 3x^2 + 2ax + b + 5\sin 2x > 0$

which will be true if 3x² + 2ax + b - 5 > 0, always if D < 0

7. We know that sinx is an increasing funciton of x in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$f\{f(x)\} = \sin(\sin x)$;

$\therefore \frac{d}{dx}\{f(f(x)\} = \cos(\sin x) \cdot \cos x \geq 0$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$\therefore$ f{f(x)} is increasing $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

8. $\frac{dy}{dx} = f'(x) \Rightarrow x(e^x - 1)(x-1)(x-2)^3(x-3)^5 = 0$

   Critical points are 0, 1, 2, 3. Consider change of sign of $\frac{dy}{dx}$ at $x = 3$

   $x < 3, \frac{dy}{dx} = -ve$ and $x > 3, \frac{dy}{dx} = +ve$

   Change is from –ve to +ve,
   Hence minimum at x = 3.
   Again minimum and maximum occur alternately.
   $\therefore$ 2nd minimum is at x = 1.

9. Obviously, at x = 0, f(x) = ∞

   $\therefore f'(0)$ does not exist.
   So, x = 0 is a critical point

   Now, $f(x) = \frac{2-x}{x^2}, 0 < x < 2, \frac{x-2}{x^2}, x \geq 2$

   At x = 2, 4 the function f(x) is not differentiable. So, they are critical points.

10. $f'(x) = (1-x^2)e^{-x^2/2}$

    For extremum, $(1-x^2)e^{-x^2/2} = 0$, ie x= 1, -1.

11. Differentiating w.r.t. x, $\frac{dy}{dx} = |x| = 1$ because the slope of y = x is 1

    $\therefore$ at (α, β), $\left.\frac{dy}{dx}\right)_{\alpha,\beta} = 1 = |\alpha|$ $\therefore$ a = 1, -1

    $\therefore$ when $\alpha = 1, \beta = \int_0^1 |t| dt = \int_0^1 t dt = \frac{1}{2}$ and

    when $\alpha = -1, \beta = \int_0^{-1} |t| dt = -\int_{-1}^{0} |t| dt = -\int_{-1}^{0} t dt = -\frac{1}{2}$

    $\therefore$ the points where the tangents are parallel to the line y = x are $\left(1, \frac{1}{2}\right)\left(-1, -\frac{1}{2}\right)$

    The tangent at $\left(1, \frac{1}{2}\right)$ is $y - \frac{1}{2} = 1(x - 1)$,

    i.e. 2x - 2y = 1

    The tangent at $\left(-1, -\frac{1}{2}\right)$ is $y + \frac{1}{2} = 1(x + 1)$, i.e. 2x - 2y + 1 = 0

12. The maximum value of f(x) = cos x + cos $(\sqrt{2}x)$ is 2 which occurs at x = 0.

    Also, there is no value of x for which this value will be attained again.

13. $f(2) = 18 \Rightarrow 8a + 4b + 2c + d = 18$ .....(i)

    $f(1) = -1 \Rightarrow a + b + c + d = -1$ ....(ii)

    f(x) has local max. at x = –1

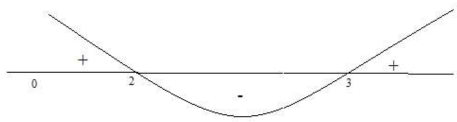
    $\Rightarrow 3a - 2b + c = 0$ .....(iii)

    $f'(x)$ has local min. at x = 0 $\Rightarrow$ b = 0.....(iv)

    Solving (i), (ii), (iii) and (iv), we get result.

    $\Rightarrow f(x) = \frac{1}{4}(19x^3 - 57x + 34)$

14. $f'(x) = e^{x^2}(x-2)(x-3)$



MATRIXMATCHING TYPE QUESTIONS

1. **Let f(x) = $(2^x - 1)(2^x - 2)$ and g(x) = 2 sin x + cos 2x in [0, π]**

Column-I	**Column-II**
A) f increases on	P) $(\log_2(3/2), \infty)$
B) f decreases on	Q) $(-\infty, \log_2(3/2))$
C) g decreases on	R) $\left(0, \frac{\pi}{6}\right)$
D) g increases on	S) $\left(\frac{5\pi}{6}, \pi\right)$

2. **Column - I gives functions which satisfy conditions of CMVT an specified interval and Column - II gives value of 'C' for which LMVT is satisfied**

Column - I	**Column - II**
A) $f(x) = x(x-2)$ **in** $[1,2]$	P) $\frac{5}{6}$
B) $f(x) = x(2-x)$ in $[0,1]$	Q) $\frac{1}{3}$
C) $f(x) = x^3 - 2x^2 - x + 3$ in $[0,1]$	R) 3
D) $f(x) = (x-1)(x-2)(x-3)$ in $[1,4]$	S) $\frac{7}{6}$
	T) A rational number

3. **Match the following**

Column - I	**Column - II**
A) $f(x) = \sin x + \cos x + 2x$ strictly increases on	P) $(3, \infty)$
B) $f(x) = \frac{x^2 + x + 1}{x^2 - x + 1}$ strictly increases on	Q) $(1, \infty)$
C) $f(x) = \frac{x^2 + x + 1}{x^2 - x + 1}$ strictly decreases on	R) $(-\infty, -1)$
D) $f(x) = \frac{x^3}{x^4 + 27}$ strictly decreases on	S) $(-1, 1)$
	T) $(-\infty, \infty)$

4. **Match the max / min value of functionin Col - I with corresponding values in Col - II**

Column - I	**Column - II**
A) Greatest value of $f(x) = \frac{x}{4 + x + x^2}$ on $[0, \infty)$ is	P) $\frac{18}{e}$
B) Maximum value of $\frac{\ln x}{x}$ in $[2, \infty)$ is	Q) $\frac{1}{e}$
C) Let $x > 0, y > 0$ & $xy = 1$ then minimum value of $\frac{3}{e^3}x + 27ey$ is	R) e
D) Perimeter of a sector is $4e$. The area of sector is maximum when its radius is	S) $\frac{1}{5}$
	T) An irrational number

KEY

01) $A \to P, B \to Q, C \to S, D \to R$
02) A-S,T , B-P, T, C-Q, T, D-R, T
03) A- P,Q,R,S,T , B- S , C- Q,R,P , D- P
04) A- S, B- Q, T, C- P, T, D- R, T

SOLUTIONS

1. A) $f''(x)>0\forall x\in R\Rightarrow f'(x)$ is an increasing function.

Now $g'(x)=-f'(4-x)+f'(2+x)$

If $g'(x)>0\Rightarrow f'(2+x)>f'(4-x)\Rightarrow 2+x>4-x$ or $x>1$

B) $f'(x)=3(x-1)(x+1)\Rightarrow f'(x)=0$
has roots x=-1, 1
f(x) =0 will have exactly one real root if f(-1) f(1) >0

$\Rightarrow(a+2)(a-2)>0\Rightarrow a<-2$ or $a>2$

C) $f'(x)=-\sin x+a^2\geq 0\,\forall x\in R$

$\Rightarrow a^2\geq\sin x\forall x\in R$

D) $f'(x)=2e^x+ae^{-x}+2a+1=2e^{-x}(e^x+a)\left(e^x+\frac{1}{2}\right)$

f(x) increases for all x if $f'(x)\geq 0\forall x\in R$

$\therefore e^x+a\geq 0\ \forall\ x\in R\Rightarrow a\geq 0$

2. $f'(c)=\frac{f(b)-f(a)}{b-a}$ for LMVT

$f'(c)=\frac{0+1}{3}$ for 'a'

$\Rightarrow 2c-2=\frac{1}{3}\Rightarrow 2c=\frac{7}{3}\Rightarrow c=\frac{7}{6}$

$f'(c)=\frac{1}{3}$ for 'b'

$\Rightarrow 2-2c=\frac{1}{3}\Rightarrow 2c=\frac{5}{3}\Rightarrow c=\frac{5}{6}$

$f'(c)=\frac{1-3}{+1}\Rightarrow 3c^2-4c-1=-2$

$3c^2-4c+1=0\Rightarrow c=1$

$3c(c-1)-1(c-1)=0$

$c=\frac{1}{3}\qquad\because c\in[0,1]$

$f'(c)=\frac{6}{3}\qquad\Rightarrow 3c^2-12c+11=2$

$\Rightarrow c^2-4c+3=0,\qquad c=3\ \because c\in[1,4]$

3. A. $f(x)=\sin x+\cos x+2x$

$f'(x)=\cos x-\sin x+2>0$

$\therefore$ Strictly increases on $(-\infty,\infty)$

B. $f(x)=\frac{x^2+x+1}{x^2-x+1}$

$$f'(x)=\frac{(2x+1)(x^2-x+1)-(x^2+x+1)(2x-1)}{(x^2-x+1)^2}$$

$$=\frac{-4x^2+2x^2+2}{(x^2-x+1)^2}=\frac{-2x^2+2}{(x^2-x+1)^2}>0$$

$\Rightarrow x\in(-1,1)$

C. $f(x)=\frac{x^2+x+1}{x^2-x+1}$ striclty decreases on $(-\infty,-1)\cup(1,\infty)$

D. $f(x)=\frac{x^3}{x^4+27}$

$$f'(x)=\frac{(x^4+27)(3x^2)-x^3(4x^3)}{(x^4+27)^2}$$

$$=\frac{81x^2-x^6}{(x^4+27)^2}\Rightarrow x^2(81-x^4)>0\Rightarrow|x|<3$$

$\Rightarrow x\in(3,\infty)\cup(-\infty,-3)$

4. A. $f(x)=\frac{x}{4+x+x^2}$ on $[0,\infty)$

$$f'(x)=\frac{(4+x+x^2)-x(2x+1)}{(4+x+x^2)^2}=0$$

$x^2+x+4-2x^2-x=0$

$4 = x^2, x = \pm 2$

if $x = 2 \Rightarrow \frac{2}{4+2+4} = \frac{2}{10} = 1/5$

$x = -2 \Rightarrow \frac{-2}{6} = -1/3$

B. $f'(x) = \frac{1-\ln x}{x^2} > 0$

$\ln x < 1 \quad x < e$

$\therefore$ Maximum value of $\frac{\ln x}{x} = \frac{1}{e}$

C. $y = 1/x \Rightarrow \frac{3}{e^3}x + \frac{27e}{x}$

$f'(x) = \frac{3}{e^3} - \frac{27e}{x^2} = 0$, $x \neq 0$

$x = 3e^2$

$f''(x) = +\frac{54e}{x^3} \Rightarrow \frac{3}{e^3} \times 3e^2 + 27 \times e \times \frac{1}{3e^2}$

$= \frac{9}{e} + \frac{9}{e} = \frac{18}{e}$

D. $R\theta + 2R = 4e$

Area $= \frac{1}{2}LR.\frac{1}{2}R^2\theta$

$\theta = \frac{4R-2R}{R} \Rightarrow \frac{1}{2}\left(4eR - 2R^2\right)$

$= 2eR - R^2 = f(R)$

$f'(R) = 2e - 2R = 0 \quad R = e$

INTEGER TYPE QUESTIONS

1. **If the chord joining the points where x = p , x = - p on the curve $y = ax^2 + bx + c$ is parallel to the tangent drawn to the curve at (α, β) then α is**
2. **Area of the triangle formed by the tangent , normal at (1,1) on the curve $\sqrt{x} + \sqrt{y} = 2$ and the y - axis is**
3. **If the curves $ax^2 + by^2 = 1$ and $a_1x^2 + b_1y^2 = 1$ may cut each other orthogonally such that $\frac{1}{a} - \frac{1}{a_1} = \lambda\left(\frac{1}{b} - \frac{1}{b_1}\right)$ then λ is equal to**
4. **The number of non - zero integral values 'a' for which the function $f(x) = x^4 + ax^3 + \frac{3x^2}{2} + 1$ is concave upward along the entire real line is**
5. **Let C be a curve defined by $y = e^{a+bx^2}$. The curve C passes trhoug the point $P(1,1)$ and the slope of the curve tan gent at P is -2. Then the value of $2a - 3b$ is**
6. **At a point $p\left(a, a^n\right)$ on the graph of $y = x^n$ in the first quadrant a normal is drawn. The normal intersets the line y-axis at the point $(0,b)$. If $\lim_{a \to 0} b = \frac{1}{2}$ then $n =$**
7. **Equation of the normal to the curve $y = (1+x)^y + \sin^{-1}(\sin^2 x)$ at x = 0 is x + y = k, then k is**
8. **If the function f(x) = x^3 + $e^{x/2}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is**

(2009)

8. The number of distinct real roots of $x^4-4x^3+12x^2+x-1=0$ is (2011)

10. Let p(x) be a real polynomial of least degree which has a local maximum at x=1 and a local minimum at x=3. If $p(1)=6$ and $p(3)=2$, then $p'(0)$ is

KEY

01) 0 02) 1 03) 1 04) 3 05) 5 06) 2
07) 1 08)2 09) 2 10) 9

SOLUTIONS

1. Points are

$\left(p, ap^2+bp+c\right), \left(-p, ap^2-bp+c\right)$

slope of the line joining the point $= \frac{2bp}{2p} = b$

$\frac{dy}{dx} = 2ax+b$

$\left(\frac{dy}{dx}\right)_{(\alpha,\beta)} = 2a\alpha+b$

$2a\alpha+b=b \Rightarrow \alpha=0$

2. Find equation of tangent and normal & then put x = 0 to evaluate vertices of triangle. Then find area of triangle.
3. Solve the curves simultaneously and apply

$\left.\frac{dy}{dx}\right|_{1st} \times \left.\frac{dy}{dx}\right|_{2nd} = -1$

4. Soluiton : **3**

$f''(x)=12x^2+6ax+3>0 \forall x\in R$

$\Rightarrow 36a^2-144<0 \Rightarrow a\in(-2,2)$

$\Rightarrow$ Number of non - zero integral values of 'a' is 3

5. $y=e^{a+bx^2}$

$1=e^{a+b}$ ($\because$ it passes through 1, 1)

$a+b=0$

$\left(\frac{dy}{dx}\right)_{(1,1)} = -2$

$e^{a+bx^2}.2bx=-2$; $e^{a+b}.2b(1)=-2$

$b=-1, a=1 \quad 2a-3b=5$

6. $y=x^n$

$\frac{dy}{dx}=nx^{n-1}=na^{n-1}$

slope of normal $= -\frac{1}{na^{n-1}}$

equation of normal

$= y-a^n = -\frac{1}{na^{n-1}}(x-1)$

Put $x=0$ to get y-intercept

$y=a^n+\frac{1}{na^{n-2}}$

$\therefore b=a^n+\frac{1}{na^{n-2}}$

$$\lim_{a\to 0} b = \begin{bmatrix} 0 & if \;\; n<2 \\ \frac{1}{2} & if \;\; n=2 \\ \infty & if \;\; n>2 \end{bmatrix}$$

7. At x = 0, y = 1

Evaluate $\left.\frac{dy}{dx}\right|_{at\, x=0\,\&\, y=1}$

Find equation of tangent at x = 0 and y = 1.

8. $f(0)=1,\ f'(x)=3x^2+\frac{1}{2}e^{x/2} \Rightarrow$ f'(g(x)) g'(x)=1

put x = 0 $\Rightarrow$ g'(1) = $\frac{1}{f'(0)}=2$

9. $f(x)=x^4-4x^3+12x^2+x-1$

clearly $f(0)=-1<0 \Rightarrow$ at least two real roots

$f'(x)=4x\left(x^2-3x+7\right)$ dont have all real roots

$\therefore$ f(x)=0 has only two real roots.

10. $p'(x)=3k\{(x-1)(x-3)\}$

$=3k\{x^2-4x+3\}$

$p(x) = k\left\{x^3 - 6x^2 + 9x\right\} + c$

$p(1) = 6 \Rightarrow 4k + c = 6$, $p(3) = 2 \Rightarrow c = 2$

$\Rightarrow k = 1 \quad \therefore p'(0) = 9k = 9$

****

SEQUENCE & SERIES

SYNOPSIS

Sequence :

A set of numbers is arranged in a definite order according to some definite rule is called a sequence. e.g. 2, 4, 6, 8, ..., is a sequence

- A sequence is a function whose domain is a set of natural numbers. If the range of a sequence is a subset of real numbers (or complex numbers), then it is called a real sequence (or complex sequence)

Series :

The sum of the terms of a sequence is called a series.

- If $a_1, a_2, a_3, ...$ is a sequence, then the expression $a_1 + a_2 + a_3 + ...$ is a series
- A series is called finite series, if it has finite number of terms. Otherwise it is called infinite series.

e.g .i) 1+3+5+ ....+21 is a finite series.

ii) 2+4+6+8+....... is an infinite series.

- Sequences following specific patterns are called progressions.

Arithmetic progression (A.P) :-

- A sequence is called an arithmetic progression, if the difference between any two consecutive terms is the same.
- A.P is of the form $a, a+d, a+2d, a+3d$ ..... where a is 1[st] term and d is common difference

General term of an A.P :

- Let 'a' be the first term and 'd' be common difference of an A.P, then its genaral term (or) n$^{\text{th}}$ term is $T_n = a + (n-1)d$
- If 'l' be the last term and 'd' be common difference of an A.P, then m^{th} term from the end $T_m' = l-(m-1)d$
- m^{th} term from the end = $(n-m+1)^{\text{th}}$ term from the beginning.

Sum to n terms of an A.P :

$$S_n = \frac{n}{2}[a+l] = \frac{n}{2}[2a+(n-1)d]$$

where $a =$ first term, $l =$ last term

$d =$ common difference

- If the sum of n terms of a sequence S_n is given, then its n^{th} term T_n can be determined by $T_n = S_n - S_{n-1}$

Properties of A.P :-

- a, b, c are in AP $\Leftrightarrow 2b = a + c$
- In a finite A.P, the sum of the terms equidistant from the begining and the end is always same and is equal to the sum of the first and last term

  i.e., $a_2 + a_{n-1} = a_3 + a_{n-2} = a_4 + a_{n-3} = a_1 + a_n$

- $(a_1 + a_2 + a_3 + ... + a_n)$

$$= \begin{cases} n \times (\text{middle term}), \text{ if } n \text{ is odd} \\ \frac{n}{2} \times (\text{sum of two middle terms}), \text{ if } n \text{ is even} \end{cases}$$

- If $a_1, a_2, a_3 ....a_n$ are in A.P then

a) $a_n, a_{n-1}, ...a_3, a_2, a_1$ are in A.P

b) $a_1 \pm \lambda,\ a_2 \pm \lambda,\ a_3 \pm \lambda;$ .... $a_n \pm \lambda$ are in A.P (where $\lambda \in R$)

c) $\lambda a_1,\ \lambda a_2,\ \lambda a_3; ....\lambda a_n$ are in A.P (where $\lambda \in R - \{0\}$)

- p^{th} term of an A.P. is 'q' and q^{th} term is 'p', then $T_{p+q} = 0$
- If m^{th} term of an A.P. is 'n' and n^{th} term is 'm' then p^{th} term is 'm+n-p'

→ If $S_p = q$ and $S_q = p$ for an A.P., then $S_{p+q} = -(p+q)$

Selection of terms in an A.P :

Number of terms	Terms	c.d
3	$a-d,\ a,\ a+d$	d
4	$a-3d, a-d, a+d, a+3d$	$2d$
5	$a-2d, a-d, a, a+d, a+2d$	d
6	$a-5d, a-3d, a-d, a+d, a+3d, a+5d$	$2d$

Some Facts about A.P :-

→ If $a_1, a_2, a_3........a_n$ and $b_1, b_2, b_3........b_n$ are two A.P's then

a) $a_1 \pm b_1,\ a_2 \pm b_2,.........$ are in AP

b) $a_1b_1,\ a_2b_2, a_3b_3.........$ and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}..........$ are not in A.P

c) If the terms of an A.P. are chosen at regular intervals, then they form an A.P

→ If a constant 'k' is added to each term of A.P., with common difference 'd', then the resulting sequence also will be in A.P., with common difference (d+k).

→ If every term is multiplied by a constant 'k', then the resulting sequence will also be in A.P., with the first term 'ka' and common difference 'kd'.

→ If n th term of the sequence $T_n = An + B$ (i.e) [Linear expression in n] then the sequence is A.P with first term is 'A+B' and common difference $A(\textit{coefficient of } n)$

→ If sum of n terms of a sequence is $S_n = An^2 + Bn + C$ (i.e.Quadratic exprssion in n) then the sequence is A.P with first term is 3A+B and common difference is 2A. Also in this sequence n^{th} term T_n = 2An + (A+B)

→ If the ratio of the sums of n terms of two A.P.'s is given then the ratio of their n^{th} terms may be obtained by replacing n with $(2n-1)$ in the given ratio.

→ If the ratio of n^{th} terms of two A.P.'s is given, then the ratio of the sums of their n terms may be obtained by replacing n with $\frac{n+1}{2}$ in the given ratio

→ Sum of the interior angles of a polygon of 'n' sides is $(n-2)180^0$

→ The n^{th} common term of two Arithmetic Series is (L.C.M of common difference of 1st series and 2nd series)(n-1)+ 1st common term of both series

Arithmetic mean (A.M) :

The Arithmetic mean A of any two numbers a and b is given by $\frac{a+b}{2}$, where a, A, b are in AP

→ If $a_1, a_2, a_3......a_n$ are n numbers then Arithmetic mean A of these numbers is given by $A = \frac{1}{n}[a_1 + a_2 + ... + a_n]$

→ The n numbers $A_1, A_2, A_3.......A_n$ are said to be Arithmetic means between a and b if $a, A_1, A_2, A_3.........A_n, b$ are in AP

Here $a =$ First term

$b = (n+2)th$ term =a+(n+1)d

then, $d = \frac{b-a}{n+1}$

$$A_1 = a + \frac{b-a}{n+1}$$

$$A_2 = a + \frac{2(b-a)}{n+1}, ...$$

$$A_n = a + \frac{n(b-a)}{n+1}$$

$$A_1 + A_2 + A_3 + ... + A_n = n\left[\frac{a+b}{2}\right]$$

Geometric Progression (G.P):- A Sequence is called a Geometric progression, if the ratio of any two consecutive terms is the same

- G.P is of the form $a,\ ar,\ ar^2,\ ar^3........,$ Where a is the first term and r is the common ratio

Genaral term of G.P:- If 'a' be the first term and 'r' be the common ratio, then general term (or) n^{th} term of G..P is $T_n = ar^{n-1}$

- The n^{th} term from the end of a finite G.P consisting of m terms = ar^{m-n}
- The n^{th} term from the end of a finite G.P with last term l and common ratio r is $l\left(\frac{l}{r}\right)^{n-1}$

Sum to n terms of a G.P :

- a) sum of n terms

$$S_n = a + ar + ar^2 + ... + ar^{n-1} = a\left(\frac{1-r^n}{1-r}\right), \text{ if } r<1$$

$$= a\left(\frac{r^n-1}{r-1}\right), \text{ if } r>1 = \text{na.}, \quad \text{if r} =1$$

b) If l be the last term of the G.P., then l=ar^{n-1},

$$S_n = \frac{a-lr}{1-r}, \text{ if } \quad r<1 \qquad = \frac{lr-a}{r-1}, \text{ if } \quad r>1$$

- If the number of terms are infinite, then the sum of G..P. is

$$S_\infty = a + ar + ar^2 + ... = \frac{a}{1-r} \text{ if } |r|<1$$

Selection of terms in G.P :

No.of	Terms	Common terms ratio
3	$\frac{a}{r}, a, ar$	r
4	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$	r^2
5	$\frac{a}{r^2}\ \frac{a}{r}\ a, ar, ar^2$	r
6	$\frac{a}{r^5}\ \frac{a}{r^3}\ \frac{a}{r}, ar, ar^3, ar^5$	r^2

Properties of G.P :-

- $a,\ b,\ c$ are in G..P $\Leftrightarrow b^2 = ac$
- In a finite G.P, the product of the terms equidistant from the begining and end is always same and is equal to the product of the first and last terms

  (i.e) $a_2 a_{n-1} = a_3 a_{n-2} = a_4 a_{n-3} .... = a_1.a_n$

- $a_1.a_2.a_3.......a_n$ =(middle term)^n ,if n is odd

  =(Product of two middle terms)$^{n/2}$, if n is even

- If $a_1, a_2, a_3, ...a_n$ are in G.P

a) $a_n, a_{n-1}, a_{n-2}, ...a_1$ are in G.P

b) $\lambda a_1, \lambda a_2, \lambda a_3, ..., \lambda a_n$ are in G.P $(\lambda \in R - \{0\})$

c) $a_1^n, a_2^n, a_3^n, ...a_n^n$ are in G.P for $n \in R$

d) $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3} .... \frac{1}{a_n}$ are in G.P

- If $a_1, a_2, a_3.....a_n$ is a G.P of non zero, non negative terms then $\log a_1,\ \log a_2, \log a_3.......\log a_n$ are in A.P and vice versa

Some facts about G.P :-

- If $a_1, a_2, a_3, ...a_n$ and $b_1, b_2, b_3, ...b_n$ are two G.P's with common ratio r_1 and r_2 respetively, then

a) $a_1 \pm b_1,\ a_2 \pm b_2,\ a_3 \pm b_3.........a_n \pm b_n$ are not in G.P

b) $a_1b_1,\ a_2b_2,\ a_3b_3,.........a_nb_n$ are in G..P with common ratio r_1r_2

c) $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3} .... \frac{a_n}{b_n}$ are in G.P with common ratio $\frac{r_1}{r_2}$

Increasing and decreasing G.P:-

- Let a, ar, ar^2, ... be G..P

a) If $a>0; r>1$ then it is an increasing G..P

b) If $a>0;\ 0<r<1$ then it is decreasing G..P

c) If $a < 0; r > 1$ then it is decreasing G.P

d) If $a < 0; 0 < r < 1$ then it is an increasing G.P

Geometric mean (G.M):- The geometric mean G of any two numbers 'a' and 'b' is given by $\sqrt{ab}$ where a, G, b are in G.P

✈ If $a_1, a_2, a_3, ...a_n$ be n numbers then geometric mean of these numbers is $(a_1.a_2.a_3.....a_n)^{\frac{1}{n}}$

✈ The n numbers $G_1, G_2, G_3, ...G_n$ are said to be geometric means between 'a' and 'b'. If $a, G_1, G_2, G_3, ..., G_n$, b are in G.P

Here $a =$ First term ; $b = (n+2)th$ term

then $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$; $G_1 = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}$;

$G_2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}$ ....; $G_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$

$G_1 G_2 G_3.........G_n = \left(\sqrt{ab}\right)^n = (GM\ of\ a, b)^n$

✈ If 'a' and 'b' are two numbers of opposite signs, the G.M. between them does not exist.

Arithmetico - Geometric progression (A.G.P): A sequence is called an arithmerico-geometric progression, if each term is the product of the corresponding terms of an A.P. and a G.P.,

✈ If a, a+d, a+2d, a+3d,.......is an A.P and b, br, br^2,.......is in G.P. then ab, (a+d)br, (a+2d) br^2, ... is an A .G .P

✈ The general form of an A.G..P is a, (a+d)r,(a+2d) r^2, ...

Genaral term of A.G.P :

Genaral term of an A .G .P is

$T_n = \left[a + (n-1)d\right].r^{n-1}$ where a = first term, d = common difference and r = common ratio.

Sum to n term of an A .G .P :

$$S_n = \begin{cases} \dfrac{a}{1-r} + dr\dfrac{(1-r^{n-1})}{(1-r)^2} - \dfrac{\left[a+(n-1)d\right]r^n}{1-r} (r \neq 1) \\ \dfrac{n}{2}\left[2a+(n-1)d\right] \quad (when\ r = 1) \end{cases}$$

✈ If the number of terms are infinite, then the sum of A.G..P is

$$S_\infty = \frac{a}{(1-r)} + \frac{dr}{(1-r)^2} \quad (when |r| < 1)$$

Eg. 21

Find the nth term of arithmetico- geometric series $1 - 3x + 5x^2 - 7x^3 + ...$

Sol: The given arithmetico-geometric series is $1 - 3x + 5x^2 - 7x^3 + ...$. The A.P. corresponding to this series is 1,3,5,7, ... and the G.P. corresponding to this series is $1, (-x), (-x)^2, (-x)^3, ...$

clearly, the nth term of the A.P.={1+(n-1)(2)}=2n-1

and the nth term of

G.P= $\left\{1(-x)^{n-1}\right\} = (-1)^{n-1}.x^{n-1}$

$\therefore$ the nth term of the given series

$= (2n-1)(-1)^{n-1}.x^{n-1} = (-1)^{n-1}(2n-1).x^{n-1}$

To Find nth term by Difference Method :

If $T_1, T_2, ..., T_n$ are terms of any series and their difference $(T_2 - T_1), (T_3 - T_2), (T_4 - T_3), ...,$ $(T_n - T_{n-1})$ are either in A.P. or in G..P., then T_n and S_n of series may be found by the method of differences. Let $S_n = T_1 + T_2 + ... + T_n$

again $S_n = T_1 + T_2 + ... + T_{n-1} + T_n$

$S_n - S_n = T_1 + (T_2 - T_1) + (T_3 - T_2) + ... + (T_n - T_{n-1}) - T_n$

$T_n = T_1 + (T_2 - T_1) + (T_3 - T_2) + ... + (T_n - T_{n-1})$

$T_n = T_1 + t_1 + t_2 + ... + t_{n-1}$

where $t_1, t_2, ..., t_{n-1}$ are terms of the new series.

Harmonic Progression (H.P): A sequence is in H.P, if the reciprocals of its terms form an A.P.

✈ In general H.P is of the form

$$\frac{1}{a},\frac{1}{a+d},\frac{1}{a+2d}........\frac{1}{a+(n-1)d}$$

where a = first term, d=common difference in A.P.

Properties of H.P :

✈ a,b,c are in $H.P \Leftrightarrow \frac{2}{b}=\frac{1}{a}+\frac{1}{c}$

✈ If $a_1,a_2,a_3,..........a_n$ are in H.P then

(a) $a_n,a_{n-1},..........a_3,a_2,a_1$ are in H.P

(b) $\lambda a_1,\lambda a_2,\lambda a_3,..........\lambda a_3$ are in H.P $(\lambda \in R)$

(c) $\frac{a_1}{\lambda},\frac{a_2}{\lambda},\frac{a_3}{\lambda}........\frac{a_n}{\lambda}$ are in H.P where $\lambda \neq 0$

(d) If a, b are the first two terms of an H.P, then the n^{th} term$=\frac{ab}{b+(n-1)(a-b)}$

(e) If m^{th} term of H.P.is 'n' and n^{th} term of H.P is 'm', then $T_r=\frac{mn}{r}$

✈ If $a_1,a_2,a_3.........a_n$ be n numbers then H.M of these numbers is $H=\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+...+\frac{1}{a_n}}$

$$\Rightarrow \frac{1}{H}=\frac{1}{n}\left[\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+........\frac{1}{a_n}\right]$$

✈ The n numbers $H_1,H_2,H_3........H_n$ are said to be harmonic means between a and b if $a,H_1,H_2,H_3...........H_n,b$ are in HP.

Here a = first term ; b = $(n+2)th$ term

If D is common difference of AP

then $D=\frac{a-b}{(n+1)ab}$; $\frac{1}{H_1}=\frac{1}{a}+\frac{a-b}{(n+1)ab}$;

$$\frac{1}{H_2}=\frac{1}{a}+\frac{2(a-b)}{(n+1)ab}........;\frac{1}{H_n}=\frac{1}{a}+\frac{n(a-b)}{(n+1)ab}$$

$$\frac{1}{H_1}+\frac{1}{H_2}+\frac{1}{H_3}+.........+\frac{1}{H_n}=\frac{n}{2}\left[\frac{1}{a}+\frac{1}{b}\right]$$

✈ If $x_1,x_2,x_3,....x_n$ are n-H.M's between a and b, then $x_1=\frac{ab(n+1)}{b(n+1)+(a-b)}$,

$$x_2=\frac{ab(n+1)}{b(n+1)+2(a-b)},..\ x_n=\frac{ab(n+1)}{b(n+1)+n(a-b)}$$

Relations between A.M, G.M, H.M:- Let A,G,H be A.M, G.M and H.M between two numbers a and b then

✈ $A=\frac{a+b}{2}$; $G=\sqrt{ab}$; $H=\frac{2ab}{a+b}$

✈ $A \geq G \geq H$

✈ A,G,H are in GP (i.e) $G^2=AH$

✈ The equation having a and b as its roots is $x^2-2Ax+G^2=0$

✈ If A,G,H are A.M, G.M, H.M between three numbers a,b,c then the equation having a,b,c as its roots is $x^3-3Ax^2+\frac{3G^3}{H}x-G^3=0$

✈ $\frac{a^n+b^n}{a^{n-1}+b^{n-1}}$ is the A.M, G.M & H.M between a and b for $n=1,\frac{1}{2},0$ respectively

✈ If A and G be the A.M. and G..M between two positive numbers, then the numbers are $A \pm \sqrt{A^2-G^2}$

✈ If the A.M. and G.M. between two numbers are in the ratio $m:n$, then the numbers are in the ratio $m+\sqrt{m^2-n^2}:m-\sqrt{m^2-n^2}$

Summation of some series of natural numbers:

- $\Sigma 1 = \sum_{k=1}^{n} 1 = 1+1+...........+1(n\ terms) = n$
- $\sum_{k=1}^{n} k = 1+2+3+...+n = \frac{1}{2}n(n+1)$
- $\sum_{k=1}^{n} k^2 = 1^2+2^2+...+n^2 = \frac{1}{6}n(n+1)(2n+1)$
- $\sum_{k=1}^{n} k^3 = 1^3+2^3+...+n^3 = \left(\sum_{k=1}^{n} k\right)^2 = (1+2+3+...+n)^2$

$$= \left[\frac{1}{2}n(n+1)\right]^2 = \frac{1}{4}n^2(n+1)^2$$

- $1+3+5+....n \text{ terms} = \sum_{k=1}^{n}(2k-1) = n^2$
- $2+4+6+....n \text{ terms} = \sum_{k=1}^{n}(2k) = n(n+1)$
- $1^2+3^2+5^2+.....+n\ terms = \sum_{k=1}^{n}(2k-1)^2$

$$= \frac{n}{3}(4n^2-1)$$

- $2^2+4^2+6^2+.....+n\ terms = \sum_{k=1}^{n}(2k)^2$

$$= \frac{2}{3}n(n+1)(2n+1)$$

- $1^3+3^3+5^3+.....+n\ terms = \sum_{k=1}^{n}(2k-1)^3$

$$= n^2(2n^2-1)$$

- Sum of n terms of series

$$1^2-2^2+3^2-4^2+5^2-...... = \begin{cases} \frac{n(n+1)}{2}, if\ n\ is\ odd \\ \frac{-n(n+1)}{2}, if\ n\ is\ even \end{cases}$$

Note: If $|x|<1$ then

1. $(1-x)^{-1} = 1+x+x^2+x^3+......$
2. $(1-x)^{-2} = 1+2x+3x^2+4x^3+......$
3. $(1-x)^{-3} = 1+3x+6x^2+10x^3+......$
4. $(1+x)^{-1} = 1-x+x^2-x^3+......$
5. $(1+x)^{-2} = 1-2x+3x^2-4x^3+......$
6. $(1+x)^{-3} = 1-3x+6x^2-10x^3+......$

General rule for finding the values of recurring decimal : Let X denote the figure which do not recur and assume they are l in number. Let Y denote recurring period of consisting of m figures. Let R denote the value of recurring decimal then $R = XYYY....$ (or)

$R = X\dot{Y}$

$\therefore 10^l R = X.YYY$ and $10^{l+m} R = XY.YYY$

$\therefore$ Subtracting we get $R = \frac{XY - X}{10^{l+m} - 10^l}$

E.g: $0.6\dot{2}\dot{3} = \frac{623-6}{990} = \frac{617}{990}$

E.g: $1.2\dot{4}\dot{3} = 1+\frac{243-2}{990} = 1+\frac{241}{990} = \frac{1231}{990}$

Sum of the products of two terms of a sequence :

To obtain the sum $\sum_{i<j} a_i a_j$, we use the identity

$$2\sum_{i<j} a_i a_j = (a_1+a_2+...+a_n)^2 - (a_1^2+a_2^2+...+a_n^2)$$

Cauchy-Schwartz's In equality If $a_1, a_2, a_3, ...a_n$ and $b_1, b_2, ...b_n$ are 2n real numbers, then $(a_1b_1+a_2b_2+...+a_nb_n)^2 \le (a_1^2+a_2^2+...+a_n^2)$ $(b_1^2+b_2^2+...+b_n^2)$ with the equality holding if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} ... = \frac{a_n}{b_n}$

Eg. 1

Find the first negative term of the sequence

$20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, ...$

Sol: The given sequence is an A.P in which first term a=20 and common difference d=-3/4.

Let the n[th] term of the given A.P. be the first negative term. Then, $a_n < 0$

$\Rightarrow a + (n-1)d < 0 \Rightarrow 20 + (n-1)(-3/4) < 0$

$\Rightarrow \frac{83}{4} - \frac{3n}{4} < 0 \Rightarrow 83 - 3n < 0 \Rightarrow 3n > 83$

$\Rightarrow n > 27\frac{2}{3} \quad \Rightarrow n = 28$

thus, 28th term of the given sequence is the first negative term.

Eg. 2

If 100 times the 100th term of an A.P with non-zero common difference equals the 50 times of 50th term, then find 150th term of this A.P. (AIEEE 2012)

Sol: $100T_{100} = 50T_{50}$ **;** 100(a+99d)=50(a+49d)

2a+198d=a+49d ; a+149d=0

$T_{150} = a + 149d = 0$

Eg. 3

How many terms are to be added to make the sum 52 in the series (-8)+(-6)+ (-4)+....?

Sol: $S_n = 52 \Rightarrow \frac{n}{2}\left[2(-8) + (n-1)2\right] = 52$

$\Rightarrow n(2n-18) = 104$

$\Rightarrow n(n-9) = 52 \Rightarrow n = 13$

Eg. 4

Let $a_1, a_2, ..., a_n$ **be the terms of an A.P.**

If $\frac{a_1 + a_2 + ... + a_p}{a_1 + a_2 + ... + a_q} = \frac{p^2}{q^2}$**,** $p \neq q$ **then find** $\frac{a_6}{a_{12}}$**.**

Sol:

$$\frac{\frac{p}{2}\left[2a_1 + (p-1)d\right]}{\frac{q}{2}\left[2a_1 + (q-1)d\right]} = \frac{p^2}{q^2} \Rightarrow \frac{\left[2a_1 + (p-1)d\right]}{\left[2a_1 + (q-1)d\right]} = \frac{p}{q}$$

$$\Rightarrow \frac{a_1 + \left(\frac{p-1}{2}\right)d}{a_1 + \left(\frac{q-1}{2}\right)d} = \frac{p}{q} \; For \; \frac{a_6}{a_{21}}, p = 11, q = 41$$

$$\Rightarrow \frac{a_6}{a_{21}} = \frac{11}{41}$$

Eg. 5

If $1, \frac{1}{2}\log_3\left(3^{1-x} + 2\right), \log_3\left(4.3^x - 1\right)$ **are in A.P, then find x.**

Sol: $1, \frac{1}{2}\log_3\left(3^{1-x} + 2\right), \log_3\left(4.3^x - 1\right)$ are in A.P

$\Rightarrow \log_3\left(3^{1-x} + 2\right) = 1 + \log_3\left(4.3^x - 1\right)$

$\Rightarrow \log_3\left(3^{1-x} + 2\right) = \log_3 3 + \log_3\left(4.3^x - 1\right)$

$\Rightarrow \log_3\left(3^{1-x} + 2\right) = \log_3 3\left(4.3^x - 1\right)$

$\Rightarrow \left(3^{1-x} + 2\right) = 3\left(4.3^x - 1\right)$

$\Rightarrow 3.3^{-x} + 2 = 12.3^x - 3$

$\Rightarrow \frac{3}{t} + 2 = 12t - 3, \left(where \;\; t = 3^x\right)$

$\Rightarrow 3 + 2t = 12t^2 - 3t \Rightarrow 12t^2 - 5t - 3 = 0$

$\Rightarrow (4t-3)(3t+1) = 0$

$\Rightarrow t = \frac{3}{4}, \frac{-1}{3} \Rightarrow 3^x = \frac{3}{4}\left(\because 3^x > 0\right)$

$\Rightarrow x = \log_3\left(\frac{3}{4}\right) = 1 - \log_3 4$

Eg. 6

If the sum of four numbers in A.P is 24 and the sum of their squares is 164 then find those numbers.

Sol: (a-3d)+(a-d)+(a+d)+(a+3d)=24

$\Rightarrow 4a = 24 \Rightarrow a = 6$

$(a-3d)^2 + (a-d)^2 + (a+d)^2 + (a+3d)^2 = 164$

$\Rightarrow 2\left(a^2 + 9d^2\right) + 2\left(a^2 + d^2\right) = 164$

$\Rightarrow a^2+5d^2=41 \Rightarrow 36+5d^2=41 \Rightarrow d=\pm 1$

required numbers are 3,5,7,9

Eg. 7

Find the n^{th} term of the sequence 5,15,29,47,69,95,...

Sol: The given sequence is not an A.P. but the successive differences between the various terms

i.e. (15-5),(29-15),(47-29),(69-47),(95-69),....

i.e. 10,14,18,22,26,..... are in A.P

Let n^{th} term of the given sequence be $t_n = an^2+bn+c \rightarrow (1)$ Putting n=1,2,3 in (1), we get

$t_1 = a+b+c \Rightarrow a+b+c=5 \rightarrow (2)$

$t_2 = 4a+2b+c \Rightarrow 4a+2b+c=15 \rightarrow (3)$

$t_3 = 9a+3b+c \Rightarrow 9a+3b+c=29 \rightarrow (4)$

Solving (2),(3),(4), we get a=2,b=4,c=-1.

$\therefore$ the n^{th} term of the given sequence is $t_n = 2n^2+4n-1$

Eg. 8

The sum of the first n terms of two A.P's are in the ratio (2n+3):(3n-1). Find the ratio of 5th terms of these A.P's.

Sol: Given that $\frac{S_n}{S_n^{'}} = \frac{2n+3}{3n-1}$

The ratio of nth terms $\frac{t_n}{t_n^{'}} = \frac{2(2n-1)+3}{3(2n-1)-1} = \frac{4n+1}{6n-4}$

$t_5 : t_5^{'} = 21:26$

Eg. 9

The interior angles of a polygon are in A.P. the smallest angle is 120^0 and the common difference is 5^0. Find the number of sides of the polygon .

Sol: Given a=120^0, d= 5^0

Sum of the interior angles of a polygon of n sides is $(n-2)180^0$

$\therefore \frac{n}{2}\left[2(120)+(n-1)5\right]=(n-2)180$

$\Rightarrow n[5n+235]=(n-2)360$

$\Rightarrow 5n(n+47)=(n-2)360$

$\Rightarrow n^2+47n=(n-2)72$

$\Rightarrow n^2-25n+144=0 \Rightarrow (n-9)(n-16)=0$

$\Rightarrow n=9 \text{ or } 16$

(Since neglecting n=16, Since that case largest angle is [120+(15)5]=195, which is not possible no longer angle of a polygon is more than 180)

$\therefore$ n=9

Eg. 10

Find 12^{th} common term of two Arithmetic Series 7+10+13+..... and 4+11+18+......... .

Sol: The n^{th} common term of between two series = (L.C.M of common difference of 1st series and 2nd series)(n-1) + 1st common term of both series.

=(L.C.M of 3,7) (12-1)+25 =21(11)+25 =256

Eg. 11

Find the number of common terms to the two sequences 17,21,25,...,417 and 16,21,26,...,466.

Sol: series 17,21,25,.,417 has common difference4 series 16,21,26,...,466 has common difference5 LCM of 4 and 5 is 20, the first common term is 21. Hence, the series is 21,41,61,...,401; which has 20 terms.

Eg. 12

If n arithmetic means are inserted between 2 and 38, then the sum of the resulting series is obatined as 200, then find the value of n.

Sol: We have

$\frac{n+2}{2}(2+38)=200 \Rightarrow n+2=10 \Rightarrow n=8$

Arithmetic mean of the m^{th} power : Let $a_1, a_2, ...a_n$ be n positive real number (not all equal) & let m be real number then $\frac{a_1^m + a_2^m + ... + a_n^m}{n}$

$$> \left(\frac{a_1 + a_2 + ... + a_n}{n}\right)^m \ \forall m \in R - [0,1]$$

$$< \left(\frac{a_1 + a_2 + ... a_n}{n}\right)^m \ \forall m \in (0,1)$$

$$= \left(\frac{a_1 + a_2 + ... + a_n}{n}\right)^m \ \forall m \in \{0,1\}$$

Eg. 13

Prove that $\sqrt{1} + \sqrt{2} + ... + \sqrt{n} < n\sqrt{\frac{n+1}{2}}$

Sol: $\frac{\sqrt{1} + \sqrt{2} + ... + \sqrt{n}}{n} < \left(\frac{1+2+3+...+n}{n}\right)^{\frac{1}{2}}$

$$< \left(\frac{n\frac{(n+1)}{2}}{n}\right)^{\frac{1}{2}} < \left(\frac{(n+1)}{2}\right)^{\frac{1}{2}}$$

$$\therefore \sqrt{1} + \sqrt{2} + ... + \sqrt{n} < n\sqrt{\frac{n+1}{2}}$$

Eg. 14

If the third term of G.P is 4, then find the product of first 5terms.

Sol: Given $t_3 = ar^2 = 4$

Product of first 5 terms =

$$(a)(ar)(ar^2)(ar^3)(ar^4) = a^5 r^{10} = (ar^2)^5 = 4^5 = 1024$$

Eg. 15

If

$(10)^9 + 2(11)^1(10)^8 + 3(11)^2(10)^7 + ... + 10(11)^9 = k(10)^9$,

then find k. [JEE MAIN 2014]

Sol:

$$k(10)^9 = 10^9 + 2(11)^1(10)^8 + 3(11)^2(10)^7 + ... + 10(11)^9$$

$$\Rightarrow k = 1 + 2\left(\frac{11}{10}\right) + 3\left(\frac{11}{10}\right)^2 + ... + 10\left(\frac{11}{10}\right)^9 \rightarrow (1)$$

$$\frac{11k}{10} = \frac{11}{10} + 2\left(\frac{11}{10}\right)^2 + 3\left(\frac{11}{10}\right)^3 + ... + 10\left(\frac{11}{10}\right)^{10} \rightarrow (2)$$

$$(1)-(2) \Rightarrow -\frac{k}{10} = 1 + \frac{11}{10} + \left(\frac{11}{10}\right)^2 + ... + \left(\frac{11}{10}\right)^9 - 10\left(\frac{11}{10}\right)^{10}$$

$$= \frac{\left(\frac{11}{10}\right)^{10} - 1}{\frac{11}{10} - 1} - 10\left(\frac{11}{10}\right)^{10} = 10\left(\frac{11}{10}\right)^{10} - 10 - 10\left(\frac{11}{10}\right)^{10}$$

$$\therefore \frac{-k}{10} = -10 \Rightarrow k = 100$$

Eg. 16

Three Possitive numbers from an increasing G.P. If the middle term in this G.P is doubled, the new number are in A.P Then find the common ratio of the G.P. [JEE-2014]

Sol: Let a, ar, ar^2 be in G.P and r>1.

Given $a, 2ar, ar^2$ are in A.P .

$$\Rightarrow 2(2ar) = a + ar^2 \Rightarrow r^2 - 4r + 1 = 0$$

$$r = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3} \quad \therefore r > 1 \Rightarrow r = 2 + \sqrt{3}$$

Eg. 17

Three numbers are in G.P. Whose sum is 70, if the extremes be each multiplied by 4 and the mean by 5, they will be in A.P. then find the sum of numbers.

Sol: Let the numbers be a, ar, ar^2 and sum=70

$$\Rightarrow a(1 + r + r^2) = 70 \rightarrow (1)$$

it is given that $4a, 5ar, 4ar^2$ are in A.P

$$\Rightarrow 2(5ar) = 4a + 4ar^2 \Rightarrow 5r = 2 + 2r^2$$

$$\Rightarrow 2r^2 - 5r + 2 = 0 \Rightarrow (2r-1)(r-2) = 0$$

$\Rightarrow r = 2, \frac{1}{2}$ put r=2 in (1), then a=10

put $r = \frac{1}{2}$ in (1), then a=40

$\therefore$ The numbers are 10,20,40 or 40,20,10.

$\therefore$ Sum of the numbers =70

Eg. 18

If the sides of a triangle are in G.P and it's larger angle is twice the smallest, then find the common ratio r satisfies the inequality.

Sol: Let the sides of a triangle be $a/r, a \text{ and } ar$, with a>0 and r>1. let α be the smallest angle. So that the largest angle is 2α. then α is opossite to the side a/r, and 2α is positive to the side ar. Applying sine rule, we get

$$\frac{a/r}{\sin\alpha} = \frac{ar}{\sin 2\alpha}$$

$$\Rightarrow \frac{\sin 2\alpha}{\sin\alpha} = r^2 \Rightarrow r^2 = 2\cos\alpha < 2$$

$$\Rightarrow r^2 < 2 \Rightarrow r < \sqrt{2}$$

$$\therefore 1 < r < \sqrt{2}$$

Eg. 19

Find the geometric mean between -9 and -16.

Sol:

Required G..M = $\sqrt{-9 \times -16} = (3i)(4i) = -12$

Eg. 20

If we insert two numbers between 3 and 81 so that the resulting sequence is G.P then find the numbers.

Sol: Let the two numbers be a and b, then 3,a,b,81 are in G.P.

$\because$ nth term $T_n = AR^{n-1}$; $81 = 3R^{4-1}$

$$\Rightarrow R^3 = \frac{81}{3} = 27 \Rightarrow R^3 = 3^3 \Rightarrow R = 3$$

$$\therefore a = AR = 3 \times 3 = 9, b = AR^2 = 3 \times 3^2 = 27$$

Eg. 21

Find the sum of upto n terms of series : 5+7+13+31+85+....

Sol: The difference between the successive terms are **2,6,18,54,.....**Clearly it is a G..P. Let T_n be the n[th] term of the given series and S_n be the sum of its n terms, then

$$S_n = 5 + 7 + 13 + 31 + .. + T_n \rightarrow (1)$$

$$S_n = 5 + 7 + 13 + \ldots + T_{n-1} + T_n \rightarrow (2)$$

Subtracting (2) from (1)

$$0 = 5 + \left[2 + 6 + 18 + 54 + \ldots + \left(T_n - T_{n-1}\right)\right] - T_n$$

$$\Rightarrow 0 = 5 + 2\frac{3^{n-1} - 1}{3 - 1} - T_n \Rightarrow T_n = 5 + \left(3^{n-1} - 1\right) = 4 + 3^{n-1}$$

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}\left(4 + 3^{k-1}\right) = \sum_{k=1}^{n} 4 + \sum_{k=1}^{n} 3^{k-1}$$

$$= 4n + \left(1 + 3 + 3^2 + \ldots. + 3^{n-1}\right) = 4n + 1\left(\frac{3^n - 1}{3 - 1}\right)$$

$$= 4n + \left(\frac{3^n - 1}{2}\right) = \frac{1}{2}\left(3^n + 8n - 1\right)$$

Eg. 22

Find the Sum to infinity of the series $1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \ldots.$ **is [AIEEE 2009]**

Sol: Let $S = 1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \ldots.$ --(1)

$$\frac{1}{3}S = \frac{1}{3} + \frac{2}{3^2} + \frac{6}{3^3} + \frac{10}{3^4} + \ldots. \text{--------(2)}$$

Subtracting (2) from (1)

$$S\left(1 - \frac{1}{3}\right) = 1 + \frac{1}{3} + \frac{4}{3^2} + \frac{4}{3^3} + \frac{4}{3^4} + \ldots.$$

$$S\left(\frac{2}{3}\right) = \frac{4}{3} + \frac{4}{3^2}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \ldots.\right)$$

$$\frac{2}{3}S = \frac{4}{3} + \frac{4}{3^2}\left(\frac{1}{1 - \frac{1}{3}}\right) = \frac{4}{3} + \frac{4}{3^2}.\frac{3}{2} = 2 \qquad \therefore \text{ S=3}$$

Eg. 23

Find the sum of the infinite series $1 + \frac{4}{3} + \frac{9}{3^2} + \frac{16}{3^3} + \ldots.\infty$ **.**

Sol: This is clearly not an AG.P Series, since 1,4,9,16.... are not in A.P. However their successive differences 4-1=3,9-4=5,16-9=7, ... are in A.P.

Let $S_\infty = 1 + \frac{4}{3} + \frac{9}{3^2} + \frac{16}{3^3} + \ldots\infty$ -(1)

$$\frac{1}{3}S_\infty = \frac{1}{3}+\frac{4}{3^2}+\frac{9}{3^3}+...\infty \quad -(2)$$

Subtracting (2) from (1)

$$\frac{2}{3}S_\infty = 1+\frac{3}{3}+\frac{5}{3^2}+\frac{7}{3^3}+....+\infty$$

$$\frac{1}{3}.\frac{2}{3}S_\infty = \frac{1}{3}+\frac{3}{3^2}+\frac{5}{3^3}+....+\infty$$

on Subtracting $\left(\frac{4}{9}\right).S_\infty = 1+\frac{2}{3}+\frac{2}{3^2}+...+\infty$

$$=1+\frac{2}{3}\left(1+\frac{1}{3}+\frac{1}{3^2}+...+\infty\right)$$

$$=1+\frac{2}{3}\left(\frac{1}{1-\frac{1}{3}}\right)=2 \quad \therefore S_\infty = \left(2\times\frac{9}{4}\right)=\frac{9}{2}$$

Eg. 24

The 5th and 11th terms of an H.P are $\frac{1}{45}$ and $\frac{1}{69}$ respectively, then find 16th term .

Sol: The 5th and 11th terms of the corresponding A.P. are 45 and 69 respectively. Let a be the first term and d be the common difference of the corresponding A.P then,5th term = a+4d=45.......(i)

and 11th term = a+10d=69......(ii)

solving equations (i) and (ii), we get a=29, d=4

$\therefore$ the 16th term of the A.P =a+15d=29+15(4)=89

hence, the 16th term of the H.P=1/89

Harmonic Mean (H.M):- The harmonic mean H of any two numbers a and b is given by $H=\frac{2ab}{a+b}$, where a,H,b are in H.P.

Eg. 25

Find two H.M's between 1/2,4/17.

Sol: Let x_1 and x_2 be two H.M's between1/2,4/17

$$\therefore \ a=\frac{1}{2} \ , \ b=\frac{4}{17} \ , \ n=2$$

$$x_1 = \frac{ab(2+1)}{b(2+1)+1(a-b)} = \frac{3ab}{a+2b} = \frac{3\left(\frac{1}{2}\right)\left(\frac{4}{17}\right)}{\left(\frac{1}{2}\right)+2\left(\frac{4}{17}\right)} = \frac{4}{11}$$

$$x_2 = \frac{ab(2+1)}{b(2+1)+2(a-b)} = \frac{3ab}{2a+b} = \frac{3\left(\frac{1}{2}\right)\left(\frac{4}{17}\right)}{2\left(\frac{1}{2}\right)+\left(\frac{4}{17}\right)} = \frac{2}{7}$$

Eg. 26

Let two numbers have arithmetic mean 9 and geometric mean 4. then find the numbers are the roots of the quadratic equation.

Sol: The A.M. of the two numbers is A=9 and the G.M of two numbers is G=4

The quadratic equation whose roots are the numbers having A.M and G.M. are A,G respectively is $x^2-2Ax+G^2=0$ .So, the required quadratic equation is $x^2-18x+16=0$

Eg. 27

Find two numbers whose arithmetic mean is 34 and geometric mean is 16.

Sol: Let the two numbers be a and b then $\frac{a+b}{2}$=34 and $\sqrt{ab}$=16

$\Rightarrow$ a+b=68 and ab=256

$$\therefore (a-b)^2=(a+b)^2-4ab$$

$$=(68)^2-4(256)=3600 \ \Rightarrow \text{a-b=60}$$

on solving a+b=68 and a-b=60, we get a=64, and b=4. thus, the required numbers are 64 and 4.

Eg. 28

The H.M. between two numbers is 16/5, their A.M. is A and G.M. is G. If 2A+G^2=26 then find the numbers.

Sol: Given H.M of a and b is $\frac{2ab}{a+b}=\frac{16}{5}$

$$\Rightarrow a+b=\frac{5ab}{8} \quad \rightarrow (1)$$

Given $2A+G^2=26 \Rightarrow 2\left(\frac{a+b}{2}\right)+ab=26$

$\Rightarrow (a+b)+ab=26 \Rightarrow \frac{5ab}{8}+ab=26 \Rightarrow ab=16$

From (1), $a+b=\frac{5}{8}(16) \Rightarrow a+b=10 \quad \rightarrow (2)$

$\therefore (a-b)^2=(a+b)^2-4ab=100-64=36$

$\therefore (a-b)=6 \rightarrow (3)$ Solving (2) and (3)

$\therefore$ a=8, b=2

Weighted Means: Let $a_1, a_2, ...a_n$ be n positive real numbers and $m_1, m_2, m_3...m_n$ be n positive rational numbers. Then we have weighted Arithmetic mean A, Weighted geometric mean G and weighted harmonic mean H as

$$A=\frac{m_1a_1+m_2a_2+.....+m_na_n}{m_1+m_2+m_3+.....+m_n},$$ and

$$G=\left(a_1^{m_1}a_2^{m_2}.....a_n^{m_n}\right)^{\frac{1}{m_1+m_2+........+m_n}}$$

$$H=\frac{m_1+m_2+.....+m_n}{\frac{m_1}{a_1}+\frac{m_2}{a_2}+.....+\frac{m_n}{a_n}}$$ Then we have

$A \geq G \geq H$. Moreover equality hold at either place $\Leftrightarrow a_1=a_2=...=a_n$

Eg. 29

If 2p+3q+4r=15, then find the maximum value of $p^3q^5r^7$.

Sol: Since

$$\frac{\frac{2p}{3}+\frac{2p}{3}+\frac{2p}{3}+\left(\frac{3q}{5}+...+\frac{3q}{5}\right)(5\ times)+\left(\frac{4r}{7}+...+\frac{4r}{7}\right)(7\ times)}{15}$$

$$\geq \sqrt[15]{\left(\frac{2p}{3}\right)^3\left(\frac{3q}{5}\right)^5\left(\frac{4r}{7}\right)^7} \quad (\because AM \geq GM)$$

$$p^3q^5r^7\frac{2^33^54^7}{3^35^57^7} \leq 1 \Rightarrow p^3q^5r^7 \leq \frac{5^57^7}{2^33^24^7}$$

Eg. 30

$1^3-2^3+3^3-4^3+....+9^3=$

Sol: $1^3-2^3+3^3-4^3+....+9^3=$

$=\left(1^3+2^3+...+9^3\right)-2\left(2^3+4^3+...8^3\right)$

$\left[\frac{9(9+1)}{2}\right]^2-2\times 2^3\left(1^3+2^3+3^3+4^3\right)$

$=2025-1600=425$

EXERCISE - I

1. **If the first term of an A.P is –1 and common difference is – 3, then 12th term is**
   1) 34 2) 32 3) –32 4) –34
2. **If the sum to n terms of an A.P. is $3n^2+5n$ while T_m=164, then value of m is**
   1) 25 2) 26 3) 27 4) 28
3. **Let T_r be the rth term of an AP for r=1, 2, ... If for some positive integers m and n we have T_m = 1/n and T_n=1/m, the T_{mn}=**
   1) – 1/mn 2) 1/m + 1/n
   3) 1 4) 0
4. **The interior angles of a polygon are in A.P. If the smallest angle is 100° and the common difference is 4°, then the number of sides is**
   1) 5 2) 7 3) 36 4) 44
5. **If a, b, c, d, e, f are in A.P., then e–c is equal to**
   1) 2(c – a) 2) 2 (d – c) 3) f – e 4) d – c
6. **If the ratio between the sums of n terms of two A.P.'s is $3n+8:7n+15$, then the ratio between their 12th terms is**
   1) 16 : 7 2) 7 :16 3) 74 : 169 4) 169 : 74
7. **If the sum of the first ten terms of an A.P is four times the sum of its first five terms, then ratio of the first term to the common difference is**
   1) 1 : 2 2) 2 : 1 3) 1 : 4 4) 4 : 1
8. **If S_n denotes the sum of n terms of an A.P., then $S_{n+3} - 3S_{n+2} + 3S_{n+1} - S_n$=**
   1) 0 2) 1 3) 3 4) 2
9. **In an A.P of 99 terms , the sum of all the odd numbered terms is 2550. Then the sum of all 99 terms is**
   1) 5039 2) 5029 3) 5019 4) 5049

10. If the first, second and the last terms of an A.P. are a,b,c respectively, then the sum of the A.P. is

1) $\frac{(a+b)(a+c-2b)}{2(b-a)}$ 2) $\frac{(b+c)(a+b-2c)}{2(b-a)}$

3) $\frac{(a+c)(b+c-2a)}{2(b-a)}$ 4) $\frac{(a+2c)(b+c+2c)}{2(b-a)}$

11. Four numbers are in arithmetic progression. The sum of first and last terms is 8 and the product of both middle terms is 15. The least number of the series is.

1) 4 2) 3 3) 2 4) 1

12. If n arthmetic means are inserted between 2 and 38, then the sum of the resulting series is obtained as 200, then the value of n is

1) 6 2) 8 3) 9 4) 10

13. If $m>1$ and $n \in N$ then

1. $\frac{1^m+2^m+...+n^m}{n} > \left(\frac{n+1}{2}\right)^m$

2. $\frac{1^m+2^m+...+n^m}{n} < \left(\frac{n+1}{2}\right)^m$

3. $\frac{1^m+2^m+...+n^m}{n} \geq 1$ 4. $\frac{1^m+2^m+...+n^m}{n} \leq 1$

14. Sum of the series

$S=1+\frac{1}{2}(1+2)+\frac{1}{3}(1+2+3)+\frac{1}{4}(1+2+3+4)+....$ **upto 20 terms is**

1) 110 2) 111 3) 115 4) 116

15. The first and second terms of a G.P are x^{-4} and x^n respectively. If x^{52} is the eighth term of the same progression, then n is equal to

1) 13 2) 4 3) 5 4) 3

16. How many terms of the series 1+3+9+ ... sum to 364?

1) 5 2) 6 3) 4 4) 3

17. If a, b and c are in G.P., then $\frac{b-a}{b-c}+\frac{b+a}{b+c}=$

1) b^2-c^2 2) ac 3) ab 4)0

18. If x, y, z are the three geometric means between 6, 54, then z =

1) $9\sqrt{3}$ 2) 18 3) $18\sqrt{3}$ 4) 27

19. H_1,H_2 are 2 H.M.'s between a, b then $\frac{H_1+H_2}{H_1.H_2}=$

1) $\frac{ab}{a+b}$ 2) $\frac{a+b}{ab}$ 3) $\frac{a-b}{ab}$ 4) $\frac{ab}{a-b}$

20. If H_1, H_2,....., H_n are n harmonic means between a and b($\neq$ a), then the value of $\frac{H_1+a}{H_1-a}+\frac{H_n+b}{H_n-b}=$

1) n + 1 2) n – 1 3)2n 4)2n + 3

21. If $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+...\infty=\frac{\pi^4}{90}$**, then** $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+...\infty$ **is equal to**

1) $\frac{\pi^2}{36}$ 2) $\frac{\pi^4}{48}$ 3) $\frac{\pi^2}{72}$ 4) $\frac{\pi^4}{96}$

22. The rational number which is equal to the number $2.\overline{357}$ with recurring decimal is

1) $\frac{2355}{1001}$ 2) $\frac{2370}{999}$ 3) $\frac{2355}{999}$ 4) $\frac{2359}{991}$

KEY

1) 4	2) 3	3) 3	4) 1	5) 2	6) 2
7) 1	8) 1	9) 4	10) 3	11) 4	12) 2
13) 1	14) 3	15) 2	16) 2	17) 4	18) 3
19) 2	20) 3	21) 4	22) 3		

SOLUTIONS

1. $t_{12}=a+(12-1)d$
2. $T_m=S_m-S_{m-1}$
3. $T_m=a+(m-1)d=\frac{1}{n}$, $T_n=a+(n-1)d=\frac{1}{m}$

   $T_m-T_n=\frac{1}{n}-\frac{1}{m}$, find $d=\frac{1}{mn}$, using T_m, find a and T_{mn}
4. Sum of interior angles of a polygon of n sides

   $=(n-2)\,180^o=\frac{n}{2}[2(100)+(n-1)4]$
5. Let A be first term and D be c.d

   e=A+4D,c=A+2D $\therefore$ e-c=2D,check with option

6. Ratio of the sums of n terms $=\dfrac{3n+8}{7n+15}$

$\therefore$ Ratio of n^{th} terms Replace n with (2n-1)

$=\dfrac{3(2n-1)+8}{7(2n-1)+15}=\dfrac{6n+5}{14n+8}$

$\Rightarrow$ Ratio of 12^{th} terms $=\dfrac{6\times12+5}{14\times12+8}=\dfrac{77}{176}=\dfrac{7}{16}$

7. $S_{10}=4S_5$

8. $(S_{n+3}-S_{n+2})-2(S_{n+2}-S_{n+1})+(S_{n+1}-S_n)$
=d-2d+d=0

9. $\dfrac{50}{2}(a_1+a_{99})=2550 \Rightarrow a_1+a_{99}=102$

sum of all the terms = $\dfrac{99}{2}[a_1+a_{99}]=5049$

10. Let there be n terms in the A.P. Then,

$c=a+(n-1)(b-a) \quad \Rightarrow n=\dfrac{b+c-2a}{b-a}$

$\therefore$Sum of n terms$=\dfrac{n}{2}(a+c)=\dfrac{(b+c-2a)(a+c)}{2(b-a)}$

11. Take A.P as (a – 3d), a – d, a+d, a + 3d

12. Total no. of terms in A.P is n + 2

given that $\left(\dfrac{n+2}{2}\right)(2+38)=200$

13.

$$\frac{1^m+2^m+...+n^m}{n}>\left(\frac{1+2+3+...+n}{n}\right)^m>\left(\frac{n+1}{2}\right)^m$$

14. $S=\Sigma\dfrac{n(n+1)}{2.n}=\Sigma\left(\dfrac{n+1}{2}\right)$

15. The common ratio of the G.P.'s x^{n+4}

$\therefore x^{52}=$ Eighth term

$\Rightarrow x^{52}=x^{-4}\left(x^{n+4}\right)^7 \Rightarrow 7n=28 \Rightarrow n=4$

16. $\dfrac{1(3^n-1)}{3-1}=364$ find n

17. $b^2=ac$ and simplifying the given

18. a = 6, $ar^4=54 \Rightarrow r=\sqrt{3}$

19. $\dfrac{1}{H_1}+\dfrac{1}{H_2}+...+\dfrac{1}{H_n}=\dfrac{n}{2}\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$

20. Use $\dfrac{1}{H_1}+\dfrac{1}{H_2}+...+\dfrac{1}{H_n}=\dfrac{n}{2}\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$,

find $\dfrac{a}{H_1}$ and $\dfrac{b}{H_n}$

21. $\dfrac{1}{1^4}+\dfrac{1}{2^4}+\dfrac{1}{3^4}+...\infty=\dfrac{\pi^4}{90}$,

$$\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+...\infty\right)+\left(\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+...\infty\right)=\frac{\pi^4}{90}$$

$$\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+...\infty\right)+\frac{1}{2^4}\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+...\infty\right)=\frac{\pi^4}{90}$$

$$\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+...\infty\right)+\frac{1}{16}\left(\frac{\pi^4}{90}\right)=\frac{\pi^4}{90}$$ **, simplfy**

22. let x=2.357357357....
1000x = 2357.357357; subtract

EXERCISE - II

1. Let the sequence $a_1,a_2,a_3,......a_n$ form an A.P

$a_1^2-a_2^2+a_3^2-a_4^2+....+a_{2n-1}^2-a_{2n}^2$ **is equal to**

1) $\dfrac{n}{2n-1}(a_1^2-a_{2n}^2)$ 2) $\dfrac{1}{2n-1}(a_1^2-a_{2n}^2)$

3) $\dfrac{n}{n+1}(a_1^2+a_{2n}^2)$ 4) $\dfrac{1}{2n+1}(a_1^2-a_{2n}^2)$

2. The sum to 101 terms of an A.P. is 1212. The middle term is

1) 6 2) 12 3) 24 4) 26

3. If $\log 2, \log(2^x-1)$ and $\log(2^x+3)$ are in AP, then the value of x is given by

1) $\dfrac{5}{2}$ 2) $\log_2 5$ 3) $\log_3 5$ 4) $\log_5 3$

4. If in AP, $a_7=9$ and if $a_1.a_2.a_7$ is least, then common difference is

1) $\dfrac{11}{30}$ 2) $\dfrac{13}{10}$ 3) $\dfrac{32}{33}$ 4) $\dfrac{33}{20}$

5. **The number of common terms in two A.P's 2,7,12,17........... 500 terms and 1,8,15,22,. .... 300 terms is**

1) 58 2) 60 3) 61 4) 63

6. **In G.P. $(p+q)^{th}$ term is m, $(p-q)^{th}$ term is n, then p^{th} term is**

1) nm 2) $\sqrt{nm}$ 3) m/n 4) $\sqrt{m/n}$

7. **If a_1, a_2, a_3 are three positive consecutive terms of a GP with common ratio K. then all values of K for which the in equality $a_3 > 4a_2 - 3a_1$, is satisfies**

1) $(1,3)$ 2) $(-\infty,1)\cup(3,\infty)$

3) $(-\infty,\infty)$ 4) $(0,\infty)$

8. **The series**

$$\frac{2x}{x+3}+\left(\frac{2x}{x+3}\right)^2+\left(\frac{2x}{x+3}\right)^3+......to\,\infty$$ **will have a definite sum when**

1) $-1<x<3$ 2) $0<x<1$

3) $x=0$ 4) $x>3$

9. **If a,b,c,d,x are real and the roots of equation $\left(a^2+b^2+c^2\right)x^2-2(ab+bc+cd)x+\left(b^2+c^2+d^2\right)=0$ real and equal, then a,b,c,d are in**

1) A.P 2) G.P 3) H.P 4) None of these

10. (666.... ndigits)^2 + **(888.....n digits) =**

1) $\frac{4}{9}(10^n-1)$ 2) $\frac{4}{9}(10^{2n}-1)$

3) $\frac{4}{9}(10^n-1)^2$ 4) $\frac{4}{9}\left(10^n-1\right)^2$

11. **Let a = 111....1(55 digits),**

$b=1+10+10^2+....+10^4$

$c=1+10^5+10^{10}+10^{15}+....+10^{50}$, **then**

1) a = b+c 2) a = bc 3) b =ac 4) c =ab

12. **The sum to infinity of $\frac{1}{7}+\frac{2}{7^2}+\frac{1}{7^3}+\frac{2}{7^4}+...$ is**

1) 1/5 2) 7/24 3) 5/48 4) 3/16

13. **If each term of an infinite G.P is twice the sum of the terms following it , then the common ratio of G.P is**

1) $\frac{1}{2}$ 2) $\frac{2}{3}$ 3) $\frac{1}{3}$ 4) $\frac{3}{2}$

14. **Sum of infinite No.of terms in G.P is 20 and sum of their squares is 100 , then the common ratio of G.P.is**

1) $\frac{1}{5}$ 2) $\frac{4}{5}$ 3) $\frac{2}{5}$ 4) $\frac{3}{5}$

15. **If 's' is the sum to infinite terms of a G.P. whose first term is 1, then the sum of n terms is**

1) $s\left(1-\left(1-\frac{1}{s}\right)^n\right)$ 2) $\frac{1}{s}\left(1-\left(1-\frac{1}{s}\right)^n\right)$

3) $1-\left(1-\frac{1}{s}\right)^n$ 4) $1+\left(1-\frac{1}{s}\right)^n$

16. **If $r>1$ and $x=a+a/r+a/r^2+......,$**

$y=b+b/r+b/r^2+......,$

And $z=c+c/r+c/r^2+......,$

Then value of xy/z^2 is

1) ab/c^2 2) abr/c 3) ab/c^2r 4) ab / c

17. **If the A.M. and G.M. of two numbers are 13 and 12 respectively then the two numbers are**

1) 8, 12 2) 8, 18 3) 10, 18 4) 12, 18

18. **If n!, 3(n!) and (n+1)! are in G.P., then n!, 5(n!) and (n+1)! are in**

1) A.P. 2) G.P. 3) H.P. 4) None

19. **If G_1 and G_2 are two geometric means and A is the arithmetic mean inserted between two positive numbers then the value of $\frac{G_1^2}{G_2}+\frac{G_2^2}{G_1}$ is**

1) A/2 2) A 3) 2A 4) 3A

20. **If $x_i>0,\quad i=1,2,3,...50$ and $x_1+x_2+x_3+...+x_{50}=50$ and minimum value of $\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+...+\frac{1}{x_{50}}$ is λ then $\lambda=$**

1) 50 2) 60 3) 40 4) 202.

21. If $A_1, A_2, A_3, ...$ belongs to A.P such that $A_1 + A_4 + A_7 + ... + A_{28} = 140$ then maximum value of $A_1.A_2....A_{28}$ *is*

1) 2^{28} 2) 7^{28} 3) $(14)^{28}$ 4) $(28)^{28}$

22. Let a,b and c be the real numbers such that $a+b+c=6$ then, the range of ab^2c^3 is

1) $(0,\infty)$ 2) $(0,1)$

3) $(0,108]$ 4) $(6,108]$

23. If none of $b_1, b_2, ...b_n$ is zero then $\left(\frac{a_1}{b_1}+\frac{a_2}{b_2}+....+\frac{a_n}{b_n}\right)^2$ is

1) $\geq \left(a_1^2+a_2^2+...+a_n^2\right)\left(b_1^{-2}+b_2^{-2}+...+b_n^{-2}\right)$

2) $\leq \left(a_1^2+a_2^2+...+a_n^2\right)\left(b_1^{-2}+b_2^{-2}+...+b_n^{-2}\right)$

3) $> \left(a_1^2+a_2^2+...+a_n^2\right)\left(b_1^{-2}+b_2^{-2}+...+b_n^{-2}\right)$

4) $< \left(a_1^2+a_2^2+...+a_n^2\right)\left(b_1^{-2}+b_2^{-2}+...+b_n^{-2}\right)$

24. **If a,b,c be the pth, qth and rth terms respectively of a G.P., then the equation $a^q b^r c^p x^2 + pqrx + a^r b^p c^q = 0$ has**

1) both roots zero

2) at least one root zero

3) no root zero 4) both roots unity

25. If –1 < a, b, c < 1 and a, b, c are in A.P. and $x = \sum_{n=0}^{\infty} a^n$, $y = \sum_{n=0}^{\infty} b^n$, $z = \sum_{n=0}^{\infty} c^n$ then x, y, z are in

1) A.P. 2) G.P. 3) H.P. 4) A.G.P

26. If $a_1, a_2, a_3, ...a_n$ are in H.P then

$$\frac{a_1}{a_2+a_3+...a_n}, \frac{a_2}{a_1+a_3+...a_n}, \frac{a_3}{a_1+a_2+...a_n}, ..., \frac{a_n}{a_2+a_3+...a_{n-1}}$$

1) A.P. 2) G.P. 3) H.P. 4) A.G.P

27 If a, 8, b are in A.P; a, 4, b are in G.P; a, x, b are in H.P then x =

1) 2 2) 1 3) 4 4) 16

28. Number of positive integral ordered pairs of (a,b) such that 6,a,b are in H.P is

1) 5 2) 6 3) 7 4) 8

29. If a, b, c are in H.P, then the value of $\frac{a+c}{a-c}$ is

1) $\frac{a}{a-b}$ 2) $\frac{a-b}{a}$ 3) $\frac{b}{a}$ 4) $\frac{a}{a+b}$

30. If $x>1, y>1, z>1$ are in G.P then $\frac{1}{1+\log x}, \frac{1}{1+\log y}, \frac{1}{1+\log z}$ are in

1) AP 2) GP 3) HP 4) AGP

31. If $a = \sum_{r=1}^{\infty}\frac{1}{r^2}, b = \sum_{r=1}^{\infty}\frac{1}{(2r-1)^2}$, then $\frac{a}{b} =$

1) $\frac{5}{4}$ 2) $\frac{4}{3}$ 3) $\frac{3}{4}$ 4) $\frac{4}{5}$

KEY

1) 1	2) 2	3) 2	4) 4	5) 2	6) 2
7) 2	8) 1	9) 2	10) 2	11) 2	12) 4
13) 3	14) 4	15) 1	16) 1	17) 2	18) 1
19) 3	20) 1	21) 3	22) 3	23) 2	24) 3
25) 3	26) 3	27) 1	28) 3	29) 1	30) 3
31) 2					

SOLUTIONS

1. $-d\left(a_1+a_2+a_3+....+a_{2n}\right)$

2. $S_{101} = 1212 \Rightarrow a+50d = 12$, middle term=$T_{\frac{n+1}{2}}$

3. t is given that

$\log 2, \log\left(2^x-1\right), \log\left(2^x+3\right)$ are in A.P.

$\Rightarrow 2\log\left(2^x-1\right) = \log 2 + \log\left(2^x+3\right)$

$\Rightarrow \left(2^x-1\right)^2 = 2\left(2^x+3\right)$

$\Rightarrow \left(2^x\right)^2 - 4\left(2^x\right) - 5 = 0$

$\Rightarrow \left(2^x-5\right)\left(2^x+1\right) = 0 \Rightarrow 2^x = 5 \Rightarrow x = \log_2 5$

4. $a_7 = 9 \Rightarrow a_1 + 6d = 9$; $D = a_1 a_2 a_7$

$= (9-6d)(9-5d)9 = 270\left[\left(d - \frac{33}{20}\right)^2 - \frac{9}{400}\right]$

is least for d=$\frac{33}{20}$

5. 2,7,12,17,......500 terms

$T_{500} = 2 + (500-1)5 = 2497$

1,8,15,22........300 terms

$T_{300} = 1 + (300-1)7 = 2094$

The common difference of common terms = 5x7=35

Common terms are 22,57,92,...........

Let last term ≤ 2094

$\Rightarrow 22 + (n-1)35 \le 2094$

$\Rightarrow n \le 60.2$

6. $ar^{p+q-1} = m$ and $a.r^{p-q-1} = n$, find mn

7. $\frac{a_2}{a_1} = \frac{a_3}{a_2} = K$ From the given in equality

$K^2 a_1 > 4a_1 K - 3a_1 > 0 \Rightarrow K^2 - 4K - 3 > 0$

8. Common ratio of given $G.P = \frac{2x}{x+3}$

For definite sum of infinite G.P., $-1 < \frac{2x}{x+3} < 1$

$\Rightarrow \frac{2x}{x+3} + 1 > 0 \, and \, \frac{2x}{x+3} - 1 < 0 \Rightarrow -1 < x < 3$

9. Roots are real and equal

$\Rightarrow (a^2+b^2+c^2)(b^2+c^2+d^2) - (ab+bc+cd)^2 = 0$

$\Rightarrow b^2 = ac, c^2 = bd, ac = bd$

$\Rightarrow a, b, c, d \, are \, in \, G.P$

10. $\left(6+6(10)+....+6(10)^{n-1}\right)^2 + \left[8+8(10)+.....8(10)^{n-1}\right]$

$= \left(\frac{2}{3}(10^n - 1)\right)^2 + \frac{8}{9}(10^n - 1)$

11. $a = 1 + 10 + 10^2 + ... + 10^{54}$

$\frac{10^{55}-1}{10-1} = \frac{10^{55}-1}{10^5-1} . \frac{10^5-1}{10-1} = bc$

12. $\left(\frac{1}{7} + \frac{1}{7^3} + \frac{1}{7^5} + ...\right) + \left(\frac{2}{7^2} + \frac{2}{7^4} + \frac{2}{7^6} + ...\right)$

$S_\infty = a / 1 - r$

13. $a_n = 2\left[a_{n+1} + a_{n+2} + a_{n+3} + ...\infty\right], \forall \, n \in N$

$ar^{n-1} = 2\left[ar^n + a.r^{n+1} + a.r^{n+2} + ...\right]$;

$ar^{n-1} = \frac{2ar^n}{1-r} \Rightarrow r = \frac{1}{3}$

14. $a + ar + ar^2 + ...\infty = 20 \Rightarrow \frac{a}{1-r} = 20$ ....(1)

$a^2 + a^2 r^2 + a^2 r^4 + ...\infty = 100$

$\Rightarrow \frac{a^2}{1-r^2} = 100$ ....(2)

from 1 and 2 we get $r = \frac{3}{5}$

15. $s = \frac{1}{1-r} \Rightarrow r = \left(1 - \frac{1}{s}\right)$

sum to n

terms=$\frac{1\left(1-\left(1-\frac{1}{s}\right)^n\right)}{1-\left(1-\frac{1}{s}\right)} = s\left(1-\left(1-\frac{1}{s}\right)^n\right)$

16. we have $x = \frac{ar}{r-1}, y = \frac{br}{r-1}, z = \frac{cr}{r-1}$

17 $\frac{a+b}{2} = 13$

$\sqrt{ab} = 12$

18. $9(n!)^2 = (n!)(n+1)! \Rightarrow n = 8$

19. from synopsis

$G_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$; $A_n = a + \frac{n(b-a)}{n+1}$

20. $\frac{x_1 + x_2 + ... + x_{50}}{50} \ge (x_1 x_2 ....x_n)^{\frac{1}{50}}$ ..(1)

$\frac{\frac{1}{x_1} + ... + \frac{1}{x_{50}}}{50} \ge \left(\frac{1}{x_1}, \frac{1}{x_2} .... \frac{1}{x_{50}}\right)^{\frac{1}{50}}$ ..(2)

$\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_{50}} \ge 50 \therefore \lambda = 50$

21. $A_1 + A_4 + A_7 + ... + A_{28} = 140$

$A_1 + A_{28} = A_4 + A_{25} = ... = A_{13} + A_{16}$

$5(A_1 + A_{28}) = 140 \Rightarrow A_1 + A_{28} = 28$

$\frac{A_1 + A_2 + ...A_{28}}{28} = 14$

$AM \geq GM$

22. $\frac{a + 2\left(\frac{b}{2}\right) + 3\left(\frac{c}{3}\right)}{6} \geq \left(a\left(\frac{b}{2}\right)^2\left(\frac{c}{3}\right)^3\right)^{\frac{1}{6}}$

$1 > \left(\frac{ab^2c^3}{108}\right)^{\frac{1}{6}} \Rightarrow ab^2c^3 \leq 108$

23. By using Cauchy-Schwartz's Inequality

$\left(a_1.\frac{1}{b_1} + a_2.\frac{1}{b_2} + ... + a_n.\frac{1}{b_n}\right)^2$

$\leq \left(a_1^2 + a_2^2 + ... + a_n^2\right)\left(\frac{1}{b_1^2} + \frac{1}{b_2^2} + ... + \frac{1}{b_n^2}\right)$

24. Product of roots = $a^{r-q}b^{p-r}c^{q-p} = 1 \neq 0$
no root is equal to zero.

25. $x = \frac{1}{1-a}, y = \frac{1}{1-b}, z = \frac{1}{1-c}$
find a, b, c
given a, b, c are in A.P.

26. $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}$........ in AP

$\frac{a_1+a_2+a_3+....+a_n}{a_1}, \frac{a_1+a_2+a_3+....+a_n}{a_2}, ..., \frac{a_1+a_2+a_3+.....+a_n}{a_n}$

are in AP

27. $a + b = 16$ and $ab = 16$ and $\frac{2}{x} = \frac{1}{a} + \frac{1}{b}$

28. 6,a,b are in H.P $\Rightarrow \frac{1}{6}, \frac{1}{a}, \frac{1}{b}$ are in A.P

$\Rightarrow \frac{2}{a} = \frac{1}{6} + \frac{1}{b} \Rightarrow b = \frac{6a}{12-a}$

$a \in \{3,4,6,8,9,10,11\}$

29. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. $\frac{2}{b} = \frac{1}{c} + \frac{1}{a} = \frac{a+c}{ac}$

$\frac{a+c}{a-c} = \frac{2ac}{b(a-c)} = \frac{\frac{2}{b}}{\frac{1}{c} - \frac{1}{a}}$

30. $y^2 = zx \Rightarrow 1 + \log x, 1 + \log y, 1 + \log z$ are in AP

31. $a = \frac{1}{1^2} + \frac{1}{3^2} + ... + \frac{1}{4}\left(\frac{1}{1^2} + \frac{1}{2^2} + ...\right)$

$a - \frac{a}{4} = b \Rightarrow \frac{a}{b} = \frac{4}{3}$

EXERCISE - III

1. The series of natural numbers are arranged as follow.

1
2 3
4 5 6
7 8 9 10
- - - — — — — —

. The sum of numbers in the nth row is

1) $\frac{n(n+1)}{2}$ 2) $\frac{n(n^2+1)}{2}$ 3) $\frac{n^2(n+1)}{2}$ 4) $\frac{n^2(n^2+1)}{2}$

2. If a,b,c,d are distinct integers in A.P. such that $d = a^2 + b^2 + c^2$, then $a + b + c + d =$

1) 0 2) 1 3) 2 4) 4

3. A person is to count 4500 currency notes. Let a_n denote the number of notes he counts in the n^{th} minute.

If $a_1 = a_2 = ... = a_{10} = 150$ and $a_{10}, a_{11}, ...,$ are in A.P. with common difference -2, then the time taken by him to count all notes is

1) 135mins 2) 24mins 3) 34mins 4) 125mins

4. A man saves Rs. 200 in each of the first 3 months of his service. In each of the subsequent months his saving increases by Rs. 40 more than the saving of immediately previous month. His total saving from the start of service will be Rs 11040 after.

[AIEEE 2011]

1) 21 months 2) 18 months

3) 19 months 4) 20 months

5. The sum of first 20 terms of the sequence 0.7,0.77,0.777,.... is. [MAINS-2013]

1) $\frac{7}{81}\left(179-10^{-20}\right)$ 2) $\frac{7}{9}\left(99-10^{-20}\right)$

3) $\frac{7}{81}\left(179+10^{-20}\right)$ 4) $\frac{7}{9}\left(99+10^{-20}\right)$

6 Sum of n terms of the series 1,3,7,15,31,.... is

1) $2^{n+1}-n-2$ 2) $2^{n}-n-2$

3) $2^{n+1}+n+2$ 4) $2^{n}-1$

7. The three successive terms of a GP will form the sides of a triangle if the common ratio satisfies the inequality $(r>1)$

1) $\left(1,\frac{\sqrt{5}+1}{2}\right)$ 2) $\left(-\infty,\frac{\sqrt{5}-1}{2}\right)\cup\left(\frac{\sqrt{5}+1}{2},\infty\right)$

3) $\left[-\sqrt{5},\sqrt{5}\right]$ 4) $\left(-\sqrt{5},\sqrt{5}\right)$

8. If a,b,c be respectively the p^{th},q^{th} and r^{th} terms of G.P then $\Delta=\begin{vmatrix}\log a & \log b & \log c\\ p & q & r\\ 1 & 1 & 1\end{vmatrix}$ **equals to**

1) 1 2) 0 3) -1 4) 2

9. If $t_r=2^{\frac{r}{3}}+2^{-\frac{r}{3}}$, **then** $\sum_{r=1}^{100}t_r^3-3\sum_{r=1}^{100}t_r+1=$

1) $\frac{2^{101}+1}{2^{100}}$ 2) $\frac{2^{101}-1}{2^{100}}$ 3) $\frac{2^{201}-1}{2^{100}}$ 4) $\frac{2^{201}+1}{2^{100}}$

10 The value of x satisfying the equation

$\left[3\left(1-\frac{1}{2}+\frac{1}{4}......to\ \infty\right)\right]^{\log_{10}x}$

$=\left[20\left(1-\frac{1}{4}+\frac{1}{16}.....\infty\right)\right]^{\log_{x}10}$ **is**

1) $\frac{1}{100}$ 2) 10 3) 1000 4) $\frac{1}{10}$

11. If $\exp\left\{(\sin^2 x+\sin^4 x+\sin^6 x+...\text{upto}\ \infty)\log_e 2\right\}$ **satisfies the equation $x^2 - 17x + 16 = 0$ then the value of** $\frac{2\cos x}{\sin x+2\cos x}(0<x<\pi/2)$ is

1) 1/2 2) 3/2 3) 5 4) 2/3

12. The length of the side of square is 'a' metre. A second square is formed by joining the middle points of the sides of the squares. Then a third square is formed by joininig the middle points of the sides of the second squares and so on. Then the sum of the area of squares which carried upto infinity is

1) a^2 2) $2a^2$ 3) $3a^2$ 4) $4a^2$

13. If $\frac{a+be^y}{a-be^y}=\frac{b+ce^y}{b-ce^y}=\frac{c+de^y}{c-de^y}$ **then a,b,c,d are in**

1) A.P. 2) G.P. 3) H.P. 4) A.G..P

14. If a,b,c,d are positive real numbers such that $a+b+c+d=2$, then $M=(a+b)(c+d)$ satisfies the relation

1) $0<M\le 1$ **2)** $1\le M\le 2$

3) $2\le M\le 3$ 4) $3\le M\le 4$

15. If n be the number of sequence a,b,c,d,e satisfying the conditions
(i) a,b,c,d,e are in A.P and G.P. both,
(ii) c= 3,7 then 'n'= ------

1) 1 2) 2 3) 5 4) 10

16. If p^{th},q^{th},r^{th} terms of an A.P are in G.P. whose common ratio is k, then the root of equation $(q-r)x^2+(r-p)x+(p-q)=0$ other than unity is

1) k 2) 2k 3) k^2 4) $\frac{1}{k}$

17. If the sum to infinity of the series $1+4x+7x^2+10x^3+.....$ is $\frac{35}{16}$ then x =

1) $\frac{1}{5}$ 2) $\frac{2}{5}$ 3) $\frac{3}{7}$ 4) $\frac{1}{7}$

18. The value of $2^{1/4}\ 4^{1/8}\ 8^{1/16}\ 16^{1/32}$ ... is

1) 2 2) 3/2 3) 1 4) 1/2

19. Let x be the arithmetic mean and y,z be the two geometric means between any two positive numbers. Then value of $\frac{y^3+z^3}{xyz}$ is

1) 2 2) 3 3) 1/2 4) 3/2

20. If a, b, c are in G.P., then the equations $ax^2+2bx+c=0$ and $dx^2+2ex+f=0$ have a common root if a/d, b/e, c/f are in

1) A.P. 2) G.P. 3) H.P. 4) A.G..P

21. Let $I_n=\int_0^{\pi/4}\tan^n x\,dx$. Then $I_2+I_4, I_3+I_5, I_4+I_6, I_5+I_7, ...$ are in

1) A.P. 2) G.P. 3) H.P. 4) A.G.P

22. Let $a_1, a_2, ...a_{10}$ be in A.P. and $h_1, h_2, ...h_{10}$ be in H.P. If $a_1 = h_1 = 2$ and $a_{10}=h_{10} = 3$, then a_4h_7 is

1) 2 2) 3 3) 5 4) 6

23. If the sytem of linear equations x + 2ay + az = 0, x + 3by + bz = 0, x+ 4cy + cz = 0 has a non-zero solution, then a, b, c are in

1) G.P. 2) H.P.
3) Satisfy a + 2b + 3c = 0 4) A.P.

24. If cos(x–y), cos x and cos (x+y) are in H.P, then value of cos x sec (y/2) is

1) $\pm\sqrt{2}$ 2) $\pm\sqrt{3}$ 3) ± 2 4) ± 1

25. If a, b, c are real and in A.P. and a^2, b^2, c^2 are in H.P., then

1) a = b = c 2) 2b = 3a + c
3) $b^2=\sqrt{ac/8}$ 4) $ab=c$

26. If 9A.M.'s and 9 H.M's be inserted between 2 and 3 and A be any A.M. and H be the corresponding H.M., then H(5-A) =

1) 10 2) 6 3) -6 4) -10

27. Suppose 'a' is a fixed real number such that $\frac{a-x}{px}=\frac{a-y}{qy}=\frac{a-z}{rz}$ if p,q, r are in AP then x,y,z all are in

1) A.P. 2) G.P. 3) H.P. 4) A.G.P.

28. a,b,c are in A.P; b,c,d are in G.P and c,d,e are in H.P. If a=2 and e=18, then the sum of all possible values of c is

1) -6 2) 6 3)12 4) 0

29. If an A.P., a G.P. and a H.P. have the same first term and same $(2n+1)$ th term and their $(n+1)^{th}$ terms are a,b,c, respectively, then the radius of the circle. $x^2+y^2+2bx+2ky+ac=0$ is

1) k 2) $|k|$ 3) $\sqrt{b^2-ac}$ 4) k^2

30. If $a, a_1, a_2, a_3, a_4, ..., a_{2n}$, b are in A.P and $a, g_1, g_2, g_3, g_4, ..., g_{2n}, b$ are in G.P and h is the H.M of a and b then $\frac{a_1+a_{2n}}{g_1g_{2n}}+\frac{a_2+a_{2n-1}}{g_2g_{2n-1}}+....+\frac{a_n+a_{n+1}}{g_ng_{n+1}}$ is equal to

1) $2n/h$ 2) $2nh$ 3) nh 4) n/h

31. If $f(x)=x^2-(a+b)x+ab$ and A and H be the A.M. and H.M. between two quantities a and b, then

1) $Af(A)=Hf(H)$
2) $Af(H)=Hf(A)$
3) $A+f(A)=H+f(H)$
4) $f(A)+H=f(H)+A$

32. If positive numbers a,b,c be in H.P. , then equation $x^2-kx+2b^{101}-a^{101}-c^{101}=0\,(k\in\mathbb{R})$ has

1) both roots positive
2) both roots negative
3) one positive & one negative root
4) both roots imaginary

33. **The value of** $\sum_{n=1}^{10}\int_0^n x.dx$ **is**

1) an even integer 2) an irrational number
3) a rational number 4) an irrational number

34. **Let** $\sum_{r=1}^{n} r^4 = f(n)$**, then** $\sum_{r=1}^{n}(2r-1)^4$ **is equal to**

1) f(2n) – 16f(n) 2) f(2n) – 7f(n)
3) f(2n–a) – 8f(n) 4) f(2n–a) – 7f(n)

35. **For** $x \in R$ **let [x] denote the greatest integer** $\le$ x. **Largest natural number n for which** $E = \left[\frac{\pi}{2}\right] + \left[\frac{1}{100}+\frac{\pi}{2}\right] + \left[\frac{2}{100}+\frac{\pi}{2}\right] .... + \left[\frac{n}{100}+\frac{\pi}{2}\right] < 43,$ **is**

1) 41 2) 42 3) 43 4) 97

36. **The sum to n terms of the series**

$$\frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + --- \text{ is}$$

1) $\frac{6n}{n+1}$ 2) $\frac{9n}{n+1}$ 3) $\frac{12n}{n+1}$ 4) $\frac{3n}{n+1}$

37. **Let** r^{th} **term of a series be given by** $T_r = \frac{r}{1-3r^2+r^4}$ **then** $\underset{n\to\infty}{Lt}\sum_{r=1}^{n} T_r =$

1) $\frac{3}{2}$ 2) $\frac{1}{2}$ 3) $\frac{-1}{2}$ 4) $\frac{-3}{2}$

38. **The sum of the first n terms of the series**

$1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + ...$ **is** $\frac{n(n+1)^2}{2}$ **when n is even. When n is odd the sum is**

1) $\frac{3n(n+1)}{2}$ 2) $\left[\frac{n(n+1)}{2}\right]^2$

3) $\frac{n(n+1)^2}{4}$ 4) $\frac{n^2(n+1)}{2}$

39. **Sum to n terms of the series**

$$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) + ... \text{ is}$$

1) $\tan^{-1}\left(\frac{n}{n+2}\right)$ 2) $\tan^{-1}\left(\frac{2n-1}{2n+2}\right)$

3) $\tan^{-1}\left(\frac{1}{3n}\right)$ 4) $\tan^{-1}\left(\frac{n}{n+1}\right)$

40. **The sum of the series**

$$\frac{1}{3^2+1} + \frac{1}{4^2+2} + \frac{1}{5^2+3} + \frac{1}{6^2+4} + ...\infty \text{ is}$$

1) $\frac{13}{36}$ 2) $\frac{13}{33}$ 3) $\frac{11}{36}$ 4) $\frac{15}{36}$

41. $\sum_{n=1}^{n} n(1-a)(1-2a)(1-3a)....\{1-(n-1)a\} =$

1) $1-(1-a)(1-2a)(1-3a)....(1-na)$

2) $a[1-(1-a)(1-2a)....(1-na)]$

3) $\frac{1}{a}[1-(1-a)(1-2a)....(1-na)]$

4) $\frac{1}{a}[1-(1-a)(1-2a)(1-3a)....\{1-(n-1)a\}]$

42. **If** $\sum_{r=1}^{n} t_r = \sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} 2, then \sum_{r=1}^{n}\frac{1}{t_r} =$

1) $\frac{n+1}{n}$ 2) $\frac{n}{n+1}$ 3) $\frac{n-1}{n}$ 4) $\frac{n}{n-1}$

43. $S_n = \sum_{n=1}^{n}\frac{n}{1+n^2+n^4}$**, then** $S_{10}.S_{20}$

1) $\frac{110}{111}.\frac{211}{421}$ 2) $\frac{110}{421}.\frac{111}{112}$

3) $\frac{110}{111}.\frac{420}{421}$ 4) $\frac{55}{111}.\frac{210}{421}$

44. **If** $b_i = 1-a_i, na = \sum_{i=1}^{n} a_i, nb = \sum_{i=1}^{n} b_i$ **, then**

$$\sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n}(a_i - a)^2$$

1) ab 2) -nab 3) nab 4) $(n+1)ab$

45. **If (1 + 3 + 5+.....+p) + (1 + 3 + 5+..+q) = (1 + 3 + 5 + ... + r) where each set of parentheses contains the sum of consecutive odd integers as shown, the smallest possible value of p + q + r, (where p > 6) is**

1) 12 2) 21 3) 45 4) 54

46. The largest term of the sequence

$\frac{1}{503}, \frac{4}{524}, \frac{9}{581}, \frac{16}{692}, ...$

1) $\frac{49}{16}$ 2) $\frac{48}{1509}$ 3) $\frac{49}{1529}$ 4) $\frac{64}{1509}$

47. Consecutive odd integers whose sum is 25^2-11^2 are

1) 23,25,27,....,49 2)25,27,29,....,51
3) 21,23,25,....,49 4) 19,21,23,.....,47

48. If $a_n = \int_0^{\frac{\pi}{2}} \frac{\sin^2 nx}{\sin^2 x} dx, then \begin{vmatrix} a_1 & a_{51} & a_{101} \\ a_2 & a_{52} & a_{102} \\ a_3 & a_{53} & a_{103} \end{vmatrix}$

1) 1 2)0 3) -1 4) 2

49. Consider the sequence 1,2,2,4,4,4,4,8,8,8,8,8,8,8,8,..... then 1025^{th} term will be

1) 2^9 2) 2^{11} 3) 2^{10} 4) 2^{12}

50. If set of two numbers $\left(\tan^{-1} x, \tan^{-1} y, \tan^{-1} z\right)$ **and** (x, y, z) **are in A.P such that** y **does not belong to the set** $\{0, -1, 1\}$ **then**

1) set $\left\{\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right\} \in GP$

2) set of numbers $\left\{\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right\} \notin A.G.P$

3) set of numbers are not identical

4) sum of squares of their differences taken pairwise is not equal to zero

KEY

1) 2 2) 3 3) 3 4) 1 5) 3 6) 1
7) 1 8) 2 9) 3 10) 1 11) 1 12) 2
13) 2 14) 1 15) 2 16) 4 17) 1 18) 1
19) 1 20) 3 21) 3 22) 4 23) 2 24) 1
25) 1 26) 2 27) 3 28) 4 29) 2 30) 1
31) 2 32) 3 33) 3 34) 1 35) 1 36) 1
37) 3 38) 4 39) 1 40) 1 41) 3 42) 2
43) 4 44) 3 45) 2 46) 3 47) 1 48) 2
49) 3 50) 1

SOLUTIONS

1. $S = 1+2+4+7+11+...+x_n ...(i)$

$S = \quad 1+2+4+7+.....+x_{n-1}+x_n ....(ii)$

$(i)-(ii) \Rightarrow 0 = 1+[1+2+3+4+....+(n-1)]-x_n$

$\therefore x_n = 1+\frac{(n-1)n}{2} = \frac{n^2-n+2}{2}$

The nth row contains n consecutive numbers with $\frac{n^2-n+2}{2}$ as the first term,

Sum $= \frac{n}{2}\left[2\left(\frac{n^2-n+2}{2}\right)+(n-1).1\right]$

2. $a+3k = a^2+(a+k)^2+(a+2k)^2$, ...$(i)$

Where $k = c.d \text{ of } A.P$

$\Rightarrow 5k^2+3(2a-1)k+3a^2-a = 0$ ...(i)

using $\Delta \geq 0$ then a=0 or -1
From (i), when a =0, $5k^2$-3k =0
then k does not exist,

if $a = -1, 5k^2-9k+4 = 0$

$\Rightarrow k = 1, \frac{4}{5} \Rightarrow k = 1$ ($\because k$ is an integer)

$\therefore a = -1, b = 0, c = 1, d = 2 \Rightarrow a+b+c+d = 2$

3. Till 10^{th} minute, the number of counted notes is 1500.

$\therefore 3000 = \frac{n}{2}[2(148)+(n-1)(-2)] = n[148-n+1]$

$\Rightarrow n^2-149n+3000 = 0$

Since n=125 is not possible, total time required is 24+10=34 minutes.

4. Total saving = 200+200+200+240+280+.... to n months= 11040

$\Rightarrow 400+\frac{n-2}{2}[400+(n-3)40] = 11040$

$\Rightarrow (n-21)(n+26) = 0 \Rightarrow n = 21$

5. $0.7+0.77+0.777+...+0.777...7$

$\Rightarrow \frac{7}{9}[0.9+0.99+.....+0.999..9]$

$\Rightarrow \frac{7}{9}\left[(1-0.1)+(1-0.01)+....+(1-0.000....1)\right]$

$\Rightarrow \frac{7}{9}\left[20-\left(\frac{1}{10}+\frac{1}{10^2}+.....+\frac{1}{10^{20}}\right)\right]$

$=\frac{7}{81}\left(179+10^{-20}\right)$

6. Let $S=1+3+7+15+31+....+T_n$ .....(1)

$S=0+1+3+7+15+....+T_{n-1}+T_n$ .....(2)

(1)-(2) $\Rightarrow T_n=\frac{1(2^n-1)}{(2-1)}=2^n-1$ and $S_n=\sum T_n$

7. Sum of two sides of a triangle > third side $a+ar>ar^2$

8. Let A be the first term and R be the common ratio of the G.P. Then,

$a=AR^{p-1}\Rightarrow \log a=\log A+(p-1)\log R...(i)$

$b=AR^{q-1}\Rightarrow \log b=\log A+(q-1)\log R....(ii)$

$c=AR^{r-1}\Rightarrow \log c=\log A+(r-1)\log R....(iii)$

Multiplying (i), (ii) and (iii) by $(q-r),(r-p)$ and $(p-q)$ respectively and adding, we get

$(q-r)\log a+(r-p)\log b+(p-q)\log c=0$

$\Rightarrow \Delta=0$

9. $\sum_{r=1}^{100} t_r^3=\sum_{r=1}^{100} 2^r+\sum_{r=1}^{100}\frac{1}{2^r}+3\sum_{r=1}^{100} t_r$

$=2^{101}-2+1-\frac{1}{2^{100}}+3\sum_{r=1}^{100} t_r$

$=\frac{2^{201}-1}{2^{100}}-1+3\sum_{r=1}^{100} t_r$

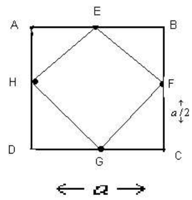
10. $\left(3\frac{1}{1+\frac{1}{2}}\right)^{\log_{10}x}=\left(\frac{20}{1+\frac{1}{4}}\right)^{\log_x^{10}}$

$2^{\log_{10}x}=\left(2^4\right)^{\log_x 10}\Rightarrow \log_{10}x=\frac{4}{\log_{10}x}$

$\therefore \log_{10}x=\pm 2\Rightarrow x=100\ or\ x=\frac{1}{100}$

11. $e^{\frac{\sin^2 x}{\cos^2 x}\times\log_e 2}=16$ or 1 ; $2^{\text{Tan}^2 x}=2^4$ or 2^0

12.



side of second square is $\frac{a}{\sqrt{2}}$, side of third square is $\frac{a}{2}$, ... sum of areas of squares

$a^2+\left(\frac{a}{\sqrt{2}}\right)^2+\left(\frac{a}{2}\right)^2+......=2a^2$

13. $\frac{2a-(a-be^y)}{a-be^y}=\frac{2b-(b-ce^y)}{b-ce^y}$

$=\frac{2c-(c-de^y)}{c-de^y}$ by law of proportion

$\Rightarrow \frac{(a-be^y)}{a}=\frac{(b-ce^y)}{b}=\frac{(c-de^y)}{c}$

$\Rightarrow a,b,c,d,$ *are in G.P.*

14. Since G.M. $\le$ A.M.

$\therefore\sqrt{[(a+b)(c+d)]}\le\frac{(a+b)+(c+d)}{2}=\frac{2}{2}=1$

Also a,b,c,d > 0 $\therefore M>0$ Thus 0<M$\le$1.

15. a,b,c,d,e are in A.P. and G.P both

$\Rightarrow a=b=c=d=e=3,7$

$\Rightarrow$ Required sequences are 3,3,3,3,3 and 7,7,7,7,7 $\Rightarrow n=2$

16. Given $k=\frac{a+(q-1)d}{a+(p-1)d}=\frac{a+(r-1)d}{a+(q-1)d}$

$=\frac{a+(q-1)d-a-(r-1)d}{a+(p-1)d-a-(q-1)d}$

$=\frac{(q-r)d}{(p-q)d}=\frac{(q-r)}{p-q}=\frac{1}{k}$

17. $S_\alpha = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

18. $2^{\frac{1}{4}}.2^{\frac{1}{4}}.2^{\frac{3}{16}}.2^{\frac{1}{8}}..... = 2^{\frac{1}{2^2}+\frac{2}{2^3}+\frac{3}{2^4}+\frac{4}{2^5}+....}$

19. Given that $x = \frac{a+b}{2}$ and a, y, z, b are in G.P.

$y^2 = az, z^2 = by$, $y^3 + z^3 = \frac{y^2}{xz} + \frac{z^2}{xy}$

20. $ax^2 + 2bx + c = 0 \Rightarrow (\sqrt{a}x + \sqrt{c})^2 = 0$

$x = -\sqrt{\frac{c}{a}}$, use in $dx^2 + 2ex + f = 0$

21. We know that $I_n + I_{n+2} = \frac{1}{n+1}$ from integration

22. let 'd' is common difference of A.P

$\therefore 3 = a_{10} = 2 + 9d \Rightarrow d = \frac{1}{9}$

let 'D' is common differnce of $\frac{1}{h_1}, \frac{1}{h_2}, ... \frac{1}{h_{10}}$

$\therefore \frac{1}{3} = \frac{1}{h_{10}} = \frac{1}{2} + \frac{9}{D} \Rightarrow D = \frac{-1}{54}$

23. det = 0

24. $\frac{2}{\cos x} = \frac{1}{\cos(x+y)} + \frac{1}{\cos(x-y)}$

25. $2b = a + c$ & $b^2 = \frac{2a^2c^2}{a^2+c^2}$

simplify, we get a = 3 $\therefore a = b = c$

26. Let A be the k^{th} A.M., then H will be the k^{th} H.M Now, $A = 2 + kd = 2 + k\left(\frac{3-2}{10}\right) = \frac{20+k}{10}$

$H = \frac{1}{2} + \frac{k\left(\frac{1}{3} - \frac{1}{2}\right)}{10} = \frac{30-k}{60}$

$\therefore A + \frac{6}{H} = 5 \Rightarrow H(5-A) = 6$

27. $p - q = q - r = k(let) \Rightarrow \frac{\frac{a}{x}-1}{p} = \frac{\frac{a}{y}-1}{q} = \frac{\frac{a}{z}-1}{r}$

$\Rightarrow \frac{\left(\frac{a}{x}-1\right)-\left(\frac{a}{y}-1\right)}{p-q} = \frac{\left(\frac{a}{y}-1\right)-\left(\frac{a}{z}-1\right)}{q-r}$

by law of proportion $\Rightarrow \frac{1}{x} - \frac{1}{y} = \frac{1}{y} - \frac{1}{z}$

28. $b = \frac{a+c}{2}, c^2 = bd, d = \frac{2ce}{c+e}$

Now , $c^2 = bd \Rightarrow c^2 = \left(\frac{a+c}{2}\right)\left(\frac{2ce}{c+e}\right)$

$\Rightarrow c^2 = ae = 36; c = 6 \ or \ -6$

29. let A be the first term, D be the common difference and B be the $(2n+1)^{th}$ term of A.P.

then $B = A + 2nD \Rightarrow D = \frac{B-A}{2n}$

$a = A + (n+1-1)D = \frac{A+B}{2}$

similarly $b = \sqrt{AB}$ and $c = \frac{2AB}{A+B}$

$\therefore b^2 = ac$ then find r.

30. $a + b = a_1 + a_{2n} = a_2 + a_{2n-1} = ...$ and

$ab = g_1 g_{2n} = g_2 g_{2n-1}....$ and $h = \frac{2ab}{a+b}$

31. We have to calculate $\frac{f(A)}{f(H)}$ and $f(A) - f(H)$

Here $A = \frac{a+b}{2}, H = \frac{2ab}{a+b}$; $\frac{(a+b)^2}{4ab} = \frac{A}{H}$

32. a,b,c are in H.P $\Rightarrow H.M. of\ a\ and\ c\ is\ b$

$\Rightarrow \sqrt{ac} > b [\because G.M > H.M]$

Since A.M

>G.M. $a^{101} + c^{101} > 2\left(\sqrt{ac}\right)^{101} > 2b^{101}$

$f(x) = x^2 - kx + 2b^{101} - a^{101} - c^{101}$

$Then\ f(-\infty) > \infty > 0,$

$f(0) = 2b^{101} - a^{101} - c^{101} < 0$

Hence equation f(x) = 0 has one root in $(-\infty, 0)$ and other in $(0, \infty)$

33. $\sum_{n=1}^{10}\left(\frac{x^2}{2}\right)_0^n = \frac{1}{2}\sum_{n=1}^{10} n^2 = \frac{n(n+1)(2n+1)}{12}$
is rational number

34. $\sum_{r=1}^{n}(2r-1)^4$ =Total sum - Even sum=

$\sum_{r=1}^{2n} r^4 - \sum_{r=1}^{n}(2r)^4 = f(2n) - 16f(n)$

35. Since 3.14 < π< 3.142, $1.57 < \frac{\pi}{2} < 1.571$

$\therefore \left[\frac{\pi}{2} + \frac{n}{100}\right] = 1$ for n = 0, 1, 2, ..., 42

the largest possible number n for which E < 43 is 41.

36. $T_n = \frac{(2n+1)6}{n(n+1)(2n+1)} = 6\left(\frac{1}{n} - \frac{1}{n+1}\right)$

$S_n = \sum T_n = \frac{6n}{n+1}$

37. $T_r = \frac{r}{(r-1)^2 - r^2} = \frac{1}{2}\left[\frac{1}{r^2 - r - 1} - \frac{1}{r^2 - r + 1}\right]$

38. If n is odd; $S_{2m+1} = \frac{2m(2m+1)^2}{2} + (2m+1)^2$

39. $\tan^{-1}\left(\frac{2-1}{1+1.2}\right) + \tan^{-1}\left(\frac{3-2}{1+2.3}\right) + \ldots$ Verify

40. $T_n = \frac{1}{n^2 + (n-2)} = \frac{1}{(n+2)(n-1)}$, where

$n = 3,4,5,\ldots\ldots \therefore S_\infty = \sum_{n=3}^{\infty}\frac{1}{3}\left[\frac{1}{n-1} - \frac{1}{n+2}\right] = \frac{13}{36}$

41. $t_n = \frac{1}{a}\left[\begin{array}{l}(1-a)(1-2a)\ldots\{1-(n-1)a\} \\ -(1-a)(1-2a)\ldots(1-na)\end{array}\right]$

Put n=1,2,3,...n and add.

42. $\sum_{k=1}^{n}\sum_{j=1}^{k}\sum_{i=1}^{j} 2 = \sum_{k=1}^{n}\sum_{j=1}^{n} 2j$

$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$

$\therefore S_n = \frac{n(n+1)(2n+1)}{3}$

$\Rightarrow t_r = S_r - S_{r-1} = \frac{r(r+1)(r+2)}{3} - \frac{(r-1)(r+1)}{3}$

$\therefore \sum_{r=1}^{n}\frac{1}{t_r} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$

43. $t_n = \frac{n}{(1+n^2)^2 - n^2} \Rightarrow S_n = \frac{1}{2}\left|1 - \frac{1}{1+n+n^2}\right|$

and $\Rightarrow S_{20} = \frac{1}{2}\left[1 - \frac{1}{421}\right] = \frac{210}{421}$

44. $\sum a_i b_i = \sum a_i(1-a_i) = na - \sum a_i^2$

$= na - \sum(a_i - a)^2 - \sum a^2 - 2a\sum(a_i - a)$

$\Rightarrow \sum a_i b_i + \sum(a_i - a)^2$ =nab

$\left(\therefore \sum b_i = \sum 1 - \sum a_i, \therefore nb = n - na\,(or)\ \text{a+b=1}\right)$

45. Sum of first n odd natural numbers $= n^2$

$\therefore \left(\frac{p+1}{2}\right)^2 + \left(\frac{q+1}{2}\right)^2 = \left(\frac{r+1}{2}\right)^2$

$\therefore p+1 = 8,\ q+1 = 6,\ r+1 = 10$

46. $T_n = \frac{n^2}{500 + 3n^3}$;

$\frac{dT_n}{dn} = \frac{n(1000 - 3n^3)}{(500 + 3n^3)^2} = 0$

$n = \left(\frac{1000}{3}\right)^{\frac{1}{3}}$ between 6 and 7

Hence T_7 is largest term

47. Let the n consecutive odd integers be $2k+1, 2k+3, 2k+5, \ldots + 2k + 2n - 1$

Given $(n+k)^2 - k^2 = 25^2 - 11^2$

$\therefore k = 11, n + k = 25 \Rightarrow n = 14$

48. $a_{n+2} + a_n - 2a_{n+1} = 0$

$a_2 + a_{102} = 2a_{52}, a_3 + a_{103} = 2a_{53}$.

49. In the given Sequence 1^{st} term is 1.

The first 2 is in term 2

The first 4 is in term 4

The first 8 is in term 8

The sequence is doubling the first number and putting that number in the sequence for however many terms it is worth, i.e 8 is in the sequence 8 times, 4 is in the sequence 4 times, because we double the number each time, we know the pattern will go

1,2,4,8,16,32,64,128,256,512,1024,.....

So that means the number 1024 will start from 1024^{th} term

$\therefore$ 1025 term is also $1024 = 2^{10}$

50. $\tan^{-1} y - \tan^{-1} x = \tan^{-1} z - \tan^{-1} y$

$\frac{y-x}{1+xy} = \frac{z-y}{1+zy}$ .... (1)

x, y, z are in AP

y - x = z - y .... (2)

from (1) and (2) $1 + xy = 1 + zy$

$x = z \quad \therefore x = y = z \therefore x, y, z$ are in A.P

JEE MAINS QUESTIONS

1.
The common difference of the A.P. $b_1, b_2, ..., b_m$ is 2 more than the common difference of A.P. $a_1, a_2, ..., a_n$. If $a_{40} = -159$, $a_{100} = -399$ and $b_{100} = a_{70}$, then b_1 is equal to:

(1) 81 (2) – 127 (3) – 81 (4) 127

2.
If $3^{2\sin 2\alpha - 1}$, 14 and $3^{4-2\sin 2\alpha}$ are the first three terms of an A.P. for some α, then the sixth term of this A.P is:

(1) 66 (2) 81 (3) 65 (4) 78

3.
Let $a_1, a_2, ..., a_n$ be a given A.P. whose common difference is an integer and $S_n = a_1 + a_2 + ... + a_n$. If $a_1 = 1$, $a_n = 300$ and $15 \le n \le 50$, then the ordered pair (S_{n-4}, a_{n-4}) is equal to :

(1) (2490, 249) (2) (2480, 249)
(3) (2480, 248) (4) (2490, 248)

4.The number of terms common to the two A.P.'s 3, 7, 11, ...,407 and 2, 9, 16, ..., 709 is _____.

5.
Let $f: R \to R$ be such that for all $x \in R$, $(2^{1+x} + 2^{1-x})$, $f(x)$ and $(3^x + 3^{-x})$ are in A.P., then the minimum value of $f(x)$ is:

(1) 2 (2) 3 (3) 0 (4) 4

6.
Let S_n denote the sum of the first n terms of an A.P. If $S_4 = 16$ and $S_6 = -48$, then S_{10} is equal to :

(1) –260 (2) –410 (3) –320 (4) –380

7.
If $a_1, a_2, a_3, ...$ are in A.P. such that $a_1 + a_7 + a_{16} = 40$, then the sum of the first 15 terms of this A.P. is :

(1) 200 (2) 280 (3) 120 (4) 150

8.
If $a_1, a_2, a_3, ... a_n$ are in A.P. and $a_1 + a_4 + a_7 + ... + a_{16} = 114$, then $a_1 + a_6 + a_{11} + a_{16}$ is equal to :

(1) 98 (2) 76 (3) 38 (4) 64

9.If the sum and product of the first three terms in an A.P.are 33 and 1155, respectively, then a value of its 11th term is

(1) –35 (2) 25 (3) –36 (4) –25

10.The sum of all natural numbers 'n' such that 100 < n < 200 and H.C.F. (91, n) > 1 is

(1) 3203 (2) 3303 (3) 3221 (4) 3121

11.If 19th term of a non-"ero A.P. is Zero, then its (49th term) :(29th term) is

(1) 4 : 1 (2) 1 : 3 (3) 3 : 1 (4) 2 : 1

12.The sum of all two digit positive numbers which whendivided by 7 yield 2 or 5 as remainder is:

13.
Let a, b, c, d and p be any non zero distinct real numbers such that $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + (b^2 + c^2 + d^2) = 0$. Then :

(1) a, c, p are in A.P . (2) a, c, p are in G.P.
(3) a, b, c, d are in G.P. (4) a, b, c, d are in A.P.

14.
Suppose that a function f: R→R satisfies $f(x+y) = f(x)f(y)$ for all x, $y \in$ R and f(a) = 3. If $\sum_{i=1}^{n} f(i) = 363$, then n is equal to _____.

15.

Let α and β be the roots of $x^2-3x+p=0$ and γ and δ be the roots of $x^2-6x+q=0$. If $\alpha, \beta, \gamma, \delta$ form a geometric progression. Then ratio $(2q+p):(2q-p)$ is :

(1) 3 : 1 (2) 9 : 7 (3) 5 : 3 (4) 33 : 31

16.

The value of $(0.16)\log_{2.5}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots \text{ to } \infty\right)$ is equal to ________ .

17.

Let a_n be the n^{th} term of a G.P. of positive terms.

If $\sum_{n=1}^{100} a_{2n+1}=200$ and $\sum_{n=1}^{100} a_{2n}=100$, then $\sum_{n=1}^{200} a_n$ is equal to :

(1) 300 (2) 225 (3) 175 (4) 150

18.

If $x=\sum_{n=0}^{\infty}(-1)^n \tan^{2n}\theta$ and $y=\sum_{n=0}^{\infty}\cos^{2n}\theta$, for $0<\theta<\frac{\pi}{4}$,

(1) x(1 + y) = 1 (2) y(1 – x) = 1
(3) y(1 + x) = 1 (4) x(1 – y) = 1

19.

The greatest positive integer k, for which 49^k+1 is a factor of the sum $49^{125}+49^{124}+\ldots+49^2+49+1$, is:

(1) 32 (2) 63 (3) 60 (4) 65

20. If m arithmetic means (A.Ms) and three geometric means (G.Ms) are inserted between 3 and 243 such that 4th A.M. is equal to 2nd G.M., then m is equal to ___________.

21.

If $1+(1-2^2\cdot 1)+(1-4^2\cdot 3)+(1-6^2\cdot 5)+\ldots\ldots+(1-20^2\cdot 19)=\alpha-220\beta$, then an ordered pair (α, β) is equal to :

(1) (10, 97) (2) (11, 103)
(3) (10, 103) (4) (11, 97)

22.

Let $f:\mathbf{R}\rightarrow\mathbf{R}$ be a function which satisfies $f(x+y)=f(x)+f(y), \forall\, x, y \in \mathbf{R}$. If $f(a)=2$ and $g(n)=\sum_{k=1}^{(n-1)} f(k), n\in\mathbf{N}$, then the value of n, for which $g(n)=20$, is:

(1) 5 (2) 20 (3) 4 (4) 9

23.

The sum, $\sum_{n=1}^{7}\frac{n(n+1)(2n+1)}{4}$ is equal to ________.

24.

The sum $\sum_{k=1}^{20}(1+2+3+\ldots+k)$ is ________.

25..Some identical balls are arranged in rows to form anequilateral triangle. The first row consists of one ball, the second row consists of two balls and so on. If 99 more identical balls are added to the total number of balls used in forming the equilateral triangle, then all these balls can be arranged in a square whose each side contains exactly 2 balls less than the number of balls each side of the triangle contains. Then the number of balls used to form the equilateral triangle is

(1) 157 2) 262 (3) 225 (4) 190

KEY

1. 3	2. 1	3. 4	4. 14	5. 2
6. 3	7. 1	8. 2	9. 4	10. 4
11. 3	12. 4	13. 3	14. 5	15. 2
16. 4	17. 4	18. 2	19.2	20. 39
21. 2	22. 1	23. 504	24. 1540	25. 4

SOLUTIONS

1.

Let common difference of series

$a_1, a_2, a_3, \ldots\ldots, a_n$ be d.

$\because a_{40} = a_1 + 39d = -159 \quad \ldots(\text{i})$

and $a_{100} = a_1 + 99d = -399 \quad \ldots(\text{ii})$

From equations (i) and (ii),

$d = -4$ and $a_1 = -3$

Since, the common difference of $b_1, b_2, \ldots\ldots, b_n$ is 2 more than common difference of $a_1, a_2, \ldots\ldots, a_n$.

$\therefore$ Common difference of $b_1, b_2, b_3, \ldots\ldots$ is (-2).

$\because b_{100} = a_{70}$

$\Rightarrow b_1 + 99(-2) = (-3) + 69(-4)$

$\Rightarrow b_1 = 198 - 279 \Rightarrow b_1 = -81$

2.

Given that $3^{2\sin 2\alpha - 1}, 14, 3^{4 - 2\sin 2\alpha}$ are in A.P.

So, $3^{2\sin 2\alpha - 1} + 3^{4 - 2\sin 2\alpha} = 28$

$$\Rightarrow \frac{3^{2\sin 2\alpha}}{3} + \frac{81}{3^{2\sin 2\alpha}} = 28$$

Let $3^{2\sin 2\alpha} = x$

$$\Rightarrow \frac{x}{3} + \frac{81}{x} = 28$$

$\Rightarrow x^2-84x+243=0 \Rightarrow x=81,\ x=3$

When $x=81 \Rightarrow \sin 2\alpha = 2$ (Not possible)

When $x=3 \Rightarrow \alpha=\dfrac{\pi}{12}$

$\therefore a=3^0=1,\ d=14-1=13$

$a_6=a+5d=1+65=66$

3.

Given that $a_1=1$ and $a_n=300$ and $d\in \mathbf{Z}$

$\therefore 300=1+(n-1)d$

$\Rightarrow d=\dfrac{299}{(n-1)}=\dfrac{23\times 13}{(n-1)},$

$\because$ d is an integer

$\therefore n-1=13$ or 23

$\Rightarrow n=14$ or 24 $\quad (\because 15\le n\le 50)$

$\Rightarrow n=24$ and $d=13$

$a_{20}=1+19\times 13=248$

$s_{20}=\dfrac{20}{2}(2+19\times 13)=2490.$

4.

First common term of both the series is 23 and common difference is $7\times 4=28$

$\because$ Last term ≤ 407

$\Rightarrow\ 23+(n-1)\times 28\le 407$

$\Rightarrow\ (n-1)\times 28\le 384$

$\Rightarrow\ n\le \dfrac{384}{28}+1$

$\Rightarrow\ n\le 14.71$

Hence, $n=14$

5.

If $2^{1-x}+2^{1+x}, f(x), 3^x+3^{-x}$ are in A.P., then

$$f(x)=\left(\frac{2^{1+x}+2^{1-x}+3^x+3^{-x}}{2}\right)$$

$$2f(x)=2\left(2^x+\frac{1}{2^x}\right)+\left(3^x+\frac{1}{3^x}\right)$$

Using AM $\ge$ GM

$f(x)\ge 3$

6.

Given, $S_4=16$ and $S_6=-48$

$\Rightarrow\ 2(2a+3d)=16 \Rightarrow 2a+3d=8 \quad$ ...(i)

And $3[2a+5d]=-48 \Rightarrow 2a+5d=-16$

$\Rightarrow\ 2d=-24 \quad$ [using equation (i)]

$\Rightarrow\ d=-12$ and $a=22$

$\therefore\ S_{10}\ \dfrac{10}{2}=(44+9(-12))=-320$

7.

Let the common difference of the A.P. is 'd'.

Given, $a_1 + a_7 + a_{16} = 40$

$\Rightarrow \quad a_1 + a_1 + 6d + a_1 + 15d = 40$

$\Rightarrow \quad 3a_1 + 21d = 40$

$\Rightarrow \quad a_1 + 7d = \frac{40}{3}$ ...(i)

Now, sum of first 15 terms of this A.P. is,

$S_{15} = \frac{15}{2}[2a_1 + 14d] = 15(a_1 + 7d)$

$= 15\left(\frac{40}{3}\right) = 200$ [Using (i)]

8.

$a_1 + a_4 + a_7 + ... + a_{16} = 114$

$\Rightarrow 3(a_1 + a_{16}) = 114$

$\Rightarrow a_1 + a_{16} = 38$

Now, $a_1 + a_6 + a_{11} + a_{16} = 2(a_1 + a_{16}) = 2 \times 38 = 76$

9.

Let three terms of A.P. are $a - d, a, a + d$

Sum of terms is, $a - d + a + a + d = 33 \Rightarrow a = 11$

Product of terms is, $(a-d)\,a\,(a+d) = 11(121 - d^2) = 1155$

$\Rightarrow 121 - d^2 = 105 \Rightarrow d = \pm 4$

if $d = 4$

$T_{11} = T_1 + 10d = 7 + 10(4) = 47$

if $d = -4$

$T_{11} = T_1 + 10d = 15 + 10(-4) = -25$

10.

$\because 91 = 13 \times 7$

Then, the required numbers are either divisible by 7 or 13.

$\therefore$ Sum of such numbers = Sum of no. divisible by 7 + sum of the no. divisible by 13 – Sum of the numbers divisible by 91

$= (105 + 112 + ... + 196) + (104 + 117 + ... + 195) - 182$

$= 2107 + 1196 - 182 = 3121$

11. Let first term and common difference of AP be a and d respectively, then

-

$t_n = a + (n-1)d$

$\therefore \quad t_{19} = a + 18d = 0$

$\therefore \quad a = -18d$

$\therefore \quad \frac{t_{49}}{t_{29}} = \frac{a + 48d}{a + 28d}$

$= \frac{-18d + 48d}{-18d + 28d} = \frac{30d}{10d} = 3$

$t_{49} : t_{29} = 3 : 1$

12.

Two digit positive numbers which when divided by 7 yield 2 as remainder are 12 terms i.e,16, 23, 30, ..., 93

Two digit positive numbers which when divided by 7 yield 5 as remainder are 13 terms i.e,12, 19, 26, ..., 96

By using AP sum of 16, 23, ..., 93, we get

$S_1 = 16 + 23 + 30 + ... + 93 = 654$

By using AP sum of 12, 19, 26, ..., 96, we get

$S_1 = 12 + 19 + 26 + ... + 96 = 702$

$\therefore$ required Sum $= S_1 + S_2 = 654 + 702 = 1356$

13.

Rearrange given equation, we get

$$(a^2p^2 - 2abp + b^2) + (b^2p^2 - 2bcp + c^2) + (c^2p^2 - 2cdp + d^2) = 0$$

$$\Rightarrow (ap-b)^2 + (bp-c)^2 + (cp-d)^2 = 0$$

$$\therefore ap - b = bp - c = cp - d = 0$$

$$\Rightarrow \frac{b}{a} = \frac{c}{b} = \frac{d}{c} \qquad \therefore a, b, c, d \text{ are in G.P.}$$

14.

$$\because f(x+y) = f(x)\cdot f(y) \qquad \forall x \in R \text{ and } f(1) = 3$$

$$\Rightarrow f(x) = 3^x \Rightarrow f(i) = 3^i$$

$$\Rightarrow \sum_{i=1}^{n} f(i) = 363$$

$$\Rightarrow 3 + 3^2 + 3^3 + ... + 3^n = 363$$

$$\Rightarrow \frac{3(3^n - 1)}{3-1} = 363 \qquad \left[\because S_n = \frac{a(r^n - 1)}{(r-1)}\right]$$

$$\Rightarrow 3^n - 1 = \frac{363 \times 2}{3} = 242$$

$$\Rightarrow 3^n = 243 = 3^5 \Rightarrow n = 5$$

15.

Let $\alpha, \beta, \gamma, \delta$ be in G.P., then $\alpha\delta = \beta\gamma$

$$\Rightarrow \frac{\alpha}{\beta} = \frac{\gamma}{\delta} \Rightarrow \left|\frac{\alpha-\beta}{\alpha+\beta}\right| = \left|\frac{\gamma-\delta}{\gamma+\delta}\right|$$

$$\Rightarrow \frac{\sqrt{9-4p}}{3} = \frac{\sqrt{36-4q}}{6}$$

$$\Rightarrow 36 - 16p = 36 - 4q \Rightarrow q = 4p$$

$$\therefore \frac{2q+p}{2q-p} = \frac{8p+p}{8p-p} = \frac{9p}{7p} = \frac{9}{7}$$

16.

$$(0.16)^{\log_{2.5}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+.....\infty\right)}$$

$$= 0.16^{\log_{2.5}\left(\frac{\frac{1}{3}}{1-\frac{1}{3}}\right)} \qquad \left[\because S_\infty = \frac{a}{1-r}\right]$$

$$= 0.16^{\log_{2.5}\left(\frac{1}{2}\right)}$$

$$= (2.5)^{-2\log_{2.5}\left(\frac{1}{2}\right)} = \left(\frac{1}{2}\right)^{-2} = 4.$$

17.

Let G.P. be a, ar, ar^2 .....

$$\sum_{n=1}^{100} a_{2n+1} = a_3 + a_5 + ... + a_{201} = 200$$

$$\Rightarrow \frac{ar^2(r^{200}-1)}{r^2-1} = 200$$

$$\sum_{n=1}^{100} a_{2n} = a_2 + a_4 + ... + a_{200} = 100$$

$$\Rightarrow \frac{ar(r^{200}-1)}{r^2-1} = 100$$

From equations (i) and (ii), $r = 2$ and

$a_2 + a_3 + ... + a_{200} + a_{201} = 300$

$\Rightarrow r(a_1 + ... + a_{200}) = 300$

$$\Rightarrow \sum_{n=1}^{200} a_n = \frac{300}{r} = 150$$

18.

$y = 1 + \cos^2\theta + \cos^4\theta + ...$

$$\Rightarrow y = \frac{1}{1-\cos^2\theta} \Rightarrow \frac{1}{y} = \sin^2\theta$$

$x = 1 - \tan^2\theta + \tan^4\theta + ...$

$$x = \frac{1}{1-(-\tan^2\theta)} = \frac{1}{\sec^2\theta} \Rightarrow x = \cos^2\theta$$

$$y = \frac{1}{\sin^2\theta} \Rightarrow y = \frac{1}{1-x}$$

$\therefore \ y(1-x) = 1$

19.

$$\frac{(49)^{126}-1}{48} = \frac{((49)^{63}+1)(49^{63}-1)}{48} \left[\because S_n = \frac{a(r^n-1)}{r-1}\right]$$

$\therefore$ K = 63

20.

Let m arithmetic mean be $A_1, A_2 ... A_m$ and G_1, G_2, G_3 be geometric mean.

The A.P. formed by arithmetic mean is,

$3, A_1, A_2, A_3, ...A_m, 243$

$$\therefore d = \frac{243-3}{m+1} = \frac{240}{m+1}$$

The G.P. formed by geometric mean

$3, G_1, G_2, G_3, 243$

$$r = \left(\frac{243}{3}\right)^{\frac{1}{3+1}} = (81)^{1/4} = 3$$

$\because A_4 = G_2$

$$\Rightarrow 3 + 4\left(\frac{240}{m+1}\right) = 3(3)^2$$

$$\Rightarrow 3 + \frac{960}{m+1} = 27 \Rightarrow m+1 = 40 \Rightarrow m = 39.$$

21.

The given series is

$$1+(1-2^2\cdot1)+(1-4^2\cdot3)+(1-6^2\cdot5)+...(1-20^2\cdot19)$$

$$S=1+\sum_{r=1}^{10}[1-(2r)^2(2r-1)]$$

$$=1+\sum_{r=1}^{10}(1-8r^3+4r^2)=1+10-\sum_{r=1}^{10}(8r^3-4r^2)$$

$$=11-8\left(\frac{10\times11}{2}\right)^2+4\times\left(\frac{10\times11\times21}{6}\right)$$

$$=11-2\times(110)^2+4\times55\times7$$

$$=11-220(110-7)$$

$$=11-220\times103=\alpha-220\beta$$

$$\Rightarrow\alpha=11,\ \beta=103$$

$$\therefore(\alpha,\beta)=(11,103)$$

22.

Given : $f(x+y)=f(x)+f(y),\ \forall x,y\in R,\ f(1)=2$

$$\Rightarrow f(2)=f(1)+f(1)=2+2=4$$

$$f(3)=f(1)+f(2)=2+4=6$$

$$f(n-1)=2(n-1)$$

Now, $g(n)=\sum_{k=1}^{n-1}f(k)$

$$=f(1)+f(2)+f(3)+....f(n-1)$$

$$=2+4+6+.....+2(n-1)$$

$$=2[1+2+3+.....+(n-1)]$$

$$=2\times\frac{(n-1)(n)}{2}=n^2-n$$

$\because g(n)=20$ (given)

So, $n^2-n=20$

$$\Rightarrow n^2-n-20=0$$

$$\Rightarrow(n-5)(n+4)=0$$

$\Rightarrow n=5$ or $n=-4$ (not possible)

23.

$$\left[\sum_{n=1}^{7}\frac{n(n+1)(2n+1)}{4}\right]\frac{1}{4}\left[\sum_{n=1}^{7}(2n^3+3n^2+n)\right]$$

$$=\frac{1}{4}\left[2\left(\frac{7.8}{2}\right)^2+3\left(\frac{7.8.15}{6}\right)+\frac{7.8}{2}\right]$$

$$\Rightarrow\ \frac{1}{4}[2\times49\times16+28\times15+28]$$

$$=\frac{1}{4}[1568+420+28]=504$$

24.

Given series can be written as

$$\sum_{k=1}^{20}\frac{k(k+1)}{2}=\frac{1}{2}\sum_{k=1}^{20}(k^2+k)$$

$$=\frac{1}{2}\left[\frac{20(21)(41)}{6}+\frac{20(21)}{2}\right]$$

$$=\frac{1}{2}\left[\frac{420\times41}{6}+\frac{20\times21}{2}\right]=\frac{1}{2}[2870+210]=1540$$

25.

Number of balls used in equilateral triangle

$$=\frac{n(n+1)}{2}$$

$\because$ side of equilateral triangle has n-balls

$\therefore$ no. of balls in each side of square is $=(n-2)$

According to the question,

$$\frac{n(n+1)}{2}+99=(n-2)^2$$

$\Rightarrow n^2+n+198=2n^2-8n+8$

$\Rightarrow n^2-9n-190=0 \Rightarrow (n-19)(n+10)=0$

$\Rightarrow n=19$

Number of balls used to form triangle

$$=\frac{n(n+1)}{2}=\frac{19\times20}{2}=190$$

STATISTICS

SYNOPSIS

FREQUENCY DISTRIBUTION

- **Class Limits:** The starting and end vlaues of each class are called the lower limit and upper limit respectively of that class.

Ex. 1) The lower limit of the class 0-9 is 0

2) The upper limit of the class 50-59 is 59

- **Class boundaries :** The average of the upper limit of a class and the lower limit of the next class is called the upper boundary of that class. The upper boundary of a class becomes the lower boundary of the next class. These boundaries are called True class limits.

Ex. 1) 1-10, 11-20, 21-30 ..... are the classes, the lower boundary of the class 11-20 is $\frac{10+11}{2}=10.5$

2) 60-69, 70-79, 80-89, 90-99 .... are the classes, the upper boundary of the class 70-79 is $\frac{79+80}{2}=79.5$

- **Class interval (or) the size of the class :** The difference between the lower limits or the upper limits of two consecutive classes is called the Class-interval (or) the size of the class.

  Ex. The class interval in the frequency distribution with the classes 1-8, 9-16, 17-24 ... length of class = 9-1 = 8

- **Mid value of the class :** Mid value of class 1-10 is $\frac{1+10}{2}=5.5$
- For over lapping classes 0-10, 10-20, 20-30 etc the class mark of the class 0-10 is $\frac{0+10}{2}=5$

3) For non over lapping class 0-19, 20-39, 40-59,...... etc the class mark of the class 20-39 is $\left(\frac{20+39}{2}\right)=29.5$

- **Measures of Central Tendency:** One of the most important objectives of statistical analysis is to get one single value that describes the characteristic of the entire data. Such a value is called the central value or an average.

  The following are the important types of averages:

  1. Arithmetic Mean **2. Geometric Mean**
  3. Harmonic mean **4. Median**
  5. Mode

  We consider these measures in three cases (i) Individual series (i.e. each individual observation is given) (ii) discrete series (i.e the observations along with number of times a particular observation called the frequency is given) (iii) continuous series (i.e. the class intervals along with their frequencies are given)

Arithmetic Mean :

- **Individual Series :** If $x_1, x_2 ....x_n$ are the values of the variable x, then the arithmetic mean usually denoted by $\overline{x}$ $or\ \mu\ or\ E(x)$ is given by

$$\overline{x}=\frac{x_1+x_2.....+x_n}{n}=\frac{1}{n}\sum_{i=1}^{n}x_i$$

Note: A.M. $(\overline{x})=A+\frac{\sum(x_i-A)}{n}$ where A is the assumed average. (For individual series)

- **Discrete Series :** If a variable takes values $x_1, x_2 ....x_n$ with corresponding frequencies $f_1, f_2 ..... f_n$ then the arithmetic mean $\overline{x}$ is given by

$$\overline{x}=\frac{f_1x_1+f_2x_2.....+f_nx_n}{f_1+f_2....+f_n}=\frac{1}{N}\sum_{i=1}^{n}f_ix_i,$$

where $N=\sum_{i=1}^{n}f_i$

- **Continuous Series :** In case of a set of data with class intervals, we cannot find the exact value of the mean because we do not know the exact values of the variables. We, therefore, try to obtain an approximate value of the mean. The method of approximate is to replace all the observed values belonging to a class by mid-value of the class. If $x_1, x_2 \ldots x_n$ are the mid values of the class intervals having corresponding frequencies $f_1, f_2 \ldots f_n$ then we apply the same formula as in discrete series.

  $$\bar{x} = \frac{1}{N}\sum_{i=1}^{n} f_i x_i, \quad N = \sum_{i=1}^{n} f_i$$

- **Combined Arithmetic Mean:** If $\bar{x}_i (i = 1, 2, \ldots, k)$ are the means of k - series of sizes $n_i (i = 1, 2, 3, \ldots, k)$ respectively, then the combined or composite mean $\bar{x}$ can be obtained by the formula :

  $$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + \ldots + n_k\bar{x}_k}{n_1 + n_2 + \ldots + n_k} = \frac{\sum n_i \bar{x}_i}{\sum n_i}$$

- **Weighted Arithmetic Mean :**

  Let $w_1, w_2, \ldots, w_n$ be the weights assigned to the values $x_1, x_2, \ldots, x_n$ respectively of a variable x, then the weighted A.M. is $\bar{x} = \dfrac{\sum w_i x_i}{\sum w_i}$.

Properties of Arithmetic Mean :

- Sum of all the deviations from arithmetic mean is zero i.e.,

  $$\sum_{i=1}^{n}\left(x_i - \bar{x}\right) = 0 \text{ (in case of individual series)}$$

  $$\sum_{i=1}^{n} f_i\left(x_i - \bar{x}\right) = 0 \text{ (in case of discrete or continuous series)}$$

- If each observation is increased or decreased by a given constant K, the mean is also increased or decreased by K

  The property is also known as effect of change of origin. K can be taken to be any number. However, to simplify the calculations, K should be taken as a value which is in the middle of the table.

- Step Deviation Method or change of scale

  If $x_1, x_2, \ldots, x_n$ are mid values of class intervals with corresponding frequencies $f_1, f_2, \ldots, f_n$ then we may change the scale by taking $d_i = \dfrac{x_i - A}{h}$, in this case.

  $$\bar{x} = A + h \times \left(\frac{1}{N}\sum f_i d_i\right) \text{(if A is assumed mean)}$$

  A and h can be any numbers but if the lengths of class intervals are equal then h may be taken as width of the class interval.

  In particular if each observation is multiplied or divided by a constant, the mean is also multiplied or divided by the same constant.

- The sum of the squared deviation of the variate from their mean is minimum i.e., the quantity $\sum(x_i - A)^2$ or $\sum f_i(x_i - A)^2$ is minimum when $A = \bar{x}$

- $E(aX + b) = aE(X) + b$ (where $E(X)$ = Mean of X)

Geometric Mean :

- In case of individual series $x_1, x_2 \ldots x_n$

  G.M. = $(x_1 x_2 \ldots x_n)^{1/n}$

  In case of discrete or continuous series

  G.M. $= \left(x_1^{f_1} x_2^{f_2} \ldots x_n^{f_n}\right)^{1/N}$, where $N = \sum_{i=1}^{n} f_i$

- **Harmonic Mean:** The harmonic mean is based on the reciprocals of the value of the variable

  $$\text{H.M.} = \frac{1}{\frac{1}{n}\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}\right)} \text{ or } \frac{1}{H} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}$$

  (Incase of Individual series)

  and $\dfrac{1}{H} = \dfrac{1}{N}\sum_{i=1}^{n} f_i \dfrac{1}{x_i}$ (in case of discrete series or continuous series)

  If $x_1, x_2, \ldots x_n > 0$ then it is known that

  A.M $\geq$ G.M $\geq$ H.M

Median :

- **Individual Series :** If the items are arranged in ascending or descending order of magnitude then the middle value is called median.

  In case of odd number of values Median = size of

$\frac{n+1}{2}th$ item. In case of even number of values

Median = average of $\frac{n}{2}$ th and $\frac{n+2}{2}$ th observation.

- **Discrete Frequency Distribution :** Arrange the data in ascending or descending order. Find the cumulative frequencies.

Apply the formula :

Median = Size of $\left(\frac{N+1}{2}\right)^{\text{th}}$ item (N is odd)

$$= \frac{1}{2}\left[\left(\frac{N}{2}\right)^{th} \text{observation} + \left(\frac{N}{2}+1\right)^{th} \text{observation}\right]$$

(N is even)

$N = \sum f_i$ = sum of frequencies

- **Continuous Frequency Distribution :** Consider the cumulative frequency (c.f.). Find $\frac{N}{2}$, where $N = \sum_{i=1}^{n} f_i$. Find the cumulative frequency (c.f.) just more than N/2. The corresponding value of x is median.In case of continuous distribution, the class corresponding to c.f. just more than $\frac{N}{2}$ is called the median class and the median is obtained by Median = $l + \frac{h}{f}\left(\frac{N}{2} - C\right)$

Where l = the lower limit of the median class;

f = the frequency of the median class;

h = the width of the median class;

C = the **c.f.** of the class preceding to the median class and

$N = \sum_{i=1}^{n} f_i$

Measures of Dispersion: Literally, dispersion means 'scatteredness'. Dispersion measures the degree of scatteredness of the variable about a central value. Different measures of dispersion are

1. Range 2. Mean-deviation 3. Quartile deviation 4. Standard deviation

- **Range:** The range is the difference between the largest and smallest observation.

Coefficient of Range = $\frac{\text{Range}}{\text{Maximum} + \text{Minimum}}$

- **Mean-deviation:** If $x_1, x_2............x_n$ are n observations then mean deviation about a point M is given by

M.D. $= \frac{1}{n}\sum |x_i - M|$ where M is mean or median or mode

In case of discrete or continuous series

M.D. $= \frac{1}{N}\sum f_i |x_i - M|, N = \sum_{i=1}^{n} f_i$

M.D. is least when taken from the median

Coefficient of Mean Deviation

$= \frac{\text{Mean Deviation}}{M}$

where M is the Mean, Median or Mode

- **Quartile Deviation:** Q.D. = $\frac{Q_3 - Q_1}{2}$, where Q_3 and Q_1, are the third quartile and the first quartile. Q_1 and Q_3 can be calculated in a similar manner as median. In fact, quartiles divides the data into four parts.

In case of **individual series**, arrange the data in ascending or descending order.

Q_1 = size of $\frac{n+1}{4}th$ and Q_3 = size of $\frac{3(n+1)}{4}th$ item

In case of **discrete frequency distribution**, Q_1 is obtained by considering cumulative frequency. Find N/4, where $N = \sum f_i$. Find the cumulative frequency (c.f.) just more than N/4. The corresponding value of x is Q_1. Similarly for obtaining Q_3, find 3N/4 and the c.f. just more than 3N/4. The corresponding value of x is Q_3, In case of continuous distribution.

$Q_i = l + \frac{h}{f}\left(i\frac{N}{4} - c\right), i = 1, 2, 3$

Where l = the lower limit of the class whose c.f. is

just more than iN/4,
f is its frequency and h is its width. C = c.f. of the class preceding to the class whose c.f. is just more than iN/4, i = 1,2,3.
Note that i = 2 will given us median.

Coefficient of Quartile Deviation $=\frac{Q_3-Q_1}{Q_3+Q_1}$

✈ **Standard Deviation:** Variance σ^2 in case of **individual series** is given by

$$\sigma^2=\frac{1}{n}\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2=\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2$$

If $x_1, x_2, \ldots\ldots, x_n$ occur with frequency $f_1, f_2 \ldots f_n$ respectively then σ^2 (variance)

$$=\frac{1}{N}\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2=\frac{1}{N}\sum f_i x_i^2-\left(\frac{1}{n}\sum f_i x_i\right)^2$$

Standard deviation = the positive square root of variance

There is no effect of change of origin on standard deviation

$$\sigma_x^2=h^2\left[\frac{1}{N}\sum f_i d_i^2-\left(\frac{1}{N}\sum f_i d_i\right)^2\right]$$

Coefficient of Standard Deviation is $\frac{\sigma}{\bar{x}}$.

Coefficient of variation = $\frac{\sigma}{\bar{x}}\times 100$

✈ **Combined variance:** If there are two samples of sizes n_1 and n_2 with $\bar{x}_1$ and $\bar{x}_2$ as their means σ_1 and σ_2 their standard deviations respectively, then the combined variance is given by

$$\sigma^2=\frac{1}{n_1+n_2}\left[n_1\sigma_1^2+n_2\sigma_2^2+\frac{n_1n_2}{n_1+n_2}(\bar{x}_1-\bar{x}_2)^2\right]$$

or $$\sigma^2=\frac{n_1\left(\sigma_1^2+d_1^2\right)+n_2\left(\sigma_2^2+d_2^2\right)}{n_1+n_2}$$

where $d_1=\bar{x}_1-\bar{x}$ and $d_2=\bar{x}_2-\bar{x}$, $\bar{x}$ being the combined mean.

Note :1

if $\vee$ (X) is variance of X then

$$V(X+a)=V(X)$$

$$V(aX)=a^2V(X)$$

$$V(aX+b)=a^2V(X)$$

$$V\left(aX+bY\right)=a^2v\left(X\right)+b^2v\left(Y\right)$$

Note :2

For a, $a+d$, $a+2d$, ..., $a+(n-1)d$,

$$\bar{x}=a+\frac{(n-1)d}{2};\ \sigma^2=\frac{n^2-1}{12}d^2$$

Note:3

1) Q.D < M.D < S.D

2) $\frac{\text{Q.D}}{10}=\frac{\text{M.D}}{12}=\frac{\text{S.D}}{15}$

EXAMPLES

1. **The weighted mean of the first n natural numbers, the weights being the corresponding numbers, is**

Sol. First n natural numbers are 1, 2, 3,...,n; whose corresponding weights are 1, 2, 3,...,n respectively.

$\therefore$ weight mean $=\frac{1\times1+2\times2+.....+n\times n}{1+2+....+n}$

$$=\frac{1^2+2^2+.....+n^2}{1+2+.....+n}$$

$$=\frac{\frac{n(n+1)(2n+1)}{6n(n+1)}}{2}=\frac{2n+1}{3}$$

2. **The weighted mean of the first n natural numbers whose weights are equal to the squares of the corresponding numbers is**

Sol. weighted mean $=\frac{1.1^2+2.2^2+....+n.n^2}{1^2+2^2+....+n^2}$

$$=\frac{\Sigma n^3}{\Sigma n^2}=\frac{\frac{n(n+1)}{2}\frac{n(n+1)}{2}}{\frac{n(n+1)(2n+1)}{6}}=\frac{3n(n+1)}{2(2n+1)}$$

3. The average salary of male employees in a firm is Rs. 5200 and that of females is Rs.4200. The mean salary of all the employees is Rs.5000. The percentage of male and female employees are respectively is

Sol. Let $x_1 = 5200, x_2 = 4200, \bar{x} = 5000$

Also, we know that $\bar{x} = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$

$\Rightarrow 5000(n_1 + n_2) = 5200n_1 + 4200n_2 \Rightarrow \dfrac{n_1}{n_2} = \dfrac{4}{1}$

$\therefore$ The percentage of male employees in the firm $= \dfrac{4}{4+1} \times 100 = 80\%$

and the percentage of female employees in the firm $= \dfrac{1}{4+1} \times 100 = 20\%$

4. If the mean of 9 observations is 100 and mean of 6 observations is 80, then the mean of 15 observations is

Sol. $n_1 = 9, \bar{x}_1 = 100 \text{ and } n_2 = 6, \bar{x}_2 = 80$

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} = \frac{9 \times 100 + 6 \times 80}{9+6} = 92$$

5. If a variate X is expressed as a linear function of two variates U and V in the form $X = aU + bV$ then the mean $\bar{X}$ of X is

Sol. we have $\Sigma X = a\Sigma U + b\Sigma V$

$$\bar{X} = \frac{1}{n}\Sigma X = a.\frac{1}{n}\Sigma U + b.\frac{1}{n}\Sigma V$$

$$\Rightarrow \bar{X} = a\bar{U} + b\bar{V}$$

6. If the arithmetic mean of the numbers $x_1, x_2, x_3....x_n$ is $\bar{x}$, then the arithmetic mean of the numbers $ax_1 + b, ax_2 + b, ax_3 + b, ...ax_n + b$, where a, b are two constants, would be

Sol. Required mean

$$= \frac{(ax_1 + b) + (ax_2 + b) + ... + (ax_n + b)}{n}$$

$$= \frac{a(x_1 + x_2 + ... + x_n)}{n} + b = a\bar{x} + b$$

7. If the mean of the numbers $27 + x, 31 + x$, $89 + x$, $107 + x, 156 + x$ is 82, then the mean of $130 + x, 126 + x,\ 68 + x,\ 50 + x, 1 + x$ is

Sol. Given

$$82 = \frac{(27+x)+(31+x)+(89+x)+(107+x)+(156+x)}{5}$$

$\Rightarrow 82 \times 5 = 410 + 5x \Rightarrow 410 - 410 = 5x \Rightarrow x = 0$

Therefore, the required mean is

$$\bar{x} = \frac{130 + x + 126 + x + 68 + x + 50 + x + 1 + x}{5}$$

$$= \frac{375 + 5x}{7} = 75$$

8. A student obtain 75%, 80% and 85% in three subjects. If the marks of another subject is added, then his average cannot be less than

Sol. Marks obtained from three subjects out of 300 is $75 + 80 + 85 = 240$

If the marks of another subject is added, then the marks will be ≥ 240 out of 400

$\therefore$ minimum average marks $= \dfrac{240}{4} = 60\%$

[when marks in the fourth subject = 0]

9. The mean of 100 items is 49. It was discovered that three items which should have been 60, 70, 80 were wrongly read as 40, 20, 50 respectively. The correct mean is

Sol. Sum of 100 items = 49 × 100 = 4900

sum of items added = 60+ 70+80=210

new sum = 4900+210–110=5000

$\therefore$ correct mean $= \dfrac{5000}{100} = 50$

10 The mean weight per student in a group of seven students is 55kg. If the individual weights of six students are 52, 58, 55, 53, 56

and 54, then the weight of the seventh student is

Sol. The total weight of seven students is 55×7 = 385kg

The sum of the weights of six students is 52+58+55+53+56+54=328kg

Hence, the weight of the seventh student is

$= 385 - 328 = 57kg$

11 The geometric mean of the numbers $3, 3^2, 3^3, ..., 3^n$ is

Sol. $\therefore G.M = \left(3.3^2......3^n\right)^{1/n}$

$= 3^{\frac{1+2+.....+n}{n}} = 3^{\frac{n(n+1)}{2n}} = 3^{\frac{n+1}{2}}$

12. Find the harmonic mean of $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ..., \frac{n}{n+1}$, occurring with frequencies 1, 2, 3,.....n, respectively.

Sol. We know that, Harmonic mean $= \frac{\sum f}{\sum\left(\frac{f}{x}\right)}$

$\sum f = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

and $\sum \frac{f}{x} = \frac{1}{1/2} + \frac{2}{2/3} + \frac{3}{3/4} + ... + \frac{n}{n/(n+1)}$

$= 2 + \frac{3\times 2}{2} + \frac{4\times 3}{3} + ... + \frac{n(n+1)}{n}$

$= 2 + 3 + 4 + ... + n + (n+1)$

Which is an arithmetic progression with a = 2 and d = 1.

By the formula of sum of n term of an A.P,

$\sum\left(\frac{f}{x}\right) = \left[\frac{n}{2}\{2a + (n-1)d\}\right]$

we have $= \frac{n}{2}\{2\times 2 + n - 1\} = \frac{n}{2}(3+n)$

$\therefore$ Harmonic mean

$= \frac{2}{n(3+n)} = \frac{n(n+1)\times 2}{n(3+n)\times 2} = \frac{n+1}{3+n}$

13. The median of a set of nine distinct observations is 20.5. If each of the last four observations of the set is increased by 2, then the median of the new set is

Sol. Since n = 9, median term $= \left(\frac{9+1}{2}\right)^{th}$ = 5th term.

Now, the last four observations are increased by 2. Since the median is the 5th observation, which remains unchanged, there will be no change in median.

14. If a variable takes the discrete value $\alpha - 4$, $\alpha - \frac{7}{2}$, $\alpha - \frac{5}{2}$, $\alpha - 3$, $\alpha - 2$, $\alpha + \frac{1}{2}$, $\alpha - \frac{1}{2}$, $\alpha + 5 (a > 0)$, then the median is

Sol. Arrange the data as follows:

$\alpha - \frac{7}{2}, \alpha - 3, \alpha - \frac{5}{2}, \alpha - 2, \alpha - \frac{1}{2},$

$\alpha + \frac{1}{2}, \alpha - 4, \alpha + 5$

median $= \frac{1}{2}$[value of 4th item+value of 5th item]

$\therefore$ median $= \frac{\alpha - 2 + \alpha - \frac{1}{2}}{2} = \alpha - \frac{5}{4}$

15. The median of distribution 83, 54, 78, 64, 90, 59, 67, 72, 70, 73 is

Sol. On arranging in ascending order, we get 54, 59, 64, 67, 70, 72, 73, 78, 83, 90

n = 10

$\therefore median = \frac{value\ of\ \frac{10}{2}^{th}\ term + value\ of\ \left(\frac{10}{2}+1\right)^{th}\ term}{2}$

$= \frac{value\ of\ 5th\ term + value\ of\ 6th\ term}{2}$

$= \frac{70+72}{2} = 71$

Mode : The mode is that value in a series of observations which occurs with greatest frequency.

In case of **individual series**, the mode is the value which occurs more frequently

In case of **discrete series**, quite often mode can be determined just by inspection i.e. by looking to that value of variable around which the items are most heavily concentrated.

In case of **continuous series**,

$$Mode = l + \frac{f_1 - f_0}{2f_1 - f_2 - f_0} \times h$$

Where $l =$ the lower limit of the modal class i.e. the class having maximum frequency; f_1 = frequency of the modal class; f_0= frequency of the class preceding the modal class;

f_2 = frequency of the class succeeding the modal class and

h = width of the modal class.

✈ Relation between Mean, Median and Mode is mean - mode = 3 (mean - median) or

Mode = 3 median - 2 mean

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The mode of the following distribution is

Class interval	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Frequency	5	8	7	12	28	20	10	10

Sol. Here, maximum frequency is 28. Thus, the class 40-50 is the modal class.

$$Mode = l + \frac{f_1 - f_0}{2f_1 - f_2 - f_0} \times h$$

$$= 40 + \frac{10(28-12)}{(2\times 28 - 12 - 20)}$$

$$= 40 + 6.666 = 46.67(approx.)$$

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If in a frequency distribution, the mean and median are 20 and 21 respectively, then its mode is approximately

Sol. mode = 3 median - 2 mean = 3 (21) -2(20) =23 .

18. f in a moderately asymmetrical distribution the mode and the mean of the data are 6λ *and* 9λ, respectively, then the median is

Sol. For a moderately skewed distribution,

mode = 3median – 2 mean

$$\Rightarrow 6\lambda = 3median - 18\lambda \Rightarrow median = 8\lambda$$

19

The quartile deviation of daily wages of in (Rs.) of 11 persons given below 140, 145, 130, 165, 160, 125, 150, 170, 175, 120, 180

Sol. The given data in ascending order of magnitude is 120, 125, 130, 140, 145, 150, 160, 165, 170, 175, 180

Here, $Q_1 = \left(\frac{n+1}{4}\right)^{th}$ term $= \frac{11+1}{4} term = 3^{rd}$ term

= 130

$$Q_3 = \frac{3(n+1)^{th}}{4} = \frac{3\times(11+1)}{4} = 9^{th} = 170$$

$$QD = \frac{Q_3 - Q_1}{2} = \frac{170-130}{2} = \frac{40}{2} = 20$$

20

The variance of the first 'n' natural numbers is

Sol. Variance

$$= (SD)^2 = \frac{1}{n}\Sigma x^2 - \left(\frac{\Sigma x}{n}\right)^2, \left(\because \bar{x} = \frac{\Sigma x}{n}\right)$$

$$= \frac{n(n+1)(2n+1)}{6n} - \left(\frac{n(n+1)}{2n}\right)^2 = \frac{n^2-1}{12}$$

21

If the M.D is 12, the value of S.D will be

Sol. We know that Q.D = $\frac{5}{6} \times M.D = \frac{5}{6} \times 12 = 10$

$$\therefore S.D = \frac{3}{2} \times Q.D = \frac{3}{2} \times 10 = 15$$

22

The mean of five observations is 4 and their variance is 5.2. If three of these observations are 1, 2 and 6, then the other two observations are

Sol. Let the two unknown items be x and y, then

Mean = 4 $\Rightarrow \frac{1+2+6+x+y}{5} = 4$

$\Rightarrow x+y=11$.....(1) and variance = 5.2

$\Rightarrow \frac{1^2+2^2+6^2+x^2+y^2}{5} - (mean)^2 = 5.2$

$41+x^2+y^2 = 5\left[5.2+(4)^2\right]$

$41+x^2+y^2 = 106$

$x^2+y^2 = 65$ ......(2)

Solving (1) and (2) for x and y, we get
x = 4, y = 7 or x = 7, y = 4.

EXERCISE - I

1. For overlapping classes 0-10, 10-20, 20-30, etc. Then the class mark of the class 0-10 is
1) 0 2) 10 3) 5 4) 6

2. Which one of the following measures is the most suitable one of central location for computing intelligence of students?
1) Mode 2) A.M. 3). G.M. 4). Median

MEAN (A.M, G.M, H.M)

3. The mean of 20 observations is 15. On checking it was found that two observations were wrongly copied as 3 and 6. If wrong observations are replaced by correct values 8 and 4, then the correct mean is
1) 15 2) 15.15 3) 16.15 4) 17

4. The mean weight of 9 items is 15. If one more item is added to the series the mean becomes 16. The value of 10th item is
1) 35 2) 30 3) 25 4) 20

5. When 15 was subtracted from each of the seven observations the following number resulted : -3,0,-2,4,6,1,1. The mean of the distribution is
1) 14 2) 15 3) 16 14) 17

6. Mean of 100 items is 49. It was discovered that three items which should have been 60, 70, 80 were wrongly read as 40, 20, 50 respectively. The correct mean is.
1) 48 2) $82\frac{1}{2}$ 3) 80 4) 50

7. If the arithmetic and harmonic means of two numbers are 4.5 and 4 respectively, then one of the number is
1) 5 2) 6 3) 7 4) 4

MEDIAN & MODE

8. If the mode of a data is 18 and the mean is 24, then median is
1) 18 2) 24 3) 21 4) 22

9. If the median of 21 observations is 40 and if the observations greater than the median are increased by 6 then the median of the new data will be
1) 40 2) 46 3) $46 + \frac{40}{21}$ 4) $46 - \frac{40}{21}$

10. Mode of the data 3, 2, 5, 2, 3, 5, 6, 6, 5, 3, 5, 2, 5 is
1) 6 2) 4 3) 3 4) 5

11. Mode of the distribution

Marks	**4**	**5**	**6**	**7**	**8**
No.of students	**3**	**5**	**10**	**6**	**1**

1) 6 2) 10 3) 8 4) 4

RANGE, Q.D, S.D AND VARIANCE

12. The range of the following set of observations 2, 3, 5, 9, 8, 7, 6, 5, 7, 4, 3 is
1) 11 2) 7 3) 5.5 4) 6

13. The quartile deviation of daily wages (in Rs.) of 7 persons given below is 12, 7, 15, 10, 17, 17, 25 is
1) 14.5 2) 5 3) 3.5 4). 4.5

14. If the standard deviation of 0,1,2,3…..9 is K, then the standard deviation of 10,11,12,13…. 19 is
1) K + 10 2) K 3) $\sqrt{10} + K$ 4) 10 K

15. The variance of the first n natural numbers is
1). $\frac{n^2-1}{12}$ 2) $\frac{n^2-1}{6}$ 3) $\frac{n^2+1}{6}$ 4) $\frac{n^2+1}{12}$

16. The mean of four observations is 3. If the sum of the squares of these observations is 48 then their standard deviation is
[EAMCET-2014]

1) $\sqrt{2}$ 2) $\sqrt{3}$ 3) $\sqrt{5}$ 4) $\sqrt{7}$

17. If $x_1, x_2, ...x_n$ are *n* observations such that $\sum_{i=1}^{n} x_i^2 = 400$ and $\sum_{i=1}^{n} x_i = 80$ then the least value of *n* is [EAMCET-2014]

1) 12 2) 15 3) 16 4) 18

18. The sum of 10 items is 12 and sum of their squares is 18, then standard deviation is

1) –3/5 2) 6/5 3) 4/5 4) 3/5

19. The mean of two samples of sizes 200 and 300 were found to be 25, 10 respectively. Their standard deviations were 3 and 4 respectively. The variance of combined sample of size 500 is

1) 64 2) 65.2 3) 67.2 4) 64.2

KEY

1) 3	2) 4	3) 2	4) 3	5) 3	6) 4
7) 2	8) 4	9) 1	10) 4	11) 1	12) 2
13) 3	14) 2	15) 1	16) 2	17) 3	18) 4
19) 3					

SOLUTIONS

1. 5 (mid value of the class)

2. **most suitable one of central location for computing intelligence of students is median**

3. given,

$\bar{X} = 15$ $\qquad \dfrac{\sum_{i=1}^{n} X_i}{20} = 15$

$\sum_{i=1}^{n} X_i = 20.15 = 300$

New mean ,

$$\frac{\sum_{i=1}^{n} X_i}{20} = \frac{300-3-6+8+4}{20} = \frac{303}{20}$$

3. $\dfrac{20\times 15-3-6+8+4}{20}$

4. $\dfrac{9.15+x}{10} = 16 \Rightarrow 135 + x = 160 \Rightarrow x = 25$

5. Mean $= \dfrac{-3+0-2+4+6+1}{7} = 1$

The mean of the original distribution=1+15=16

6. $\dfrac{4900-40-20-50+60+70+80}{100} = \dfrac{5000}{100} = 50$

7. $\dfrac{a+b}{2} = 4.5, \dfrac{2ab}{a+b} = 4, \quad a+b = 9,$

ab = 18 $\Rightarrow$ a = 6

8. Mode = 3 median - 2 mean, 18 = 3 (median) -2 (24), Median $= \dfrac{66}{3} = 22$

9. upon change of axis median doesnot change so the new median will be 40

10. 3, 3, 3, 2, 2, 2, 5, 5, 5, 5, 5, 6, 6 mode = 5

11. The number having maximum frequency

12. 9 - 2 = 7 (Range = max-min)

13. $\dfrac{Q_3 - Q_1}{2}$ where $Q_1 = 10, Q_3 = 17$

14. K

15. $\dfrac{n^2-1}{12}$

16. $\bar{x} = 3, \sum x_i^2 = 48$; $\sigma^2 = \dfrac{1}{n}\sum x_i^2 - (\bar{x})^2$

$= \dfrac{1}{4} \times 48 - 9 = 12 - 9 = 3$; $\sigma = \sqrt{3}$

$\sum_{i=1}^{n} X_i^2 = 400, \sum_{i=1}^{n} X_i = 80$

$$\Rightarrow \sigma^2 = n\frac{\sum_{i=1}^{n} X_i^2}{n} - \left(\frac{\sum_{i=1}^{n} X_i^2}{n}\right)^2 \geq 0$$

$$\Rightarrow \frac{400}{n} - \left(\frac{80}{n}\right)^2 \geq 0 \Rightarrow \frac{400}{n} \geq \frac{6400}{n^2}$$

$\Rightarrow n \geq 16$

17. Given , $\sigma^2 \geq 0$

18. $\sum x_i = 12, \sum x_i^2 = 18, n = 10$

$$S.D = \sqrt{\frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2}$$

$$S.D = \sqrt{\frac{18}{10} - \left(\frac{12}{10}\right)^2} = \sqrt{\frac{180-144}{(10)^2}} = \frac{6}{10} = \frac{3}{5}$$

19. Combined mean $\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$; $d_1 = \bar{x}_1 - \bar{x}$,

$d2 = \bar{x}_2 - \bar{x}$ and use the formula Variance of combined data $= \sigma^2 = \frac{n_1\left(\sigma_1^2 + d_1^2\right) + n_2\left(\sigma_2^2 + d_2^3\right)}{\left(n_1 + n_2\right)}$

$n_1 = 200, n_2 = 300$ $\quad \bar{x}_1 = 25, \bar{x}_2 = 10$

$\sigma_1 = 3, \sigma_2 = 4$

$$\bar{X} = \frac{200X25 + 300X10}{500} = \frac{50+30}{5}$$

$$= 16\sigma^2 = \frac{200(9+81) + 300(16+36)}{500}$$

$$= \frac{18000 + 15600}{500} = \frac{336}{5} = 67.5$$

EXERCISE - II

MEAN (A.M, G.M, H.M)

1. The geometric mean of 10 observations on a certain variable was calculated as 16.2. It was later discovered that one of the observations was wrongly recorded as 12.9; infact it was 21.9. The correct geometric mean is

1) $\left(\frac{(16.2)^9 \times 21.9}{12.9}\right)^{1/10}$ 2) $\left(\frac{(16.2)^{10} \times 21.9}{12.9}\right)^{1/10}$

3) $\left(\frac{(16.2)^{10} \times 12.9}{21.9}\right)^{1/10}$ 4) $\left(\frac{(16.2)^{11} \times 21.9}{12.9}\right)^{1/11}$

2. The A.M. of the observations 1.3.5, 3.5.7, 5.7.9, ..., (2n-1)(2n+1)(2n+3) is $(\forall n \in N)$

1) $2n^3 + 6n^2 + 7n - 2$ 2) $n^3 + 8n^2 + 7n - 2$

3) $2n^3 + 5n^2 + 6n - 1$ 4) $2n^3 + 8n^2 + 7n - 2$

3. The mean weight of 9 items is 15. If one more item is added tot he series the mean becomes 16. The value of 10th item is

1) 35 2) 30 3) 25 4) 20

4. The mean marks got by 300 students in the subject of statistics was 45. The mean of the top 100 of them was found to be 70 and the mean of the last 100 was known to be 20, then the mean of the remaining 100 students is

1) 45 2) 58 3) 68 4) 88

5. The average marks of boys in a class is 52 and that of girls is 42. The average marks of boys and girls combined is 50. The percentage of boys in the class is

[AIEEE - 2007]

1) 60 2) 40 3) 20 4) 80

6. Mean of 'n' items is $\bar{x}$. If these n items are successively increased by 2, 2^2, 2^3, ... 2^n, then the new mean is

1) $\bar{x} + \frac{2^{n+1}}{n}$ 2) $\bar{x} + \frac{2^{n+1}}{n} - \frac{2}{n}$

3) $\bar{x}+\frac{2^n}{n}$ 4) $\bar{x}+2^n$

7. The frequency distribution of discrete data given below, the frequency x against value 0 is missing.

Variable x :	**0**	**1**	**2**	**3**	**4**	**5**
Frequency f :	**x**	**20**	**40**	**40**	**20**	**4**

If the mean is 2.5, then the missing frequency x will be____

1) 0 2) 1 3) 3 4) 4

8. The minimum value of $(x-6)^2+(x+3)^2+(x-8)^2+(x+4)^2+(x-3)^2$ is

1) 114 2) 141 3) 104 4) 2

9. Product of n positive numbers is unity. The sum of these numbers cannot be less than

1) 1 2) n 3) n^2 4) 2

10. An automobile driver travels from plane to hill station 100 km distance at an average speed of 30 km per hour. He then makes the return trip at average speed of 20 km per hour. What is his average speed over the entire distance (200 km)?

1) 25 km/hr 2) 24.6 km/hr
3) 24 km/hr 4) 24.5 km/hr

11. If A.M. = 24.5, G.M. = 24.375 then H.M. =

1) 24 2) 24.125 3) 24.5 4) 24.25

MEDIAN & MODE

12. The minimum value of $|x-6|+|x+3|+|x-8|+|x+4|+|x-3|$ is

1) 11 2) 21 3) 31 4) 42

13. If in a frequency distribution, the mean and median are 21 and 22 respectively, then its mode is approximately

1) 20.5 2) 22.0 3) 24.0 4) 25.5

M.D, S.D & VARIANCE

14. Mean deviation of the series a, a+d, a + 2d,-----------, a + 2nd from its mean is

1) $\frac{(n+1)d}{(2n+1)}$ 2) $\frac{nd}{2n+1}$

3) $\frac{(2n+1)d}{n(n+1)}$ 4) $\frac{n(n+1)d}{2n+1}$

15. If mean deviation through median is 15 and median is 450, then coefficient of mean deviation is

1) 1/30 2) 30 3) 15 4) 45

16. The mean and S.D. of 1, 2, 3, 4, 5, 6 is

1) 3, 3 2) $\frac{7}{2},\sqrt{\frac{35}{12}}$ 3) $\frac{7}{2},\sqrt{3}$ 4) $\frac{35}{12}$

17. If the S.D. of n observations $x_1, x_2, ..., x_n$ **is 4 and another set of n observations $y_1, y_2, ...,y_n$ is 3 the S.D. of n observations $x_1-y_1, x_2-y_2, ..., x_n-y_n$ is**

1) 1 2) $2/\sqrt{3}$ 3) 5 4) 7

18. The variance of first 10 multiples of 3 is

1) 64.25 2) 54.25 3) 70.25 4) 74.25

19. Let r be the range and $S^2=\frac{1}{n-1}\sum(x_i-\bar{x})^2$. If $S^2\le r^2k$ then k is equal to

1) $\frac{1}{n-1}$ 2) $\frac{n}{n-1}$ 3) $\frac{n+1}{2(n-1)}$ 4) $\frac{1}{2(n-1)}$

20. The mean of the numbers a, b, 8, 5, 10 is 6 and the variance is 6.80, then which of the following gives possible values of a and b (AIEEE-2008)

1) a = 0, b = 7 2) a = 5, b = 2
3) a = 1, b = 6 4) a= 3, b = 4

21. Suppose a population A has 100 observations 101, 102....., 200 and another population B has 100 observations 151, 152, ... 250. If V_A and V_B represent the variances of the two populations, respectively, respectively, then V_A/V_B is

1) 1 2) 9/4 3) 4/9 4) 2/3

22. If the mean deviation about the median of the numbers a, 2a, ..., 50a is 50, then $|a|$ equal to

1) 4 2) 5 3) 2 4) 3

23. The variance of first 50 even natural numbers is

1) $\frac{833}{4}$ 2) 833 3) 437 4) $\frac{437}{4}$

24. All the students of a class performed poorly in Mathematics. The teacher decided to give grace marks of 10 to each of the students. Which of the following statistical measures will not change even after the grace marks were given? [JEE MAIN-2013]

1) mode 2) variance 3) mean 4) median

KEY

1) 2	2) 4	3) 3	4) 1	5) 4	6) 2
7) 4	8) 1	9) 2	10) 3	11) 4	12) 2
13) 3	14) 4	15) 1	16) 2	17) 3	18) 4
19) 2	20) 4	21) 1	22) 1	23) 2	24) 2

SOLUTIONS

1. $\left(\frac{(16.2)^{10}\times 21.9}{12.9}\right)^{\frac{1}{10}}$

2. $\frac{\Sigma(2n-1)(2n+1)(2n+3)}{n}=\frac{\Sigma(4n^2-1)(2n+3)}{n}$

$=\frac{\Sigma(8n^3+12n^2-2n-3)}{n}$

$=2n^3+8n^2+7n-2$

3. 15 x 9 = 135
16 x 10 = 160
10th item is 160 - 135 = 25

5. no. of boys = b, no. of girls = g
$(b\times 52)+(g\times 42)=(b+g)50\Rightarrow b:g=4:1$

6. $\overline{X}+\frac{2(2^n-1)}{n}=\overline{X}+\frac{2^{n+1}}{n}-\frac{2}{n}$

7. $\frac{\sum f_i x_i}{\sum f_i}=2.5$

8. Minimum value obtained at the mean of 6, -3, 8, -4, 3

9. A.M. $\geq G.M$

$\frac{x_1+x_2+x_3+........+x_n}{n}\geq\sqrt[n]{x_1x_2........x_n}$,

$\frac{x_1+x_2+.......x_n}{n}\geq 1\Rightarrow\sum_{i=1}^{n}x_i\geq n$

10. $V(Average)=\frac{2V_1V_2}{V_1+V_2}=\frac{2\times 30\times 20}{30+20}=24\ km/hr$

11. $G^2=A.H$

12. Minimum value obtained at median of -4, -3, 3, 6, 8

13. Mode = 3 Median - 2 Mean

14. Mean $\overline{x}=a+nd$

$M.D.=\frac{1}{(2n+1)}\sum\left|x_i-\overline{x}\right|=\frac{n(n+1)}{(2n+1)}d$

15. $\frac{M.D}{Median}$

16. $\overline{x}=\frac{\sum x_i}{n},\sigma^2=\left(\frac{n^2-1}{12}\right)d^2$

17. $V(aX+bY)=a^2V(X)+b^2V(Y)$

18. $\sigma^2=\left(\frac{n^2-1}{12}\right)d^2$ where n = 10, d = 3

19. range 'r' and variance related by, $\sigma^2\leq r^2$

$\frac{\Sigma(x_i-\overline{x})^2}{n}=\sigma^2\Rightarrow\Sigma(x_i-\overline{x})^2=n\sigma^2$,

20. $\frac{a+b+8+5+10}{5}=6\Rightarrow a+b=7$

now use verification for variance

21. $V_B(x)=V_A(X+50)=V_A(X)$

22. $\mathrm{M.D}=\frac{\sum|x_i-M|}{n},M=\frac{51a}{2}$ and $n=50\Rightarrow|a|=4$

23. $\bar{X} = \frac{\sum x_i}{n} = \frac{2+4+...100}{50} = 51$

$\text{variance} = \frac{1}{n}\sum x_1^2 - (\bar{x})^2$

$= \frac{1}{50}(2^2+4^2+...+100^2) - (51)^2 = 833$

24. Median will go up by 2 and S.D. will remain same.

JEE MAINS QUESTIONS

1.

Consider the data on x taking the values 0, 2, 4, 8, ..., 2^n with frequencies ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$ respectively. If the mean of this data is $\frac{728}{2^n}$, then n is equal to _____.

2.

If for some $x \in \mathbf{R}$, the frequency distribution of the marks obtained by 20 students in a test is :

Marks	2	3	5	7
Frequency	$(x+1)^2$	$2x-5$	x^2-3x	x

then the mean of the marks is :

(1) 3.2 (2) 3.0 (3) 2.5 (4) 2.8

3.

The mean and the median of the following ten numbers in increasing order 10, 22, 26, 29, 34, x, 42, 67, 70, y are 42 and 35 respectively, then $\frac{y}{x}$ is equal to

(1) 9/4 (2) 7/2 (3) 8/3 (4) 7/3

4..The mean and variance of 8 observations are 10 and 13.5,respectively. If 6 of these observations are 5, 7, 10, 12, 14,15, then the absolute difference of the remaining twoobservations is :

(1) 9 (2) 5 (3) 3 (4) 7

5.
If a variance of the following frequency distribution :

Class	10-20	20-30	30-40
Frequency	2	x	2

is 50, then x is equal to ____________.

6.
For the frequency distribution :

Variate (x) :	x_1	x_2	$x_1 ... x_{15}$
Frequency (f) :	f_1	f_2	$f_3 ... f_{15}$

where $0 < x_1 < x_2 < x_3 < ... < x_{15} = 10$ and $\sum_{i=1}^{15} f_i > 0,$ the standard deviation **cannot** be :

(1) 4 (2) 1 (3) 6 (4) 2

7. If the variance of the terms in an increasing A.P., b1,b2 b3, b4, , , , ..., is 90, then the common difference ofthis A.P. is ___________

8.
The mean and the standard deviation (s.d.) of 10 observations are 20 and 2 respectively. Each of these 10 observations is multiplied by p and then reduced by q, where p ¹ 0 and q ¹ 0. If the new mean and new s.d. become half of their original values, then q is equal to:

(1) –5 (2) 10 (3) –20 (4) –10

9.. The mean and variance of 20 observations are found tobe 10 and 4, respectively. On rechecking, it was foundthat an observation 9 was incorrect and the correctobservation was 11. Then the correct variance is:

1) 3.99 (2) 4.01 (3) 4.02 (4) 3.98

10 .If the variance of the first n natural numbers is 10 and thevariance of the first m even natural numbers is 16, thenm + n is equal to

11. If the mean and variance of eight numbers 3, 7, 9, 12, 13,20, x and y be 10 and 25 respectively, then .x .y is equal to
_________.

KEY

1) 6 2) 4 3) 4 4) 4 5) 4 6) 3

7) 3 8) 3 9) 1 10) 18 11) 52

SOLUTIONS

1.

$$\text{Mean} = \frac{\Sigma x_i f_i}{\Sigma f_i} = \frac{0\cdot {}^nC_0 + 2\cdot {}^nC_1 + 2^2\cdot {}^nC_2 + ... + 2^n\cdot {}^nC_n}{{}^nC_0 + {}^nC_1 + ... + {}^nC_n}$$

To find sum of numerator consider

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n \quad ...(i)$$

Put $x = 2 \Rightarrow 3^n - 1 = 2\cdot {}^nC_1 + 2^2\cdot {}^nC_2 + ... + 2^n\cdot {}^nC_n$

To find sum of denominator, put $x = 1$ in (i), we get

$$2^n = {}^nC_0 + {}^nC_1 + ... + {}^nC_n$$

$$\therefore \frac{3^n - 1}{2^n} = \frac{728}{2^n} \Rightarrow 3^n = 729 \Rightarrow n = 6$$

2.

Number of students are,

$$(x+1)^2 + (2x-5) + (x^2 - 3x) + x = 20$$

$$\Rightarrow 2x^2 + 2x - 4 = 20 \Rightarrow x^2 + x - 12 = 0$$

$$\Rightarrow (x+4)(x-3) = 0 \Rightarrow x = 3$$

$\therefore$

Marks	2	3	5	7
No. of students	16	1	0	3

$$\text{Average marks} = \frac{32+3+21}{20} = \frac{56}{20} = 2.8$$

3.

Ten numbers in increasing order are

$10, 22, 26, 29, 34, x, 42, 67, 70, y$

$$\text{Mean} = \frac{\sum x_i}{n} = \frac{x+y+300}{10} = 42 \Rightarrow x+y = 120$$

$$\text{Median} = \frac{T_5 + T_6}{2} = 35 = \frac{34+x}{2} \Rightarrow x = 36 \text{ and } y = 84$$

$$\text{Hence, } \frac{y}{x} = \frac{84}{36} = \frac{7}{3}$$

4.

Let the two remaining observations be x and y.

$$\because \bar{x} = \frac{5+7+10+12+14+15+x+y}{8}$$

$$\Rightarrow 10 = \frac{63+x+y}{8}$$

$$\Rightarrow x + y = 80 - 63$$

$$\Rightarrow x + y = 17 \quad ...(i)$$

$$\because \text{var}(x) = 13.5$$

$$= \frac{25+49+100+144+196+225+x^2+y^2}{8} - (10)^2$$

$$\Rightarrow x^2 + y^2 = 169 \quad ...(ii)$$

From (i) and (ii) we get

$(x, y) = (12, 5)$ or $(5, 12)$

So, $|x - y| = 7$.

5.

x_i	15	25	35
f_i	2	x	2

$$\overline{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{30 + 70 + 25x}{4 + x} = 25$$

$$\sigma^2 = \frac{\Sigma f_i x_i^2}{\Sigma f_i} - (\overline{x})^2$$

$$\Rightarrow 50 = \frac{450 + 625x + 2450}{4 + x} - 625$$

$$\Rightarrow 675 = \frac{2900 + 625x}{4 + x} \Rightarrow 50x = 200$$

$$\therefore x = 4$$

6.

If variate varries from a to b then variance

$$\text{var}(x) \le \left(\frac{b-a}{2}\right)^2$$

$$\Rightarrow \text{var}(x) < \left(\frac{10-0}{2}\right)^2$$

$$\Rightarrow \text{var}(x) < 25$$

$\Rightarrow$ standard deviation < 5

It is clear that standard deviation cann't be 6.

7.

$$\text{Variance} = \frac{\sum_{i=1}^{11} b_i^2}{11} - \left(\frac{\sum_{i=1}^{11} b_i}{11}\right)^2$$

Let common difference of A.P. be d

$$= \frac{\sum_{r=0}^{10} (b_1 + rd)^2}{11} - \left(\frac{\sum_{r=0}^{10} (b_1 + rd)}{11}\right)^2$$

$$= \frac{11b_1^2 + 2b_1 d\left(\frac{10 \times 11}{2}\right) + d^2\left(\frac{10 \times 11 \times 21}{6}\right)}{11} - \left(\frac{11b_1 + \frac{10 \times 11}{2} d}{11}\right)^2$$

$$= (b_1^2 + 10b_1 d + 35d^2) - (b_1 + 5d)^2 = 10d^2$$

$\because$ Variance = 90 (Given)

$$\Rightarrow 10d^2 = 90 \Rightarrow d = 3.$$

9.

Let $x_1, x_2, \ldots, x_{20}$ be 20 observations, then

$$\text{Mean} = \frac{x_1 + x_2 + \ldots + x_{20}}{20} = 10$$

$$\Rightarrow \frac{\sum_{i=1}^{20} x_i}{20} = 10 \qquad \text{...(i)}$$

$$\text{Variance} = \frac{\Sigma x_i^2}{n} - (\overline{x})^2$$

$$\Rightarrow \frac{\Sigma x_i^2}{20} - 100 = 4 \qquad \text{...(ii)}$$

$\Sigma x_i^2 = 104 \times 20 = 2080$

Actual mean $= \dfrac{200 - 9 + 11}{20} = \dfrac{202}{20}$

Variance $= \dfrac{2080 - 81 + 121}{20} - \left(\dfrac{202}{20}\right)^2$

$= \dfrac{2120}{20} - (10.1)^2 = 106 - 102.01 = 3.99$

8.

Let $\bar{x}$ and σ be the mean and standard deviations of given observations.

If each observation is multiplied with p and then q is subtracted.

New mean $(\bar{x}_1) = p\bar{x} - q$

$\Rightarrow\ 10 = p(20) - q$ ...(i)

and new standard deviations $\sigma_1 = |p|\,\sigma$

$\Rightarrow\ 1 = |p|(2) \ \Rightarrow\ |p| = \dfrac{1}{2} \Rightarrow\ p = \pm\dfrac{1}{2}$

If $p = \dfrac{1}{2}$, then $q = 0$ (from equation (i))

If $p = -\dfrac{1}{2}$, then $q = -20$

10.

Var $(1, 2, \ldots\ldots, n) = 10$

$\Rightarrow\ \dfrac{1^2 + 2^2 + \ldots\ldots + n^2}{n} - \left(\dfrac{1 + 2 + \ldots\ldots + n}{n}\right)^2 = 10$

$\Rightarrow\ \dfrac{(n+1)(2n+1)}{6} - \left(\dfrac{n+1}{2}\right)^2 = 10$

$\Rightarrow\ n^2 - 1 = 120 \qquad \Rightarrow\ n = 11$

Var $(2, 4, 6, \ldots\ldots, 2m) = 16 \Rightarrow$ Var $(1, 2, \ldots\ldots, m) = 4$

$\Rightarrow\ m^2 - 1 = 48 \Rightarrow m = 7$

$\Rightarrow\ m + n = 18$

11.

Mean $= \bar{x} = \dfrac{3 + 7 + 9 + 12 + 13 + 20 + x + y}{8} = 10$

$\Rightarrow\ x + y = 16$ ...(i)

Variance $= \sigma^2 = \dfrac{\Sigma(x_i)^2}{8} - (\bar{x})^2 = 25$

$\sigma^2 = \dfrac{9 + 49 + 81 + 144 + 169 + 400 + x^2 + y^2}{8} - 100 = 25$

$\Rightarrow\ x^2 + y^2 = 148$ ...(ii)

From eqn. (i), $(x + y)^2 = (16)^2$

$\Rightarrow\ x^2 + y^2 + 2xy = 256$

Using eqn. (ii), $148 + 2xy = 256$

$\Rightarrow\ xy = 52$