MATHEMATICS - IA

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MATHEMATICAL REASONING

SYNOPSIS

In a mathematical language, there are two kinds of reasoning inductive and deductive. We have already discussed the inductive reasoning in the context of mathematical induction. In this chapter, we shall discuss some fundamentals of deductive reasoning.

- **Sentence:** We communicate our ideas or thoughts with the help of sentences in particular languages. Following types of sentences are normally used.

(i) Assertive Sentence : A sentence that makes an assertion is called an 'assertive sentence or a declarative sentence'.
Eg: New Delhi is the capital of India

(ii) Imperative Sentence: A sentence that expresses a request or a command is called an imperative sentence.
Eg: please give me a glass of water

(iii) Exclamatory Sentence: A sentence that expresses some strong feeling is called an exclamatory sentence.
Eg: Oh God! what a beautiful scene

(iv) Interrogative sentence : A sentence that asks some question is called an interrogative sentence.
Eg: To which state do you belong?

(v) Optative sentence: A sentence that expresses a wish is called an optative sentence.
Eg: God bless you.

- **Statement (or) Proposition :** A sentence is called a mathematically acceptable statement if it is either true(T) or false(F) but not both.
Eg: Natural numbers are always positive
Statements are usually dentoted by the letters p,q,r,......etc.
- The truthness or falsity of a statement is called its truth value. Truthness of a statement is denoted by T. while its falsity is denoted by F.
- **True statements :** Eg: (i) 2012 is a leap year, (ii)The sum of all interior angles of a triangle is 180^0.
- **False statements :**
Eg: (i) All prime numbers are odd integers.
(ii) Two plus two is five.
- **Not a statement :** Eg:(i) Mathematics is difficult.
(ii) Tomorrow is Sunday.
- **Simple Statement :** Any statement or proposition whose truth value does not explicitly depend on another statement is called a simple statement.

Eg: Sun rises in the east. Its truth value is T

- **Compound statement**: A statement which is made up of two or more simple statements using the connectives "and($\wedge$)", "or($\vee$)", "implies($\Rightarrow$)", "if and only if($\Leftrightarrow$)" etc... is called a compound statement. In this case each statement is called a component statement.
Eg: This book is for mathematics and its target is Jee-mains
- **Sub-Statement:** The simple statements which form a compound statement are known as its sub-statements or components or constituents.
- If p, q, r ...... are sub-statements of a compound statement S then the compound statement can be written as $S(p,q,r....)$.
- Compound statement is that its truth value is completely determined by the truth values of the sub-statements together with the way in which they are connected to form the compound statement.
- **Open Statement:** A sentence which contains one or more variables such that when certain values are given to the variable it becomes a statement, is called an open statement.
Eg: "He is a great man" is an open sentence because in the sentence "He" can be replaced by any person.

- **Eg:** Which of the following statement is/are open statement(s)? (1) Ram eats a mango.

  (2) Krishna goes to school (3) He lives in India

  (4) Anil and Anuj are good friends.

Sol: In a given options, only option (3) is an open statement, because in this sentence 'he' can be specifed to any person.

- **Truth Table :** A table that shows the relationship between the truth value of compound statement, S(p,q,r,...) and the truth values of its substatements p,q,r,...etc.., is called the truth tables of statement S.

(i) For a single statement p, number of rows=2^1=2

P
T
F

(ii) For two statements p and q, number of rows=2^2=4

p	q
T	T
T	F
F	T
F	F

(iii) For the three statements p,q,r,

Number of Rows =2^3=8

p	q	r
T	F	F
T	F	T
T	T	F
T	T	T
F	F	F
F	F	T
F	T	F
F	T	T

- **Note:** If a compound statement has simply *n* substatements, then there are 2^n rows representing logical possibilities.

Eg. 1

If there are 6 simple statements, then for making a table, find number of rows

Sol: We know that, if compound statements has n substatements, then there are 2^n rows in a table. Here, n=6 $\therefore$ Total number of rows = 2^6=64

- **Basic logical connectives or logical operators :** Two or more statements are combined to form a compound statement by using symbols. These symbols are called logical connectives. Logical connectives are given below.

Connective	Symbol	Nature of the compound statement formed by using the connective
and	$\wedge$	Conjunction
or	$\vee$	Disjunction
If .... then	$\Rightarrow$ or $\rightarrow$	Implication or conditional
If and only if (iff)	$\Leftrightarrow$ or $\leftrightarrow$	Equivalence or Bi-conditional or Bi-implication
not	~	Negation

- **Negation(~) :** The process of forming the contradictory of a given statement is called negation.
- If p is a statement, then the negation of p is also a statement denoted by ~p.
- Eg :- p: New Delhi is a city, then ~p: It is false that New Delhi is a city. (or) New Delhi is not a city

- **Negation Truth table :**

p	~p
T	F
F	T

- **Conjunction($\wedge$):** Any two simple statements can be connected by the word 'and' to form a compound statement called the conjunction of the original statements.
  Let **p** and **q** be two statements. The conjunction of **p** and **q** is denoted by $p \wedge q$, read as p and q

- ## Truth table for Conjunction :

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- **Eg:-** **p**: Two is an even number.
  q: Two is a prime number.
  $p \wedge q$: Two is an even number and a prime number.
- **Disjunction (Alternation)($\vee$)** : Any two statements can be connected by the word 'or' to form a compound statement called the 'disjunction' of the original statements.
  Let p and q be two statements the disjunction of p and q is denoted by $p \vee q$, read as p or q.
- ## Truth table for disjunction :

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- Eg:**p**: Two is an even number.
  q: Two is a prime number.
  $p \vee q$: Two is an even number or a prime number.
- **Conditional (or) Implication**: Two statements connected by the connective phrase 'if .... then' give rise to a compound statement which is known as an implication or a conditional statement.
  If p and q are two statements forming the implication of 'if p then q' then the implication denoted by '$p \Rightarrow q$ or '$p \rightarrow q$. p is called the 'antecedent' and q is called 'consequent'.
- $p \Rightarrow q$ read as p implies q, q if p, p is sufficient for q, q is necessary for p.
- $p \Rightarrow q$ is the statement that is false when p is true and q is false and true otherwise.
- ## Truth table for Conditional :

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- **Eg:-** **p**: An integer is a multiple of 9.
  q: An integer is a multiple of 3.
  $p \rightarrow q$:If an integer is a multiple of 9 then it is a multiple of 3.
- **Bi-implication** $\left(\leftrightarrow (or) \Leftrightarrow\right)$**:** A statement is a biconditional statement if it is the conjunction of two conditional statements one converse to the other. If p, q are two statements, then the compound statement $p \Rightarrow q$ and $q \Rightarrow p$ is called a biconditional statement or an equivalence and is denoted by $p \Leftrightarrow q$, $(p \Rightarrow q) \wedge (q \Rightarrow p)$.
- $p \leftrightarrow q$ is the statement that is the true when p and q have the same truth value and otherwise false.
- ## Truth table for Bi-implication :

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- **Eg:- p**: A number is divisible by 3.
  q: Sum of the digits of a number is divisible by 3. $p \leftrightarrow q$:A number is divisible by 3 if and only if the sum of its digits is divisible by 3.
- **Converse:** Let p,q be two statements. "If q then p" is called the converse of " if p then q".
  Thus the converse of $p \Rightarrow q$ is $q \Rightarrow p$.
- **Inverse :** Let p,q be two statements. " if ~p then ~q" is called the inverse of " if p then q". Thus the inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$.
- **Contrapositive** :The statement $\sim q \rightarrow \sim p$ is called the contrapositive of $p \rightarrow q$.

✈ Eg:- **p**: x is an even integer. **q:** x^2 is divisible by 4.

(i) $p \to q$: If x is even integer then x^2 is divisible by 4.

(ii) $q \to p$: x^2 is divisible by 4 then x is even.

(iii) $\sim p \to \sim q$ **:** If x is not even integer then x^2 is not divisible by 4.

(iv) $\sim q \to \sim p$ **:** if x^2 is not divisible by 4 then x is not an even integer.

✈ **Converse, inverse and contra positive of a conditional :** Suppose p, q are two statements such that $p \Rightarrow q$ then

i) Converse is $q \Rightarrow p$

ii) Inverse is $\sim p \Rightarrow \sim q$

iii) Contra positive is $\sim q \Rightarrow \sim p$

✈ **Truth Table :**

p	q	Conditional $p \Rightarrow q$	Converse $q \Rightarrow p$	Inverse $\sim p \Rightarrow \sim q$	Contra positive $\sim q \Rightarrow \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

✈ **Tautology, Contradiction** : (i) A compound statement that is **always true** is called a tautology.

(ii) A compound statement that is **always false** is called a contradiction or fallassy.

Eg:- Let **p** be a statement

✈ **Truth table :**

P	~P	$p \vee \sim P$	$p \wedge \sim P$
T	F	T	F
F	T	T	F

(i) $p \vee \sim p$ is a tautology

(ii) $p \wedge \sim p$ is a contradiction

✈ **Logical Equivalence**: The statements **p** and **q** are called logically equivalent if they have the same entries in the last column of the truth tables.

✈ **Eg:-** (i) $\sim (p \vee q)$ and $(\sim p) \wedge (\sim q)$ are logically equivalent.

(ii) $p \to q$ and $\sim p \vee q$ are logically equivalent.

(iii) $\boxed{p \Rightarrow q \equiv \sim q \Rightarrow \sim p}$

(iv) $\boxed{\sim (p \to q) \equiv p \wedge \sim q}$

p	q	$p \vee q$	$\sim (p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

p	q	$\sim p$	$(\sim p) \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

✈ The phrases, 'for all', 'for some', 'for no', 'for every' and 'there exists atleast one' convey the idea of quantity and refer to some specific collection of numbers or objects. these phrases quantify the variable in open sentences. they are called quantifiers.

✈ The quantifier 'for all' or 'for every' is called the universal quantifier and is denoted by $\forall$. The quantifier 'some' or 'there exists atleast one' is called existantial quantifier and is denoted by the symbol $\exists$.

✈ **Algebra of statements :**

1. **Idempotent laws** : For any statement **p**,
   (i) $p \vee p \equiv p$ (ii) $p \wedge p \equiv p$
2. **Commutative laws**: For any statements **p** and **q**
   (i) $p \vee q \equiv q \vee p$ (ii) $p \wedge q \equiv q \wedge p$
3. **Associative laws**: For any three statements **p,q,r**
   (i) $(p \vee q) \vee r \equiv p \vee (q \vee r)$ (ii) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
4. **Distributive Laws:** For any three statements **p,q,r,**
   (i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
   (ii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
5. **DeMorgan's laws**: If **p** and **q** are two statements, then

(i) $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$

(ii) $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$

6. **Identity laws**: If t and c denote a tautology and a contradiction respectively, then for any statement **p**,
(i) $p \wedge t \equiv p$ (ii) $p \vee c \equiv p$
(iii) $p \vee t \equiv t$ (iv) $p \wedge c \equiv c$
7. **Complement laws:** For any statement **p**,
(i) $p \vee (\sim p) \equiv t$ (ii) $p \wedge (\sim p) \equiv c$
8. **Law of contrapositive** : For any two statements p and q,
(i) $p \Rightarrow q \equiv \sim q \Rightarrow \sim p$
9. **Involution Laws**: For any statement **p**, we have $\sim(\sim p) \equiv p$.

EXERCISE - I

CONJUNCTION - DISJUNCTION

1. The disjunction of the statement, " It is raining"; The sun is shining", is

1) It is raining and the sun is shining
2) It is raining or the sun is shining
3) It is raining and the sun is not shining
4) It is not raining or the sun is not shining

2. $"5+7=10 \text{ and } 4+3=7"$. **Write the statement using the appropriate connective**

1) $5+7=10 \vee 4+3=7$
2) $5+7=10 \wedge 4+3=7$
3) $5+7=10 \Rightarrow 4+3=7$
4) $5+7=10 \Leftrightarrow 4+3=7$

3. "7 is odd or 7 is prime". Write the statement using the appropriate connective

1) 7 is odd $\vee$ 7 is prime
2) 7 is odd $\wedge$ is prime
3) 7 is odd $\Rightarrow$ 7 is prime
4) 7 is odd $\Leftrightarrow$ 7 is prime

IMPLICATION & BI-IMPLICATION

4. The truth value of "if 3 + 2 = 5 then 1x 0=0" is

1) T 2) F 3) T or F 4) T and F

5. p: he is hard working q: he will win. The symbolic form of "if he is hard working then he will win", is

1) $p \vee q$ 2) $p \wedge q$
3) $p \Rightarrow q$ 4) $q \Rightarrow p$

NEGATION, INVERSE, CONVERSE, CONTRAPOSITIVE, TAUTOLOGY, CONTRADICTION

6. "Earth is a planet", the negation of this statement is

1) The earth is round
2) The earth is not round
3) The earth revolves round the sun
4) The earth is not a planet

7. The truth table of $(\sim p) \wedge q$ **is**

1)

p	q	(~p) ∧ q
T	T	F
T	F	F
F	T	T
F	F	F

2)

p	q	(~p) ∧ q
T	T	F
T	F	T
F	T	F
F	F	F

3)

p	q	(~p) ∧ q
T	T	F
T	F	F
F	T	F
F	F	F

4)

p	q	$(\sim p) \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

8. The truth table of $(\sim p) \Rightarrow q$ is

1)

p	q	$(\sim p) \Rightarrow q$
T	T	F
T	F	T
F	T	T
F	F	T

2)

p	q	$(\sim p) \Rightarrow q$
T	T	T
T	F	T
F	T	T
F	F	F

3)

p	q	$(\sim p) \Rightarrow q$
T	T	F
T	F	T
F	T	F
F	F	T

4)

p	q	$(\sim p) \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

9. Negation of the compound proposition: If the examination is difficult, then I shall pass if I study hard

1) The examination is difficult and I study hard and I shall pass
2) The examination is difficult and I study hard but I shall not pass
3) The examination is not difficult and I study hard and I shall pass
4) All of these

10. Which of the following is not logically equivalent to the proposition : " A real number is either rational or irrational"

1) If a number is neither rational nor irrational then it is not real
2) If a number is not a rational or not an irrational, then it is not real
3) If a number is not real, then it is neither rational nor irrational
4) If a number is real, then it is rational or irrational

11. If $p \rightarrow (q \vee r)$ is false, then the truth values of p, q, r are respectively

1) T, F, F 2) F, F, F 3) F, T, T 4) T, T, F

12. The proposition $(p \rightarrow \sim p) \wedge (\sim p \rightarrow p)$ is

1) a tautology 2) a contradiction
3) neither a tautology nor a contradiction
4) a tautology and a contradiction

13. Let p be the proposition: Mathematics is interesting and let q be the proposition that Mathematics is difficult, then the symbol $p \wedge q$ means

1) Mathematics is interesting implies that Mathematics is difficult
2) Mathematics is interesting implies and is implied by Mathematics is difficult
3) Mathematics is interesting and Mathematics is difficult
4) Mathematics is interesting or Mathematics is difficult

14. The negation of the compound proposition $p \leftrightarrow \sim q$ is logically equivalent to

1) $p \leftrightarrow q$ 2) $(p \rightarrow q) \wedge (\sim q \rightarrow p)$
3) $(\sim q \rightarrow p) \vee (\sim p \rightarrow q)$
4) $(\sim p \wedge \sim q) \vee (q \wedge \sim p)$

KEY

1) 2	2) 2	3) 1	4) 1	5) 3	6) 4
7) 1	8) 2	9) 2	10) 2	11) 1	12)2
13) 3	14) 1				

SOLUTIONS

1. It is raning or the sun is shining
2. 5+7 = 10 " $\wedge$ " 4+3 = 7
3. 7 is odd " $\vee$ " 7 is prime
4. p is true , q is true , p $\Rightarrow$ q is true
5. p $\Rightarrow$ q
6. The earth is "not" a planet
7. ~ p is true , q is true $\Rightarrow$ ~ p $\wedge$ q is true
8. ~ p is true , q is false $\Rightarrow$ ~ p $\Rightarrow$ q is false
9. The examination is difficult and I study hard and I shall pass
10. If a number is not a rational or not a irrational , then it is not a real number
11. Take p is true , q is false and r is false

    q $\vee$ r is false

12. $(p \to \sim p) \wedge (\sim p \to p)$ **is always false so it is contradiction**

13. Mathematics is interesting and Mathematics is difficult

14.

p	q	$p \leftrightarrow \sim q$	$\sim (p \leftrightarrow \sim q)$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	F	F
F	T	F	F	F
F	F	T	T	T

thefore, $\sim (p \leftrightarrow \sim q) = p \leftrightarrow q$

EXERCISE - II

CONJUNCTION &DISJUNCTION

1. Let there be two propositions
p: I take only bread and butter in breakfast.
q: I do not take any thing in breakfast.
Then the compound proposition "I take only bread and butter in breakfast or I do not take any thing" is represented by

1) $p \wedge q$ 2) $p \vee q$ 3) $p \to q$ 4) $p \leftrightarrow q$

NEGATION, INVERSE, CONVERSE, CONTRAPOSITIVE, TAUTOLOGY, CONTRADICTION

2. The negation of the proposition: " If a number is divisible by 15, then it is divisible by 5 or 3:.

1) If a number is divisible by 15, then it is not divisible by 5 and 3.
2) A number is divisible by 15, and it is not divisible by 5 and 3.
3) A number is divisible by 15, then it is not divisible by 5 or 3.
4) A number is not divisible by 15 or it is not divisible by 5 and 3.

3. Negation of the statement $\sim p \to (q \vee r)$

1) $\sim p \to (q \vee r)$ 2) $p \vee (q \wedge r)$
3) $\sim p \wedge (\sim q \wedge \sim r)$ 4) $p \wedge (q \vee r)$

4. The converse of "if in a triangle ABC, AB =AC then $\angle B = \angle C$" is

1) If in a triangle ABC, $\angle B = \angle C$ then $AB = AC$
2) If in a triangle ABC, $AB \neq AC, then\ \angle B \neq \angle C$
3) If in a triangle ABC, $\angle B \neq \angle C$ then $AB \neq AC$
4) If in a triangle ABC, $\angle B \neq \angle C$, then $AB = AC$

5. The inverse of the proposition $(p \wedge \sim q) \to r$ is

1) $\sim r \to \sim p \vee q$ 2) $\sim p \vee q \to \sim r$
3) $r \to p \wedge \sim q$ 4) $\sim p \to (p \wedge r)$

6. The contrapositive of " if two triangles are congruent then they are similar" is

1) If two triangles are similar then they are congruent

2) If two triangles are not congruent then they are not similar

3) If two triangles are not similar then they are not congruent

4) If two triangles are similar then they are not congruent

7. **"The diagonals of a rhombus are perpendicular". The contrapositive of the above statement is**

1) If the figure is not a rhombus, then its diagonals are not perpendicular.

2) If the diagonals are perpendicular, then the figure is a rhombus.

3) If the diagonals are not perpendicular, then the figure is a rhombus

4) If the diagonals are not perpendicular, then the figure is not a rhombus

8. **The contrapositive of** $(\sim p \wedge q) \rightarrow (q \wedge \sim r)$ **is**

1) $(p \vee \sim q) \rightarrow (\sim q \vee p)$

2) $(\sim q \vee r) \rightarrow (\sim p \vee q)$

3) $(\sim q \vee r) \rightarrow (p \vee \sim q)$

4) $(\sim p \vee r) \rightarrow (\sim p \wedge \sim r)$

9. **Consider the following propositions**

p: I have the raincoat

q: I can walk in the rain.

The proposition " If I have the raincoat, then I can walk in the rain" is represented by

1) $p \rightarrow q$ 2) $q \rightarrow p$

3) $p \rightarrow \sim q$ 4) $p \leftrightarrow q$

10. **Given that water freezes below zero degree celsius.**

Consider the following statements :

p: Water freeze this morning,

q: This morning temperature was below $0^0 C$.

Which of the following is correct?

1) p and q are logically equivalent

2) p is the inverse of q

3) p is the converse of q

4) p is the contrapositive of q

11. **Consider the following proposition**

p: I take medicine, q: I can sleep. Then, the compound statement $\sim p \rightarrow \sim q$ **means**

1) If I do not take medicine, then I cannot sleep

2) If do not take medicine, then I can sleep

3) I take medicine iff I can sleep

4) If take medicine if I can sleep.

12. **The statement** $\sim p \vee q$ **is equivalent to**

1) $p \rightarrow q$ 2) $\sim p \rightarrow q$

3) $\sim p \rightarrow \sim q$ 4) $p \rightarrow \sim q$

13. **If p: 4 is an even prime number**

q: 6 is a divisor of 12 and

r: the HCF of 4 and 6 is 2, then which one of the following is true ?

1) $p \wedge q$ 2) $(p \vee q) \wedge \sim r$

3) $\sim (q \wedge r) \vee p$ 4) $\sim p \vee (q \wedge r)$

KEY

1) 2 2) 2 3) 3 4) 1 5)2 6) 3
7) 4 8) 3 9) 1 10)1 11) 1 12) 1
13)4

SOLUTIONS

1. **"I take only bread and butter in breakfast or I do not take any thing" is also represented as**
p" ∨ ' q

2. $\sim (p \Rightarrow q) \equiv p \wedge \sim q$
A number is divisible by 15 , and it is not divisible by 5 and 3

3. apply the formula $\sim (p \Rightarrow q) \equiv p \wedge \sim q$
as $\sim (p \Rightarrow q \vee r)$

$\equiv \sim p \wedge (\sim q \wedge \sim r)$

4. The converse of $p \Rightarrow q$ is $q \Rightarrow p$

5.The inverse of $[(p \wedge \sim q) \Rightarrow r] \equiv \sim (p \wedge \sim q) \rightarrow \sim r \equiv (\sim p \vee q) \Rightarrow \sim r$

6. The contra positive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$

7. contra positive of $p \Rightarrow q$ is the diagnols are not perpendicular , then the figure is not a rhombus

8. The contra positive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$

$$(\sim q \vee r) \to (p \vee \sim q)$$

9. $p \Rightarrow q$

10. p and q are logically equivalent

11. If I do not take medicine, then I cannot sleep

12.

p	q	$\sim p$	$\sim p \vee q$	$p \leftrightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$$\sim p \vee q \equiv p \leftrightarrow q$$

13. 4 is not a even prime number or 6 is a divisor of 12 and the H.C.F of 4 and 6 is 2 i.e

$\sim p \vee (q \wedge r)$

JEE MAINS QUESTIONS

1. The Boolean expression

$\sim (p \vee q) \vee (\sim p \wedge q)$ is equivalent to : **[2018]**

(1) p (2) q (3) ~q (4) ~p

2. If $p \to (\sim p \vee \sim q)$ is false, then the truth values of p and q are respectively. [2018]

(1) T, F (2) F, F (3) F, T (4) T, T

3. If $(p \wedge \sim q) \wedge (p \wedge r) \to \sim p \vee q$ is false, then the truth values of p, q and r are respectively [2018]

(1) F, T, F (2) T, F, T (3) F, F, F (4) T, T, T

4. If the truth value of the statement $p \to (\sim q \vee r)$ is false (F), then the truth values of the statements p, q, r are respectively. [2019]

(1) T, T, F (2) T, F, F (3) T, F, T (4) F, T, T

5. If $p \Rightarrow (q \vee r)$ is false, then the truth values of p, q, r are respectively: [2019]

(1) F, T, T (2) T, F, F (3) T, T, F (4) F, F, F

6. Which one of the following Boolean expressions is a tautology ? [2019]

(1) $(p \wedge q) \vee (p \wedge \sim q)$ (2) $(p \vee q)$ $(p \vee \sim q)$

(3) $(p \vee q) \wedge (p \vee \sim q)$ (4) $(p \vee q) \wedge (\sim p \vee \sim q)$

7. The Boolean expression

$((p \wedge q) \vee (p \vee \sim q)) \wedge (\sim p \wedge \sim q)$ is equivalent to : [2019]

(1) $p \wedge q$ (2) $p \wedge \sim q$

(3) $\sim p \wedge \sim q$ (4) $p \vee \sim q$

8. The expression $\sim(\sim p \rightarrow q)$ is logically equivalent to : [2019]

(1) $\sim p \wedge \sim q$ (2) $p \wedge \sim q$

(3) $\sim p \wedge q$ (4) $p \wedge q$

9. If the Boolean expression $(p \oplus q) \wedge (\sim p \odot q)$ is equivalent to $p \wedge q$, where $\oplus, \odot \in \{\wedge, \vee\}$ then the ordered pair $(\oplus, \odot)$ is: [2019]

(1) $(\vee, \wedge)$ (2) $(\vee, \vee)$

(3) $(\wedge, \vee)$ (4) $(\wedge, \wedge)$

10.The contrapositive of the statement "If you are born inIndia, then you are a citi”en of India", is [2019]

(1) If you are not a citi”en of India, then you are not born
in India.
(2) If you are a citi”en of India, then you are born in India.
(3) If you are born in India, then you are not a citi”en of
India.
(4) If you are not born in India, then you are not a citizen of India.

11.Contrapositive of the statement “If two numbers are not equal, then their squares are not equal”. is :

[2019]

(1) If the squares of two numbers are not equal, then the numbers are equal.
(2) If the squares of two numbers are equal, then the numbers are not equal.
(3) If the squares of two numbers are equal, then the numbers are equal.
(4) If the squares of two numbers are not equal, then the numbers are not equal.

12. The negation of the Boolean expression $p \vee (\sim p \wedge q)$ is equivalent to : [2020]

(1) $p \wedge \sim q$ (2) $\sim p \wedge \sim q$

(3) $\sim p \vee \sim q$ (4) $\sim p \vee q$

13. Given the following two statements :

$(S_1) : (q \vee p) \rightarrow (p \leftrightarrow \sim q)$ is a tautology.

$(S_2) : \sim q \wedge (\sim p \leftrightarrow q)$ is a fallacy. Then : [2020]

(1) both (S1) and (S2) are correct
(2) only (S1) is correct
(3) only (S2) is correct
(4) both (S1) and (S2) are not correct

14 The proposition $p \rightarrow \sim (p \wedge \sim q)$ is equivalent to : [2020]

(1) q (2) $(\sim p) \vee q$

(3) $(\sim p) \wedge q$ (4) $(\sim p) \vee (\sim q)$

15. Let p, q, r be three statements such that the truth value of $(p \wedge q) \rightarrow (\sim q \vee r)$ is F. Then the truth values of p, q, r are respectively : [2020]

(1) T, F, T (2) T, T, T (3) F, T, F (4) T, T, F

16. If $p \rightarrow (p \wedge \sim q)$ is false, then the truth values of p and q are respectively: [2020]

(1) F, F (2) T, F (3) T, T (4) F, T

KEY

1) 4	2) 4	3) 2	4) 1	5) 2	6) 1
7) 3	8) 1	9) 3	10) 1	11) 3	12) 2
13) 4	14) 2	15) 4	16) 3		

SOLUTIONS

1.

$$\sim(p \vee q) \vee (\sim p \wedge q)$$
$$\Rightarrow (\sim p \wedge \sim q) \vee (\sim p \wedge q)$$
$$\Rightarrow \sim p \wedge (\sim q \vee q)$$
$$\Rightarrow \sim p \wedge t \equiv \sim p$$

2.

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$p \to (\sim p \vee \sim q)$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

From the truth table,

$p \to (\sim p \vee \sim q)$ is false only when p and q both are true.

3. As the truth table for the

$(p \wedge \sim q) \wedge (p \wedge r) \to \sim p \vee q$ is false, t

then only possible values of (p, q, r) is (T, F, T)

4.

Given statement $p \to (\sim q \vee r)$ is False.

$\Rightarrow$ p is True and $\sim q \vee r$ is False

$\Rightarrow$ p is True and $\sim q$ is False and r is False

$\therefore$ truth values of $p, q\ r$ are T, T, F respectively.

5.

For $p \Rightarrow q \vee r$ to be F.

r should be F and $p \Rightarrow q$ should be F

for $p \Rightarrow q$ to be F, $p \Rightarrow$ T and $q \Rightarrow$ F

$p, q, r \equiv$ T, F, F

6.

By truth table :

p	q	$\sim q$	$p \vee \sim q$	$\sim p$	$p \wedge \sim q$	$p \vee q$	$p \to p \vee q$	$p \wedge q$	$(p \wedge q) \to p$	$\sim p \vee q$
T	T	F	T	F	F	T	T	T	T	T
T	F	T	T	F	T	T	T	F	T	F
F	T	F	F	T	F	T	T	F	T	T
F	F	T	T	T	F	F	T	F	T	T

$(p \wedge q) \to (\sim p) \vee q$	$(p \vee q) \to (p \vee (\sim q))$
T	T
T	T
T	F
T	T

7.

Consider the Boolean expression

$((p \wedge q) \vee (p \vee \sim q)) \wedge (\sim p \wedge \sim q)$

$= (p \vee \sim q) \wedge (\sim p \wedge \sim q)$

$= ((p \vee \sim q) \wedge \sim p) \wedge ((p \vee \sim q) \wedge \sim q)$

$= ((p \wedge \sim p) \vee (\sim q \wedge \sim p)) \wedge \sim q$

$= (\sim p \wedge \sim q) \wedge \sim q = (\sim p \wedge \sim q)$

8.

$\sim(\sim p \rightarrow Q) \equiv \sim(p \vee q) \equiv \sim p \wedge \sim q$

9.

Check each option

(a) $(p \vee q) \wedge (\sim p \wedge q) = (\sim p \wedge q)$

(b) $(p \vee q) \wedge (\sim p \vee q) = q$

(c) $(p \wedge q) \wedge (\sim p \vee q) = p \wedge q$

(d) $(p \wedge q) \wedge (\sim p \wedge q) = F$

10.

S: ''If you are born in India, then you are a citizen of India.''

Contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$

So contrapositive of statement S will be :

''If you are not a citizen of India, then you are not born in India.''

11.

Contrapositive of "If A then B" is "If ~B then ~A". Hence contrapositive of "If two numbers are not equal, then their squares are not equal" is "If squares of two numbers are equal, then the two numbers are equal".

12.

Negation of given statement $= \sim(p \vee (\sim p \wedge q))$

$= \sim p \wedge \sim(\sim p \wedge q) = \sim p \wedge (p \vee \sim q)$

$= (\sim p \wedge q) \vee (\sim p \wedge \sim q)$

$= F \vee (\sim p \wedge \sim q) = \sim p \wedge \sim q$

13.

The truth table of both the statements is

p	q	~p	~q	q ∨ p	p↔~q	(S_1)	~p↔q	(S_2)
T	T	F	F	T	F	F	F	F
T	F	F	T	T	T	T	T	T
F	T	T	F	T	T	T	T	F
F	F	T	T	F	F	T	F	F

$\therefore$ S_1 is not tautology and

S_2 is not fallacy.

Hence, both the statements (S_1) and (S_2) are not correct.

14.

p	q	$\sim q$	$p \wedge \sim q$	$\sim p$	$p \to \sim(p \wedge \sim q)$	$\sim p \vee q$
T	T	F	F	F	T	T
T	F	T	T	F	F	F
F	T	F	F	T	T	T
F	F	T	F	T	T	T

$\therefore p \to \sim(p \wedge \sim q)$ is equivalent to $\sim p \vee q$

15.

$(p \wedge q) \to (\sim q \vee r)$

$= \sim(p \wedge q) \vee (\sim q \vee r)$

$= (\sim p \vee \sim q) \vee (\sim q \vee r)$

$= (\sim p \vee \sim q \vee r)$

$\because (\sim p \vee \sim q \vee r)$ is false, then $\sim p$, $\sim q$ and r all these must be false.

$\Rightarrow p$ is true, q is true and r is false.

16.

p	q	$\sim q$	$p \wedge \sim q$	$p \to (p \wedge \sim q)$
T	T	F	F	F
T	F	T	T	T
F	T	F	F	T
F	F	T	F	T

****

SET THEORY

SYNOPSIS

- **Object :** In our mathematical language, every thing in this universe ,whether living or non living is called an object.
- **Set :** A set is a well defined collection of objects. The objects in a set are called its members or elements.
- **Well defined:** Well defined is for a given object, it is possible to determine, whether that object belongs to the given collection or not. The following collections constitute a set:

1) The vowels in english alphabets : a,e,i,o,u
2) All prime numbers
3) All rivers flowing in india.
4) The collection of all prime numbers less than20.

- **Not well defined:** The collection of all beautiful girls of india is not a set,since the term 'beautiful' is vague and it is not well defined.similarly 'rich persons', 'honest persons' ,'good players', 'young men' , 'yesterday', etc., do not form sets.
- **Notations:** The sets are usually denoted by capital letters A, B, C, etc. The members or elements of the set are denoted by lower-case letters a, b, c, etc.If x is a member of the set A, we write $x \in A$ (read as x belongs to A) and if x is not a member of the set A, we write $x \notin A$ (read as x does not belong to A). If x and y both belong to A, we write x, y $\in$ A. some examples of sets used particularly in mathematics

  N: The set of all natural numbers

  I or Z : The set of all integers

  Q :The set of all rational numbers

  R :The set of all real numbers

  Z^+ : The set of all positive integers

  Q^+ : The set of all positive rational numbers

  R^+ : The set of all positive real numbers

- **Representation of a Set :** Usually, sets are represented in the following two ways.

1. Roster form or Tabular form.
2. Set Builder form or Rule Method.

- **Roster form :** In this form, all elements of a set are listed ,the elements are being separated by commas and are enclosed within brackets {}(curly brackets). For example, the set A of all odd natural numbers less than 10 in the roster form is written as A = {1,3,5,7,9}

1) In roster form, every element of the set is listed only once.
2) The order in which the elements are listed is immaterial.

Eg 1: Each of the following sets denotes the same set {1, 2, 3}, {3, 2, 1}, {1, 3, 2}.

Eg 2: Roster form or tabular form of set of all letters in the word 'MATHEMATICS' is given by { M,A,T,H,I,E,C,S}

Note : (i) All infinite sets cannot be described in the roster form

(ii)The set of real numbers cannot be described in this form , because these elements of the set do not follow any particular pattern.

- **Set - Builder form :** In this form, All the elements of a set possess a single common property or characterstic property which is not possessed by any element outside the set.

  write a variable (say x) representing any member of the set followed by colon(:) or slash (/) which is follwed by a property satisfied by each member of the set. i.e., A set is denoted as { x : x satisfies p(x) where p(x) is the common property}.

  For example, the set A of all prime numbers less than 10 in the set-builder form is written as

  A = {x / x is a prime number less than 10}

The symbol '/' stands for the words 'such that'. Sometimes, we use the symbol ':' in place of the symbol '/'.

Eg : Set builder form of {a,e,i,o,u} is
V= { x : x is a vowel in english alphabet }

✈ Classification (or) Types of Sets :

✈ Empty Set or Null Set or void set :

A set which has no elements is called the null set or empty set or void set. It is denoted by the symbol ϕ or { }. For example, each of the following is a null set.

Eg 1 : Let A = { x : 1< x <2, x is a natural number} then A is the empty set because there is no natural number between 1and 2.

Eg 2: The set of all real numbers whose square is -1.

Eg 3: The set of all rational numbers whose square is 2.

Note : A set consisting of atleast one element is called a non-empty set.

✈ **Singleton Set :** A set having only one element is called a singleton set.

Eg 1 : {0},{ ϕ } are singleton sets, which contains only one element.

Eg 2 : Let $A = \{x : x \in N$ and $x^2 - 9 = 0\}$ then $A = \{3\}$, which is a singleton set.

But { $x : x \in Z$ and $x^2 - 9 = 0$ } = {-3 ,3 }is not a singleton set.

✈ **Finite and Infinite Sets :** A set which is empty or consists of finite number of elements is called a finite set. Otherwise, it is called an infinite set. For example, the set of all days in a week is a finite set. Where as, the set of all integers, denoted by Z={..., -2, -1, 0, 1, 2, ...} or {x / x is an integer}, is an infinite set.

✈ Cardinal Number (or) Order of a set :

The number of distinct elements in a finite set A is called the cardinal number of the set A and is denoted by n(A) or O(A) or $|A|$.

Eg : If A = { 2,4,6,8,10,12} then, n(A) = 6.

✈ **Equal Sets :** If A and B are two sets such that every member of A is a member of B and every member of B is a member of A, then we say that A and B are equal, we write as A = B. Otherwise the sets are said to be unequal and we write as $A \neq B$.

Eg 1 : A = {1,2,3 }, B = { 3,1,2 } Then A=B

Eg 2 : A set does not change if one or more elements of the set is repeated.

A= { 1,2,3} B = { 2,2,1,3,3} are equal sets.That is why we generally do not repeat any element in describing a set.

Note : A = { 1,2,3} B = { 1,3,4} Then $A \neq B$

✈ **Equivalent Sets:** Two finite sets A and B are said to be equivalent, if n(A) = n(B).

Clearly, equal sets are equivalent but equivalent sets need not be equal.

For example, the sets A={4, 5, 3, 2} and B={1, 6, 8, 9} are equivalent but are not equal.

✈ **Subset and Superset:** Let A and B be any two sets. If every element of A is an element of B, then A is called a subset of B and we write $A \subseteq B$.

If $A \subseteq B$, then ,B is called superset of A and we write $B \supseteq A$.

✈ **Proper Subset:** If $A \subseteq B$ and $A \neq B$ then A is called a proper subset of B and we write $A \subset B$ (read as A is a proper subset of B or B is a proper superset of A)

Eg : The set Q of rational numbers is proper subset of real number set R.

In two sets one is a subset of the other ,then the sets are called comparable sets.

✈ Properties of subset :

1) Every set is a subset and a superset of itself.
2) The empty set is the subset of every set.
3) If A is a set with n(A)=m, then the number of subsets of A is 2^m and the number of proper subsets of A is $2^m - 1$.

Note : If A,B,C are any three sets,then

i) $A \subseteq B, B \subseteq C \Rightarrow A \subseteq C$

ii) $A \subseteq B, B \subset C \Rightarrow A \subset C$

iii) $A \subset B, B \subseteq C \Rightarrow A \subset C$

iv) $A \subset B, B \subset C \Rightarrow A \subset C$

✈ **Power Set :** The set of all subsets of a given set A is called power set of A and is denoted by P(A). Clearly, if A has n elements, then its power set P(A) contains exactly 2^n elements.

For example, if A={1, 2, 3}, then $P(A) = \{\phi,$ {1}, {2}, {3}, {1, 2}, {1,3},{2, 3}, {1, 2, 3}}.

- **Universal Set :** If there are some sets under consideration , then there happens to be a set which is a superset of each one of the given sets. Such a set is known as the universal set for those sets,We shall denote by U or μ.

Eg 1 : A = {1,2,3} B = { 2,3,4,5} C ={6,7} Then we consider U={ 1,2,3,4,5,6,7} as its one of the universal sets.

Eg 2 : In the study of two dimensional geometry, the set of all points in the XY- plane is called universal set.

- **Disjoint sets :** If two sets A and B are such that they do not have any elements in common i.e., $A \cap B = \phi$,then A,B are said to be disjoint sets.

  Eg:A= { X : X is odd number},
  B = {X : X is even number } then A,B have no common elements.

- **Venn Diagram :** In order to express the relationship among sets in perspective,we represent them pictorially by means of diagrams is called Venn Diagram. In these diagrams ,the universal set is represented by a rectangular region and the subsets by circles inside the rectangle.We represent disjoint sets by disjoint circles and intersecting sets by intersecting circles.

Operations on Sets :

- **Union of Two Sets :** The union of two sets A and B, written as $A \cup B$ (read as A union B), is the set consisting of all the elements which are either in A or in B or in both. Thus,

  $$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

  Clearly (i) $x \in A \cup B \Rightarrow x \in A$ or $x \in B$

  (ii) $x \notin A \cup B \Rightarrow x \notin A$ and $x \notin B$.

  For example, if A={a, b, c, d} and B = {c,d,e,f}, then $A \cup B$={a, b, c, d, e, f}.

- **Intersection of Two Sets :** The intersection of two sets A and B, written as $A \cap B$ (read as A intersection B) is the set consisting of all the common elements of A and B. Thus,

  $$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

  Clearly (i) $x \in A \cap B \Rightarrow x \in A$ and $x \in B$

  (ii) $x \notin A \cap B \Rightarrow x \notin A$ or $x \notin B$.

  For example, if A={a, b, c, d} and B = {c,d,e,f} then $A \cap B$={c, d}.

- **Difference of Two Sets :** If A and B are two sets, then their difference $A - B$ or $A \setminus B$ (or) $\frac{A}{B}$ is defined as :

  $$A - B = \{x : x \in A \text{ and } x \notin B\}$$

  Similarly, $B - A = \{x : x \in B \text{ and } x \notin A\}$

  For example, if A={1,2,3,4} and B={1,3,5,7,9}, then A–B={2,4} and B – A = {5,7,9}.

- **Important Results :** In general
  1. $A - B \neq B - A$
  2. The sets A-B, B-A and $A \cap B$ are disjoint sets.
  3. $A - B \subseteq A$ and $B - A \subseteq B$
  4. $A - \phi = A$ and $A - A = \phi$

- **Symmetric Difference of Two sets :** The symmetric difference of two sets A and B, denoted by $A \Delta B$, is defined as

  $$A\Delta B = (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$

Eg : If A={1, 2, 3, 4, 5} and B={1, 3, 5, 7, 9} then

$$A\Delta B = (A - B) \cup (B - A)$$

$$= \{2,4\} \cup \{7,9\} \; = \{2,4,7,9\}$$

- **Complement of a Set :** If U is a universal set and A is a subset of U, then the complement of A is the set which contains those elements of U, which are not contained in A and is denoted by A' or A^c. Thus, $A' = \{x : x \in U \text{ and } x \notin A\}$

Eg : If U={1, 2, 3, 4, ...} and A = {2, 4, 6, 8, ...}, then, A'={1, 3, 5, 7, ...}.

Properties of complement sets :

i) $U' = \phi$ ii) $\phi' = U$

iii) $A \cup A' = U$ iv) $A \cap A' = \phi$

v) $(A')' = A$, law of double complementation.

vi) $(A\cup B)' = A'\cap B'$ and

$(A\cap B)' = A'\cup B'$ are called demorgan laws

Algebra of sets :

i) **Idempotent Laws :** For any set A, we have

a) $A\cup A = A$ b) $A\cap A = A$

ii) **Commutative Laws :** For any two sets A and B, we have

a) $A\cup B = B\cup A$ b) $A\cap B = B\cap A$

iii) **Identity Laws :** For any set A, U is universal set, we have

a) $A\cup\phi = A$ b) $A\cap\phi = \phi$

c) $A\cup U = U$ d) $A\cap U = A$

iv) **Associative Laws :** For any three sets A, B and C, we have

a) $A\cup(B\cup C) = (A\cup B)\cup C$

b) $A\cap(B\cap C) = (A\cap B)\cap C$

v) **Distributive Laws:** For any three sets A, B and C, we have

a) $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

b) $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

For any three sets A, B and C

i) $A-(B\cup C) = (A-B)\cap(A-C)$

ii) $A-(B\cap C) = (A-B)\cup(A-C)$

✈ For any two sets A and B, we have

a) $P(A)\cap P(B) = P(A\cap B)$

b) $P(A)\cup P(B)\subseteq P(A\cup B)$, where $p(A)$ is the power set of A.

✈ If A and B are any two sets then

i) $A\subset B \Rightarrow A\cup B = B,\ A\cap B = A$

ii) $A-B = A-(A\cap B) = (A\cup B)-B$

iii) $(A-B)\cup(A\cap B) = A$

iv) $(A-B)\cup(B-A)\cup(A\cap B) = A\cup B$

v) $(A-B)\cup(B-A) = (A\cup B)-(A\cap B)$

vi) $A\subseteq A\cup B$, $B\subseteq A\cup B$,

$A\cap B\subseteq A$, $A\cap B\subseteq B$

vii) $A-B = A\cap B'$

viii) $(A-B)\cup B = A\cup B$

ix) $(A-B)\cap B = \phi$

x) $A\subseteq B \Leftrightarrow B'\subseteq A'$

xi) $A-B = B'-A'$

xii) $A-B = B-A \Leftrightarrow A = B$

xiii) $A\cup B = A\cap B \Leftrightarrow A = B$

✈ **Properties on symmetric difference :**

A,B,C are any three sets

i) $A\Delta\phi = A$ ii) $A\Delta A = \phi$

iii) $A\Delta B = A\Delta C \Rightarrow B = C$

iv) $A\Delta B = B\Delta A$ v) $(A\Delta B)\Delta C = A\Delta(B\Delta C)$

vi) $A\cap(B\Delta C) = (A\cap B)\Delta(A\cap C)$

✈ **Some important results on cardinal numbers :** If A, B and C are finite sets and U be the finite universal set, then

i) $n(A') = n(U)-n(A)$

ii) $n(A\cup B) = n(A)+n(B)-n(A\cap B)$

iii) $n(A\cup B) = n(A)+n(B)$,

where A and B are disjoint non-empty sets

iv) $n(A-B) = n(A\cap B') = n(A)-n(A\cap B)$

$= n(A\cup B)-n(B)$

v) $n(B-A) = n(A^1\cap B) = n(B)-n(A\cap B)$

$= n(A\cup B)-n(A)$

vi) $n(A'\cap B') = n(A\cup B)' = n(U)-n(A\cup B)$

vii) $n(A'\cup B') = n(A\cap B)' = n(U)-n(A\cap B)$

viii) $n(A\cap B) = n(A\cup B)-n(A\cap B')-n(A'\cap B)$

ix) $n(A\cup B\cup C) = n(A)+n(B)+n(C)$

$-n(A\cap B)-n(B\cap C)-n(C\cap A)+n(A\cap B\cap C)$

x) If A_1, A_2, A_3 .... A_n are pair-wise disjoint sets, then $n(A_1\cup A_2\cup A_3\cup....\cup A_n)$

$= n(A_1)+n(A_2)+n(A_3)+....+n(A_n)$

✈ $n(A\Delta B)$ = number of elements which belong to exactly one of A or B.

$$n(A\Delta B)=n\{(A-B)\cup(B-A)\}$$
$$=n(A)+n(B)-2n(A\cap B)$$
$$=n(A\cup B)-n(A\cap B)$$

✈ A and B are two sets and $n(A)=p$, $n(B)=q$,

Then (i) $\min\{n(A\cup B)\}=\max\{p,q\}$

(ii) $\max\{n(A\cup B)\}=p+q$,

(iii) $\min\{n(A\cap B)\}=0$

(iv) $\max\{n(A\cap B)\}=\min\{p,q\}$

✈ No. of elements in exactly one of the sets A,B,C

$$=n(A)+n(B)+n(C)-2n(A\cap B)-$$
$$2n(B\cap C)-2n(A\cap C)+3n(A\cap B\cap C)$$

✈ No. of elements in exactly two of the sets A,B,C

$$=n(A\cap B)+n(B\cap C)$$
$$+n(C\cap A)-3n(A\cap B\cap C)$$

EXERCISE - I

1. **Which of the following not a well defined collection of objects**
   1) The set of Natural Numbers
   2) Rivers of India
   3) Various kinds of Triangles
   4) Five most renowned Mathematicians of the world.
2. **Write the solution set of the equation $x^2+x-6=0$ in roster form**
   1) {2, -3} 2) $\{-1,-2\}$ 3) $\{1,2\}$ 4) $\{-1,2\}$
3. **Write the set A={1,4,9,16,25......}in set builder form**
   1) $\{x:x=n^2 \text{ where } n\in N\}$
   2) $\{x:x=n^2 \text{ where } n\in W\}$
   3) $\{x:x=n^2 \text{ where } n\in Z\}$
   4) $\{x:x=n^2 \text{ where } n\in Q\}$
4. **Which of the following is not empty set**
   1) A={x : 1<x<2, x is a natural number between 1 and 2}
   2) B = $\{x:x^2-2=0$ and x is rational$\}$
   3) C = $\{x:x$ is even prime number >2$\}$
   4) D = $\{x:x^2=0$ and x is integer$\}$
5. **If A={x/x is a letter in the word "ACCOUNTANCY"}then cardinality of A is**
   1) 5 2) 6 3) 7 4) 8
6. **Let F_1 be the set of all parallelograms, F_2 be the set of rectangles, F_3 be the set of rhombuses, F_4 be the set of squares and F_5 be the set of trapeziums in a plane then F_1=**
   1) $F_2\cap F_3$ 2) $F_2\cup F_3\cup F_4$
   3) $F_3\cup F_4\cup F_5$ 4) $F_3\cap F_1$
7. **If the set of factors of a whole number 'n' including 'n' itself but not '1' is denoted by F(n). If $F(16)\cap F(40)=F(x)$ then 'x' is**
   1) 4 2) 8 3) 6 4) 10
8. **If A is the set of the divisors of the number 15, B is the set of prime numbers smaller than 10 and C is the set of even numbers smaller than 9, then $(A\cup C)\cap B$ is the set**
   1){1, 3, 5} 2){1, 2, 3} 3){2, 3, 5} 4){2, 5}
9. **Let A={1,2,3,4,5,6}, B={2,4,6,8}then A - B =**
   1) {1,3,5} 2) {8} 3) {2,4,6} 4) ϕ
10. **Let $A=\{1,2,3,4\}$ and $B=\{2,3,4,5,6\}$, then $A\Delta B=$**
    1) $\{2,3,4\}$ 2) $\{1\}$ 3) $\{5,6\}$ 4) $\{1,5,6\}$
11. **Let U={1,2,3,4,5,6}, A={2,3},B={3,4,5} then $A^1\cap B^1=$ __**
    1) {1,2} 2) {1,6} 3){1,5} 4){1,4}
12. **In a class of 35 students,24 like to play cricket and 16 like to play football also each student likes to play at least one of the two games. How many students like to play both cricket and football?**
    1) 3 2) 4 3) 5 4) 6

13. In a group of 70 people, 37 like coffee, 52 like tea and each person like atleast one of the two drinks. The number of persons liking both coffee and tea is

1) 16 2) 13 3) 19 4) 20

14. If n(X)=28, n(Y)=32, n($X \cup Y$)=50 then $n(X \cap Y) =$

1) 6 2) 7 3) 8 4)10

15. If n(A)=50, n(B)=20 and $n(A \cap B) = 10$ **then** $n(A \Delta B)$ **is**

1) 50 2) 60 3) 70 4) 40

KEY

01) 4	02) 1	03) 1	04) 4	05) 3	06) 2
07) 2	08) 3	09) 1	10) 4	11) 2	12) 3
13) 3	14) 4	15) 1			

SOLUTIONS

1. For determining a mathematicians most renowned may vary from person to person

2. $x^2 + x - 6 = 0 \Rightarrow x = 2, -3$

3. $1^2 = 1, 2^2 = 4, 3^2 = 9....$ all are square of natural numbers.

4. 1) $A = \phi$ because there are no natural numbers between 1 and 2.

   2) $B = \phi$ because $x^2 = 2 \Rightarrow x = \sqrt{2}$ not a rational number

   3) $c = \phi$ because there is only one even prime2

5. Different letters of the word ACCOUNTANCY is $\{A, C, O, U, N, T, Y\}$; Cardinality of A = 7.

6. Since every rectangle, rhombus and square is a parallelogram so $F_1 = F_2 \cup F_3 \cup F_4 \cup F_1$

7. $F(16) = \{2,4,8,16\}, F(40) = \{2,4,8,20,40\}$

   $F(16) \cap F(40) = \{2,4,8\} = F(8)$

   $F(x) = F(8) \Rightarrow x = 8$

8. $A = \{1,3,5,15\}, B = \{2,3,5,7\}, C = \{2,4,6,8\}$

   $\therefore A \cup C = \{1,2,3,4,5,6,7,8,15\}$

   $(A \cup C) \cap B = \{2,3,5\}$

9. A-B= {1,2,3,4,5,6}-{2,4,6,8,}={1,3,5}

10. $A \Delta B = \left\{ x : x \in \frac{A}{B} \text{ or } x \in \frac{B}{A} \right\}$

11. $A^1 = U - A = \{1,4,5,6\}$, $B^1 = U - B = \{1,2,6\}$

    $A^1 \cap B^1 = \{1,6\}$

12. n(C)=24, n(F)=16, n($C \cup F$)=35

    n($C \cap F$)=n(c)+n(F)-n($C \cup F$)

    = 24 +16 – 35 = 40 – 35 = 5

13. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

    we have, $70 = 37 + 52 - n(A \cap B)$

14. n($X \cap Y$)=n(x) +n(Y)-n(X U Y)

    = 28 + 32 - 50 = 10

15. $n(A \Delta B) = n(A \cup B) - n(A \cap B)$

    $= n(A) + n(B) - 2n(A \cap B) = 50$

EXERCISE - II

1. Let X = {1, 2, 3, 4, 5, 6}. If n represent any member of X, then roster form of $n \in X$ but $2n \notin X$

1) {2, 3, 5, 6} 2) {5, 6}
3) {4, 5, 6} 4) {3}

2. Two finite sets have m and n elements. If total number of subsets of the first set is 56 more than that of the total number of subsets of the second. The values of *m* and *n* respectively are

1) 7,6 2) 6,3 3)5,1 4) 8, 7

3. If $A=\{8^n-7n-1:n\in N\}$ and $B=\{49(n-1):n\in N\}$ then

1) $A\subset B$ 2) $B\subseteq A$ 3) $A=B$ 4) $A\subseteq B$

4. If $A=\{\phi,\{\phi\}\}$ then the power set of A is

1) A 2) $\{\phi,\{\phi\},A\}$ 3) $\{\phi,\{\phi\},\{\{\phi\}\},A\}$ 4) $\{\phi\}$

5. The smallest set A such that $A\cup\{1,2\}=\{1,2,3,5,9\}$ is

1) $\{2,3,5\}$ 2) $\{3,5,9\}$ 3) $\{1,2,5,9\}$ 4) $\{1,2\}$

6. If sets A and B are defined as

$A=\{(x,y):y=e^x, x\in R\}$

$B=\{(x,y):y=x, x\in R\}$, **then**

1) $B\subset A$ 2) $A\subset B$
3) $A\cap B=\phi$ 4) $A\cup B=A$

7. If $aN=\{ax:x\in N\}$ then $3N\cap 7N=$

1) 21N 2) 10N 3) 4N 4) 5N

8. If $A=\{1,2,3,4,5,6\}$, $B=\{1,2\}$, then $\dfrac{A}{\left(\dfrac{A}{B}\right)}=$

1) A 2) ϕ 3) $A\cap B$ 4) $A\cup B$

9. If $n(U)=700$, $n(A)=200$, $n(B)=300$, $n(A\cap B)=100$, then $n(A'\cap B')$ is equal to

1) 400 2) 240 3) 300 4) 500

10. If $n(U)=48, n(A)=28, n(B)=33$ and $n(B-A)=12$, then $n(A\cap B)^C$ is

1) 27 2) 28 3) 29 4) 30

11. If $n(A\cap B^C)=5, n(B\cap A^C)=6$, $n(A\cap B)=4$ then the value of $n(A\cup B)$ is

1) 18 2) 15 3) 16 4) 17

12. Let n(A-B)=25+X, n(B-A)=2X and $n(A\cap B)=2X$. If n(A)=2(n(B)) then 'X' is

1) 4 2) 5 3) 6 4) 7

13. Of the members of three athletic teams in a school 21 are in the cricket team, 26 are in the hockey team and 29 are in the football team. Among them, 14 play hockey and cricket, 15 play hockey and foot ball, and 12 play foot ball and cricket. Eight play all the three games. The total number of members in the three athletic teams is

1) 43 2) 76 3) 49 4) 53

14. If sets A and B have 3 and 6 elements each, then the minimum number of elements in $A\cup B$ is

1) 3 2) 6 3) 9 4) 18

15. If $n(U)=60, n(A)=21, n(B)=43$ then greatest value of $n(A\cup B)$ and least value of $n(A\cup B)$ are

1) 60, 43 2) 50, 36 3) 70, 44 4) 60, 38

KEY

01) 3	02) 2	03) 1	04) 3	05) 2	06) 3
07) 1	08) 3	09) 3	10) 1	11) 2	12) 2
13) 1	14) 2	15) 1			

SOLUTIONS

1. $A=\{n/n\in X \text{ but } 2n\notin X\}\Rightarrow n=4,5,6$

2. $2^m-2^n=56$

3. $8^n=(7+1)^n$

$={}^nc_0 7^n+{}^nc_1 7^{n-1}+........+{}^nc_{n-2}7^2+{}^nc_{n-1}7+{}^nc_n$

$={}^nc_0 7^n+{}^nc_1 7^{n-1}+........+{}^nc_{n-2}49+7n+1$

$8^n - 7n - 1 = 49\left[{}^n c_0 7^{n-2} + {}^n c_1 7^{n-3} + ... + {}^n c_{n-2}\right]$

$8^n - 7n - 1$ is a multiple of 49 for all $n \in N$

$\therefore$ A contains elements which are multiple of 49 and clearly B contains all multiples of 49.

$\therefore A \subset B$

4. no of subsets $= 2^n$

5. Since $A \cup \{1,2\} = \{1,2,3,5,9\}$ {given}

$\Rightarrow A = \{3,5,9\}$ atleast

6. The graph of $y = e^x$ *and* $y = x$ do not intersect

7. $3N = \{3x : x \in N\}, 7N = \{7x : x \in N\}$

$\Rightarrow 3N \cap 7N = 21N$

8. $\frac{A}{B} = A - B = \{3,4,5,6\}$, $\frac{A}{\left(\frac{A}{B}\right)} = \{1,2\}$

9. $n\left(A' \cap B'\right) = n(A \cup B)' = n(U) - n(A \cup B)$

10. $n(U) = 48, n(A) = 28, n(B) = 33$,

n(B – A) = 12

$n(A \cap B) = n(B) - n(B - A) = 33 - 12 = 21$

$n(A \cap B)^C = n(U) - n(A \cap B) = 48 - 21 = 27$

11. $n\left(A \cap B^C\right) = 5, n\left(B \cap A^C\right) = 6$,

$n(A \cap B) = 4$

$n(A) = n(A \cap B) + n\left(A \cap B^C\right) = 4 + 5 = 9$

$n(B) = n(A \cap B) + n\left(B \cap A^C\right) = 6 + 4 = 10$

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

12. $n(A - B) = 25 + x, n(B - A) = 2x$,

$n(A \cap B) = 2x, n(A) = n(A - B) + n(A \cap B)$

$= 25 + x + 2x = 25 + 3x$

$n(B) = n(B - A) + n(A \cap B) = 2x + 2x = 4x$

$n(A) = 2n(B) \Rightarrow 25 + 3x = 2(4x)$

$\Rightarrow 5x = 25 \Rightarrow x = 5$

13. $n(C) = 21$, $n(H) = 26$, $n(F) = 29$,

$n(H \cap C) = 14$, $n(H \cap F) = 15$

$n(F \cap C) = 12$, $n(F \cap C \cap H) = 8$

Total no.of players = $n(C \cup H \cup F) = 43$

14. $n(A \cup B) \geq \max\{3,6\} = 6$

15. $n(U) = 60, n(A) = 21, n(B) = 43$

Greatest value of $n(A \cup B) = n(U) = 60$

Least value of $n(A \cup B) = n(B) = 43$

EXERCISE - III

1. Let A and B be two sets then

$(A \cup B)^C \cup \left(A^C \cap B\right) =$

1) A^C 2) B^C 3) ϕ 4) U

2. A set contains $(2n+1)$ elements. The number of subsets of this set containing more than n elements is equal to

1) 2^{n-1} 2) 2^n 3) 2^{n+1} 4) 2^{2n}

3. From 50 students taking examinations in Mathematics, Physics and Chemistry, each of the students has passed in at least one of the subject, 37 passed Mathematics, 24 Physics and 43 Chemistry. At most 19 passed Mathematics and Physics, at most 29

Mathematics and Chemistry and at most 20 Physics and Chemistry. Then the largest possible number that could have passed all three examinations is

1) 16 2) 14 3) 18 4) 15

4. Which is the simplified representation of $(A^1 \cap B^1 \cap C) \cup (B \cap C) \cup (A \cap C)$ where A,B,C are subsets of set X

1) A 2) B 3) C 4) $X \cap (A \cup B \cup C)$

5. The set $(A \cup B \cup C) \cap (A \cap B' \cap C')' \cap C'$ is equal to

1) $B \cap C'$ 2) $A \cap C$ 3) $B \cup C'$ 4) $A \cap C'$

6. If $P = \{x \in R : f(x) = 0\}$ and $Q = \{x \in R : g(x) = 0\}$ then $P \cup Q$ is

1) $\{x \in R : f(x) + g(x) = 0\}$

2) $\{x \in R : f(x).g(x) = 0\}$

3) $\{x \in R; (f(x))^2 + (g(x))^2 = 0\}$

4) $\{x \in R; x > 1\}$

7. Suppose $A_1, A_2 ..., A_{30}$ are thirty sets each with five elements and $B_1, B_2, ...B_n$ are n sets each with three elements such that $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$. If each element of S belongs to exactly ten of the A_i's and exactly 9 of the B_j's, then the value of n is

1) 15 2) 135 3) 45 4) 90

8. If $aN = \{ax / x \in N\}$ and $bN \cap cN = dN$, where $b, c \in N$ are relatively prime, then

1) $d = bc$ 2) $c = bd$ 3) $b = cd$ 4) none

9. A survey show that in a city that 63% of the citizens like tea where as 76% like coffee. If x% like both tea and coffee, then

1) $x = 63$ 2) $x = 39$

3) $50 \le x \le 63$ 4) $39 \le x \le 63$

**10. An investigator interviewed 100 students to determine their preferences for the three drinks:
milk (M), coffee(C) and tea (T). He reported the following : 10 students had all the three drinks M,C,T; 20 had M and C only: 30 had C and T;25 had M and T;12 had M only;5 had C only;8 had T only. Then how many did not take any of the three drinks**

1) 20 2) 3 3) 36 4) 42

11. In a college of 300 students , every students reads 5 newspapers and every newspaper is read by 60 students. The number of newspapers is

1) atleast 30 2) atmost 20

3) exactly 25 4) atmost10

12. In a class of 55 students the numbers of students studying different subjects are 23 in mathematics, 24 in physics, 19 in chemistry, 12 in mathematics and physics, 9 in mathematics and chemistry 7 in physics and chemistry and 4 in all the three subjects. the numbers of students who have taken exactly one subject is

1) 6 2) 13 3) 16 4) 22

13. Out of 800 boys in a school. 224 played cricket 240 played hockey and 336 played basketball. of the total, 64 played both basketball and hockey, 80 played cricket and basketball and 40 played cricket and hockey, 24 played all the three games. The numbers of boys who did not play any game is

1) 128 2) 216 3) 240 4) 160

**14. In a certian town 25% families own a phone and 15% own a car, 65% families own neither a phone nor a car. 2000 families own both a car and a phone. Consider the following statements in this regard.
i. 10% families own both a car and a phone
ii. 35% families own either a car or a phone
iii. 40,000 families live in the town.
Which of the above statements are correct?**

1) i and ii 2) i and iii 3) ii and iii 4) i,ii and iii

15. In a battle 70% of the combatants lost one eye, 80% an ear, 75% an arm, 85% a leg, x% lost all the four limbs the minimum value of x is

1) 10 2) 12 3) 15 4) 5

KEY

01) 1 02) 4 03) 2 04) 3 05) 1 06) 2
07) 3 08) 1 09) 4 10) 1 11) 3 12) 4
13) 4 14) 3 15) 1

SOLUTIONS

1. $\left(A\cup B\right)^{C}\cup\left(A^{C}\cup B\right)$

$=\left(A^{C}\cap B^{C}\right)\cup\left(A^{C}\cup B\right)$

$=\left(A^{C}\cup A^{C}\right)\cap\left(A^{C}\cup B\right)\cap\left(B^{C}\cup A^{C}\right)\cap\left(B^{C}\cup B\right)$

$=A^{C}\cap\left[A^{C}\cup\left(B\cap B^{C}\right)\right]\cap U$

$=A^{C}\cap\left(A^{C}\cup\phi\right)\cap U$

$=A^{C}\cap A^{C}\cap U=A^{C}\cap U=A^{C}$

2. Let the original set contains 2n+1 elements, then subsets of this set containing more than n elements means subsets containing (n+1) elements, (n+2) elements .... (2n+1) elements.
$\therefore$ Required number of subsets

$={}^{2n+1}c_{n+1}+{}^{2n+1}c_{n+2}+.....+{}^{2n+1}c_{2n}+{}^{2n+1}c_{2n+1}$

$={}^{2n+1}c_{n}+{}^{2n+1}c_{n+1}+.....+{}^{2n+1}c_{1}+{}^{2n+1}c_{0.}$

$={}^{2n+1}c_{0}+{}^{2n+1}c_{1}+.....+{}^{2n+1}c_{n-1}+{}^{2n+1}c_{n.}$

$=\frac{1}{2}\left[\left(1+1\right)^{2n+1}\right]=\frac{1}{2}\left(2^{2n+1}\right)=2^{2n}$.

3. The given conditions can be expressed as

$n\left(M\cup P\cup C\right)=50, n\left(M\right)=37, n\left(p\right)=24,$

$n\left(C\right)=43, n\left(M\cap P\right)\le 19, n\left(M\cap C\right)\le 29$

and $n\left(P\cap C\right)\le 20$.

$n\left(M\cup P\cup C\right)=$

$n\left(M\right)+n\left(P\right)+n\left(C\right)-n\left(M\cap P\right)$

$-n\left(M\cap C\right)-n\left(P\cap C\right)+n\left(M\cap P\cap C\right)$

$\Rightarrow n\left(M\cap P\cap C\right)\le n\left(M\cap P\right)+n\left(M\cap C\right)$

$+n\left(P\cap C\right)-54$

Therefore, the number of students is at most 19+29+20-54=14

4. $\left(A^{1}\cap B^{1}\cap C\right)\cup\left(B\cap C\right)\cup\left(A\cap C\right)=C$

draw venn diagram.

5. $\left(A\cup B\cup C\right)\cap\left(A\cap B'\cap C'\right)'\cap C'=\left(A\cup B\cup C\right)$

$\cap\left(A'\cup B\cup C\right)\cap C'=$

$\left[\left(A\cap A'\right)\cup\left(B\cup C\right)\right]\cap C'=\left(B\cup C\right)\cap C'$

$=\left(B\cap C'\right)\cup\left(C\cap C'\right)=B\cap C'$

6. $f\left(x\right).g\left(x\right)=0\Rightarrow$ either $f\left(x\right)=0$ or $g\left(x\right)=0$.

7. $S=\bigcup_{i=1}^{30}A_i\Rightarrow n\left(S\right)=\frac{1}{10}\left(5\times 30\right)=15$

Again, $S=\bigcup_{j=1}^{n}B_j\Rightarrow n\left(S\right)=\frac{1}{9}\left(3\times n\right)=\frac{n}{3}$.

Thus $\frac{n}{3}=15\Rightarrow n=45$

8. We have $bN=\left\{bx\middle|x\in N\right\}=$ the set of positive integral multiples of b and $cN=\left\{cx\middle|x\in N\right\}=$ the set of positive integral multiples of c.
$\therefore bN\cap cN=$ the set of positive integral multiples of bc $=bcN$ [$\because$ b and c are relatively prime] Hence, $d=bc$

9. Let the population of the city be 100
Let A denote the set of citizens who like tea and B denote the set of citizens who like coffee.

$\therefore n\left(A\right)=63$ and $n\left(B\right)=76$

$\therefore n\left(A\cup B\right)=n\left(A\right)+n\left(B\right)-n\left(A\cap B\right)$ and

$n\left(A\cup B\right)\le 100\Rightarrow 63+76-n\left(A\cap B\right)\le 100$

$\Rightarrow 63+76-n\left(A\cap B\right)\le 100$

$\Rightarrow 39\le n\left(A\cap B\right)\to\left(1\right)$

Also $n\left(A\cap B\right)\le n\left(A\right)$ and $n\left(A\cap B\right)\le n\left(B\right)$

$\Rightarrow n\left(A\cap B\right)\le 63$ and $n\left(A\cap B\right)\le 76$

$\Rightarrow n\left(A\cap B\right)\le 63\to\left(2\right)$

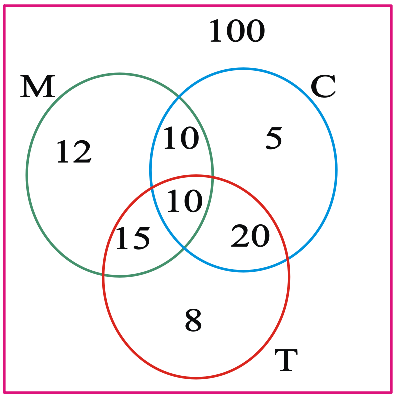
From (1) and (2) $:39\le n\left(A\cap B\right)\le 63$

$\Rightarrow 39\le x\le 63$

10. $S=100$. The numbers can be read from the fig, number of people who did not take any drink

$=100-\left\{12+5+8+10+20+15+10\right\}$

$=100-80=20$



11. If n is the required number of newspapers then
$n \times 60 = 300 \times 5 \Rightarrow n = 25$

12. $n(M) = 23, n(P) = 24, n(C) = 19,$

$n(M \cap C) = 9,$

$n(P \cap C) = 7, n(M \cap P \cap C) = 4.$

We have to find , $n(M \cap P' \cap C'),$

$n(P \cap M' \cap C'), n(C \cap M' \cap P')$

Now $n(M \cap P' \cap C') = n\left[M \cap (P \cup C)'\right]$

$= n(M) - n[M \cap (P \cup C)]$

$= n(M) - n[(M \cap P) \cup (M \cup C)]$

$= n(M) - n(M \cap P) - n$

$(M \cap C) + n(M \cap P \cap C)$

$= 23 - 12 - 9 + 4 = 27 - 21 = 6.$

$n(P \cap M' \cap C') = n\left[P \cap (M \cup C)'\right]$

$= n(P) - n[P \cap (M \cup C)]$

$= n(P) - n[(P \cap M) \cup (P \cap C)]$

$= n(P) - n[P \cap M] -$

$n(P \cap C) + n(P \cap M \cap C)$

$= 24 - 12 - 7 + 4 = 9$, $n(C \cap M' \cap P') =$

$n(C) - n(C \cap P) - n(C \cap M) + n(C \cap P \cap M)$

$= 19 - 7 - 9 + 4 = 23 - 16 = 7$

13. $n(C) = 224, n(H) = 240, n(B) = 336.$

$n(H \cap B) = 64,$

$n(B \cap C) = 80, n(H \cap C) = 40,$

$n(C \cap H \cap B) = 24$

$n(C^C \cap H^C \cap B^C) = n\left[(C \cup H \cup B)^C\right] =$

$n(U) - n(C \cup H \cup B)$

$$= 800 - \begin{bmatrix} n(C) + n(H) + n(B) - n(H \cap C) \\ -n(H \cap B) - n(C \cap B) + n(C \cap H \cap B) \end{bmatrix}$$

$= 800 - [224 + 240 + 336 - 64 - 80 - 40 + 24]$

$= 800 - [824 - 184] = 984 - 824 = 160.$

14. $n(P) = 25\%, n(C) = 15\%$

$n(P^C \cap C^C) = 65\%,\ \ n(P \cap C) = 2000$

Since $n(P^c \cap C^c) = 65\% \Rightarrow n(P \cup C)^C$

$= 65\% \Rightarrow n(P \cup C) = 35\%$, Now

$n(P \cup C) = n(P) + n(C) - n(P \cap C)$

$\Rightarrow 35 = 25 + 15 - n(P \cap C)$

$\Rightarrow n(P \cap C) = 40 - 35 = 5.$

Thus $n(P \cap C) = 5\%$. But $n(P \cap C) = 2000$

$\therefore 5\%$ of the total $= 2000 \Rightarrow$

Total numbers of families $= \dfrac{2000 \times 100}{5} = 40000.$

Since $n(P \cup C) = 35\%$ and total number of families =40,000 and $n(P \cap C) = 5\%$.

$\therefore (ii)$ *and* (iii) are correct.

15. Minimum value of
$x = 100 - (30 + 20 + 25 + 15) = 100 - 90 = 10.$

JEE MAINS QUESTIONS

1.Set A has m elements and set B has n elements. If the totalnumber of subsets of A is 112 more than the total numberof subsets of B, then the value of $m \times n$ is [2020]

2.A survey shows that 73% of the persons working in anoffice like coffee, whereas 65% like tea. If x denotes thepercentage of them, who like both coffee and tea, then xcannot be : [2020]

(1) 63 (2) 36 (3) 54 (4) 38

3.A survey shows that 63% of the people in a city readnewspaper A whereas 76% read newspaper B. If x% of thepeople read both the newspapers, then a possible value ofx can be : [2019]

(1) 29 (2) 37 (3) 65 (4) 55

4.In a class of 140 students numbered 1 to 140, all evennumbered students opted Mathematics course, thosewhose number is divisible by 3 opted Physics course andthose whose number is divisible by 5 opted Chemistrycourse. Then the number of students who did not opt forany of the three courses is: [2019]

(1) 102 (2) 42 (3) 1 (4) 38

KEY

1) 28 2) 2 3) 4 4) 4

SOLUTIONS

1.

$$2^m = 112 + 2^n \Rightarrow 2^m - 2^n = 112$$

$$\Rightarrow 2^n(2^{m-n} - 1) = 2^4(2^3 - 1)$$

$$\therefore m = 7,\ n = 4 \Rightarrow mn = 28$$

2.

Given, $n(C) = 73,\ n(T) = 65,\ n(C \cap T) = x$

$$\therefore 65 \geq n(C \cap T) \geq 65 + 73 - 100$$

$$\Rightarrow 65 \geq x \geq 38 \Rightarrow x \neq 36.$$

3.

Let $n(U) = 100$, then $n(A) = 63, n(B) = 76$

$n(A \cap B) = x$

Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B) \leq 100$

$$= 63 + 76 - x \leq 100$$

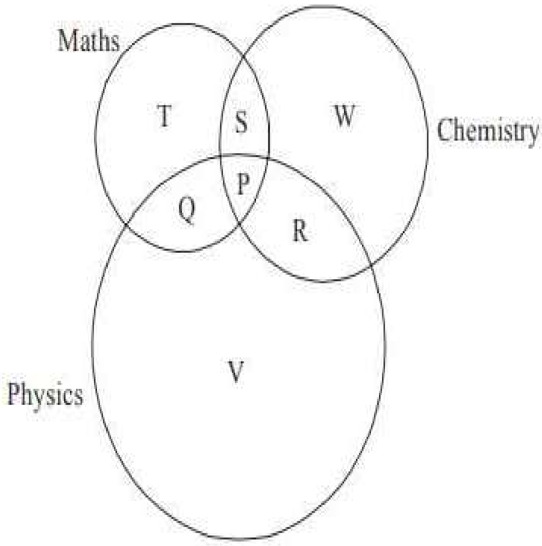
$$\Rightarrow x \geq 139 - 100 \Rightarrow x \geq 39$$

$$\because n(A \cap B) \leq n(A)$$

$$\Rightarrow x \leq 63$$

$$\therefore 39 \leq x \leq 63$$

4.



$$P = \{30, 60, 90, 120\}$$

$$\Rightarrow n(P) = 4$$

$$Q = \{6n : n \in N, 1 \le n \le 23\} - P$$

$$\Rightarrow n(Q) = 19$$

$$R = \{15n : n \in N, 1 \le n \le 9\} - P$$

$$\Rightarrow n(R) = 5$$

$$S = \{10n : n \in N, 1 \le n \le 14\} - P$$

$$\Rightarrow n(S) = 10$$

$$n(T) = 70 - n(P) - n(Q) - n(S) = 70 - 33 = 37$$

$$n(V) = 46 - n(P) - n(Q) - n(R) = 46 - 28 = 18$$

$$n(W) = 28 - n(P) - n(R) - n(S) = 28 - 19 = 9$$

$\Rightarrow$ Number of required students

$$= 140 - (4 + 19 + 5 + 10 + 37 + 18 + 9)$$

$$= 140 - 102 = 38$$

RELATIONS

SYNOPSIS

✈ **Cartesian product of sets:** Let P and Q be two non empty sets. The set of all ordered pairs (a,b) such that $a \in P$ and $b \in Q$ is called the cartesian product of set P with set Q and is denoted by $P \times Q$.

Thus $P \times Q = \{(a,b) : a \in P \text{ and } b \in Q\}$

If P or Q is a null set then $P \times Q$ will also a null set.

Note: (i) Two ordered pairs are equal , if and only if the corresponding first elements are equal and the second elements are also equal.
(ii) If n(A) = p, n(B)=q ,then n(A x B) = pq

✈ $A \times A \times A$ = {(a,b,c) : a,b,c $\in$ A}. Here (a,b,c) is called an ordered triplet.

Eg : If $A = \{1,2,3\}$ and $B = \{a,b\}$ then

i) $A \times B = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$

ii) $B \times A = \{(a,1),(b,1),(a,2),(b,2),(a,3),(b,3)\}$

iii) $A \times A = \{(1,1),(1,2),(1,3),(2,1),$
$(2,2),(2,3),(3,1),(3,2),(3,3)\}$

iv) $B \times B = \{(a,a),(a,b),(b,a),(b,b)\}$

✈ **Relation :** Let A and B be two non empty sets. Then any subset of $A \times B$ is called a relation R from A to B. Here A is called domain and B iscalled codomain of R
The set of the all second elements in a relation R from a set A to set B is called the range of the relation R.

Note: 1) Range $\subseteq$ Codomain

2) A relation may be represented algebrically either by the roster method or by the set builder method.

3) The total number of relations that can be defined from a set A to set B is the number of possible subsets of $A \times B$.

If $n(A) = p$ and $n(B) = q$ then $n(A \times B) = pq$ and the total number of relations is 2^{pq}.

Eg 1 : If $A = \{2,3,5,6\}$ and R be a relation "divides" on A such that a Rb $\Leftrightarrow$ a divides b
$\therefore R = \{(2,2)(2,6)(3,3)(3,6)(5,5)(6,6)\}$

Eg 2 : If $A = \{1,2,3\}, B = \{a,b,c\}$ and
$R = \{(1,a)(1,c)(2,b)\}$ Then
Domain of $R = \{1,2\}$, Range of $R = \{a,b,c\}$

Eg 3 : Let R be a relation on the set N of natural numbers defined by $a + 3b = 12$ Then
$R = \{(9,1)(6,2)(3,3)\}$
$\therefore$ Domain of $R = \{9,6,3\}$ and Range of $R = \{1,2,3\}$

✈ **Inverse Relation:** Let A and B be two sets and let R be a relation from a set A to B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by
$R^{-1} = \{(b,a);(a,b) \in R\}$.

Also dom(R)=range (R^{-1}) and
range (R)=dom(R^{-1})

Eg : If $A = \{1,2,3\}$ and $B = \{a,b,c\}$
if $R = \{(1,a),(2,a),(3,b),(3,c)\}$
then $R^{-1} = \{(a,1),(a,2),(b,3),(c,3)\}$

- ## Types of relations :

Void Relation: The relation having no ordered pairs is called a void relation and is denoted by ϕ

Universal Relation: Let A be a set then $A \times A$ is called universal relation on A

Note: The void and the universal relations on a set A are respectively the smallest and the largest relations on A.

Eg : If $A=\{1,2\}$ then universal relation in A is

$$AXA=\{(1,1),(1,2),(2,1),(2,2)\}$$

Identity Relation: Let A be a set. Then the relation $\{(a,a): a \in A\}$ is called the identity relation on A and is denoted by I_A.

Eg : If $A=\{1,2,3\}$ then

$I_A=\{(1,1),(2,2),(3,3)\}$ is the identity relation on A.

Reflexive Relation: A relation R on a set A is said to be reflexive if every element of A is related to itself. Thus R is reflexive $\Leftrightarrow (a,a) \in R \, \forall a \in A$

A relation R on a set A is not reflexive if there exists an element $a \in A$ such that $(a,a) \notin R$.

Note: Every identity relation is reflexive but every reflexive relation need not be an identity relation.

Eg : If $A=\{1,2,3\}$ and $R=\{(1,1),(2,2)\}$ then R is not reflexive since $3 \in A$ but $(3,3) \notin R$

- **Symmetric relation :** A relation R on a set A is said to be a **symmetric relation** iff $(a,b) \in R \Rightarrow (b,a) \in R$ for all $a,b \in A$

  i.e $aRb \Rightarrow bRa$ for all $a,b \in A$

  Eg : If $A=\{2,4,6,8\}$ and

  $R=\{(2,4)(4,2)(4,6)(6,4)\}$ then R is symmetric.

- **Transitive Relation:** Let A be any set. A relation R on A is said to be transitive relation iff $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$ for all $a,b,c \in A$ i.e aRb and bRc $\Rightarrow$ aRc for all $a,b,c \in A$

- **Antisymmetric Relation :** Let A be anyset. A relation R on set A is said to be antisymmetric relation iff $(a,b) \in R$ and $(b,a) \in R \Rightarrow a=b$ for all $a,b \in A$.

  Note: If $(a,b) \in R$ and $(b,a) \notin R$ then still R is an anti symmetric relation

- **Equivalence Relation:** A relation R on a set A is said to be an equivalence relation on A iff it is i) reflexive ii) symmetric iii) transitive

  Note: i) The least equivalence relation on a given set A is the identity relation on A.

  ii) The greatest equivalence relation is universal relation.

- **Ordered Relation:** A relation R is called ordered if R is transitive but not an equivalence relation.

- **Partial Order Relation:** A relation R is called partial order relation if R is reflexive, transitive and anti symmetric at the same time.

  Eg : Let $A=\{1,2,3\}$ we defined

  $R=\{(1,1),(2,2),(3,3)\}$ then R is both equivalence relation and partial order relation

- **Composition of relations** : Let R and S be two relations from sets A to B and B to C respectively. Then we can define a relation SoR from A to C such that

  $(a,c) \in SoR \Leftrightarrow \exists b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. This relation is called the composition of R and S. In general $RoS \neq SoR$. Also

  $(SoR)^{-1} = R^{-1}oS^{-1}$.

  Eg : Let $A=\{1,2,3\}$ $B=\{x,y\}$ $C=\{a,b,c\}$

  Let $R=\{(1,x),(1,y),(3,y)\} \subseteq A \times B$

  $S=\{(x,a),(x,b),(y,b),(y,c)\} \subseteq B \times C$ Then

  $SoR=\{(1,a),(1,b),(1,c),(3,b),(3,c)\} \subseteq A \times C$

  because $(1,x) \in R$ and $(x,a) \in S \Rightarrow (1,a) \in SoR$

✈ **Congruence Modulo (m):** Let m be any positive integer. The integer 'a' is said to be congruent to 'b' of modulo 'm'if (a–b) is divisible by m. we write

$a \equiv b(\text{mod}\, m)$ Thus

$a \equiv b(\text{mod}\, m) \Leftrightarrow (a-b)$ is divisible by m

Eg : i) $17 \equiv 2(\text{mod}\,5)$ as $17-2=15$ which is divisible by 5.

ii) $4 \equiv 14(\text{mod}\,5) \Rightarrow 4-14 \equiv (\text{mod}\,5) \Rightarrow -10$ is divisible by 5.

Note: Congruence modulo m is always an equivalence relation.

✈ **Some facts on relations :** Let A be a finite set and $n(A)=n$, then

(i) the number of elements in $A \times A$ is n^2

(ii) the number of relations from A to A is 2^{n^2}

(iii) number of reflexive relations from A to A is 2^{n^2-n}

(iv) number of symmetric relations from A to A is $2^{\frac{n^2+n}{2}}$

(v) number of relations from A to A which are not symmetric is $2^{n^2}-2^{\frac{n(n+1)}{2}}$

(vi) number of relations from A to A which are both reflexive and symmetric is $2^{\frac{n^2-n}{2}}$

(vii) number of relations from A to A which are symmetric but not reflexive is $2^{\frac{n(n+1)}{2}}-2^{n^2-n}$

✈ (i) Total number of relations from the set A to set B is $2^{n(A)n(B)}$

(ii) Let A and B be two non-empty sets having n elements in common then number of elements common in $(A \times B) \cap (B \times A) = n \times n = n^2$

✈ (i) Let A be a finite set. If $B_1, B_2, ...B_n$ are non-empty subsets of A such that $B_1 \cup B_2 \cup .... \cup B_n = A$ and $B_i \cap B_j = \phi$ for $i \neq j$ then $P=\{B_1, B_2, ..., B_n\}$ is called a partition of A.

Eg : Let $A=\{1,2,3,4,5,6,7\}$

suppose $B_1=\{1,5\}, B_2=\{2,4,7\}, B_3=\{3,6\}$

and Clearly $B_1 \cup B_2 \cup B_3 = S$ and

$B_1 \cap B_2 = \phi,\ B_2 \cap B_3 = \phi$

$\therefore P=\{B_1, B_2, B_3\}$ is a partition of the set A.

✈Some facts on relations :

(i) The intersection of two equivalence relations on a set A is an equivalence relation on set A.

(ii) Inverse of an equivalence relation is an equivalence relation.

(iii) The union of two equivalence relation on a set is not necesarily an equivalence relation on the set.

(iv) The identity relation on a non-empty set is always equivalence relation i.e. it is reflexive , symmetric as well as transitive.

(v) The identity relation on a set (non-empty) is always antisymmetric relation.

(vi) The universal relation on a non-empty set is always equivalence relation.

(vii) If R be a relation from as set X to Y and R be a relation Y to Z then $(RoS)^{-1} = S^{-1}oR^{-1}$

(viii) The number of equivalence relations on a finite set A is equal to number of partitions of A.

(ix) If R is an equivalence relation on a set A then R^{-1} is also an equivalence relation.

✈ Some Results on Cartesian Product of sets :

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(iii) $A \times (B - C) = (A \times B) - (A \times C)$

(iv) If A and B are two non-empty sets,then $A \times B = B \times A \Leftrightarrow A = B$

(v) If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$

(vi) $A \subseteq B \Rightarrow A \times C \subseteq B \times C$ for any set C

(vii) $A \subseteq B$ and $C \subseteq D \Rightarrow A \times C \subseteq B \times D$

(viii) $(A\times B)\cup(C\times D)\subseteq(A\cup C)\times(B\cup D)$

(ix) $(A\times B)\cap(C\times D)=(A\cap C)\times(B\cap D)$

(x) $(A\times B)\cap(B\times A)=(A\cap B)\times(B\cap A)$

(xi) Let A and B be two non-empty sets having n elements in common,then $A\times B$ and $B\times A$ have n^2 elements in common.

Eg. 1

Let $A=\{1,2,3,4,6\}$ let R be the relation on A defined by $\{(a,b):a,b\in A,$ b is exactly divisible by a$\}$ a) write R in roster form b) find the domain of R c) find the range of R

Sol: a) $R=\{(1,1),(1,2),(1,3),(1,4),(1,6),(2,4),(2,6),$

$(2,2),(4,4),(6,6),(3,3),(3,6)\}$

b) Domain of $R=\{1,2,3,4,6\}$

c) Range of $R=\{1,2,3,4,6\}$

Eg. 2

For real numbers x and y we write x Ry iff $x-y+\sqrt{2}$ is an irrational number.Then the relation R is Reflexive.

Sol: Given that in the set of real numbers R ,

xRy iff $x-y+\sqrt{2}$ is irrational.

Reflexive: $x-x+\sqrt{2}=\sqrt{2}$ (irrational).

for every real x.

Eg. 3

A relation R defined on the set of integers $R=\{(a,b):$ a divides b; $a,b\in Z\}$ then R is__

Sol: R is not reflexive since $(0,0)\notin R$, since 0 does not divide 0.

R is not symmetric , since 2/4 but 4/2.

R is transitive since

a/b,b/c $\Rightarrow b=an, c=bm,\ \ n,m\in Z$

$\Rightarrow c=a(nm)\ \Rightarrow a/c.$

$\therefore R$ is neither reflexive, nor symmetric but only transitive.

Eg. 4

Let N be the set of natural numbers. A relation $R\subseteq N\times N$ is defined by 'x divides y' is anti symmetric

Sol: xRy , $yRx\Rightarrow x$ divides y, y divides x $\Rightarrow x=y$.

Eg. 6

Let $R=\{(2,3)(3,4)\}$ be a relation defined on the set $A=\{1,2,3,4\}$ The minimum number of ordered pairs required to be added in R so that enlarged relation becomes an equivalance relation is

Sol: Given $R=\{(2,3),(3,4)\}$

To make it reflexive,enlarge R as following

$R=\{(1,1),(2,2),(3,3),(4,4),(2,3),(3,4)\}$

Hence four more ordered pairs are added.

To make it symmetric, enlarge R as following

$R=\{(1,1),(2,2),(3,3),(4,4),$

$(2,3),(3,4),(3,2),(4,3)\}$

Hence two more ordered pairs are added.

Finally to make it transitive, we enlarge R to

$\{(1,1),(2,2),(3,3),(4,4),(2,3),(3,2),(3,4),$

$(4,3),(2,4)(4,2)\}$. Hence two more ordered pairs are added. $\therefore$ Total 8 ordered pair must be added to make the relation R an equivalence.

Eg. 7

The congruent solution of

$8x\equiv 6(\bmod 14)$ **is**

(a) $x=6,7$ **(b)** $x=6,13$

(c) $x=2,13$ **(d)** $x=2,3$

Sol : Given , $8x\equiv 6(\bmod 14)$

$\therefore\ \lambda=\dfrac{8x-6}{14}$, where $\lambda\in I$

$\therefore\ 8x=14\lambda+6\Rightarrow x=\dfrac{14\lambda+6}{8}$

$\Rightarrow x=\dfrac{7\lambda+3}{4}=\dfrac{4\lambda+3(\lambda+1)}{4}$

$x=\lambda+\frac{3}{4}(\lambda+1)$, where $\lambda\in I$

and here greatest common divisor of 8 and 14 is 2. So, there are two required solution for $\lambda=3$ and $\lambda=7\Rightarrow x=6,13.$

Hence, (b) is the correct answer.

Eg. 8

The relation " congruence modulo m" on the set Z of all integer is an equivalence relation

Sol: Let $a\in I$ then $a-a=0=0\times m$

i) $\Rightarrow a-a$ is divisible by m, $\Rightarrow a\equiv a(\bmod m)$

$\Rightarrow R$ is reflexive

ii) $a,b\in Z$ such that $a\equiv b(\bmod m)$

as $a\ -\ b$ is divisible by m

$\Rightarrow a-b=\lambda m,\forall\lambda\in Z$

$\Rightarrow(b-a)\equiv(-\lambda)m\Rightarrow(b-a)$ is divisible by m.

$\Rightarrow b\equiv a(\bmod m)$ $\Rightarrow R$ is symmetric on Z

iii) Let $a,b,c\in Z$ such that

$a\equiv b(\bmod m).b\equiv c(\bmod m)$

$\therefore a\equiv b(\bmod m)\Rightarrow a-b$ is divisible by m

$\therefore a-b=\lambda_1 m$ for some

similarly $b-c=\lambda_2 m$ for some $\lambda_2\in Z$

By (i) and (ii) we have

$a-c=(\lambda_1+\lambda_2)m=km$ for some $k\in Z$

$\therefore a-c$ is divisible by m

$\therefore\ a\equiv c(\bmod m)$

$\therefore$ Congruence modulo m is transitive on Z

As the congruence modulo m is reflexive, symmetric, transitive so it is an equivalence relation on Z.

EXERCISE - I

DOMAIN, RANGE AND NUMBER OF RELATIONS

1. **If P={1,2}, Then P x P x P is**
   1) {(1,1,1,),(1,1,2),(1,2,1),(1,2,2),(2,1,1),(2,1,2),(2,2,1),(2,2,2,)}
   2) {(1,1,1),(1,2,2),(1,2,4)}
   3) {(1,1,3)}
   4) All the above
2. **If** $A=\{x:x^2-5x+6=0\}$**,** $B=\{2,4\}$**,** $C=\{4,5\}$**, then** $A\times(B\cap C)$ **is**
   1) $\{(2,4),(3,4)\}$
   2) $\{(4,2),(4,3)\}$
   3) $\{(2,4),(3,4),(4,4)\}$
   4) $\{(2,2),(3,3),(4,4),(5,5)\}$
3. **If** $A=\{x:x^2-5x+6=0\}$**,** $B=\{1,2\}$ **and** $C=\{4,5\}$ **then** $(A-B)\times(A-C)=$
   1) $\{(2,3)\}$
   2) $\{(1,2)\}$
   3) $\{(1,2),(2,3)\}$
   4) $\{(3,2)(3,3)\}$
4. **A={1,2,3,4,}, relation R on A is defined by R={(x,y) / x<y and** $\left|x^2-y^2\right|<9$ **;x,y**$\in$**A} then R=**
   1) {(1,1) (2,2) (3,3) (4,4)}
   2) {(2,1) (3,2) (3,2) (4,3)}
   3) {(1,2) (1,3) (2,3) (3,5)}
   4) {(1,2) (1,3) (2,3) (3,4)}
5. **Let A={1,2,3,............,14}.**
   Define a relation R from A to A by
   R={$(x,y):3x-y=0;\ x,y\in A$**}.**
   Then do main of R is
   1) {3,6,9,12}
   2) {3,6}
   3) {1,2,3,...........,14}
   4) {1,2,3,4}
6. **The domain and range of relation** $\mathrm{R}=\{(\mathrm{x},\mathrm{y})/\mathrm{x},\mathrm{y}\in\mathrm{N},\mathrm{x}+2\mathrm{y}=5\}$ **is**
   1) {1, 3}, {2, 1}
   2) {2, 1}, {3, 2}
   3) {1, 3}, {1, 1}
   4) {1, 2}, {1, 3}
7. **If R ={(x,y): x,y** $\in$ **Z ,** $x^2+y^2\leq 4$**} is a relation in Z, then domain of R is**
   1) {0,1,2}
   2) {0,-1,-2}
   3) {-2,-1,0,1,2}
   4) {1,2,3}

8. If R = {(x,y): x, y $\in$ N, y is the remainder when x is divided by 7}.Then sum of all numbers in range of R

1) 14 2) 21 3) 28 4) 12

9. Write the relation $R=\{(x,x^3): x$ is prime number less than 10$\}$ in roaster form

1) $R=\{(2,8),(3,27),(5,125),(7,343)\}$

2) $R=\{(2,4),(3,9),(5,25),(7,49)\}$

3) $R=\{(2,2),(3,3),(5,5),(7,7)\}$

4) $R=\{(2,8),(3,9),(5,25),(7,343)\}$

10. A relation R is defined in the set of integers I as follows $(x,y)\in R$ iff $x^2+y^2=9$, which of the following is true?

1) $R=\{(0,3),(0,-3),(3,0),(-3,0)\}$

2) Domain of R = {–3, 0, 3}

3) Range of R = {–3, 0, 3}

4) All the above

TYPES OF RELATIONS

11. R = {(a,b): a,b $\in$ N, a+b is even} is

1) reflexive 2) Symmetric
3) both 1,2 4) none of 1,2

12. Let $X=\{1,2,3\}$ and $R=\{(1,1),(2,2),(3,3),(2,3)\}$ be a relation on X. Then which one is not true

1) R is reflexive 2) R is transitive
3) R is antisymmetric 4) R is symmetric

13. Let $A=\{a,b,c\}$ and $R=\{(a,a),(b,b),(a,b),(b,a),(b,c)\}$ be a relation on A, then R is

1) reflexive 2) symmetric
3) transitive 4) not reflexive

14. The relation R={(1,1)(2,2)(3,3)} on the set {1,2,3} is

1) Symmetric only 2) Reflexive only
3) Transitive only 4) An equivalence

15. Which of the following are not equivalence relations on I ?

1) aRb if $a+b$ is an even integer

2) aRb if $a-b$ is an even integer

3) aRb if $a<b$ 4) aRb if $a=b$

16. Total number of equivalence relations defined in the set $S=\{a,b,c\}$ is

1) 5 2) 3! 3) 2^3 4) 3^3

17. Let $R=\{(1,3),(4,2),(2,4),(2,3),(3,1)\}$ be a relation on the set A = {1, 2, 3, 4}. The relation R is [AIE-2004]

1) a function 2) transitive
3) not symmetric 4) reflexive

18. Let R = {(3, 3), (6, 6), (9, 9), (12, 12), (6, 12), (3, 9), (3, 12), (3, 6)} be a relation on the set A = {3, 6, 9, 12}. The relation is [AIE-2005]

1) reflexive and symmetric only
2) an equivalence relation
3) reflexive only
4) reflexive and transitive only

19. In the set Z of all integers, which of the following relation R is not an equivalence relation?

1) x R y : if x $\le$ y 2) x R y : if x= y
3) x R y : if x-y is an even integer
4) x R y : if x $\equiv$ y (mod 3)

20. Which of the following is an equivalence relation ?

1) $x<y$ 2) $x>y$
3) $x-y$ is divisible by 5 4) x divides y

21. Let W denote the words in the English dictionary. Define the relation R by $R=\{(x,y)\in W\times W$ / the words *x* and *y* have atleast one letter in common}Then R is

1) reflexive, symmetric and not transitive
2) reflexive, symmetric and transitive
3) reflexive, not symmetric and transitive
4) not reflexive, symmetric and transitive

INVERSE RELATION

22. If $R=\{(x,y)\mid x\in N, y\in N, x+3y=12\}$ then R^{-1} is

1) {(2, 9),(2, 6),(3, 3)} 2) {(3, 1),(2, 4),(3, 6)}
3) {(3,3),(2,6),(1,9)} 4) {(1,3), (1,6), (1,9)}

KEY

01) 1	02) 1	03) 4	04) 4	05) 4	06) 1
07) 3	08) 2	09) 1	10) 4	11) 3	12) 4
13) 4	14) 4	15) 3	16) 1	17) 3	18) 4
19) 1	20) 3	21) 1	22) 3		

SOLUTIONS

1. $p \times p \times p = \{(1,1,1,),(1,1,2),$
$(1,2,1),(1,2,2),(2,1,1),$
$(2,1,2),(2,2,1),(2,2,2,)\}$

2. We have, $A=\{2,3\}$, $B=\{2,4\}$ and
$C=\{4,5\} \therefore B\cap C=\{4\}$
$\Rightarrow A\times(B\cap C)=\{(2,4),(3,4)\}$

3. $x^2-5x+6=0 \Rightarrow A=\{2,3\}$

4. Since x<y and $\left|x^2-y^2\right|<9$
$R=\{(1,2),(1,3),(2,3),(3,4)\}$ x<y

5. $\because 3x-y=0 \Rightarrow y=3x$
$\therefore R=\{(1,3),(2,6),(3,9),(4,12)\}$

6. $x+2y=5 \Rightarrow x\in N, y\in N$.
Domain = Set of values of x = {1, 3}
Range = Set of values of y = {1, 2}

7. $x^2+y^2\le 4$ represents all points interior to the circle $x^2+y^2=4$
hence $-2\le x\le 2$ *and* $-2\le y\le 2$ integral values of x *are* $-2,-1,0,1,2$

8. Range = {0,1,2,3,4,5,6}

9. $R=\{(2,8),(3,27),(5,125),(7,343)\}$

10. $y=\pm\sqrt{9-x^2}$
$\Rightarrow R=\{(0,3),(0,-3),(3,0),(-3,0)\}$

11. $a+b$ even if both a,b are even or both a,b are odd

12. $(3,2)\notin R$

13. $(c,c)\notin R$

14. It is an equivalance relation

15. The relations given in OPTIONS (1), (2) and (4) are clearly reflexive, symmetric and transitive On the other hand, the relation of OPTION (C) is transitive but neither reflexive nor symmetric.

16. The smallest equivalence relation is the identity relation $R_1=\{(a,a),(b,b),(c,c)\}$.
Then two ordered pairs of two distinct elements can be added to give three more equivalence relation
$R_2=\{(a,a),(b,b),(c,c),(a,b),(b,a)\}$
Similarly R_3 and R_4. Finally the largest equivalence relation i.e., the universal relation R_5 = {(a, a), (b,b), (c,c),(a, b),(b, a),(a, c),(c,a), (b,c), (c,b)}

17. $(2,4),(2,3)\in R \Rightarrow 2$ has two images
$\Rightarrow$ R is not a reflexive
$(1,1)\notin R \Rightarrow R$ is not reflexive
$(2,3)\in R,(3,2)\notin R \Rightarrow R$ is not symmetric

18. $A=\{3,6,9,12\}$ and R={$(3,3)$, $(6,6)$, $(9,9)$, $(12,12)\}\in R \Rightarrow$ R is reflexive $(6,12)\in R$ but
$(12,6)\notin R \Rightarrow$ R is not symmetric
$(3,6)\in R,(6,12)\in R \Rightarrow (3,12)\in R$
$\therefore$ R is transitive

19. If R is relation defined by x R y : then R is not an equivalence relation.

20. xRx since ; 0 is divisible by 5 if xRy , x-y is divisible by 5, y-x is also divisible by 5 if xRy & yRz, $x-y=5k_1$; $y-z=5k_2$; $x-z=5(k)$

21. Let $W=\{CAT, TOY, YOU.......\}$
clearly R is reflexive and symmetric but not transitive (CAT R TOY, TOY R YOU does not implies CAT R YOU.

22. $x+3y=12x=3, y=3$, $x=6, y=2$ $x=9$
$y=1$ $R=\{(3,3),(6,2),(9,1)\}$
$R^{-1}=\{(3,3),(2,6),(1,9)\}$

EXERCISE - II

DOMAIN, RANGE AND NUMBER OF RELATIONS

1 If $A=\{(1,2)\}, B=\{(3,4)\}$,then $A\times(B\times\phi)=$

1) ϕ 2) A 3) B 4) {1,3}

2. R, S are relations from N × N to Z × Z by $R=\{(x-y, y-x): x, y\in N\}$, $S=\{(x-y, x+y): x, y\in Z\}$, Then number of elements in R ∩ S

1) 0 2) 1 3) 2 4) infinite

3. A={1,2,3,5}, B={4,6,9}. Define a relation R from A to B by R={(x,y);the difference between x and y is odd; x ∈ A, y ∈ B}. Then R is

1) {(1,2),(1,6),(2,9),(3,4),(3,5)}
2) {(1,4),(1,6),(2,9),(3,4),(3,6),(5,4),(5,6)}
3) {(1,6),(1,7),(1,8),(2,9),(2,4),(2,9)}
4) {(1,5),(1,6),(1,7),(6,4),(6,9),(6,2)}

4. Given A={1,2,3,4,5}, S={(x,y): x ∈ A, y ∈ A}. Then the ordered pair which satisfy x+y>8 then R is ....

1) {(5,5) (5,3)} 2) {(4,5) (5,4) (5,5)}
3) {(5,4) (5,6)} 4) {(5,4) (5,3)}

5. The domain and range of

$$R=\{(x,y)/y=x+\frac{6}{x}, x, y\in N \text{ and } x<6\}$$

1) Domain = {1, 2}, Range = {7, 5}
2) Domain = {1, 2, 3}, Range = {7 , 5}
3) Domain = {1}, Range = {7, 5}
4) Domain = {1, 2}, Range = {7}

6. If the number of relations on a finite set A having 'n' elements is 2^{16}, then 'n' equal to

1) 15 2) 17 3) 4 4) 8

7. If a relation R is defined on the set Z of integers as follows $(x,y)\in R \Leftrightarrow x^2+y^2=25$. Then domain of R =

\ 1) $\{3,4,5\}$ 2) $\{0,3,4,5\}$

3) $\{0,\pm 3,\pm 4,\pm 5\}$ 4) $\{0,\pm 5\}$

TYPES OF RELATIONS

8. Let R be the relation on the set of all real numbers defined by a R b if $|a-b|\le 1$. Then R is

1) reflexive and symmetric
2) symmetric only 3) transitive only
4) anti - symmetric only

9. Let $R=\{(x,y): x, y\in A; x+y=5\}$ where A = {1,2,3,4,5} then

1) R is not reflexive, symmetric and not transitive
2) R is an equivalence relation
3) R is reflexive, symmetric but not transitive
4) R is not reflexive, not symmetric but transitive

10. On the set of natural numbers N, the relation R is defined by xRy iff x + y = 100 is

1) reflexive 2) not reflexive
3) equivalence 4) not symmetric

11. On the set of all vectors in space the relation R is defined by $\bar{a}R\bar{b} \Leftrightarrow \bar{a}.\bar{b}$ is scalar is

1) symmetric 2) not symmetric
3) not reflexive 4) both 2 and 3

12. If A={ 1, 2,3} Then a relation reflexive but not Symmetric on A is

1) {(1,1), (1,2)} 2) {(1,1),(1,2),(2,1),(2,2)}
3) {(1,1),(2,2),(3,3)}
4) {(1,1),(2,2),(3,3),(2,3)}

13. The correct statement of the following is

1) The relation 'less than' on Z is anti-symmetric
2) The relation 'sister of' on the members of a family is transitive
3) the relation 'relatively prime' on N is reflexive
4) The relation 'perpendicular' on a set of lines in a plane is transitive

14. If A ={ 1,2,3} ,R ={ (1,2),(1,1),(2,3)} Then minimum number of elements may be adjoined with the elements of R so that it may become transitive is

1) 0 2) 1 3) 2 4) 3

INVERSE RELATION

15. **If R is a relation is "greater than or equal to" from A ={1,2,3,4} to B ={4,5,6}, then R^{-1} =**

1) {(4,4)} 2) ϕ 3) A x B 4) R

KEY

01) 1	02) 1	03) 2	04) 2	05) 2	06) 3
07) 3	08) 1	09) 1	10) 2	11) 1	12) 4
13) 2	14) 2	15) 1			

SOLUTIONS

1. $B\times\phi=\phi \Rightarrow A\times(B\times\phi)=\phi$
2. y - x = x+ y only if x = 0 But $\notin$ N
3. both x and y are odd
4. 4+5=9>8, 5+4=9>8, 5+5=10>8
5. $R=\{(1,7),(2,5),(3,5)\}$
6. $n^2=16$
7. $x^2+y^2=25$ represents all points on the circle $x^2+y^2=25$ Hence $-5\le x\le 5, -5\le y\le 5$. $\therefore$ integral values of x are $0,\pm3,\pm4,\pm5$.
8. $0\le 1 \Rightarrow$ aRa $|a-b|\le 1 \Rightarrow |b-a|\le 1$
9. R={(1,4),(4,1),(2,3),(3,2)}
10. $(1,1)\notin R$
11. $\bar{a}.\bar{b}=\bar{b}.\bar{a}$
12. $\{(1,1),(2,2),(3,3)(2,3)\}$ is reflexive not symmetric as $(2,3)\in R$ but $(3,2)\notin R$
13. xRy, yRz $\Rightarrow$ xRz
14. (1,3) need to be adjoined to make the relation transitive
15. $R=\{(4,4)\}$

EXERCISE - III

1. **Let A be the set of first 10 natural numbers and let** $R=\{(x,y)/x\in A, y\in N \text{ and } x+2y=10\}$ **then** $n\{dom(R^{-1})\}=$

   1) 4 2) 5 3) 8 4) 10

2. **Let** $X=\{1,2,3,4,5\}$**, the number of different ordered pairs (Y,Z) that can be formed such that** $Y\subseteq X$, $Z\subseteq X$ **and** $Y\cap Z$ **is empty, is**

   1) 3^5 2) 2^5 3) 5^3 4) 5^2

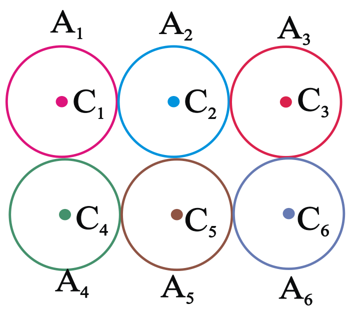
3. **The relation R defined on** $A=\{1,2,3\}$ **by aRb. if** $|a^2-b^2|\le 5$**, which of the following is false.**

   1) $R=\{(1,1),(2,2),(3,3),(2,1),(1,2),(2,3),(3,2)\}$
   2) $R^{-1}=R$ 3) Domain of R = {1, 2, 3}
   4) Range of R = {5}

4. **Let R be a relation such that R={(1,4),(3,7),(4,5),(4,6),(7,6)}, then** $(R^{-1}oR)^{-1}=$

   1) {(1,1),(3,3),(4,4),(7,7),(4,7),(7,4),(4,3)}
   2) {(1,1),(3,3), (4,4),(7,7)(4,7),(7,4)}
   3) {(1,1),(3,3),(4,4)} 4) ϕ

5. **Let** $A=\{A_1,A_2,A_3,A_4,A_5,A_6\}$ **be the set of six unit circles with centres** $C_1,C_2,C_3...C_6$ **arranged as shown in the diagram.The relation R on A is defined by** $(A_i,A_j)\in R \Leftrightarrow C_iC_j\le 2\sqrt{2}$ **then**



   1) R is symmetric, transitive but not reflexive
   2) R is only transitive
   3) R is symmetric, reflexive but not transitive
   4) R is neither reflexive nor transitive but is

symmetric.

6. **The relation R on the set of natural numbers N is defined as** $xRy \Leftrightarrow x^2 - 4xy + 3y^2 = 0$**,** $x, y \in N$ **then R is**

1) reflexive but not symmetric and not transitive
2) symmetric but not reflexive and not transitive
3) transitive but not reflexive and not symmetric
4) equivalence relation

7. **Which of the following relations is not transitive**

1) $(a,b) \in R_1 \Leftrightarrow a \le b, a,b \in z$
2) $(x,y) \in R_2 \Leftrightarrow x$ divides y *if* $x,y \in z$
3) $(x,y) \in R_3 \Leftrightarrow |x|+|y| = 1$, for $x,y \in R$
4) $(x_1,y_1) \in R_4 \Leftrightarrow l_1$ parallel to l_2.
where l_1, l_2 are lines

8. **A relation R on the set of non zero complex numbers is defined by** $z_1 R z_2 \Leftrightarrow \frac{z_1 - z_2}{z_1 + z_2}$ **is real, then** R **is**

1) Reflexive
2) Symmetric
3) Transitive
4) Equivalence

9. **S is a relation over the set R of all real numbers and it is given by (a, b)** $\in$ **S** $\Leftrightarrow$ **ab** $\ge$ **0. Then S is**

1) symmetric and transitive only
2) reflexive and symmetric only
3) a partial relation
4) an equivalance relation

10. **Let R be the real line. consider the following subsets of the plane** $R \times R$.

$S = \{(x,y) : y = x+1$ **and** $0 < x < 2\}$,

T = {(x,y) : x-y is an integer}.Which one of the following is true ? [AIE-2008]

1) S is an equivalence relation on R but T is not
2) T is an equivalence relation on R but S is not
3) Neither S nor T is an equivalence relation on R
4) Both S and T are equivalence relations on R

11. **Consider the following relations**

R = {$(x,y)/x,y$ **are real numbers and x = wy for some rational number w};**

S=$\left\{\left(\frac{m}{n}, \frac{p}{q}\right) / m,n,p \text{ and } q\right.$ **are integers such that** $n,q \ne 0$ **and qm=pn** $\left.\right\}$**,then [AIE-2010]**

1) neither R or S is an equivalence relation.
2) S is an equivalence relation but R is not an equivalence relation
3) R and S both are equivalence relations.
4) R is an equivalence relation but S is not an equivalence relation.

KEY

01) 1	02) 1	03) 4	04) 2	05) 3	06) 1
07) 3	08) 4	09) 4	10) 2	11) 2	

SOLUTIONS

1. $R = \{(2,4),(4,3),(6,2),(8,1)\}$

2. For any element x_i present in X, 4 cases arises while making subsets Y and Z.

Case 1 : $x_i \in Y, x_i \in Z \Rightarrow Y \cap Z \ne \phi$

Case 2 : $x_i \in Y, x_i \notin Z \Rightarrow Y \cap Z = \phi$

Case 3 : $x_i \notin Y, x_i \in Z \Rightarrow Y \cap Z = \phi$

Case 4 : $x_i \notin Y, x_i \notin Z \Rightarrow Y \cap Z = \phi$

For every element, number of ways=3 for which $Y \cap Z = \phi \Rightarrow$ total number of ways=$3\times3\times3\times3\times3$ [$\because$ no.of elements in set X=5]=$(3)^5$

3. $R = \{(1,1),(1,2),(2,1),(2,2),(2,3),(3,2),(3,3)\}$

4. R={(1,4),(3,7),(4,5),(4,6),(7,6)}
R^{-1}={(4,1),(7,3),(5,4),(6,4),(6,7)}

5. $(C_i, C_i) = 0 \le 2\sqrt{2}$ R is reflexive

$(A_i, A_j) \in R \Rightarrow C_i C_j \le 2\sqrt{2} \Rightarrow C_j C_i \le 2\sqrt{2}$
$\therefore (A_j, A_i) \in R$

R is symmetric

if $(A_1, A_2) \in R, (A_2, A_3) \in R \Rightarrow (A_1, A_3) \notin R$

6. Given xRy iff $x^2 - 4xy + 3y^2 = 0$

If y = x , then $x^2 - 4x^2 + 3x^2 = 0$

Hence xRx $\forall x \in N$

$\therefore R$ is reflexive,

R is not symmetric, because $3 R 1 \not\Rightarrow 1R3$

R is not transitive,because.9R3 and 3R1 $\not\Rightarrow$ 9R1

7. $(0.6, 0.4) \in R_3, (0.4, -0.6) \in R_3 \, but \, (0.6, -0.6) \notin R_3$

8. i) $z_1 R z_1 \Rightarrow \dfrac{z_1 - z_1}{z_1 + z_1} \forall z_1 \in C \Rightarrow 0$ is real

$\therefore$ R is a reflexive.

ii) $z_1 R z_2 \Rightarrow \dfrac{z_1 - z_2}{z_1 + z_2}$ is real

$\Rightarrow -\left(\dfrac{z_2 - z_1}{z_2 + z_2}\right)$ is real $\Rightarrow \left(\dfrac{z_2 - z_1}{z_2 + z_1}\right)$ is real

$\Rightarrow z_2 R z_1 \forall z_1, z_2 \in C \therefore$ R is a symmatric

iii) let $z_1 = a_1 + ib_1, z_2 = a_2 + ib_2$ and $z_3 = a_3 + ib_3$

when $a_1, b_1, a_2, b_2, a_3 b_3 \in R$

now $z_1 R z_2 \Rightarrow \dfrac{z_1 - z_2}{z_1 + z_2}$ is real

$\Rightarrow \dfrac{(a_1 - a_2) + i(b_1 - b_2)}{(a_1 + a_2) + i(b_1 + b_2)} \times \dfrac{(a_1 + a_2) - i(b_1 + b_2)}{(a_1 + a_2) - i(b_1 + b_2)}$

is real

$\Rightarrow (a_1 + a_2)(b_1 - b_2) - (a_1 - a_2)(b_1 + b_2) = 0$

(for purely real, imaginary part = 0)

$\Rightarrow \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$,similarly $z_2 R z_3 \Rightarrow \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$

$z_1 R z_2$ and $z_2 R z_3 \Rightarrow \dfrac{a_1}{b_1} = \dfrac{a_2}{b_2}$ and $\dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$

$\Rightarrow \dfrac{a_1}{b_1} = \dfrac{a_3}{b_3}$ $\quad z_1 R z_3$

R is transitive.

hence R is an equivalence relation.

9. Reflexivity : For any a $\in$ R, we have

$a^2 =$ aa $\geq 0 \Rightarrow$ (a,a) $\in$ S

Thus, (a,a) $\in$ s for all a $\in$ R.

So, S is a reflexive relation on R.

Symmetry: Let (a, b) $\in$ S. Then

(a, b) $\in$ S $\Rightarrow$ ab $\geq 0 \Rightarrow$ ba $\geq 0 \Rightarrow$ (b,a) $\in$ S

Thus (a, b) $\in$ S $\Rightarrow$ (b,a) $\in$ S for all a,b $\in$ R.

So S is symmetric relation on R.

Transitivity: Let a, b, c $\in$ S. such that

(a, b) $\in$ S and (b, c) $\in$ R $\Rightarrow$ ab ≥ 0 and bc $\geq 0 \Rightarrow$ a, b, c are of same sign.

$\Rightarrow$ ac $\geq 0 \Rightarrow$ (a, c) $\in$ R.

(a, b) $\in$ S $\Rightarrow$ (b, c) $\in$ S $\Rightarrow$ (a, c) $\in$ S.

So, S is a transitive relation on R.

Hence, S is an equivalence relation on R.

10. $S = \{(x, y); y = x + 1, 0 < x < 2\} \Rightarrow$

S is not symmetric

$T = \{(x, y) : x - y \text{ is an integer}\}$

$\Rightarrow$ clearly T is an equivalence

11. xRy need not implies yRx, then clearly

S: $\dfrac{m}{n} s \dfrac{p}{q} \Leftrightarrow qm = pn$, $\dfrac{m}{n} s \dfrac{m}{n}$ reflexive

$\dfrac{m}{n} s \dfrac{p}{q} \Rightarrow \dfrac{p}{q} s \dfrac{m}{n}$ symmetric

$\dfrac{m}{n} s \dfrac{p}{q}, \dfrac{p}{q} s \dfrac{r}{s} \Rightarrow qm = pn, ps = rq$

$\Rightarrow ms = rn$ transitive

s is an equivalence relation

JEE MAINS QUESTIONS

1.

Let R_1 and R_2 be two relations defined as follows :

$R_1 = \{(a, b) \in \mathbf{R}^2 : a^2 + b^2 \in Q\}$ and

$R_2 = \{(a, b) \in \mathbf{R}^2 : a^2 + b^2 \notin Q\}$, where Q is the set of all rational numbers. Then : [2020]

(1) Neither R1 nor R2 is transitive.
(2) R2 is transitive but R1 is not transitive.
(3) R1 is transitive but R is not transitive.
(4) R1and R2are both transitive.

2.

If $R = \{(x, y) : x, y \in \mathbf{Z}, x^2 + 3y^2 \leq 8\}$ is a relation on the set of integers $\mathbf{Z}$, then the domain of R^{-1} is :

(1) $\{-2, -1, 1, 2\}$ (2) $\{0, 1\}$ [2019]
(3) $\{-2, -1, 0, 1, 2\}$ (4) $\{-1, 0, 1\}$

KEY

1) 1 2) 4

SOLUTIONS

1.

For R_1 let $a = 1+\sqrt{2}, b = 1-\sqrt{2}, c = 8^{1/4}$

$aR_1b \Rightarrow a^2 + b^2 = (1+\sqrt{2})^2 + (1-\sqrt{2})^2 = 6 \in Q$

$bR_1c \Rightarrow b^2 + c^2 = (1-\sqrt{2})^2 + (8^{1/4})^2 = 3 \in Q$

$aR_1c \Rightarrow a^2 + c^2 = (1+\sqrt{2})^2 + (8^{1/4})^2 = 3+4\sqrt{2} \notin Q$

$\therefore R_1$ is not transitive.

For R_2 let $a = 1+\sqrt{2}, b = \sqrt{2}, c = 1-\sqrt{2}$

$aR_2b \Rightarrow a^2 + b^2 = (1+\sqrt{2})^2 + (\sqrt{2})^2 = 5+2\sqrt{2} \notin Q$

$bR_2c \Rightarrow b^2 + c^2 = (\sqrt{2})^2 + (1-\sqrt{2})^2 = 5-2\sqrt{2} \notin Q$

$aR_2c \Rightarrow a^2 + c^2 = (1+\sqrt{2})^2 + (1-\sqrt{2})^2 = 6 \in Q$

$\therefore R_2$ is not transitive.

2.

Since, $R = \{(x, y) : x, y \in \mathbf{Z}, x^2 + 3y^2 \leq 8\}$

$\therefore R = \{(1, 1), (2, 1), (1, -1), (0, 1), (1, 0)\}$

$\Rightarrow D_{R^{-1}} = \{-1, 0, 1\}$

ADDITION OF VECTORS

SYNOPSIS

- **Definition of Vector and Scalar :**
- **Scalar:** A physical quantity which has only magnitude is called a scalar.
  Examples: Length,volume, speed, time.
- **Vector:** A vector is a physical quantity which has both magnitude and direction . Geometrically a directed line segment is called a vector.
  Examples: Force, Velocity, acceleration.
  Note: All real numbers are scalars.
- **Notation:** Vectors are denoted by directed line segments such as $\overline{AB}\,\overline{CD}$.... or by $\overline{a},\overline{b}$ ... If $\overline{AB}$ is a vector then A is called initial point and B is called the terminal point or final point and the direction of $\overline{AB}$ is from A to B. The magnitude of $\overline{AB}$ is denoted by $\left|\overline{AB}\right|$ or **AB** and, is the distance between the points A and B.
- **Types of Vectors :**
- **Position Vector:** Let O be a fixed point (called the origin) and let P be any point. If $\overline{OP}=\overline{r}$ then $\overline{r}$ is called the position vector of P with respect to O.
- **Null Vector:** A vector having zero magnitude (arbitrary direction) is called the null (zero) vector. It is denoted by $\overline{0}$.
  Note (i) A zero vector can be regarded as having any direction for all mathematical calculations.
  (ii) A non-zero vector is called a proper vector
- **Unit Vector:** A vector whose magnitude is equal to one unit is called a unit vector.
- **Localised vector** :A vector is a localised vector, if the vector is specified by giving either initial point or terminal point (or) if a vector is specified by fixing atleast one of its ends is called a localised vector.
- **Free Vector**: When a vector is specified by not fixing initial point or terminal point or both, is called a free vector, i.e., a free vector does not have specific initial point or terminal point or both.
  Note: (i) A vector $\overline{a}$ means we are free to choose initial or terminal point anywhere. Once initial point is fixed at A then terminal point is uniquely fixed at B such that $\overline{AB}=\overline{a}$

(ii) A free vector is subjected to parallel displacement without changing the magnitude and direction.

(iii) In general vectors are considered to be free vectors unless they are localised.

- Let $\overline{a}$ be a nonzero vector then

(i) Unit vector in the direction of $\overline{a}=\dfrac{\overline{a}}{|\overline{a}|}$

(ii) Unit vector in the direction opposite to that of $\overline{a}$ is $-\dfrac{\overline{a}}{|\overline{a}|}$

(iii) Unit vectors parallel to $\overline{a}=\pm\dfrac{\overline{a}}{|\overline{a}|}$.

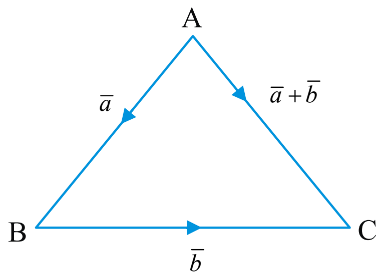
(iv) The vectors having magnitude m units and parallel to $\overline{a}=\pm\dfrac{m\overline{a}}{|\overline{a}|}$.

- **Equal Vectors:** Two vectors $\overline{a}$ and $\overline{b}$ are equal if they have the same magnitude $\left(i.e\,|\overline{a}|=|\overline{b}|\right)$ and they are in the same direction.
- Let $\overline{a}$ and $\overline{b}$ be the position vectors of the points A and B respectively. Then $\overline{AB}=$ (position vector of B) - (position vector of A) i.e., $\overline{AB}=\overline{b}-\overline{a}$

- **Co-Initial Vectors :** Vectors having the same initial point are called coinitial vectors. The vectors $\overline{OA}, \overline{OB}, \overline{OC}...$ are coinitial vectors.
- **Co-Terminal Vectors:** The vectors having the same terminal point are called the co-terminal vectors. The vectors $\overline{AO}, \overline{BO}, \overline{CO}...$ are co-terminal vectors.

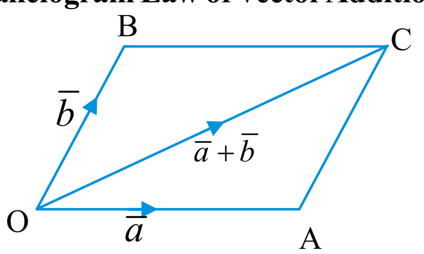
Addition of Vectors :

Triangle law of Vector Addition :



If $\overline{AB}=\overline{a}$ and $\overline{BC}=\overline{b}$ are two non-zero vectors are represented by two sides of a triangle ABC then the resultant (sum) vector is given by the closing side $\left(\overline{AC}\right)$ of the triangle in opposite direction. i.e., $\overline{AC}=\overline{AB}+\overline{BC}=\overline{a}+\overline{b}$

Parallelogram Law of Vector Addition :



If $\overline{a}$ and $\overline{b}$ are two adjacent sides of the parallelogram, then their sum (resultant) $\overline{a}+\overline{b}$ represents the diagonal of the parallelogram through the common points. It is known as parallelogram law of vector addition.

i.e., $\overline{OC}=\overline{OA}+\overline{OB}=\overline{a}+\overline{b}$

Properties of Addition of Vectors :

i) Addition of vectors is commutative

i.e., $\overline{a}+\overline{b}=\overline{b}+\overline{a}$

ii) Addition of vectors is associative

i.e., $\left(\overline{a}+\overline{b}\right)+\overline{c}=\overline{a}+\left(\overline{b}+\overline{c}\right)$

iii) There exists a vector $\overline{0}$ such that $\overline{a}+\overline{0}=\overline{0}+\overline{a}=\overline{a}$. Then $\overline{0}$ is called the additive identity vector.

iv) To each vector $\overline{a}$ there exists a vector $-\overline{a}$ such that $\overline{a}+\left(-\overline{a}\right)=\left(-\overline{a}\right)+\overline{a}=\overline{0}$. Then $-\overline{a}$ is called the additive inverse of $\overline{a}$.

- **Scalar Multiplication of Vectors:** Let $\overline{a}$ be a nonzero vector and let m be a scalar. Then m$\overline{a}$ is a scalar multiplication of $\overline{a}$ by m.

Note: i) The direction of m$\overline{a}$ is along $\overline{a}$ if m>o.

ii) The direction of m$\overline{a}$ is opposite to that of $\overline{a}$ if m<o.

- **Properties of Scalar Multiplication of Vectors:** If $\overline{a}, \overline{b}$ are vectors & m, n are scalars, then the magnitude (length) of m$\overline{a}$ is $|m|$ times that of $\overline{a}$.

Note: (i) $m(n\overline{a})=n(m\overline{a})=(mn)\overline{a}$

(ii) $(m+n)\overline{a}=m\overline{a}+n\overline{a}$

(iii) $m(\overline{a}+\overline{b})=m\overline{a}+m\overline{b}$

- If $\overline{a}$ and $\overline{b}$ are any two vectors,then

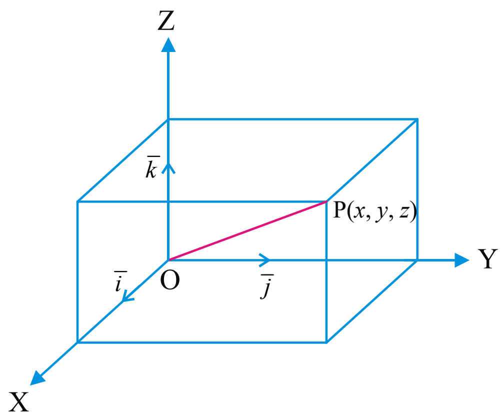
(i) $\left|\overline{a}+\overline{b}\right|\le\left|\overline{a}\right|+\left|\overline{b}\right|$

(ii) $\left|\overline{a}-\overline{b}\right|\le\left|\overline{a}\right|+\left|\overline{b}\right|$

(iii) $\left|\overline{a}-\overline{b}\right|\ge\left|\left|\overline{a}\right|-\left|\overline{b}\right|\right|$

(iv) If $\overline{a}$ and $\overline{b}$ are like vectors, then $\left|\overline{a}+\overline{b}\right|=\left|\overline{a}\right|+\left|\overline{b}\right|$

Components of a space Vector :



- Let $\overline{i}, \overline{j}, \overline{k}$ be unit vectors acting along the positive directions of x, y and z axes respectively, then position vector of any point P in the space is $\overline{OP}=x\overline{i}+y\overline{j}+z\overline{k}$. Here (x, y, z) are called scalar components of vector $\overline{OP}$ along respective axes and $x\overline{i}$, $y\overline{j}$, $z\overline{k}$ are called vector components of $\overline{OP}$ along respective axes &

$\left|\overline{OP}\right| = \sqrt{x^2 + y^2 + z^2}$

- **Section Formula :** If the position vectors of the points A, B w.r.t. O are $\overline{a}$ and $\overline{b}$ and if the point C divides the line segment $\overline{AB}$ in the ratio $m:n$ internally $(m>0, n>0)$, then the positionvector of C is $\overline{OC} = \dfrac{m\overline{b} + n\overline{a}}{m+n}$.
- If C is an external point that divides $A(\overline{a})$, $B(\overline{b})$ in the ratio $m:n$ externally then

$\overline{OC} = \dfrac{m\overline{b} - n\overline{a}}{m-n}$ $(m \neq n)$ and $(m, n > 0)$

- The point C (mid point) divides $A(\overline{a})$, $B(\overline{b})$ in the ratio $1:1$, then $\overline{OC} = \dfrac{\overline{a} + \overline{b}}{2}$.
- **Points of trisection :** Two points which divide a line segment in the ratio $1:2$ and $2:1$ are called the points of trisection.
- The position vector of the centroid G of the triangle ABC with vertices $\overline{a}, \overline{b}, \overline{c}$ is $\dfrac{\overline{a} + \overline{b} + \overline{c}}{3}$

and the incentre $I = \dfrac{a\overline{a} + b\overline{b} + c\overline{c}}{a+b+c}$, where $a = BC, b = CA$ and $c = AB$.

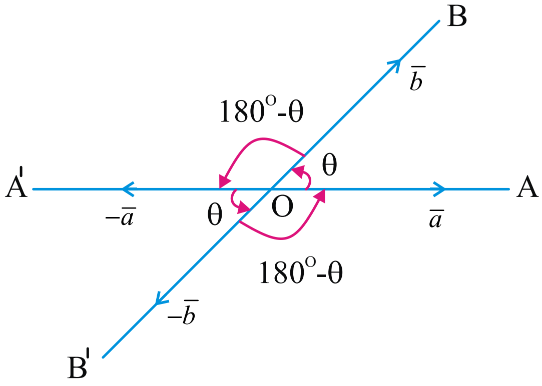
- In ΔABC if D, E, F are the mid points of the sides BC, CA, AB respectively and G is the centroid then (i) $\overline{GA} + \overline{GB} + \overline{GC} = \overline{0}$

(ii) $\overline{AD} + \overline{BE} + \overline{CF} = \overline{0}$.

(iii) $\overline{OA} + \overline{OB} + \overline{OC} = \overline{OD} + \overline{OE} + \overline{OF} = 3\overline{OG}$

- If $\overline{a}, \overline{b}, \overline{c}$ and $\overline{d}$ are the position vectors of the vertices A, B, C and D respectively of a tetrahedron ABCD then the position vector of its centroid is $\dfrac{\overline{a} + \overline{b} + \overline{c} + \overline{d}}{4}$

- **Angle between two vectors :** If $\overline{OA} = \overline{a}, \overline{OB} = \overline{b}$ be two non-zero vectors and $\angle AOB = \theta$, $0^0 \leq \theta \leq 180^0$ is defined as the angle between $\overline{a}$ and $\overline{b}$ and is written as $(\overline{a}, \overline{b})$.



Note: If $\overline{a} = \overline{0}$ or $\overline{b} = \overline{0}$, then angle between $\overline{a}$ and $\overline{b}$ is undefined.

Note: (i) $(\overline{a}, \overline{b}) = 0 \Leftrightarrow \overline{a}$ and $\overline{b}$ are like vectors.

(ii) $(\overline{a}, \overline{b}) = \dfrac{\pi}{2} \Leftrightarrow \overline{a}$ and $\overline{b}$ are orthogonal vectors.

(iii) $(\overline{a}, \overline{b}) = \pi \Leftrightarrow \overline{a}$ and $\overline{b}$ are unlike vectors.

(iv) $(\overline{a}, \overline{b}) = (\overline{b}, \overline{a})$ and $(\overline{a}, \overline{b}) = (-\overline{a}, -\overline{b})$

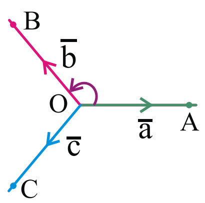
(v) $(m\overline{a}, n\overline{b}) = (\overline{a}, \overline{b})$(if m,n have same signs).

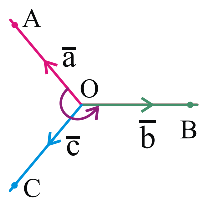
(vi) $(m\overline{a}, n\overline{b}) = 180^0 - (\overline{a}, \overline{b})$

(if m,n have opposite signs).

- **Right handed and left handed triads**

Let $\overline{OA} = \overline{a}, \overline{OB} = \overline{b}, \overline{OC} = \overline{c}$ be three non-coplanar vectors. Viewing from the point C, if the rotation of $\overline{OA}$ to $\overline{OB}$ does not exceed angle $180°$ in anti-clock sense, then $\overline{a}, \overline{b}, \overline{c}$ are set to form a **right handed system of vectors** and we say simply that $(\overline{a}, \overline{b}, \overline{c})$ is right handed vector triad. If $(\overline{a}, \overline{b}, \overline{c})$ is not a right handed system, then it is called **left handed system.**





- **Direction cosines and Directionratios of a vector:** Let $\bar{i},\bar{j},\bar{k}$ be an unit vector traid in the right handed system and $\bar{r}$ is a vector. If $\alpha=(\bar{r},\bar{i}),\beta=(\bar{r},\bar{j})$ and $\gamma=(\bar{r},\bar{k})$, then $\cos\alpha,\cos\beta,\cos\gamma$ are called the direction cosines of $\bar{r}$ denoted by l,m,n respectively. The numbers proportional to direction cosines of a given vector, i,e., kl,km,kn are called the direction ratios of that vector for $k\in R-\{0\}$
- **Some important results**: If l,m,n are the direction cosines of a line, then
$$l^2+m^2+n^2=1\cdot$$
- If $\overline{OP}=\bar{r}$ and P is the ordered triad (x,y,z) then $x=r\cos\alpha=lr, y=r\cos\beta=mr$ and $z=r\cos\gamma=nr$.
- The direction cosines of the vectors $\bar{i},\bar{j},\bar{k}$ are respectively $(1,0,0),(0,1,0),(0,0,1)$.

Linear Combination of Vectors :

- **Linear Combination:** Let $\bar{a}_1,\bar{a}_2,...,\bar{a}_n$ be n vectors and let $\bar{r}$ be any vector. Then $\bar{r}=x_1\,\bar{a}_1+x_2\,\bar{a}_2+x_3\,\bar{a}_3+....+x_n\,\bar{a}_n$ is called a linear combination of the vectors $\bar{a}_1,\bar{a}_2,...,\bar{a}_n$.
- **Collinear Vectors:** Vectors which lie on a line or on parallel lines are called collinear vectors (whatever be their magnitudes).

Note:(i) Two vectors $\bar{a}$ and $\bar{b}$ are collinear if and only if $\bar{a}=m\bar{b}$ or $\bar{b}=n\bar{a}$ where m ,n are scalars (real numbers).

(ii) Let the vectors $\bar{a}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k}$, $\bar{b}=b_1\bar{i}+b_2\bar{j}+b_3\bar{k}$ are collinear if and only if
$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$$

(iii) The Vectors $\bar{a},\bar{b}$ are collinear vectors iff $(\bar{a},\bar{b})=0^0$ or $180°$

- Let $\bar{a}$ and $\bar{b}$ be two non collinear vectors and let $\bar{r}$ be any vector coplanar with them.Then $\bar{r}=x\bar{a}+y\bar{b}$ and the scalars x and y are unique in the sense that if $\bar{r}=x_1\bar{a}+y_1\bar{b}$ and $\bar{r}=x_2\bar{a}+y_2\bar{b}$ then $x_1=x_2$ and $y_1=y_2$.
The vector equation $\bar{r}=x\bar{a}+y\bar{b}$ implies that the vectors $\bar{r},\bar{a},\bar{b}$ are coplanar.
- Let $\bar{a}$ and $\bar{b}$ be two noncollinear vectors. If $l_1\bar{a}+m_1\bar{b}=l_2\bar{a}+m_2\bar{b}$ then $l_1=l_2$ and $m_1=m_2$.
- If $\bar{a}$ and $\bar{b}$ are two noncollinear vectors and $x\bar{a}+y\bar{b}=\bar{0}$ then the scalars $x=0$ and $y=0$.
- **Collinearity of Three Points:** Let $A(\bar{a}),B(\bar{b}),C(\bar{c})$ be three points. A necessary and sufficient condition for these points to be collinear is that there exist three scalars x,y,z not all zero such that $x\bar{a}+y\bar{b}+z\bar{c}=\bar{0}$ and $x+y+z=0$
- The points $A(\bar{a}),B(\bar{b}),C(\bar{c})$ are collinear if and only if $\overline{AB}=m\overline{AC}$ or $\overline{AC}=n\overline{AB}$,where m and n are scalars.
- **Coplanar Vectors :** Vectors which lie on a plane or parallel to the same plane are called coplanar vectors. The vectors $\bar{a},\bar{b},\bar{c}$ are coplanar if their exist scalar x,y,z not all zero such that $x\bar{a}+y\bar{b}+z\bar{c}=\bar{0}$.
- A necessary and sufficient condition for four points $A(\bar{a}),B(\bar{b}),C(\bar{c}),D(\bar{d})$ to be coplanar is that there exist four scalars x,y,z,t not all zero such that $x\bar{a}+y\bar{b}+z\bar{c}+t\bar{d}=\bar{0}$ and $x+y+z+t=0$
- Let $\bar{a},\bar{b},\bar{c}$ be three noncoplanar vectors then the vector equation $x\bar{a}+y\bar{b}+z\bar{c}=\bar{0}$ implies that $x=y=z=0$.
- Let $\bar{a},\bar{b},\bar{c}$ three non-coplanar vectors. Then any vector $\bar{r}$ can be expressed as a linear combination of $\bar{a},\bar{b},\bar{c}$ i.e.,

$\overline{r} = x\overline{a} + y\overline{b} + z\overline{c}$. Further the scalars x, y, z are unique in the sense that if

$\overline{r} = x_1\overline{a} + y_1\overline{b} + z_1\overline{c}$ and ,

$\overline{r} = x_2\overline{a} + y_2\overline{b} + z_2\overline{c}$ then

$x_1 = x_2, y_1 = y_2$ and $z_1 = z_2$.

✈ **Linearly Dependent (L.D) and Independent (L.I) Vectors:** A system of vectors $\overline{a}_1, \overline{a}_2, ..., \overline{a}_n$ is linearly dependent if there exists a system of scalars $x_1, x_2, ..., x_n$ not all zero (atleast one of the scalar is non zero) such that

$x_1\overline{a}_1 + x_2\overline{a}_2 + ... + x_n\overline{a}_n = \overline{0}$

✈ The system of vectors $\overline{a}_1, \overline{a}_2, ..., \overline{a}_n$ is linearly independent if every relation of the form

$x_1\overline{a}_1 + x_2\overline{a}_2 + ... + x_n\overline{a}_n = \overline{0}$ implies that

$x_1 = x_2 = ... = x_n = 0$

✈ **Note**: i) Any two non collinear vectors are L.I.

ii) Any two collinear vectors are L.D.

iii) Any three non co-planar vectors are L.I.

iv) Any three co-planar vectors are L.D.

v) Any four vectors in 3D-space are L.D.

vi) Any super set of L.D. system of vectors is also L.D.

vii) Any Sub set of L.I. system of vectors is also L.I.

✈ To verify that three given vectors

$\overline{a} = a_1\overline{i} + a_2\overline{j} + a_3\overline{k}$, $\overline{b} = b_1\overline{i} + b_2\overline{j} + b_3\overline{k}$ and

$\overline{c} = c_1\overline{i} + c_2\overline{j} + c_3\overline{k}$ are linearly independent or linearly dependent,

find $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

i) If $\Delta \neq 0$, then they are linearly independent.

ii) If $\Delta = 0$, then they are linearly dependent.

✈ Definition of Orthonormal Vectors :

If $\overline{i}, \overline{j}, \overline{k}$ are three vectors such that

$|\overline{i}| = |\overline{j}| = |\overline{k}| = 1$ and $(\overline{i}, \overline{j}) = (\overline{j}, \overline{k}) =$

$(\overline{k}, \overline{i}) = 90^0$ then $\overline{i}, \overline{j}, \overline{k}$ are called Orthonormal Vectors

✈ Vector Equation of Lines :

(i) Vector equation to a line passing through the point A($\overline{a}$) and parallel to the vector $\overline{b}$ is $\overline{r} = \overline{a} + t\overline{b}\ (t \in R)$

(ii) Vector equation to a line passing through two points A($\overline{a}$) and B($\overline{b}$) is $\overline{r}$ = (1-t)$\overline{a}$ + t$\overline{b}$ $(t \in R)$

(iii) Vector equation to a line passing through the origin (position vector $\overline{0}$) and parallel to the vector $\overline{a}$ is $\overline{r}$ = t$\overline{a}$. $(t \in R)$

✈(i) The cartesian equation to a line passing through the point $A = (x_1, y_1, z_1)$ and parallel to the vector (a,b,c) is $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$.

(ii) The cartesian equation to a line passing through two points $A = (x_1, y_1, z_1)$ and

B=(x_2, y_2, z_2) is $\dfrac{x - x_1}{x_1 - x_2} = \dfrac{y - y_1}{y_1 - y_2} = \dfrac{z - z_1}{z_1 - z_2}$ or

$\dfrac{x - x_2}{x_1 - x_2} = \dfrac{y - y_2}{y_1 - y_2} = \dfrac{z - z_2}{z_1 - z_2}$

(iii) The cartesian equation of a line passing through the origin (0,0,0) and the point

(x_1, y_1, z_1) is $\dfrac{x}{x_1} = \dfrac{y}{y_1} = \dfrac{z}{z_1}$.

✈(i) The vector equation to the X-axis is $\overline{r} = \lambda\overline{i}$, λ is a scalar.

(ii) The cartesian equation to the X-axis is $\dfrac{x}{1} = \dfrac{y}{0} = \dfrac{z}{0}$.

✈ Vector Equation of a Plane :

(i) Vector equation to a plane passing through the point $A(\overline{a})$ and parallel to the vectors $\overline{b}$ and $\overline{c}$ is

$\overline{r} = \overline{a} + t\overline{b} + s\overline{c}\ (t, s \in R)$

(ii) Vector equation to a plane passing through the points $A(\overline{a})$ and $B(\overline{b})$ and parallel to $\overline{c}$ is

$\overline{r} = (1 - t)\overline{a} + t\overline{b} + s\overline{c}\ \ (t, s \in R)$

(iii) Vector equation to a plane passing through t h e points $A(\overline{a})$ and $B(\overline{b})$ and $C(\overline{c})$ is

$\overline{r} = (1 - t - s)\overline{a} + t\overline{b} + s\overline{c}\ \ (t, s \in R)$

(iv) Vector equation to a plane passing through the origin (**0**) and parallel to the vectors $\bar{b}$ and $\bar{c}$ is $\bar{r} = t\bar{b} + s\bar{c}$.(where t, s are parameters)

→ **Standard Results :** If AD is the internal bisector of the angle A, then D divides BC in the ratio AB : AC. then $\overline{AD} = \dfrac{|\overline{AC}|\,\overline{AB} + |\overline{AB}|\,\overline{AC}}{|\overline{AB}| + |\overline{AC}|}$

→ Let $\overline{OA} = \bar{a}$ and $\overline{OB} = \bar{b}$. Then a vector along

a) The internal bisector of the angle $\angle AOB$ is

$$\lambda\left(\frac{\bar{a}}{|\bar{a}|} + \frac{\bar{b}}{|\bar{b}|}\right),\ \lambda \text{ is a scalar.}$$

b) The external bisector of the angle $\angle AOB$ is

$$\lambda\left(\frac{\bar{a}}{|\bar{a}|} - \frac{\bar{b}}{|\bar{b}|}\right),\ \lambda \text{ is a scalar.}$$

→(i) Vector equation of the internal bisector of an angle between two vectors $\bar{b}$ and $\bar{c}$ with vertex at $\bar{a}$ is $\bar{r} = \bar{a} + t\left(\dfrac{\bar{b}}{|\bar{b}|} + \dfrac{\bar{c}}{|\bar{c}|}\right)$, where t is a scalar.

(ii) Vector equation of the external angular bisector is

$$\bar{r} = \bar{a} + t\left(\frac{\bar{b}}{|\bar{b}|} - \frac{\bar{c}}{|\bar{c}|}\right) \text{ where t is a scalar.}$$

→i) If $A(a_1, b_1, c_1)$, $B(a_1, b_1, c_2)$ and $C(a_1, b_2, c_1)$ are the vertices of a $\Delta\, ABC$ then,

(i) Circumcentre is $= \left(a_1, \dfrac{b_1 + b_2}{2}, \dfrac{c_1 + c_2}{2}\right)$

(ii) Orthocentre is $A(a_1, b_1, c_1)$

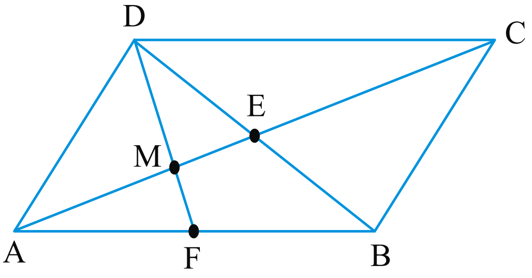
→ In $\Delta\, ABC$ if 'S' is the circumcentre and 'O' is the orthocentre then

(i) $\overline{SA} + \overline{SB} + \overline{SC} = \overline{SO}$

(ii) $\overline{OA} + \overline{OB} + \overline{OC} = 2\,\overline{OS}$

(iii) $\overline{AO} + \overline{OB} + \overline{OC} = 2\overline{AS}$

→ **Parallelogram:** Let ABCD be a parallelogram



i) Diagonal $\overline{AC} = \overline{AB} + \overline{BC}$ and diagonal $\overline{BD} = \overline{BC} + \overline{CD} = -\overline{AB} + \overline{BC}$

ii) AC and BD are diagonals of a parallelogram then,

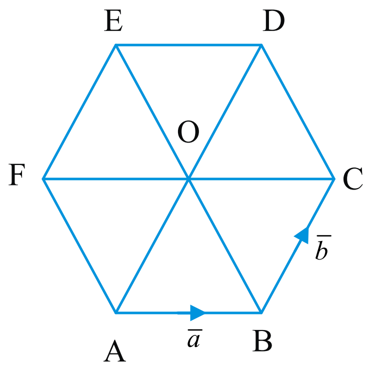
a) $\overline{AB} = \dfrac{\overline{AC} - \overline{BD}}{2}$ b) $\overline{BC} = \dfrac{\overline{AC} + \overline{BD}}{2}$

iii) If E is the point of intersection of diagonals then $\overline{OA} + \overline{OB} + \overline{OC} + \overline{OD} = 4\,\overline{OE}$ (diagonals bisect each other)

iv) If F is midpoint of AB and DF intersects AC at M then (i) AM : MC = 1 : 2 and
(ii) DM : MF = 2:1.

→ **Regular Hexagon :**

Consider regular hexagon ABCDEF.



a) If $\overline{AB} = \bar{a}$ and $\overline{BC} = \bar{b}$ then

i) $\overline{AC} = \overline{AB} + \overline{BC} = \bar{a} + \bar{b}$

ii) $\overline{AD} = 2\overline{BC} = 2\bar{b}$ iii) $\overline{CD} = \bar{b} - \bar{a}$

iv) $\overline{DE} = -\overline{AB} = -\bar{a}$ v) $\overline{EF} = -\overline{BC} = -\bar{b}$

vi) $\overline{FA} = -\overline{CD} = \bar{a} - \bar{b}$

b) If 'O' is the centre of the hexagon then $\overline{AB} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF} = 3\overline{AD} = 6\overline{AO}$

→ Let A,B be two points Such that $\bar{a} = \overline{OA}$, $\bar{b} = \overline{OB}$ than the point C where $\overline{OC} = p\bar{a} + q\bar{b}$ lies

(i) inside $\triangle$OAB, if p>0, q>0 and p+q<1

(ii) outside $\triangle$OAB but inside $\angle AOB$ if p>0, q>0 and p+q>1

(iii) outside $\triangle$OAB but inside $\angle OAB$ if p<0, q>0 and p+q<1

(iv) outside $\triangle$OAB but inside $\angle OBA$ if p>0, q<0 and p+q<1

(v) outside $\triangle$OAB if p<0, q>0 and p+q>1

Rotation of a vector about an axis :

- Let $\overline{a} = (a_1, a_2, a_3)$. If the system is rotated about

i) x-axis through an angle α, then the new components of $\overline{a}$ are

$(a_1, a_2\cos\alpha + a_3\sin\alpha, -a_2\sin\alpha + a_3\cos\alpha)$

ii) y-axis through an angle α, then the new components of $\overline{a}$ are

$(-a_3\sin\alpha + a_1\cos\alpha, a_2, a_3\cos\alpha + a_1\sin\alpha)$

iii) z-axis through an angle α, then the new components of $\overline{a}$ are

$(a_1\cos\alpha + a_2\sin\alpha, -a_1\sin\alpha + a_2\cos\alpha, a_3)$

- **Ceva's theorem:** If the lines joining any point P to the vertices A,B,C of a triangle meet the opposite sides in D,E,F respectively,then

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = 1$$

- **Menelaus' theorem :** If a transversal cuts the sides BC,CA,AB of a triangle in D,E,F respectively,then $\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = -1$

EXAMPLES

1. **$\overline{a} = \overline{i} + 2\overline{j} + 2\overline{k}$ and $\overline{b} = 3\overline{i} + 6\overline{j} + 2\overline{k}$, then vector in the direction of $\overline{a}$ and having magnitude as $|\overline{b}|$ is**

Sol: The required vector is

$$\frac{|\overline{b}|}{|\overline{a}|}\overline{a} = \frac{7}{3}\left(\overline{i} + 2\overline{j} + 2\overline{k}\right)$$

2. If $\overline{a}, \overline{b}$ are the position vectors of A,B respectively and C is a point on AB produced such that AC = 3AB, then the position vector of C is

Sol: Let the position vector for C be $\overline{c}$, then B divides AC internally in the ratio 1:2, therefore

$$\overline{b} = \frac{2\overline{a} + \overline{c}}{2+1} \Rightarrow \overline{c} = 3\overline{b} - 2\overline{a}.$$

3. **The points with position vectors $60\overline{i} + 3\overline{j}, 40\overline{i} - 8\overline{j}, a\overline{i} - 52\overline{j}$ are collinear then a = .**

Sol: Let P,Q and R be the collinear points with position vectors $60\overline{i} + 3\overline{j}, 40\overline{i} - 8\overline{j}$ and $a\overline{i} - 52\overline{j}$ respectively. Then $\overline{PQ} = \lambda\,\overline{QR}$ for some scalar λ

$\Rightarrow -20\overline{i} - 11\overline{j} = \lambda\left((a-40)\overline{i} - 44\overline{j}\right)$

$\Rightarrow \lambda(a-40) = -20, -44\lambda = -11$

$\Rightarrow \lambda = 1/4$ and $\lambda(a-40) = -20$

$\Rightarrow a - 40 = -80 \Rightarrow a = -40$

4. **Let $\overline{a}, \overline{b}$ and $\overline{c}$ be three non-zero vectors which are pairwise non-collinear. If $\overline{a} + 3\overline{b}$ is collinear with $\overline{c}$ and $\overline{b} + 2\overline{c}$ is collinear with $\overline{a}$, then $\overline{a} + 3\overline{b} + 6\overline{c}$ is [AIE-2011]**

Sol: *As* $\overline{a} + 3\overline{b}$ is collinear with $\overline{c}$, $\overline{a} + 3\overline{b} = \lambda\overline{c}$... (1)

As $\overline{b} + 2\overline{c}$ *is* collinear with $\overline{a}$, $\overline{b} + 2\overline{c} = \mu\overline{a}$...(2)

from (1) we get $\overline{a} + 3\overline{b} + 6\overline{c} = (\lambda + 6)\overline{c}$...(3)

from (2) we get $\overline{a} + 3\overline{b} + 6\overline{c} = (1 + 3\mu)\overline{a}$...(4)

since $\overline{a}$ is not collinear with $\overline{c}$,

$\lambda + 6 = 1 + 3\mu = 0$

from (3) or (4) $\overline{a} + 3\overline{b} + 6\overline{c} = \overline{0}$

5. If $\bar{a},\bar{b},\bar{c}$ are three non-coplanar vectors such that $\bar{a}+\bar{b}+\bar{c}=\alpha\bar{d}$ and $\bar{b}+\bar{c}+\bar{d}=\beta\bar{a}$ then $\bar{a}+\bar{b}+\bar{c}+\bar{d}$ is

Sol: We have,

$\bar{a}+\bar{b}+\bar{c}=\alpha\bar{d}$ and $\bar{b}+\bar{c}+\bar{d}=\beta\bar{a}$

Let $\bar{a}+\bar{b}+\bar{c}+\bar{d}=(\alpha+1)\bar{d}$ and

$\bar{a}+\bar{b}+\bar{c}+\bar{d}=(\beta+1)\bar{a}$

$\Rightarrow(\alpha+1)\bar{d}=(\beta+1)\bar{a}$, If $\alpha\neq-1$, then

$(\alpha+1)\bar{d}=(\beta+1)\bar{a}\Rightarrow\bar{d}=\dfrac{\beta+1}{\alpha+1}\bar{a}$

$\therefore\bar{a}+\bar{b}+\bar{c}=\alpha\bar{d}$

$$\Rightarrow\bar{a}+\bar{b}+\bar{c}=\alpha\left(\frac{\beta+1}{\alpha+1}\right)\bar{a}$$

$$\Rightarrow\left(1-\frac{\alpha(\beta+1)}{\alpha+1}\right)\bar{a}+\bar{b}+\bar{c}=\bar{0}$$

$\Rightarrow\bar{a},\bar{b},\bar{c}$ are coplanar. Which is a contradiction to the given condition

$\therefore\alpha=-1\Rightarrow\bar{a}+\bar{b}+\bar{c}+\bar{d}=\bar{0}$

6. If the vectors $\bar{a}$ and $\bar{b}$ are linearly independent satisfying

$$\left(\sqrt{3}\tan\theta+1\right)\bar{a}+\left(\sqrt{3}\sec\theta-2\right)\bar{b}=\bar{0},$$

then the most general values of θ are

Sol: $\sqrt{3}\tan\theta+1=0\;and\;\sqrt{3}\sec\theta-2=0$

$$\Rightarrow\theta=\frac{11\pi}{6}\;\Rightarrow\theta=2n\pi+\frac{11\pi}{6},n\in Z$$

7. The vertices of a triangle are $A(1,-1,-3),B(2,1,-2)$ and $C(-5,2,-6)$. The length of the bisector of its interior angle at the vertex A is

Sol: The bisector divides BC in the ratio AB:AC i.e $\sqrt{6}:3\sqrt{6}\;or\;1:3$ at point D.Therefore the position vector of D is $\frac{1}{4}\bar{i}+\frac{5}{4}\bar{j}-3\bar{k}$ and

$\overline{AD}=\frac{3}{4}\bar{i}+\frac{9}{4}\bar{j}+0\bar{k}$.

Hence, $\left|\overline{AD}\right|=\sqrt{\frac{9}{16}+\frac{81}{16}}=\frac{3}{4}\sqrt{10}$

8. If the vector $-\bar{i}+\bar{j}-\bar{k}$ bisects the angle between the vector $\bar{c}$ and the vector $3\bar{i}+4\bar{j}$, then the unit vector in the direction of $\bar{c}$ is

Sol : Let $x\bar{i}+y\bar{j}+z\bar{k}$ be the unit vector along $\bar{c}$.

Since $-\bar{i}+\bar{j}-\bar{k}$ bisects the angle between $\bar{c}$ and $3\bar{i}+\bar{j}$. Therefore,

$$\lambda\left(-\bar{i}+\bar{j}-\bar{k}\right)=\left(x\bar{i}+y\bar{j}+z\bar{k}\right)+\frac{3\bar{i}+4\bar{j}}{5}$$

$$\Rightarrow x+\frac{3}{5}=-\lambda,y+\frac{4}{5}=\lambda\text{ and }z=-\lambda$$

Now, $x^2+y^2+z^2=1$

($\because x\bar{i}+y\bar{j}+z\bar{k}$ is a unit vector)

$$\Rightarrow\left(-\lambda-\frac{3}{5}\right)^2+\left(\lambda-\frac{4}{5}\right)^2+\lambda^2=1$$

$$\Rightarrow\lambda=0\;or\;\lambda=\frac{2}{15}.$$

But $\lambda\neq0$. Because $\lambda=0$ implies that the given vectors are parallel.

$$\therefore\lambda=\frac{2}{15}\Rightarrow x=-\frac{11}{15},y=\frac{-10}{15}\;and\;z=\frac{-2}{15}$$

Hence,

$$x\bar{i}+y\bar{j}+z\bar{k}=-\frac{1}{15}\left(11\bar{i}+10\bar{j}+2\bar{k}\right).$$

9. **The vector $\overline{c}$, directed along the internal bisector of the angle between the vectors $\overline{a} = 7\overline{i} - 4\overline{j} - 4\overline{k}$ and $\overline{b} = -2\overline{i} - \overline{j} + 2\overline{k}$ with $|\overline{c}| = 5\sqrt{6}$, is**

Sol : The required vector $\overline{c}$ is given by $\overline{c} = \lambda\left(\frac{\overline{a}}{|\overline{a}|} + \frac{\overline{b}}{|\overline{b}|}\right)$

$$= \lambda\left\{\frac{1}{9}\left(7\overline{i} - 4\overline{j} - 4\overline{k}\right) + \frac{1}{3}\left(-2\overline{i} - \overline{j} + 2\overline{k}\right)\right\}$$

or, $\overline{c} = \frac{\lambda}{9}\left(\overline{i} - 7\overline{j} + 2\overline{k}\right)$

$$\Rightarrow |\overline{c}| = \pm\frac{\lambda}{9}\sqrt{1+49+4} = \pm\frac{\lambda}{9}\sqrt{54}$$

But $|\overline{c}| = 5\sqrt{6}$ (given).

$$\therefore \pm\frac{\lambda}{9}\sqrt{54} = 5\sqrt{6} \Rightarrow \lambda = \pm 15$$

Hence,

$$\overline{c} = \pm\frac{15}{9}\left(\overline{i} - 7\overline{j} + 2\overline{k}\right) = \pm\frac{5}{3}\left(\overline{i} - 7\overline{j} + 2\overline{k}\right)$$

EXERCISE - I

1. **If $\overline{AB} = 2\overline{i} - 3\overline{j} + \overline{k}$, $\overline{CB} = \overline{i} + \overline{j} + \overline{k}$, $\overline{CD} = 4\overline{i} - 7\overline{j}$ then $\overline{AD} =$**
   1) $5\overline{i} + 11\overline{j} - \overline{k}$ 2) $5\overline{i} - 11\overline{j}$
   3) $5\overline{i} + 11\overline{j}$ 4) $-5\overline{i} + 11\overline{j}$
2. **If the vectors $\overline{a} = x\overline{i} + 2\overline{j} + z\overline{k}$ and $\overline{b} = 2\overline{i} + y\overline{j} + \overline{k}$ are equal, then $(x, y, z) =$**
   1) (2, 1, 2) 2) (2, 2, 1) 3) (1, 2, 2) 4) (2, 1, 1)
3. **The unit vector in the direction of $2\overline{i} + 3\overline{j} + \overline{k}$ is**
   1) $\pm\frac{1}{\sqrt{14}}\left(2\overline{i} + 3\overline{j} + \overline{k}\right)$ 2) $-\frac{1}{\sqrt{14}}\left(2\overline{i} + 3\overline{j} + \overline{k}\right)$
   3) $\frac{1}{\sqrt{14}}\left(2\overline{i} + 3\overline{j} + \overline{k}\right)$ 4) $\sqrt{14}\left(2\overline{i} + 3\overline{j} + \overline{k}\right)$
4. **The vector $(\cos\alpha\cos\beta)\overline{i} + (\cos\alpha\sin\beta)\overline{j} + \sin\alpha\,\overline{k}$ is**
   1) Null vector 2) unit vector
   3) parallel to $(\overline{i} + \overline{j} + \overline{k})$
   4) a vector parallel to $(2\overline{i} + \overline{j} - \overline{k})$
5. **The unit vector in the opposite direction of the vector $\overline{a} = -6\overline{i} + 3\overline{j} - 2\overline{k}$ is**
   1) $\frac{1}{7}(-6\overline{i} + 3\overline{j} - 2\overline{k})$ 2) $\frac{1}{7}(6\overline{i} - 3\overline{j} - 2\overline{k})$
   3) $\frac{1}{7}(6\overline{i} - 3\overline{j} + 2\overline{k})$ 4) $\frac{1}{7}(-6\overline{i} + 3\overline{j} + 2\overline{k})$
6. **If $\overline{a} = (2,1,-1), \overline{b} = (1,-1,0)$, $\overline{c} = (5,-1,1)$ then the unit vector parallel to $\overline{a} + \overline{b} - \overline{c}$, but in the opposite direction is**
   1) $-\frac{1}{3}\left(2\overline{i} - \overline{j} + 2\overline{k}\right)$ 2) $\frac{1}{3}\left(2\overline{i} - \overline{j} + 2\overline{k}\right)$
   3) $\frac{1}{3}\left(2\overline{i} + \overline{j} - 2\overline{k}\right)$ 4) $-\frac{1}{3}\left(2\overline{i} - \overline{j} - 2\overline{k}\right)$
7. **If the vectors $\overline{a} = -2\overline{i} + 3\overline{j} + y\overline{k}$ and $\overline{b} = x\overline{i} - 6\overline{j} + 2\overline{k}$ are collinear, then the value of x + y is ....**
   1) 4 2) 5 3) -3 4) 3
8. **Let $\overline{a}$, $\overline{b}$ be two noncollinear vectors .If $\overline{OA}$=(x+4y)$\overline{a}$+(2x+y+1)$\overline{b}$, $\overline{OB}$=(y-2x+2) $\overline{a}$+ (2x-3y-1)$\overline{b}$ and $3\overline{OA} = 2\overline{OB}$, then (x,y)=**
   1) (1,2) 2) (1,-2) 3) (2,-1) 4) (-2,-1)
9. **Three points whose position vectors are $x\overline{i} + y\overline{j} + z\overline{k}$, $\overline{i} + z\overline{j}$ and $-\overline{i} - \overline{j}$ are collinear, then relation between x, y, z is**
   1) $x - 2y = 1, z = 0$ 2) $x + y = 1, z = 0$
   3) $x - y = 1, z = 0$ 4) $x + 2y = 1, z = 0$
10. **The position vectors of four points P,Q,R,S are $2\overline{a} + 4\overline{c}$, $5\overline{a} + 3\sqrt{3}\overline{b} + 4\overline{c}$, $-2\sqrt{3}\overline{b} + \overline{c}$ and $2\overline{a} + \overline{c}$ respectively, then**
    1) $\overline{PQ}$ is parallel to $\overline{RS}$

2) $\overline{PQ}$ is not parallel to $\overline{RS}$

3) $\overline{PQ}$ is equal to $\overline{RS}$

4) $\overline{PQ}$ is parallel and equal to $\overline{RS}$

11. If the position vectors of the points P,Q,R and S are respectively $2\overline{i}+4\overline{k}$ **,** $5\overline{i}+3\sqrt{3}\overline{j}+4\overline{k}$ **,** $-2\sqrt{3}\overline{j}+\overline{k}$ **and** $2\overline{i}+\overline{k}$ **, then** $\dfrac{|\overline{PQ}|}{|\overline{RS}|}$ **is**

1) $\frac{2}{3}$ 2) $\frac{3}{2}$ 3) $\frac{1}{3}$ 4) $\frac{3}{4}$

12. If $3\overline{i}+3\overline{j}+\sqrt{3}\overline{k}$ **,** $\overline{i}+\overline{k}$ **,**
$\sqrt{3}\overline{i}+\sqrt{3}\overline{j}+\lambda\overline{k}$ are coplanar then λ is ...

1) 1 2) 2 3) 3 4) 4

13. If the vectors $2\overline{i}+3\overline{j}-6\overline{k}, 6\overline{i}-2\overline{j}+3\overline{k}$ **,** $3\overline{i}-6\overline{j}-2\overline{k}$ **and** $\overline{i}+\overline{j}-\lambda^2\overline{k}$ **are coplanar then** $31\lambda^2-233$ **is**

1) 0 2) 2 3) 1 4) 3

14. Let $\overline{a}=\overline{i}+\overline{j}$ **,** $\overline{b}=\overline{j}+\overline{k}$ **and** $\overline{c}=\alpha\overline{a}+\beta\overline{b}$ **. If the vectors** $\overline{i}-2\overline{j}+\overline{k}$ **,** $3\overline{i}+2\overline{j}-\overline{k}$ **and** $\overline{c}$ **are coplanar then** $\frac{\alpha}{\beta}=$

1) 1 2) 2 3) 3 4) -3

15. If $2\overline{i}+3\overline{j}-6\overline{k}, 6\overline{i}-2\overline{j}+3\overline{k}, 3\overline{i}-6\overline{j}-2\overline{k}$ **represent the sides of a triangle, then the perimeter of the triangle is**

1) 6 2) 7 3) 14 4) 21

16. If G is the centroid of ΔABC, $\overline{GA}+\overline{BG}+\overline{GC}=$

1) $2\overline{GB}$ 2) $2\overline{GA}$ 3) $\overline{O}$ 4) $2\overline{BG}$

17. If $\overline{i}+2\overline{j}+3\overline{k}$, $3\overline{i}+2\overline{j}+\overline{k}$ **, are sides of a parallelogram, then a unit vector parallel to one of the diagonals is**

1) $\frac{1}{\sqrt{3}}(\overline{i}+\overline{j}+\overline{k})$ 2) $\frac{1}{\sqrt{3}}(\overline{i}-\overline{j}+\overline{k})$

3) $\frac{1}{\sqrt{3}}(\overline{i}+\overline{j}-\overline{k})$ 4) $\frac{1}{\sqrt{3}}(-\overline{i}+\overline{j}+\overline{k})$

18. Let ABC be a triangle and let D,E be the midpoints of the sides AB,AC respectively, then $\overline{BE}+\overline{DC}=$

1) $\overline{BC}$ 2) $\frac{1}{2}\overline{BC}$ 3) $\frac{3}{2}\overline{BC}$ 4) $\frac{3}{4}\overline{BC}$

19. The adjacent sides of a parallelogram are $2\overline{i}+4\overline{j}-5\overline{k}$ **and** $\overline{i}+2\overline{j}+3\overline{k}$ **then the unit vector parallel to a diagonal is**

1) $\frac{-\overline{i}+2\overline{j}+8\overline{k}}{\sqrt{69}}$ 2) $\frac{3\overline{i}+6\overline{j}+2\overline{k}}{7}$

3) $\frac{-\overline{i}+2\overline{j}-8\overline{k}}{\sqrt{69}}$ 4) $\frac{-\overline{i}-2\overline{j}+8\overline{k}}{\sqrt{69}}$

20. The vector $\overline{i}+x\overline{j}+3\overline{k}$ **is rotated through an angle** θ **and doubled in magnitude, then it becomes** $4\overline{i}+(4x-2)\overline{j}+2\overline{k}$ **. The value of** x **is**

1) $\left\{-\frac{2}{3},2\right\}$ 2) $\left\{\frac{1}{3},2\right\}$

3) $\left\{\frac{2}{3},0\right\}$ 4) $\{2,7\}$

21. If the position vectors of P and Q are $\overline{i}+2\overline{j}-7\overline{k}$ **and** $5\overline{i}-3\overline{j}+4\overline{k}$ **respectively then the cosine of the angle between** $\overline{PQ}$ **and z-axis is**

1) $\frac{4}{\sqrt{162}}$ 2) $\frac{11}{\sqrt{162}}$ 3) $\frac{5}{\sqrt{162}}$ 4) $\frac{-5}{\sqrt{162}}$

22. If a vector $\overline{a}$ **of magnitude 50 is collinear with vector** $\overline{b}=6\overline{i}-8\overline{j}-\frac{15}{2}\overline{k}$ **and makes an acute angle with positive z-axis then**

1) $\overline{a}=4\overline{b}$ 2) $\overline{a}=-4\overline{b}$

3) $\bar{b} = 4\bar{a}$ 4) $\bar{a} = -2\bar{b}$

23. **If $\bar{a}$ and $\bar{b}$ are position vectors of A and B respectively, then the position vector of a point C in $\overline{AB}$ produced such that $\overline{AC} = 2015\,\overline{AB}$ is**

1) $2014\bar{a} - 2015\bar{b}$ 2) $2014\bar{b} + 2015\bar{a}$

3) $2015\bar{b} + 2014\bar{a}$ 4) $2015\bar{b} - 2014\bar{a}$

24. **If $3\bar{a} + 4\bar{b} - 7\bar{c} = \bar{0}$ then the ratio in which $C(\bar{c})$ divides the join of $A(\bar{a})$ and $B(\bar{b})$ is**

1) 1 : 2 2) 2 : 3 3) 3 : 2 4) 4 : 3

25. **The ratio in which $\bar{i} + 2\bar{j} + 3\bar{k}$ divides the join of $-2\bar{i} + 3\bar{j} + 5\bar{k}$ and $7\bar{i} - \bar{k}$ is**

1) -3 : 2 2) 1 : 2 3) 2 : 3 4) -4 : 3

26. **A point $C = \dfrac{5\bar{a}+4\bar{b}-5\bar{c}}{3}$ divides the line joining $A = \bar{a} - 2\bar{b} + 3\bar{c}$ and B in the ratio 2:1, then $\overline{AB}$ is**

1) $\bar{a}+3\bar{b}-4\bar{c}$ 2) $2\bar{a}-3\bar{b}+4\bar{c}$

3) $\bar{a}+5\bar{b}-7\bar{c}$ 4) $2\bar{a}+3\bar{b}-4\bar{c}$

27. **The position vectors of the points A,B,C are respectively $\bar{a},\bar{b},\bar{c}$. If P divides $\overline{AB}$ in the ratio 3:4 and Q divides $\overline{BC}$ in the ratio 2 : 1 both externally then $\overline{PQ}$ =**

1) $\bar{b}+\bar{c}-2\bar{a}$ 2) $2(\bar{b}+\bar{c}-2\bar{a})$

3) $4\bar{a}-\bar{b}-\bar{c}$ 4) $\dfrac{-2\bar{a}-\bar{b}-\bar{c}}{2}$

28. **If C is the mid point of AB and P is any point out side AB then (AIE-2005)**

1) $\overline{PA} + \overline{PB} + 2\overline{PC} = \bar{0}$ 2) $\overline{PA} + \overline{PB} + \overline{PC} = \bar{0}$

3) $\overline{PA} + \overline{PB} = 2\overline{PC}$ 4)

29. **If a straight line makes an with each of the positive x, y and z-axis, a vector parallel to that line is**

1) $\bar{i}$ 2) $\bar{i} + \bar{j}$ 3) $\bar{j} + \bar{k}$ 4) $\bar{i} + \bar{j} + \bar{k}$

30. **If $\bar{e} = l\bar{i} + m\bar{j} + n\bar{k}$ is a unit vector ,the maximum value of lm + mn + nl is**

1) $-\dfrac{1}{2}$ 2) 0 3) 1 4) $\dfrac{3}{2}$

31. **If $(x, y, z) \neq 0$ and $(\bar{i} + \bar{j} + 3\bar{k})x$ $+(3\bar{i} - 3\bar{j} + \bar{k})y$ $+(-4\bar{i} + 5\bar{j})z$ $= \alpha(x\bar{i} + y\bar{j} + z\bar{k})$, then $\alpha =$**

1) 0, 1 2) –1, 0 3) 0, 2 4) –2, 0

32. **If the vectors $\bar{a} + \bar{b} + \bar{c}$, $\bar{a} + \lambda\bar{b} + 2\bar{c}$, $-\bar{a} + \bar{b} + \bar{c}$ are linearly dependent then λ =**

1) 1 2) 2 3) 3 4) 4

33. **If the vectors $\bar{a}+1343\bar{b}+\bar{c}, -\bar{a}+\bar{b}+\bar{c}$ and $\bar{a}+\mu\bar{b}+2\bar{c}$ are linearly dependent then μ =**

1) 2014 2) 2015 3) 2016 4) 0

34. **If the vectors $\bar{a} = 3\bar{i} + \bar{j} - 2\bar{k}$, $\bar{b} = -\bar{i} + 3\bar{j} + 4\bar{k}$, $\bar{c} = 4\bar{i} - 2\bar{j} - 6\bar{k}$ form the sides of the triangle then length of the median bisecting the vector $\bar{c}$ is**

1) $\sqrt{12}$ units 2) $\sqrt{6}$ units

3) $2\sqrt{6}$ units 4) $2\sqrt{3}$ units

35. **If O is the circumcentre and O$^{|}$ is the orthocentre of a triangle ABC and if $\overline{AP}$ isthe circumdiameter then $\overline{AO'} + \overline{O'B} + \overline{O'C} =$**

1) $\overline{OA}$ 2) $\overline{O'A}$ 3) $\overline{AP}$ 4) $\overline{AO}$

36. **Let G and G$^{|}$ be the centroids of the triangles ABC and A$^{|}$ B$^{|}$ C$^{|}$ respectively.Then $\overline{AA'}+\overline{BB'}+\overline{CC'} =$**

1) $2\overline{GG^{1}}$ 2) $3\overline{G^{1}G}$ 3) $3\overline{GG^{1}}$ 4) $\dfrac{3}{2}\overline{GG^{1}}$

37.Let O be the origin and A, B be two points. and $\bar{p},\bar{q}$ are vectors represented by $\overline{OA}$ and $\overline{OB}$ and their magnitudes are p,q. The unit vector bisecting the angle AOB is

1) $\dfrac{\dfrac{\overline{p}}{p}+\dfrac{\overline{q}}{q}}{\left|\dfrac{\overline{p}}{p}\right|+\left|\dfrac{\overline{q}}{q}\right|}$ 2) $\dfrac{\dfrac{\overline{p}}{p}+\dfrac{\overline{q}}{q}}{\left|\dfrac{\overline{p}}{p}\right|-\left|\dfrac{\overline{q}}{q}\right|}$ 3) $\dfrac{\dfrac{\overline{p}}{p}+\dfrac{\overline{q}}{q}}{\left|\dfrac{\overline{p}}{p}+\dfrac{\overline{q}}{q}\right|}$ 4) $\dfrac{\overline{p}+\overline{q}}{2}$

38. The vector $\mathbf{a\overline{i}+b\overline{j}+c\overline{k}}$ is a bisector of the angle between the vectors $\overline{i}+\overline{j}$ and $\overline{j}+\overline{k}$ if

1) $\mathbf{a=b}$ 2) $\mathbf{a=c}$
3) $\mathbf{a+b=c}$ 4) $\mathbf{a=b=c}$

39. Let ABCDEF be a regular hexagon, If $\overline{AD}=x\overline{BC}$ and $\overline{CF}=y\overline{BA}$ then $(x+y)^2+8=$

1) 24 2) -4 3) 2 4) -24

40. Let ABCDEF be a regular hexagon. Then $\overline{AB}+\overline{AC}+\overline{AD}+\overline{EA}+\overline{FA}=$

1) $2\overline{AB}$ 2) $3\overline{AB}$ 3) $4\overline{AB}$ 4) $\overline{AB}$

41. Cartesian equation of the plane $\overline{r}=(1+\lambda-\mu)\overline{i}+(2-\lambda)\overline{j}+(3-2\lambda+2\mu)\overline{k}$ is

1) $2x+y=5$ 2) $2x-y=5$
3) $2x+z=5$ 4) $2x-z=5$

42. The vector equation of the plane through the points (1,-2,-3) and parallel to the vectors (2,-1,3) and (2,3,-6) is $\overline{r}=$

1) $(1+2t+2s)\overline{i}-(2+t-3s)\overline{j}-(3-3t+6s)\overline{k}$
2) $(1+2t+2s)\overline{i}+(2+t+3s)\overline{j}-(3+3t+6s)\overline{k}$
3) $(1+2t+2s)\overline{i}+(2+t+3s)\overline{j}+(3+3t+6s)\overline{k}$
4) $(1+2t+2s)\overline{i}+(2+t-3s)\overline{j}+(3+3t+6s)\overline{k}$

KEY

01) 2	02) 2	03) 3	04) 2	05) 3	06) 2
07) 4	08) 3	09) 1	10) 1	11) 2	12) 1
13) 1	14) 4	15) 4	16) 4	17) 1	18) 3
19) 4	20) 1	21) 2	22) 2	23) 4	24) 4
25) 2	26) 3	27) 2	28) 2	29) 4	30) 3
31) 2	32) 2	33) 2	34) 2	35) 3	36) 3
37) 3	38) 2	39) 1	40) 3	41) 3	42) 1

HINTS

1. $\overline{AD}=\overline{AB}-\overline{CB}+\overline{CD}$
2. Two vectors are equal then corresponding components are equal.
3. Unit vector in the direction of $\overline{a}=\dfrac{\overline{a}}{|\overline{a}|}$.
4. $\sqrt{\cos^2\alpha\cos^2\beta+\cos^2\alpha\sin^2\beta+\sin^2\alpha}$
$=\sqrt{\cos^2\alpha+\sin^2\alpha}$
5. Required vector $=-\dfrac{\overline{a}}{|\overline{a}|}$
6. $\overline{a}+\overline{b}-\overline{c}=-2\overline{i}+\overline{j}-2\overline{k}$
$\left|\overline{a}+\overline{b}-\overline{c}\right|=3$, Required vector $=-\dfrac{\overline{a}+\overline{b}-\overline{c}}{\left|\overline{a}+\overline{b}-\overline{c}\right|}$
7. $\dfrac{x}{-2}=\dfrac{-6}{3}=\dfrac{2}{y}$

8.Given $\overline{OA}=(x+4y)\overline{a}+(2x+y+1)\overline{b}$

$\overline{OB}=(y-2x+2)\overline{a}+(2x-3y-1)\overline{b}$ and

$\overline{3OA}=\overline{2OB}$

$2(y-2x+2)\overline{a}+2(2x-3y-1)\overline{b}$
$=3(x+4y)\overline{a}+3(2x+y+1)\overline{b}$

two vector are equal coefficient of a,b are equal

$\Rightarrow 2(y-2x+2)=3(x+4y)$

$7x+10y-4=0-\boxed{1}$

$2(2x-3y-1)=3(2x+y+1)$

$2x+9y+5=0-\boxed{2}$

solving $\boxed{1}$ *and* $\boxed{2}$

$\boxed{1}\times\boxed{2}\Rightarrow 14x+20y-8=0$

$\boxed{2}\times 7\Rightarrow 14x+63y+35=0$

-43y-43=0

y = -1 substituing in $\boxed{1}$

7x-14=0

x=2

(x,y) = (2,-1)

9. $\overline{OA}=(x,y,z), \overline{OB}=(1,z,0), \overline{OC}=(-1,-1,0)$

$\overline{AB}=(1-x, z-y, -z)$

$\overline{AC}=(-1+x, -(1+y), -z)$

$\overline{AB}=\lambda\overline{AC} \Rightarrow 1-x=-\lambda(1+x)$

$z-y=-\lambda(1+y)$, $-z=-\lambda z$

$\lambda \neq 0, \Rightarrow z=0 \Rightarrow 2y=x-1$

10. $\overline{PQ}=3\left(\bar{a}+\sqrt{3}\bar{b}\right)$ and $\overline{RS}=2\left(\bar{a}+\sqrt{3}\bar{b}\right)$

$\overline{PQ}$ is parallel to $\overline{RS}$

11. $\left|\overline{PQ}\right|=\sqrt{9+27}=6$, $\left|\overline{RS}\right|=\sqrt{4+12}=4$

12. $\begin{vmatrix} 3 & 3 & \sqrt{3} \\ 1 & 0 & 1 \\ \sqrt{3} & \sqrt{3} & \lambda \end{vmatrix}=0$

13. Given vectors

$\overline{OA}=2\bar{i}+3\bar{j}-6\bar{k}, \overline{OB}=6\bar{i}-2\bar{j}+3\bar{k}$

$\overline{OC}=3\bar{i}-6\bar{j}-2\bar{k}, \overline{OD}=\bar{i}+\bar{j}-\lambda^2\bar{k}$

are coplaner

$\Rightarrow \overline{AB}=\overline{OB}-\overline{OA}=4\bar{i}-5\bar{j}+9\bar{k}$

$\overline{AC}=\overline{OC}-\overline{OA}=\bar{i}-9\bar{j}+4\bar{k}$

$\overline{AD}=\overline{OD}-\overline{OA}=-\bar{i}-2\bar{j}+\left(6-A^2\right)\bar{k}$

are coplanar $\Rightarrow \begin{vmatrix} 4 & -5 & 9 \\ 1 & -9 & 4 \\ -1 & -2 & 6-\lambda^2 \end{vmatrix}=0$

$\Rightarrow 4\left(-6\left(6-\lambda^2\right)+8\right)+5\left(6-\lambda^2+4\right)+9\left(-2-9\right)=0$

$\Rightarrow -216+36\lambda^2+32-5\lambda^2+50-99=0$

$31\lambda^2-233=0$

14. We have $\bar{c}=\alpha(\bar{i}+\bar{j})+\beta(\bar{j}+\bar{k})$

$\begin{vmatrix} 1 & -2 & 1 \\ 3 & 2 & -1 \\ \alpha & \alpha+\beta & \beta \end{vmatrix}=0 \Rightarrow \dfrac{\alpha}{\beta}=-3$

15. AB=BC=CA=7 $\therefore$ perimeter $=21$

16. Let $\overline{OA}=\bar{a}$ $\overline{OB}=\bar{b}, \overline{OC}=\bar{c}$

Given G is centroid or traingle ABC

$\overline{OG}=\dfrac{\overline{QA}+\overline{OB}+\overline{OC}}{3}=\dfrac{\bar{a}+\bar{b}+\bar{c}}{3}$

$3\overline{OG}=\bar{a}+\bar{b}+\bar{c}$

Now

$\overline{GA}+\overline{GB}+\overline{GC}=\overline{OA}-\overline{OG}+\overline{OB}-\overline{OG}+\overline{OC}-\overline{OG}$

17. $\overline{AC}=4\bar{i}+4\bar{j}+4\bar{k}$, $\dfrac{\overline{AC}}{\left|\overline{AC}\right|}=\dfrac{\bar{i}+\bar{j}+\bar{k}}{\sqrt{3}}$

18. Let $A=\bar{0}$ $\overline{AB}=\bar{b}$ $\overline{AC}=\bar{c}$ $\overline{AD}=\dfrac{\bar{b}}{2}$ $\overline{AE}=\dfrac{\bar{c}}{2}$

Now

$\overline{BE}+\overline{DC}=\overline{AE}-\overline{AB}+\overline{AC}-\overline{AD}=\dfrac{\bar{C}}{2}-\bar{b}+\bar{c}-\dfrac{\bar{b}}{2}=\dfrac{3}{2}$

19. $\bar{a}=2\bar{i}+4\bar{j}-5\bar{k}, \bar{b}=\bar{i}+2\bar{j}+3\bar{k}$

$\overline{AC}=\bar{a}+\bar{b}$ $\overline{BD}=\bar{b}-\bar{a}$

20. $2\left|\bar{i}+x\bar{j}+3\bar{k}\right|=\left|4\bar{i}+(4x-2)\bar{j}+2\bar{k}\right|$

$4\left(x^2+10\right)=20+\left(4x-2\right)^2$

$\Rightarrow 3x^2-4x-4=0$, $x=2,\dfrac{-2}{3}$

21. $\overline{PQ}=4\bar{i}-5\bar{j}+11\bar{k}$ direction cosines of

$\overline{PQ}=\dfrac{4}{\sqrt{162}}\bar{i}-\dfrac{5}{\sqrt{162}}\bar{j}+\dfrac{11}{\sqrt{162}}\bar{k}$

22. $\bar{a}=\lambda(6\bar{i}-8\bar{j}-\dfrac{15}{2}\bar{k})$;

$|\bar{a}|=50 \Rightarrow |\lambda|\sqrt{36+64+\dfrac{225}{4}}=50$

$\Rightarrow |\lambda|\left(\frac{25}{2}\right)=50 \Rightarrow |\lambda|=4 \quad \Rightarrow \lambda=\pm 4$

$\overline{a}=-4\overline{b}$ ($\because \overline{a}$ makes acute angle with z-axis)

23. $\overline{AC}=2015\overline{AB}$, $\overline{c}-\overline{a}=2015\left(\overline{b}-\overline{a}\right)$

24. $\overline{c}=\dfrac{3\overline{a}+4\overline{b}}{7}$

25. $-(1+2):(1-7)=-3:-6=1:2$

26. $\overline{OC}=\dfrac{\overline{OA}+2\overline{OB}}{3}$

27. $\overline{OP}=\dfrac{3\overline{OB}-4\overline{OA}}{3-4}$ $\quad \overline{OQ}=\dfrac{2\overline{OC}-\overline{OB}}{1}$

28. C is mid point of AB

$\Rightarrow \overline{PC}=\dfrac{\overline{PA}+\overline{PB}}{2} \Rightarrow \overline{PA}+\overline{PB}=2\overline{PC}$

29. $\overline{i}+\overline{j}+\overline{k}$ is parallel to $\frac{1}{\sqrt{3}}\left(\overline{i}+\overline{j}+\overline{k}\right)$.

30. $l^2+m^2+n^2-lm-mn-nl\geq 0$

31. Given

Equating coefficients of $\overline{i},\overline{j},\overline{k}$ on both sides

$x+3y-4z=\alpha x$

$$\Rightarrow x(1-\alpha)+3y-4z=0$$
$$x-y(\alpha+3)+5z=0$$
$$3x+y-\alpha z=0$$

$$\Rightarrow \begin{vmatrix} 1-\alpha & 3 & -4 \\ 1 & -(\alpha+3) & 5 \\ 3 & 1 & -\alpha \end{vmatrix}=0$$

$$\Rightarrow (1-\alpha)\left(\alpha(\alpha+3)-5\right)-3(-\alpha-15)-4\left(1+3(\alpha+3)\right)=0$$

$$(1-\alpha)\left(\alpha^2+3\alpha-5\right)+3\alpha+45-4-12\alpha-3b=0$$

$$\Rightarrow \alpha^2+3\alpha-5-\alpha^3-3\alpha^2+5\alpha-9\alpha+5=0$$

$$\Rightarrow -\alpha^3-2\alpha^2-\alpha+0=0$$

$$-\alpha\left(\alpha^2+2\alpha+1\right)=0$$

$$\alpha=0(or)\alpha=-1$$

32. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & \lambda & 2 \\ -1 & 1 & 1 \end{vmatrix}=0$

33. $\begin{vmatrix} 1 & 1343 & 1 \\ -1 & 1 & 1 \\ 1 & \mu & 2 \end{vmatrix}=0$

34. Length of the median through the vertex C=

$\frac{1}{2}\sqrt{2(a^2+b^2)-c^2}$

$=\frac{1}{2}\sqrt{2(14+26)-56}=\frac{1}{2}\sqrt{24}=\sqrt{6}$ units

35. $\overline{AO'}+\overline{O'B}+\overline{O'C}=2\overline{AO'}=\overline{AP}$

36. $\overline{AA'}+\overline{BB'}+\overline{CC'}=$

$\left(\overline{OA'}+\overline{OB'}+\overline{OC'}\right)-\left(\overline{OA}+\overline{OB}+\overline{OC}\right)$

37. $\overline{r}=t\left(\dfrac{\overline{a}}{|\overline{a}|}+\dfrac{\overline{b}}{|\overline{b}|}\right)$

38. $\overline{\gamma}=t\left(\dfrac{\overline{a}}{|\overline{a}|}\times\dfrac{\overline{b}}{|\overline{b}|}\right)$

$$a\overline{i}+b\overline{j}+c\overline{k}=t\left(\frac{\overline{i}+\overline{k}}{\sqrt{2}}+\frac{\overline{j}+\overline{k}}{\sqrt{2}}\right)$$

$$\Rightarrow a\overline{i}+b\overline{j}+c\overline{k}=\frac{t}{\sqrt{2}}\left(\overline{i}+2\overline{j}+\overline{k}\right)$$

$$\overline{a}=\frac{t}{\sqrt{2}},\ \overline{c}=\frac{t}{\sqrt{2}} \qquad \Rightarrow \overline{a}=\overline{c}$$

39. $\overline{AD}=2\overline{BC} \Rightarrow x=2$

$\overline{CF}=2\overline{BA} \Rightarrow y=2 \quad \Rightarrow (x+y)^2+8=24$

40. $\overline{FA}+\overline{AC}=\overline{FC}=2\overline{AB}$ and

$\overline{EA}+\overline{AD}=\overline{ED}=\overline{AB}$

41. $\overline{r}=(\overline{i}+2\overline{j}+3\overline{k})+\lambda(\overline{i}-\overline{j}-2\overline{k})+\mu(-\overline{i}+2\overline{k})$

plane passing through A(1, 2, 3) and parallel to

$\bar{b} = \bar{i} - \bar{j} - 2\bar{k}$ and $\bar{c} = -\bar{i} + 2\bar{k}$

42.
$\bar{a} = (1,-2,-3) = \bar{i} - 2\bar{j} - 3\bar{k}$

$\bar{b} = (2,-1,3) = 2\bar{i} - \bar{j} + 3\bar{k}$

$\bar{c} = (2,3,-6) = 2\bar{i} + 3\bar{j} - 6\bar{k}$

Vector equation of plane passing through $\bar{a}$ parallel to $\bar{b}$ and $\bar{c}$ is

$$\bar{\gamma} = \bar{a} + t\bar{b} + \delta\bar{c} = (\bar{i} - 2\bar{j} - 3\bar{k}) + t(2\bar{i} - \bar{j} + 3\bar{k}) + \delta(2\bar{i} + 3\bar{j} - 6\bar{k})$$

$$= \bar{i}(1 + 2t + 2s) - \bar{j}(2 + t - 3s) - \bar{k}(3 - 3t + 6s)$$

EXERCISE - II

1. **If** $10\bar{i}+3\bar{j}$ **,** $12\bar{i}-5\bar{j}$ **and** $\lambda\bar{i}+11\bar{j}$ **are the position vectors of three collinear points. Then λ is**

1) 4 2) 8 3) 12 4) 22

2. **P,Q and R are three points with position vectors** $\bar{i} + \bar{j}, \bar{i} - \bar{j}$ **and** $a\bar{i} + b\bar{j} + c\bar{k}$ **respectively. If P,Q and R are collinear, then**

1) a = b = c = 0

2) a = 1, b, c are any real numbers

3) a = b = c = 1

4) a =1 , c = 0 and b is any real number

3. **Let** $\bar{f}(t) = [t]\bar{i} - (t - [t])\bar{j} + [t+1]\bar{k}$ **be a vector. Where** $[.]$ **is a greatest integer function. If** $\bar{f}\left(\frac{5}{4}\right)$ **and** $\bar{i} + \lambda\bar{j} + \mu\bar{k}$ **are parallel vectors then** $(\lambda, \mu) =$

1) (1, 1) 2) $\left(\frac{-1}{4}, 2\right)$ 3) $\left(\frac{1}{2}, 2\right)$ 4) $\left(\frac{1}{4}, 4\right)$

4. **If the points** $\bar{a} + \bar{b}, \bar{a} - \bar{b}, \bar{a} + k\bar{b}$ **are collinear, then**

1) k has only one real value

2) k has two real value 3) k has all real values

4) k has finite number of real values

5. **The vectors** $\mathbf{2\bar{i} + 3\bar{j}, 5\bar{i} + 6\bar{j}}$ **and** $\mathbf{8\bar{i} + \lambda\bar{j}}$ **have their initial point at** $\mathbf{(1,1)}$**.The value of λ so that the vectors terminate on one straight line is**

1) 3 2) 6 3) 9 4) 12

6. $p\bar{i} + 3\bar{j} + 4\bar{k}$ **and** $\sqrt{q}\,\bar{i} + 5\bar{k}$ **are two vectors where** $p, q \geq 0$ **are two scalars. Then the lengths of the vector are equal for**

1) all values of (p, q)

2) only finite number of values of (p, q)

3) infinite number of values of (p,q)

4) No value of (p,q)

7. If $\bar{a},\bar{b},\bar{c}$ are non coplanar vectors and λ is a real number, then the vectors $\bar{a}+2\bar{b}+3\bar{c},\lambda\bar{b}+4\bar{c}$ and $(2\lambda-1)\bar{c}$ are non coplanar for (AIE-2004)

1) all except two values of λ

2) all except one value of λ

3) for all values of λ 4) no value of λ

8. If a is a non-zero scalar, then the vectors $\bar{\alpha}=a\bar{i}+2a\bar{j}-3a\bar{k}$, $\bar{p}=(2a+1)\bar{i}$ $+(2a+3)\bar{j}+(a+1)\bar{k}$ and $\bar{r}=(3a+5)\bar{i}+(a+5)\bar{j}+(a+2)\bar{k}$ are

1) coplanar if a < 0 2) coplanar if a > 0

3) always coplanar 4) never coplanar

9. Let $A=2\bar{i}+4\bar{j}-\bar{k}$; $B=4\bar{i}+5\bar{j}+\bar{k}$. If the centroid G of the triangle ABC is $3\bar{i}+5\bar{j}-\bar{k}$ then the position vector of C is

1) $3\bar{i}-6\bar{j}+3\bar{k}$ 2) $3\bar{i}-6\bar{j}-3\bar{k}$

3) $3\bar{i}-6\bar{j}+2\bar{k}$ 4) $3\bar{i}+6\bar{j}-3\bar{k}$

10. $\bar{a}$ and $\bar{b}$ are non collinear vectors. $\bar{u}=x\bar{a}+2y\bar{b},\bar{v}=-2y\bar{a}+3x\bar{b}$ and $\bar{w}=4\bar{a}-2\bar{b}$, are vectors such that $2\bar{u}-\bar{v}=\bar{w}$. Then

1) $x=\frac{4}{7},y=\frac{6}{7}$ 2) $x=\frac{10}{7},y=\frac{4}{7}$

3) $x=\frac{8}{7},y=\frac{2}{7}$ 4) $x=6,y=4$

11. A point $C=\frac{5\bar{a}+4\bar{b}-5\bar{c}}{3}$ divides the line joining the points $A=\bar{a}-2\bar{b}+3\bar{c}$ and B in the ratio 2:1, then the position vector of B is

1) $\bar{a}+3\bar{b}-4\bar{c}$ 2) $2\bar{a}-3\bar{b}+4\bar{c}$

3) $2\bar{a}+3\bar{b}+4\bar{c}$ 4) $2\bar{a}+3\bar{b}-4\bar{c}$

12. In the $\Delta OAB,M$ is the mid point of AB, C is a point on OM, such that $2\overline{OC}=\overline{CM}$. X is a point on the side OB such that $\overline{OX}=2\overline{XB}$. The line XC is produced to meet OA in Y. Then $\frac{OY}{YA}=$

1) $\frac{1}{3}$ 2) $\frac{2}{7}$ 3) $\frac{3}{2}$ 4) $\frac{2}{5}$

13. If $\bar{r}=3\bar{i}+2\bar{j}-5\bar{k}$,$\bar{a}=2\bar{i}-\bar{j}+\bar{k}$, $\bar{b}=\bar{i}+3\bar{j}-2\bar{k}$ and $\bar{c}=-2\bar{i}+\bar{j}-3\bar{k}$ such that $\bar{r}=\lambda\bar{a}+\mu\bar{b}+v\bar{c}$. Then

1) $\mu,\frac{\lambda}{2}$, v are in A.P. 2) $\lambda,\mu,$ v are in A.P

3) $\lambda,\mu,$ v are in H.P. 4) $\lambda,\mu,$ v are in G.P.

14. If $\overline{OA}=3\bar{i}+\bar{j}-\bar{k}$, $|\overline{AB}|=2\sqrt{6}$ and AB has the direction ratios 1,-1,2 then |OB| =

1) $\sqrt{35}$ 2) $\sqrt{41}$ 3) $\sqrt{26}$ 4) $\sqrt{55}$

15. If $\bar{a}=\bar{i}+\bar{j}+\bar{k},\bar{b}=4\bar{i}+3\bar{j}+4\bar{k}$ $\bar{c}=\bar{i}+\alpha\bar{j}+\beta\bar{k}$ be linearly dependent vectors and $|\bar{c}|=\sqrt{3}$ then

1) $\alpha=1,\beta=-1$ 2) $\alpha=1,\beta=\pm1$

3) $\alpha=-1,\beta=\pm1$ 4) $\alpha=\pm1,\beta=1$

16. x-component of $\bar{a}$ is twice its y-component. If the magnitude of the vector is $5\sqrt{2}$ and it makes an angle of 135^0 **with z-axis then the vector is**

1) $(2\sqrt{3},\sqrt{3},-3)$ 2) $(2\sqrt{6},\sqrt{6},-6)$

3) $(2\sqrt{5},\sqrt{5},-5)$ 4) $(\sqrt{6},\sqrt{6},-6)$

17. Given three vectors $\bar{a}=6\bar{i}-3\bar{j},\bar{b}=2\bar{i}-6\bar{j}$ and $\bar{c}=-2\bar{i}+21\bar{j}$ such that $\bar{\alpha}=\bar{a}+\bar{b}+\bar{c}$. Then the resolution of the vector $\bar{\alpha}$ into components with respect to $\bar{a}$ and $\bar{b}$ given by

1) $3\bar{a}-2\bar{b}$ 2) $3\bar{b}-2\bar{a}$ 3) $2\bar{a}-3\bar{b}$ 4) $\bar{a}-3\bar{b}$

18. A vector $\bar{a}$ has components 2p and 1 with respect to a rectangular cartesian coordinate system. This system is rotated through a certain angle about the origin in the counter clock - wise sense. If, with respect to the new system, $\bar{a}$ has components p + 1 and 1 then

1) $p=0$ 2) $p=0$ or $p=-1/3$

3) $p=-1$ or $p=1/2$

4) $p=1$ or $p=-1/3$

ANGULAR BISECTORS

19. If $4\bar{i}+7\bar{j}+8\bar{k}$, $2\bar{i}+3\bar{j}+4\bar{k}$ and $2\bar{i}+5\bar{j}+7\bar{k}$ are the position vectors of the vertices A,B and C of triangle ABC, the position vector of the point where the bisector of $\angle A$ meets BC is

1) $\frac{2}{3}(-6\bar{i}-8\bar{j}-6\bar{k})$ 2) $\frac{2}{3}(6\bar{i}+8\bar{j}+6\bar{k})$

3) $\frac{1}{3}(6\bar{i}+13\bar{j}+18\bar{k})$ 4) $2(\bar{i}+\bar{j}+\bar{k})$

20. Let A (4,7,8) B (2,3,4) and C (2,5,7) be the position vectors of the vertices of a ΔABC. Then the length of internal angular bisector of angle A is

1) $\frac{3}{2}\sqrt{34}$ 2) $\frac{2}{3}\sqrt{34}$ 3) $\frac{1}{2}\sqrt{34}$ 4) $\frac{1}{3}\sqrt{34}$

21. **If $A_1A_2......A_n$ is a regular polygon. Then the vector $\overline{A_1A_2}+\overline{A_2A_3}+...+\overline{A_nA_1}$ is equal to**

1) $\bar{0}$ 2) $n\left(\overline{A_1A_2}\right)$

3) $n\left(\overline{OA_1}\right)$ (O is the centre) 4) $(n-1)\left(\overline{A_1A_2}\right)$

22. If ABCDE is a regular pentagon and $\overline{AB}+\overline{AE}+\overline{BC}+\overline{DC}+\overline{ED}=(\lambda-1)(\overline{AC})$ then $\lambda=$

1) 1 2) 2 3) 3 4) 4

23. P is a point on the line through the point A whose position vector is $\bar{a}$ and the line is parallel to the vector $\bar{b}$. If $PA=6$, then the position vector of P is

1) $\bar{a}+6\bar{b}$ 2) $\bar{a}\pm\frac{6}{|\bar{b}|}\bar{b}$

3) $\bar{a}-6\bar{b}$ 4) $\bar{b}+\frac{6}{|\bar{a}|}\bar{a}$

KEY

01) 2	02) 4	03) 2	04) 3	05) 3	06) 3
07) 1	08) 4	09) 4	10) 2	11) 4	12) 2
13) 1	14) 1	15) 4	16) 3	17) 3	18) 4
19) 3	20) 2	21. 1	22) 3	23) 2	

SOLUTIONS

1. $\begin{vmatrix} \lambda & 11 & 1 \\ 10 & 3 & 1 \\ 12 & -5 & 1 \end{vmatrix}=0$

2. $\overline{AB}=-2\bar{j}$, $\overline{AC}=(a-1)\bar{i}+(b-1)\bar{j}+c\bar{k}$ and $\overline{AB}=t\,\overline{AC}$

3. $\bar{f}\left(\frac{5}{4}\right)=\bar{i}-\frac{1}{4}\bar{j}+2\bar{k}$

4. $\overline{OA}=\bar{a}+\bar{b}$, $\overline{OC}=\bar{a}+k\bar{b}$
$\overline{OB}=\bar{a}-\bar{b}$, $\overline{AB}=-2\bar{b}$
$\overline{AC}=(k+1)\bar{b}$ $\Rightarrow$ k is any real number

5. $\begin{vmatrix} 2 & 3 & 1 \\ 5 & 6 & 1 \\ 8 & \lambda & 1 \end{vmatrix}=0$

6. $\sqrt{p^2+9+16}=\sqrt{q+25}$ $\Rightarrow p^2=q$

7. $\begin{vmatrix} 0 & 0 & 2\lambda-1 \\ 0 & \lambda & 4 \\ 1 & 2 & 3 \end{vmatrix}=0\Rightarrow(2\lambda-1)\{0-\lambda\}=0$

$\Rightarrow\lambda=0,\ \lambda=\frac{1}{2}$, Vectors are coplanar for values of $\lambda = 0, \frac{1}{2}$, Vectors are non coplanar for all except two values of λ.

8. $\begin{vmatrix} a & 2a & -3a \\ (2a+1) & 2a+3 & a+1 \\ 3a+5 & a+5 & a+2 \end{vmatrix} \neq 0$

9.C=3G-A-B

10. $2(x\overline{a} + 2y\overline{b}) - (-2y\overline{a} + 3x\overline{b}) = 4\overline{a} - 2\overline{b}$

$2x + 2y = 4 \Rightarrow x + y = 2, \ 4y - 3x = -2$

11. $\overline{OC} = \dfrac{\overline{OA} + 2\overline{OB}}{3}$

12. $\overline{OA} = \overline{a}, \overline{OB} = \overline{b}, \ \therefore \overline{OM} = \dfrac{\overline{a} + \overline{b}}{2}$

$\therefore \overline{OC} = \frac{1}{6}\left(\overline{a} + \overline{b}\right)$

$\overline{OX} = \frac{2}{3}\overline{b}$ equation of $\overline{CX}$ is

$\overline{r} = (1-t)\frac{2}{3}\overline{b} + \frac{t}{6}\left(\overline{a} + \overline{b}\right)$..... (1)

equation of the line $\overline{OA}$ is, $\overline{r} = s\overline{a}$ ..... (2)

From (1) and (2) $s = \frac{t}{6}$ and $\frac{2}{3} - \frac{t}{2} = 0$,

$\therefore t = \frac{4}{3}$ or $s = \frac{2}{9}$, $\therefore \overline{OY} = \frac{2}{9}\overline{a}$, $\therefore \dfrac{\overline{OY}}{\overline{YA}} = \dfrac{2}{7}$

13. $\overline{r} = \boldsymbol{\lambda}\overline{a} + \boldsymbol{\mu}\overline{b} + \mathbf{v}\overline{c}$

Compare like vectors, $\mu = 1, v = 2, \lambda = 3$.

Hence, $\mu, \frac{\lambda}{2}, v$ are in A.P.

14. $\overline{AB} = \left|\overline{AB}\right|.\left(D.C.s \ of \ \overline{AB}\right)$

$\Rightarrow 2\sqrt{6} = t\sqrt{6} \Rightarrow t = 2$

15. $\begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 4 \\ 1 & \alpha & \beta \end{vmatrix} = 0; \ \alpha^2 + \beta^2 = 2$

16. Let $\overline{a} = 2x\overline{i} + x\overline{j} + z\overline{k}$

$\sqrt{5x^2 + z^2} = 5\sqrt{2}$

Also $\cos 135^0 = \dfrac{z}{\sqrt{5x^2 + z^2}} = \dfrac{z}{5\sqrt{2}} = \dfrac{-1}{\sqrt{2}}$

$\Rightarrow z = -5 \quad \Rightarrow x = \sqrt{5}$

Required vector $= 2\sqrt{5}\overline{i} + \sqrt{5}\overline{j} - 5\overline{k}$

17. $\overline{\alpha} = \overline{a} + \overline{b} + \overline{c} = 6\overline{i} + 12\overline{j}$

let $\overline{\alpha} = x\overline{a} + y\overline{b}$. Then

$6x + 2y = 6$ and $-3x - 6y = 12$

18. $\sqrt{4p^2 + 1} = \sqrt{(p+1)^2 + 1}$

19.Given $\overline{OA} = 4\overline{i} + 7\overline{j} + 8\overline{k}, \ \overline{OB} = 2\overline{i} + 3\overline{j} + 4\overline{k}$ and

$\overline{OC} = 2\overline{i} + 5\overline{j} + 7\overline{k}$

$\overline{AB} = \overline{OB} - \overline{OA} = -2\overline{i} - 4\overline{j} - 4\overline{k}$

$\overline{AC} = \overline{OC} - \overline{OA} = -2\overline{i} - 2\overline{j} - \overline{k}$

$\left|\overline{AB}\right| = \sqrt{4 + 16 + 16} = 6$

$\left|\overline{AC}\right| = \sqrt{4 + 4 + 1} = 3$

D position vector D divides BC in the ratio 6:3=2:1 internally

$\overline{OB} = 2\overline{i} + 3\overline{j} + 4\overline{k} \quad \overline{OC} = 2\overline{i} + 5\overline{j} + 7\overline{k}$

$\overline{OD} = \dfrac{2\left(\overline{OC}\right) + 1\left(\overline{OB}\right)}{2+1} = \dfrac{6\overline{i} + 13\overline{j} + 18\overline{k}}{3}$

$D; \overline{BC} = AB : AC$

20. $AB = 6, AC = 3$, D divides $\overline{BC}$ the ratio 2:1

21. Let O be the centre.

$\overline{A_1A_{i+1}} = \overline{OA_{i+1}} - \overline{OA_i}$

$\sum_{i=1}^{n-1} \overline{A_1A_{i+1}} = \sum_{i=1}^{n-1}\left[\overline{OA_{i+1}} - \overline{OA_i}\right] = \overline{0}$

22. $\overline{AB} + \overline{AE} + \overline{BC} + \overline{DC} + \overline{ED} = (\lambda - 1)\overline{AC}$

$\Rightarrow \overline{AB} + \overline{BC} + \overline{AE} + \overline{ED} + \overline{DC} = (\lambda - 1)\overline{AC}$

$\overline{OB}-\overline{OA}+\overline{OC}-\overline{OB}+\overline{OE}-\overline{OA}+\overline{OD}-\overline{OE}+\overline{OC}-\overline{OD}$

$=(\lambda-1)\overline{AC} \qquad \overline{AC}+\overline{AC}=(\lambda-1)\overline{AC}$

$\lambda-1=2$

$\lambda=3$ $\overline{AB}+\overline{AE}+\overline{BC}+\overline{DC}+\overline{ED}$

$=\left(\overline{AB}+\overline{BC}\right)+\left(\overline{AE}+\overline{ED}+\overline{DC}\right)$

23. Equation of line $\overline{r}=\overline{a}+t\overline{b}$

$$PA=6\Rightarrow|\overline{r}-\overline{a}|=6\Rightarrow|t\overline{b}|=6 \Rightarrow t=\pm\frac{6}{|\overline{b}|}$$

EXERCISE - III

1. In ΔABC, P,Q,R are points on $\overline{BC},\overline{CA},\overline{AB}$ respectively, dividing them in the ratio 1:4, 3:2 and 3 : 7. The point S divides AB in the ratio 1:3. Then $\dfrac{|\overline{AP}+\overline{BQ}+\overline{CR}|}{|\overline{CS}|}=$

1) $\frac{1}{5}$ 2) $\frac{2}{5}$ 3) $\frac{5}{2}$ 4) $\frac{7}{10}$

2. ABCD is a parallelogram and P is a point on the segment $\overline{AD}$ dividing it internally in the ratio 3:1 the line $\overline{BP}$ meets the diagonal AC in Q. Then the ratio AQ:QC is

1) 3:4 2) 4:3 3) 3:2 4) 2:3

3. In ΔOAB, E is the mid point of AB and F is a point on OA such that $OF=2FA$. If C is the point of intersection of OE and BF, then find the ratios OC : CE and BC : CF are

1) $1:4;\ \ 3:2$ 2) $4:1;\ \ 3:2$

3) $4:1;\ \ 1:2$ 4) $4:1;\ \ 2:3$

4. If $\overline{b}$ is the vector whose initial point divides the joining $5\overline{i}$ and $5\overline{j}$ in the ratio $\lambda:1$ and terminal point is at origin. If $|\overline{b}|\le\sqrt{37}$ then $\lambda\in$

1) $(-\infty,-6]\cup\left[-\frac{1}{6},\infty\right)$

2) $(-\infty,-3)\cup\left[-\frac{1}{4},\infty\right)$

3) $(-\infty,0)\cup\left(\frac{1}{2},\infty\right)$ 4) $\left[-6,-\frac{1}{6}\right]$

5. If $\overline{DA}=\overline{a}$; $\overline{AB}=\overline{b}$ and $\overline{CB}=k\overline{a}$ where k > 0 and X, Y are the midpoints of DB and AC respectively such that $|\overline{a}|=17$ and $|\overline{XY}|=4$, then k =

1) $\frac{8}{17}$ 2) $\frac{9}{17}$ 3) $\frac{11}{17}$ 4) $\frac{4}{17}$

6. **The vectors** $\bar{a}(x)=\cos x\bar{i}+\sin x\bar{j}$, $\bar{b}(x)=x\bar{i}+\sin x\bar{j}$ **are collinear for :**

1) Unique value of x such that $0<x<\frac{\pi}{6}$

2) Unique value of x such that $\frac{\pi}{6}<x<\frac{\pi}{3}$

3) No value of x

4) Infinitely many values of x in $0<x<\frac{\pi}{2}$

7. **Let a line cut the sides PQ,PS and diagonal PR of a parallelogram at** Q_1, R_1 **and** S_1 **respectively such that** $PQ_1=\lambda_1 PQ, PR_1=\lambda_3 PR$ **and** $PS_1=\lambda_2 PS$, **then**

1) $\frac{1}{\lambda_1}+\frac{1}{\lambda_2}=\frac{-1}{\lambda_3}$ 2) $\frac{1}{\lambda_1}+\frac{1}{\lambda_2}=\frac{1}{\lambda_3}$

3) $\frac{1}{\lambda_1}-\frac{1}{\lambda_2}=\frac{-1}{\lambda_3}$ 4) $\frac{1}{\lambda_1}-\frac{1}{\lambda_2}=\frac{1}{\lambda_3}$

8. **The value of** λ **so that the points P, Q, R, S on the sides OA, OB, OC and AB of a regular tetrahedron are coplanar. When** $\frac{OP}{OA}=\frac{1}{3}$; $\frac{OQ}{OB}=\frac{1}{2},\frac{OR}{OC}=\frac{1}{3}$ **and** $\frac{AS}{AB}=\lambda$ **is**

1) $\lambda=\frac{1}{2}$ 2) $\lambda=-1$ 3) $\lambda=0$ 4) $\lambda=2$

9. **A unit tangent vector at t = 2 on the curve** $x=t^2+2; y=4t-5; z=2t^2-6t$ **is**

1) $\frac{1}{\sqrt{3}}(\bar{i}+\bar{j}+\bar{k})$ 2) $\frac{1}{3}(2\bar{i}+2\bar{j}+\bar{k})$

3) $\frac{1}{\sqrt{6}}(2\bar{i}+\bar{j}+\bar{k})$ 4) $\frac{1}{3}(\bar{i}+\bar{j}+\bar{k})$

10. **'O' is the origin in the Cartesian plane. From the origin 'O' take point A in the North-East direction such that** $|\overline{OA}|$**=5, B is a point in the North-West direction such that** $|\overline{OB}|=5$. **Then** $|\overline{OA}-\overline{OB}|$ **is**

1) 25 2) $5\sqrt{2}$ 3) $10\sqrt{5}$ 4) $\sqrt{5}$

11. **Let O be the origin of the coordinate system in the Cartesian plane,** $\overline{OP}$ **and** $\overline{OR}$ **be vectors making angles** 45^0 **and** 135^0 **respectively with the positive directions of the X-axis (i.e., in the counter clock wise). Rectangle OPQR is completed and M is the midpoint of PQ. If the line** $\overline{OM}$ **meets the diagonal PR at T, and** $|\overline{OP}|=3, |\overline{OR}|=4$, **then** $\overline{OT}$ **is**

1) $\frac{1}{2}(\bar{i}+\bar{j})$ 2) $\frac{2}{3}(\bar{i}+5\bar{j})$

3) $\frac{\sqrt{2}}{3}(\bar{i}-5\bar{j})$ 4) $\frac{\sqrt{2}}{3}(\bar{i}+5\bar{j})$

12. **Let AC be an arc of a circle , subtending a right angle at the centre O. The point B divides the arc AC in the ratio 1:2 If** $\overline{OA}=\bar{a}$ **and** $\overline{OB}=\bar{b}$, **then** $\overline{OC}$ **in terms of** $\bar{a}$ **and** $\bar{b}$ **is**

1) $\bar{b}-\sqrt{3}\bar{a}$ 2) $2\bar{b}-\bar{a}$ 3) $2\bar{b}-\sqrt{3}\bar{a}$ 4) $\bar{b}-\bar{a}$

13. **In a triangle ABC, if** $A=(0,0); B=(3,3\sqrt{3}); \quad C=(-3\sqrt{3},3)$ **then the vector of magnitude** $2\sqrt{2}$ **units directed along** $\overline{AO}$, **where O is the circumcentre of triangle ABC is**

1) $(1-\sqrt{3})\bar{i}+(1+\sqrt{3})\bar{j}$ 2) $\sqrt{3}\bar{i}+2\bar{j}$

3) $\bar{i}-\sqrt{3}\bar{j}$ 4) $\bar{i}+2\bar{j}$

14. **Vectors** $\bar{a}=-4\bar{i}+3\bar{k}; \bar{b}=14\bar{i}+2\bar{j}-5\bar{k}$ **are laid off from one point.Vector** $\bar{d}$**,which is being laid off from the same point dividing theagle between vectors** $\bar{a}$ **and** $\bar{b}$ **in equal halves and having the magnitude** $\sqrt{6}$**,is**

1) $\bar{i}+\bar{j}+2\bar{k}$ 2) $\bar{i}-\bar{j}+2\bar{k}$

3) $\bar{i}+\bar{j}-2\bar{k}$ 4) $2\bar{i}-\bar{j}-2\bar{k}$

15. **The triangle ABC is defined by the vertices A = (0,7,10), B = (-1,6,6) and C = (-4,9,6) let D be the foot of the altitude from B to the side AC . Then** $\overline{BD}$ **is**

1) $\bar{i}+2\bar{j}+2\bar{k}$ 2) $-\bar{i}+2\bar{j}+2\bar{k}$

3) $\bar{i}+2\bar{j}-2\bar{k}$ 4) $\bar{i}-2\bar{j}+2\bar{k}$

16. The equation to the altitude of the triangle formed by (1, 1, 1), (1, 2, 3), (2, -1, 1) through (1, 1, 1) is

1) $\bar{r}=\left(\bar{i}+\bar{j}+\bar{k}\right)+t\left(\bar{i}-3\bar{j}-2\bar{k}\right)$

2) $\bar{r}=\left(\bar{i}+\bar{j}+\bar{k}\right)+t\left(3\bar{i}+\bar{j}+2\bar{k}\right)$

3) $\bar{r}=\left(\bar{i}+\bar{j}+\bar{k}\right)+t\left(\bar{i}-\bar{j}+2\bar{k}\right)$

4) $\left|\bar{r}\right|=5$

17. Image of the point P with position vector $7\bar{i}-\bar{j}+2\bar{k}$ in the line whose vector equation is $\bar{r}=9\bar{i}+5\bar{j}+5\bar{k}+\lambda\left(\bar{i}+3\bar{j}+5\bar{k}\right)$ has the position vector

1) $-9\bar{i}+5\bar{j}+2\bar{k}$ 2) $9\bar{i}+5\bar{j}-2\bar{k}$

3) $9\bar{i}-5\bar{j}-2\bar{k}$ 4) $9\bar{i}+5\bar{j}+2\bar{k}$

18. If $A\left(2\bar{i}-\bar{j}-3\bar{k}\right), B\left(4\bar{i}+\bar{j}-\bar{k}\right), C\left(\bar{i}-3\bar{j}+2\bar{k}\right)$ and $D\left(\bar{i}-\bar{j}-2\bar{k}\right)$ then the vector equation of the plane parallel to $\overline{ABC}$ and passing through the centroid of the tetrahedron ABCD is

1) $\bar{r}=\left(2\bar{i}-\bar{j}-\bar{k}\right)+s\left(2\bar{i}+2\bar{j}+2\bar{k}\right)+t\left(\bar{i}-\bar{k}\right)$

2) $\bar{r}=\left(2\bar{i}-\bar{j}-3\bar{k}\right)+s\left(\bar{i}+\bar{j}+\bar{k}\right)+t\left(\bar{i}-\bar{k}\right)$

3) $\bar{r}=\left(2\bar{i}-\bar{j}-\bar{k}\right)+s\left(\bar{i}+\bar{j}+\bar{k}\right)+t\left(\bar{i}+2\bar{j}-5\bar{k}\right)$

4) $\bar{r}=\left(2\bar{i}-\bar{j}-\bar{k}\right)+s\left(\bar{i}+\bar{j}-\bar{k}\right)+t\left(\bar{i}+2\bar{j}+5\bar{k}\right)$

19. ABCDEF be a regular hexagon in the xy plane and $\overline{AB}=4\bar{i}$. Then $\overline{CD}=$

1) $6\bar{i}+2\sqrt{3}\bar{j}$ 2) $2\left(-\bar{i}+\sqrt{3}\bar{j}\right)$

3) $2\left(\bar{i}+\sqrt{3}\bar{j}\right)$ 4) $2\left(\bar{i}-\sqrt{3}\bar{j}\right)$

20. OABCDE is a regular hexagon of side 2 units in the xy-plane O being the origin and OA taken along the x-axis. A point P is taken on a line parallel to z-axis through the centre of hexagon at a distance of 3 units from O. The vector $\overline{AP}$ is

1) $-\bar{i}+\sqrt{3}\bar{j}+\sqrt{5}\bar{k}$ 2) $\bar{i}+\sqrt{3}\bar{j}+\sqrt{5}\bar{k}$

3) $\bar{i}-\sqrt{3}\bar{j}+\sqrt{5}\bar{k}$ 4) $-\bar{i}-\sqrt{3}\bar{j}+\sqrt{5}\bar{k}$

21. 3 forces are applied to a vertex of a cube which are 1, 2 and 3 in magnitude and are directed along the diagonals of the faces of the cube meeting in that vertex. The magnitude of the resultant of these forces is

1) 3 2) 4 3) 5 4) 6

22. A man starts at the origin of the coordinate axes and walks a distance of 3 units in the North-East direction and then walks distance of 4 units in the North-West direction to reach the point P. Then $\overline{OP}$ equals

1) $\frac{1}{\sqrt{2}}\left(-\bar{i}+\bar{j}\right)$ 2) $\frac{\bar{i}+\bar{j}}{\sqrt{2}}$

3) $\frac{\bar{i}-\bar{j}}{\sqrt{2}}$ 4) $\frac{1}{\sqrt{2}}\left(-\bar{i}+7\bar{j}\right)$

KEY

01) 2	02) 1	03) 2	04) 1	05) 2	06) 2
07) 2	08) 4	09) 2	10) 2	11) 4	12) 3
13) 1	14) 1	15) 2	16) 3	17) 2	18) 3
19) 2	20) 1	21) 3	22) 4		

SOLUTIONS

1. $\overline{OP}=\frac{\overline{OC}+4\overline{OB}}{5}, \overline{OQ}=\frac{3\overline{OA}+2\overline{OC}}{5},$

$\overline{OR}=\frac{3\overline{OB}+7\overline{OA}}{10}, \overline{OS}=\frac{\overline{OB}+3\overline{OA}}{4}$

$\overline{AP}+\overline{BQ}+\overline{CR}=\overline{OP}-\overline{OA}+$

$\overline{OQ}-\overline{OB}+\overline{OR}-\overline{OC}$

$=\frac{3\overline{OA}+\overline{OB}-4\overline{OC}}{10}=\frac{4\overline{OS}-4\overline{OC}}{10}=\frac{2}{5}\overline{CS}$

2. Take $\overline{AB}=\overline{b}$ and $\overline{AD}=\overline{d}$ so that $\overline{AP}=\frac{3}{4}\overline{d}$ and $\overline{AC}=\overline{b}+\overline{d}$. Equation of the line $\overline{BP}$ is

$\overline{r}=(1-t)\overline{b}+t\left(\frac{3}{4}\overline{d}\right)$ and the equation of the line $\overline{AC}$ is $\overline{r}=s\left(\overline{b}+\overline{d}\right)$

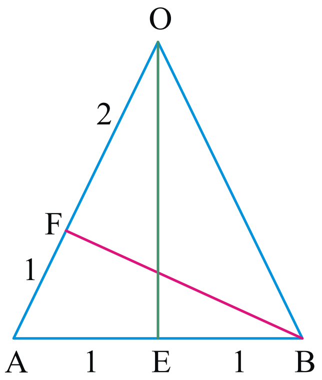
Equating the coefficients of $\overline{b},\overline{d}$

$$1-t=s=\frac{3t}{4} \Rightarrow 4=7t \;or\; t=\frac{4}{7} \;and\; s=\frac{3}{7}$$

$\Rightarrow \overline{AQ}=\frac{3}{7}\left(\overline{b}+\overline{d}\right)$ and hence

$AQ:QC=3:4$

3.



4. $\overline{b}=\dfrac{5\overline{i}+5\lambda\overline{j}}{\lambda+1}$ and $\left|\overline{b}\right|\le\sqrt{37}$

$$|6\lambda+1|\,|\lambda+6|\ge 0 \Rightarrow \lambda\in(-\infty,-6]\cup\left[-\frac{1}{6},\infty\right)$$

5. $\overline{XY}=\dfrac{(1-k)\overline{a}}{2} \Rightarrow \left|\overline{XY}\right|=4 \Rightarrow k=\dfrac{9}{17}$

6. For collinearity, $\dfrac{cosx}{x}=\dfrac{sinx}{sinx}$

$\Rightarrow \cos x=x$

now, draw the graphs of $y=\cos x$ and $y=x$, then observe

7. $\overline{PR}=\overline{PQ}+\overline{PS}$ (parallelogram law)

$$\frac{\overline{PR_1}}{\lambda_3}=\frac{\overline{PQ_1}}{\lambda_1}+\frac{\overline{PS_1}}{\lambda_2}$$

Since Q_1, R_1 and S_1 points are collinear,

$$\Rightarrow \frac{1}{\lambda_3}-\frac{1}{\lambda_1}-\frac{1}{\lambda_2}=0 \Rightarrow \frac{1}{\lambda_3}=\frac{1}{\lambda_1}+\frac{1}{\lambda_2}$$

8. $\overline{OA}=\overline{a}, \overline{OB}=\overline{b}, \overline{OC}=\overline{c}$

$\overline{AB}=\overline{b}-\overline{a}$, $\overline{OP}=\frac{1}{3}\overline{a}$

$\overline{OQ}=\frac{1}{2}\overline{b}$, $\overline{OR}=\frac{1}{3}\overline{c}$,

$$\overline{OS}=\frac{\overline{a}+\lambda\overline{b}}{1+\lambda}$$

P, Q, R, S are coplanar points

$\Rightarrow \overline{PQ}, \overline{PR}, \overline{RS}$ are coplanar vector $\Rightarrow \lambda=2$

9. $\overline{r}=\left(t^2+2\right)\overline{i}+(4t-5)\overline{j}+\left(2t^2-6t\right)\overline{k} \Rightarrow$ find $\dfrac{d\overline{r}}{dt}$

10. $BA^2=OA^2+OB^2=25+25=50$

Then $\left|\overline{OA}-\overline{OB}\right|=5\sqrt{2}$

11. $$\overline{OP}=\left(3\cos\frac{\pi}{4}\right)\overline{i}+\left(3\sin\frac{\pi}{4}\right)\overline{j}=\frac{3}{\sqrt{2}}\left(\overline{i}+\overline{j}\right)$$

$$\overline{OR}=\left(4\cos\frac{3\pi}{4}\right)\overline{i}+\left(4\sin\frac{3\pi}{4}\right)\overline{j}=\frac{4}{\sqrt{2}}\left(-\overline{i}+\overline{j}\right)$$

$=2\sqrt{2}\left(-\overline{i}+\overline{j}\right)$ Now

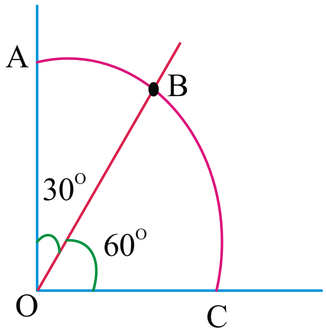
$$\overline{OP}+\overline{PQ}=\overline{OP}+\overline{OR}=\frac{1}{\sqrt{2}}\left(-\overline{i}+7\overline{j}\right)$$

$$\therefore \overline{OM}=\frac{\frac{3}{\sqrt{2}}\left(\overline{i}+\overline{j}\right)+\frac{1}{\sqrt{2}}\left(-\overline{i}+7\overline{j}\right)}{2}$$

$$=\frac{2\overline{i}+10\overline{j}}{2\sqrt{2}}=\frac{\overline{i}+5\overline{j}}{\sqrt{2}}$$

Now $PT:TR=1:2 \Rightarrow \overline{OT}=\dfrac{\sqrt{2}}{3}\left(\overline{i}+5\overline{j}\right)$

12.



Given that $\overline{OA}, \overline{OB}$ and $\overline{OC}$ are coplanar , then $\overline{OB}$ can be written as the linear combination of two non-colinear vectors $\overline{OA}$ and $\overline{OC}$. Let $\overline{OC}$ and $\overline{OA}$ be taken a long x and y axes respectively, and $|\overline{OA}| = |\overline{OB}| = |\overline{OB}| = r$ where r be the radius, then we can write

$$\overline{OB} = r(\cos 60°)\frac{\overline{OC}}{|\overline{OC}|} + r(\cos 30°)\frac{\overline{OA}}{|\overline{OA}|}$$

$$\bar{b} = \frac{1}{2}\overline{OC} + \frac{\sqrt{3}}{2}\bar{a} \Rightarrow \overline{OC} = 2\bar{b} - \sqrt{3}\bar{a}$$

13. Right angled triangle.

$$\overline{AO} = \frac{(3-3\sqrt{3})}{2}\bar{i} + \frac{(3\sqrt{3}+3)}{2}\bar{j}$$

$$= \frac{3}{2}\left\{(1-\sqrt{3})\bar{i} + (1+\sqrt{3})\bar{j}\right\}$$

Vector of magnitude $2\sqrt{2}$ units along

$$AO = (1-\sqrt{3})\bar{i} + (1+\sqrt{3})\bar{j}$$

14. $\bar{a}$ = A vector $\bar{V}$ bisecting the angle between $\bar{a}$ and $\bar{b}$ is $\bar{V} = \bar{a} + \bar{b} = \dfrac{2\bar{i} + 2\bar{j} + 4\bar{k}}{15}$

Required vector $= \sqrt{6}\bar{v}$

15. B divides $\overline{AC}$ in the ratio BA : BC

16. LetA , $A(1,1,1), B(1,2,3), C(2,-1,1)$

Then $AB = AC = \sqrt{5}$

Midpoint of BC is $D = \left(\frac{3}{2}, \frac{1}{2}, 2\right)$ and

$AD \perp BC$,

JEE MAINS QUESTIONS

1. If $\bar{\alpha} = (\lambda - 2)\bar{a} + \bar{b}$ and $\bar{\beta} = (4\lambda - 2)\bar{a} + 3\bar{b}$ be two given vectors where vectors $\bar{a}$ and $\bar{b}$ are non-collinear. the value of λ for which vectors $\bar{\alpha}$ and $\bar{\beta}$ are collinear is [2018]

1)4 2)3 3)-3 4)-4

2. Let $\sqrt{3}\bar{i} + \bar{j}, \bar{i} + \sqrt{3}\bar{j}$ and $\beta\bar{i} + (1-\beta)\bar{j}$ repectivily be the position vectors of the points A,B and C with respect to the origin 'O'. If the distance of 'c' from the bisector of the acute angle between OA and OB is $\frac{3}{\sqrt{2}}$ then the sum of all possible values of β is [2019]

1) 4 2) 1 3) 2 4) 3

3. The sum of the distinct realvalues of μ for which the vectors, $\mu\bar{i} + \bar{j} + \bar{k}$, $\bar{i} + \mu\bar{j} + \bar{k}$, $\hat{i} + \hat{j} + \mu\bar{k}$ are co-planar is [2020]

1)2 2) -1 3) 0 4) 1

4. Let $\bar{a}, \bar{b}, \bar{c} \in R$ be such that $a^2 + b^2 + c^2 = 1$. If

$$a\cos\theta = b\cos\left(\theta + \frac{2\pi}{3}\right) = c\cos\left(\theta + \frac{4\pi}{3}\right)$$

where $\theta = \frac{\pi}{9}$ then the angle between the vectors $a\bar{i} + b\bar{j} + c\bar{k}$ and $b\bar{i} + c\bar{j} + a\bar{k}$ is [2020]

1) $\frac{\pi}{2}$ 2) $\frac{2\pi}{3}$ 3) $\frac{\pi}{9}$ 4) 0

KEY

1. 4 2. 2 3. 2 4. 1

SOLUTIONS

1. $\overline{\alpha} = \lambda \overline{\beta}$

$(\lambda - 2)\overline{a} + \overline{b} = \mu\left((4\lambda - 2)\overline{a} + 3\overline{b}\right)$ comparing the coefficient of $\overline{a}$ and $\overline{b}$ on both sides

$\lambda - 2 = \mu(4\lambda - 2), -\boxed{1}$ 1=3 μ μ =1/3 sub in $-\boxed{1}$

$$\lambda - 2 = \frac{1}{3}(4\lambda - 2)$$

$$3\lambda - 6 = 4\lambda - 2 \quad -4 = \lambda$$

2. Given

$$\overline{OA} = \sqrt{3}\overline{i} + \overline{j}, \overline{OB} = \overline{i} + \sqrt{3}\overline{j}, \overline{OC} = \beta\overline{i} + (1 - \beta)\overline{j}$$

the bisector of OA and OB is x-y = 0

$$\Rightarrow \left|\frac{\beta - (1 - B)}{\sqrt{2}}\right| = \frac{3}{\sqrt{2}} \Rightarrow 2\beta - 1 = \pm 3$$

$\beta = 2 (or)\ \beta = -1$ sum = 1

3. Given vectors

$\mu\overline{i} + \overline{j} + \overline{k}, \overline{i} + \mu\overline{j} + \overline{k}, \hat{i} + \hat{j} + \mu\overline{k}$ are coplanar

$$\Rightarrow \begin{vmatrix} \mu & 1 & 1 \\ 1 & \mu & 1 \\ 1 & 1 & \mu \end{vmatrix} = 0$$

$$\mu(\mu^2 - 1) - 1(\mu - 1) + 1(1 - \mu) = 0$$

$$(\mu - 1)(\mu^2 + \mu - 1 - 1) = 0$$

$$(\mu - 1)(\mu - 1)(\mu + 2) = 0 \quad \mu = 1, \mu = -2$$

sum = -1

$$a\cos\theta = b\cos\left(\theta + \frac{2\pi}{3}\right) = c\cos\left(\theta + \frac{4\pi}{3}\right) = k$$

$$a = \frac{k}{\cos\theta}, b = \frac{k}{\cos\left(\theta + \frac{2\pi}{3}\right)}, c = \frac{k}{\cos\left(\theta + \frac{4\pi}{3}\right)}$$

$$ab + bc + ca = k^2 \frac{\left[\cos\left(\theta + \frac{4\pi}{3}\right) + \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\theta\right]}{\cos\theta\cos\left(\theta + \frac{2\pi}{3}\right)\cos\left(\theta + \frac{4\pi}{3}\right)}$$

$$= k^2 \frac{\left(\cos\theta + 2\cos(\pi + \theta)\cos\frac{\pi}{3}\right)}{\cos\theta.\cos\left(\theta + \frac{2\pi}{3}\right)\cos\left(\theta + \frac{4\pi}{3}\right)}$$

$$= k^2 \frac{\left(\cos\theta - 2.\frac{1}{2}\cos\theta\right)}{\cos\theta - \cos\left(\theta + \frac{2\pi}{3}\right)\cos\left(\theta + \frac{4\pi}{3}\right)} = 0$$

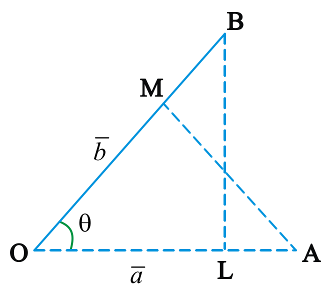
$$\cos\varphi = \frac{\left(a\overline{i} + b\overline{j} + c\overline{k}\right).\left(b\overline{i} + c\overline{j} + ak\right)}{\sqrt{a^2 + b^2 + c^2}\sqrt{b^2 + c^2 + a^2}} = \frac{ab + bc + ca}{1} = 0$$

$$\phi = \frac{\pi}{2}$$

DOT PRODUCT OF VECTORS

SYNOPSIS

- Let $\bar{a}$ and $\bar{b}$ be two nonzero vectors and θ be the angle between them, then the scalar product or dot product of $\bar{a}$ and $\bar{b}$ denoted by $\bar{a}\cdot\bar{b}$ is defined as $\bar{a}.\bar{b} = |\bar{a}|\,|\bar{b}|\cos\theta$
- If $(\bar{a},\bar{b}) = \theta$, then $\cos\theta = \dfrac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|}$
- Let $\bar{i}$, $\bar{j}$, $\bar{k}$ be unit orthogonal vectors i.e $|\bar{i}| = |\bar{j}| = |\bar{k}| = 1$ and $(\bar{i},\bar{j}) = (\bar{j},\bar{k}) = (\bar{k},\bar{i}) = \dfrac{\pi}{2}$,
  then $\bar{i}\cdot\bar{i} = \bar{j}\cdot\bar{j} = \bar{k}\cdot\bar{k} = 1$
  $\bar{i}\cdot\bar{j} = \bar{j}\cdot\bar{k} = \bar{k}\cdot\bar{i} = 0$
  $\bar{j}\cdot\bar{i} = \bar{k}\cdot\bar{j} = \bar{i}\cdot\bar{k} = 0$
- If $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}, \bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$ Then
  (i) $\bar{a}.\bar{b} = a_1b_1 + a_2b_2 + a_3b_3$
  (ii) $\bar{a}.\bar{a} = a_1^2 + a_2^2 + a_3^2$
  (iii) $\bar{a}.\bar{a} = |\bar{a}|^2$
- Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$ and let $(\bar{a},\bar{b}) = \theta$ then
  $$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$
- Let $(\bar{a},\bar{b}) = \theta$, then
  (i) $\theta < 90° \Leftrightarrow \bar{a}.\bar{b} > 0$ (ii) $\theta > 90^0 \Leftrightarrow \bar{a}.\bar{b} < 0$
  (iii) $\theta = 90^0 \Leftrightarrow \bar{a}.\bar{b} = 0$ ($\bar{a}$ is perpendicular to $\bar{b}$)
- If $\bar{a}$ and $\bar{b}$ are parallel and are in the same direction ($\bar{a},\bar{b}$ are like vectors) then $\bar{a}.\bar{b} = |\bar{a}||\bar{b}|$
- If $\bar{a}$ and $\bar{b}$ are parallel and are in the opposite direction ($\bar{a},\bar{b}$ are unlike vectors) then $\bar{a}.\bar{b} = -|\bar{a}||\bar{b}|$
- Let $\bar{a},\bar{b}$ are two unit vectors and $\bar{a}.\bar{b} = \cos\theta$
- If $\bar{a},\bar{b}$ are any two vectors, then $\bar{a}.\bar{b} = 0$ either $\bar{a} = \bar{0}$ (or) $\bar{b} = \bar{0}$ (or) $\bar{a} \perp \bar{b}$. However if $\bar{a},\bar{b}$ are two non-zero perpendicular vectors, then $\bar{a}.\bar{b} = 0$
- **Geometrical interpretation of scalar product :**
  Let $\bar{a}$ and $\bar{b}$ be two vectors represented by $\overline{OA}$ and $\overline{OB}$ respectively. Let θ be the angle between $\overline{OA}$ and $\overline{OB}$. Draw $\overline{BL} \perp \overline{OA}$ and $\overline{AM} \perp \overline{OB}$. From triangle OBL and OAM, We have $OL = OB\cos\theta$ and $OM = OA\cos\theta$. Here OL and OM are known as projections of $\bar{b}$ on $\bar{a}$ and $\bar{a}$ on $\bar{b}$.
  $\bar{a}.\bar{b} = |\bar{a}||\bar{b}|\cos\theta = |\bar{a}|(OB\cos\theta) = |\bar{a}|(OL)$
  = (magnitude of $\bar{a}$) (projection of $\bar{b}$ on $\bar{a}$)
  again $\bar{a}.\bar{b} = |\bar{a}||\bar{b}|\cos\theta = |\bar{b}|(|\bar{a}|\cos\theta)$
  $= |\bar{b}|(OA\cos\theta) = |\bar{b}|(OM)$
  = (magnitude of $\bar{b}$) (projection of $\bar{a}$ *on* $\bar{b}$)



- Let $\bar{a}$ and $\bar{b}$ be two nonzero vectors. Then Component vector of $\bar{b}$ on $\bar{a}$ (or) orthogonal projection of $\bar{b}$ on $\bar{a}$ is $\dfrac{(\bar{b}.\bar{a})\bar{a}}{|\bar{a}|^2}$

- Component vector of $\overline{a}$ on $\overline{b}$ (or) orthogonal projection of $\overline{a}$ on $\overline{b}$ is $\frac{(\overline{a}.\overline{b})\overline{b}}{|\overline{b}|^2}$
- The orthogonal projection of $\overline{b}$ in the direction perpendicular to that of $\overline{a}$ is $\overline{b}-\frac{(\overline{b}.\overline{a})\overline{a}}{|\overline{a}|^2}$
- The length of the orthogonal projection of $\overline{b}$ on $\overline{a}$ is $\left|\frac{(\overline{a}.\overline{b})}{|\overline{a}|}\right|$
- The length of the orthogonal projection of $\overline{a}$ on $\overline{b}$ is $\left|\frac{(\overline{a}.\overline{b})}{|\overline{b}|}\right|$
- The scalar product is commutative i.e., $\overline{a}\cdot\overline{b}=\overline{b}\cdot\overline{a}$
- The scalar product is distributive over vector addition i.e., $\overline{a}\cdot(\overline{b}+\overline{c})=\overline{a}\cdot\overline{b}+\overline{a}\cdot\overline{c}$, $(\overline{b}+\overline{c})\cdot\overline{a}$ $=\overline{b}\cdot\overline{a}+\overline{c}\cdot\overline{a}$
- $(l\overline{a}).\overline{b}=\overline{a}.(l\overline{b})=l(\overline{a}.\overline{b})$ where l is a scalar
- $\overline{a}.\overline{a}\ge 0; |\overline{a}.\overline{b}|\le|\overline{a}||\overline{b}|; |\overline{a}-\overline{b}|\ge||\overline{a}|-|\overline{b}||$

  $|\overline{a}+\overline{b}|\le||\overline{a}|+|\overline{b}||$
- Cauchy schwartz in equality : Let a_1,a_2,a_3 and b_1,b_2,b_3 be real numbers. Then $(a_1b_1+a_2b_2+a_3b_3)^2\le$ $(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$ and equality holds If $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$
- $|\overline{a}+\overline{b}|^2=|\overline{a}|^2+|\overline{b}|^2+2\overline{a}.\overline{b}$
- $|\overline{a}-\overline{b}|^2=|\overline{a}|^2+|\overline{b}|^2-2\overline{a}.\overline{b}$
- $|\overline{a}+\overline{b}+\overline{c}|^2=|\overline{a}|^2+|\overline{b}|^2+|\overline{c}|^2+2(\overline{a}.\overline{b}+\overline{b}.\overline{c}+\overline{c}.\overline{a})$
- Let l_1,m_1,n_1 be the direction cosines of $\overline{a}$ and let l_2,m_2,n_2 be the direction cosines of $\overline{b}$ and let $(\overline{a},\overline{b})=\theta$ then $Cos\theta=l_1l_2+m_1m_2+n_1n_2$
- The vector equation to the plane which is at a distance of p units from the origin and $\hat{n}$ is a unit vector perpendicular to the plane is $\overline{r}.\hat{n}=p$
- If the origin lies on the plane then its equation is $\overline{r}.\overline{n}=0$
- The vector equation of a plane passing through the point $A(\overline{a})$ and perpendicular to the vector $\overline{n}$ is $(\overline{r}-\overline{a}).\overline{n}=\overline{0}$
- In a parallelogram, if its diagonals are equal then it is a rectangle.
- In a parallelogram, the sum of the squares of the diagonals is equal to the sum of the squares of the sides.
- If $\overline{F}$ be the force and $\overline{s}$ be the displacement inclined at an angle θ with the direction of the force, then work done $\overline{F}.\overline{S}$
- If a constant force $\overline{F}$ acting on a particle displaces it from A to B, then work done, $W=\overline{F}.\overline{AB}$
- If $\overline{F}$ is the resultant of the forces $\overline{F_1},\overline{F_2}......\overline{F_n}$ then work done in displacing the particle from A to B is $W=(\overline{F_1}+\overline{F_2}+....+\overline{F_n}).\overline{AB}$
- If a number of forces are acting on a particle, the sum of the work done by the seperate forces is equal to the work done by the resultant force.
- A line makes angles $\alpha,\beta,\gamma,\delta$, with the four diagonals of a cube then $\cos^2\alpha+\cos^2\beta+$ $\cos^2\gamma+\cos^2\delta=4/3$
- If $\overline{r}$ is any vector then $\overline{r}=(\overline{r}.\overline{i})\overline{i}+(\overline{r}.\overline{j})\overline{j}+(\overline{r}.\overline{k})\overline{k}$.
- If $\overline{a},\overline{b}$ are two vectors then

  i) $\overline{a}.\overline{a}\ge 0$ ii) $|\overline{a}.\overline{b}|\le|\overline{a}||\overline{b}|$

  iii) $|\overline{a}+\overline{b}|\le|\overline{a}|+|\overline{b}|$ iv) $|\overline{a}-\overline{b}|\ge||\overline{a}|-|\overline{b}||$
- The cartesian equation of the plane passing through the point $A(x_1,y_1,z_1)$ and perpendicular to the vector $\overline{m}=a\overline{i}+b\overline{j}+c\overline{k}$ is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$.

✈ The equation of the plane passing through the point $A(x_1, y_1, z_1)$ and whose normal has d.r.s a,b,c is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$.

✈ Angle between any two diagonals of a cube is $\cos^{-1}(1/3)$.

✈ Angle between a diagonal of a cube and a diagonal of a face of the cube which are passes through the same corner is $\cos^{-1}\sqrt{2/3}$.

✈ Angle between a diagonal of a cube and edge of a cube is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Angle between a line and a plane :

✈i) The angle between a line and a plane is the complement of the angle between the line and normal to the plane. If θ is the anlge between a line $\overline{r}=\overline{a}+t\overline{b}$ and a plane $\overline{r}.\overline{m}=d$ then $\cos\left(90^0-\theta\right)=\sin\theta=\frac{\overline{b}.\overline{m}}{|\overline{b}||\overline{m}|}$.

✈ If θ is the angle between the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ and the plane $ax+by+cz+d=0$

then i) $\sin\theta=\frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$

ii) If the line is perpendicular to the plane then $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$.

iii) If the line is parallel to the plane then $al+bm+cn=0$

✈ The perpendicular distance of the plane $\overline{r}.\overline{n}=\overline{a}.\overline{n}$. from the origin is $\frac{|\overline{a}.\overline{n}|}{|\overline{n}|}$

✈ Angle between the planes $\overline{r}.\overline{n}_1=p_1, \overline{r}.\overline{n}_2=p_2,\ then\ \ Cos\theta=\frac{\overline{n}_1.\overline{n}_2}{|\overline{n}_1||\overline{n}_2|}$

Eg : 1

If $\overline{a}=\overline{i}-\lambda\overline{j}+2\overline{k}$ and $\overline{b}=8\overline{i}+6\overline{j}-\overline{k}$ are perpendicular then λ is

Sol: Let $\overline{a}=\overline{i}-\lambda\overline{j}+2\overline{k}$ and $\overline{b}=8\overline{i}+6\overline{j}-\overline{k}$ Since, $\overline{a},\overline{b}$ are at right angles, $\overline{a}.\overline{b}=0$

$\Rightarrow(\overline{i}-\lambda\overline{j}+2\overline{k}).(8\overline{i}+6\overline{j}-\overline{k})=0$

$\Rightarrow 8-6\lambda-2=0\Rightarrow\lambda=1$

Eg : 2

Orthogonal projection of $\overline{b}=2\overline{i}-3\overline{j}+6\overline{k}\ on\ \overline{a}=\overline{i}+2\overline{j}+2\overline{k}$ is

Sol: Orthogonal projection

$=\frac{[(2\overline{i}-3\overline{j}+6\overline{k}).(\overline{i}+2\overline{j}+2\overline{k})](\overline{i}+2\overline{j}+2\overline{k})}{|\overline{i}+2\overline{j}+2\overline{k}|^2}$

$=\frac{(2-6+12)}{9}(\overline{i}+2\overline{j}+2\overline{k})=\frac{8}{9}(\overline{i}+2\overline{j}+2\overline{k})$

Eg : 3

If $\overline{a}=2\overline{i}+\overline{j}+2\overline{k}, \overline{b}=5\overline{i}-3\overline{j}+\overline{k}$ then orthogonal projection of $\overline{a}$ on $\overline{b}$ is

Sol: Orthogonal projection of $\overline{a}$ on $\overline{b}$ is $\frac{(\overline{a}.\overline{b})\overline{b}}{|\overline{b}|^2}=$

$=\frac{(10-3+2)(5\overline{i}-3\overline{j}+\overline{k})}{25+9+1}=\frac{9(5\overline{i}-3\overline{j}+\overline{k})}{35}$

Eg : 4

The orthogonal projection of $\overline{b}=3\overline{i}+2\overline{j}-5\overline{k}$ on a vector perpendicular to $\overline{a}=2\overline{i}-\overline{j}+2\overline{k}$ is

Sol: Orthogonal projection of $\overline{b}\ on\ \overline{a}$

$=3\overline{i}+2\overline{j}-5\overline{k}-$

$\frac{(3\overline{i}+2\overline{j}-5\overline{k}).(2\overline{i}-\overline{j}+2\overline{k})}{4+1+4}(2\overline{i}-\overline{j}+2\overline{k})$

$=3\overline{i}+2\overline{j}-5\overline{k}-\frac{6-2-10}{9}(2\overline{i}-\overline{j}+2\overline{k})$

$=(3\overline{i}+2\overline{j}-5\overline{k})+\frac{2}{3}(2\overline{i}-\overline{j}+2\overline{k})$

$=\frac{13\overline{i}+4\overline{j}-11\overline{k}}{3}$

Eg : 5

The length of orthogonal projection of $\bar{a} = 2\bar{i} - 3\bar{j} + \bar{k}$ on $\bar{b} = 4\bar{i} - 4\bar{j} + 7\bar{k}$ is

Sol: The length of the orthogonal projection of $\bar{a}$

on $\bar{b}$ is $\left|\frac{(\bar{a}.\bar{b})}{|\bar{b}|}\right| = \frac{|8+12+7|}{\sqrt{16+16+49}} = \frac{27}{9} = 3$

Eg : 6

The vector equation of the plane passing through the point $(3,-2,1)$ and perpendicular to the vector $(4,7,-4)$ is

Sol. $\left\{\bar{r} - \left(3\bar{i} - 2\bar{j} + \bar{k}\right)\right\}.\left(4\bar{i} + 7\bar{j} - 4\bar{k}\right) = 0$

$\Rightarrow \bar{r}.\left(4\bar{i} + 7\bar{j} - 4\bar{k}\right) =$

$\left(3\bar{i} - 2\bar{j} + \bar{k}\right).\left(4\bar{i} + 7\bar{j} - 4\bar{k}\right) = 12 - 14 - 4$

$\Rightarrow \bar{r}.\left(4\bar{i} + 7\bar{j} - 4\bar{k}\right) = -6$

Eg : 7

The angle between the line

$\bar{r} = (-i + 3j + 3k) + t(2i + 3j + 6k)$

and the plane $\bar{r}.(-\bar{i} + \bar{j} + \bar{k}) = 5$ is

Sol: $\sin\theta = \frac{(2\bar{i} + 3\bar{j} + 6\bar{k}).(-\bar{i} + \bar{j} + \bar{k})}{\left|2\bar{i} + 3\bar{j} + 6\bar{k}\right|\left|-\bar{i} + \bar{j} + \bar{k}\right|} =$

$\frac{-2+3+6}{\sqrt{4+9+36}\sqrt{1+1+1}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Eg : 8

Angle between the planes

$\bar{r}.(2\bar{i} - \bar{j} + \bar{k}) = 3$ **and** $\bar{r}.(\bar{i} + \bar{j} + 2\bar{k}) = 4$ **is**

Sol: Let $\bar{a} = 2\bar{i} - \bar{j} + \bar{k}$

and $\bar{b} = \bar{i} + \bar{j} + 2\bar{k}$

If θ is the angle between the planes then

$\cos\theta = \cos(\bar{a},\bar{b}) = \frac{\bar{a}.\bar{b}}{|\bar{a}|\,|\bar{b}|} =$

$\frac{2-1+2}{\sqrt{4+1+1}\sqrt{1+1+4}} = \frac{3}{6} = \frac{1}{2} \Rightarrow \theta = 60°$

EXERCISE - I

1. **If** $\bar{A}.(\bar{B}+\bar{C}) = \bar{B}.(\bar{C}+\bar{A}) = \bar{C}.(\bar{A}+\bar{B}) = 0$ **and** $|\bar{A}| = 3, |\bar{B}| = 4\ and\ |\bar{C}| = 5$ **then** $|\bar{A} + \bar{B} + \bar{C}| =$

1) 5 2) $5\sqrt{2}$ 3) $5/\sqrt{2}$ 4) $\sqrt{2}$

2. $\bar{a},\bar{b},\bar{c}$ **are three vectors, such that** $\bar{a} + \bar{b} + \bar{c} = \bar{0}, |\bar{a}| = 1, |\bar{b}| = 2, |\bar{c}| = 3$**,then** $\left(\bar{a}.\bar{b} + \bar{b}.\bar{c} + \bar{c}.\bar{a}\right)^2$ **is equal to**

1) – 7 2) 49 3) 7 4) 1

3. **If** $|\mathbf{a}| = 3, |\mathbf{b}| = 4\ \ and\ |\mathbf{a} - \mathbf{b}| = 5$ **then** $|\mathbf{a} + \mathbf{b}| =$ **(EAM 1994)**

1) 6 2) 5 3) 4 4) 3

4. **If** $|\bar{a}| = 1, |\bar{b}| = 2, |\bar{a} - \bar{b}|^2 + |\bar{a} + 2\bar{b}|^2 = 20$**,** **then** $\left(\bar{a},\bar{b}\right) =$

1) $\frac{\pi}{3}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{6}$ 4) $\frac{2\pi}{3}$

5. **If θ is acute angle and the vector $(\sin\theta)\mathrm{i} + (\cos\theta)\mathrm{j}$ is perpendicular to the vector $\bar{\mathrm{i}} - \sqrt{3}\bar{\mathrm{j}}$ then θ= (EAM-2000)**

1) $\frac{\pi}{6}$ 2) $\frac{\pi}{5}$ 3) $\frac{\pi}{4}$ 4) $\frac{\pi}{3}$

6. **Let $|\bar{a}| = 3$ and $|\bar{b}| = 4$. The value of 'μ' for which the vectors $\bar{a} + \mu\bar{b}$ and $\bar{a} - \mu\bar{b}$ are perpendicular is...**

1) $\frac{3}{4}$ 2) $\frac{2}{3}$ 3) $\pm\frac{3}{4}$ 4) $-\frac{2}{3}$

7. **If $\bar{a}+\bar{b}$ is perpendicular to $\bar{b}$ and $\bar{a}+2\bar{b}$ is perpendicular to $\bar{a}$ then.**

1) $|\bar{a}| = |\bar{b}|$ 2) $|\bar{a}| = \sqrt{2}\,|\bar{b}|$

3) $|\bar{b}| = \sqrt{2}\,|\bar{a}|$ 4) $|\bar{a}| = |\bar{b}|\sqrt{3}$

8. $(\bar{a} - \bar{d}).(\bar{b} - \bar{c}) + (\bar{b} - \bar{d}).(\bar{c} - \bar{a}) + (\bar{c} - \bar{d}).(\bar{a} - \bar{b}) =$

1) $\bar{0}$ (null vector) 2) 0

3) $\bar{a}.\bar{b}+\bar{c}.\bar{d}$ 4) $\bar{a}.\bar{c}+\bar{b}.\bar{d}$

9. **If two out of the 3 vectors $\bar{a},\bar{b},\bar{c}$ are unit vectors , $\bar{a}+\bar{b}+\bar{c}=\bar{0}$ and $2(\bar{a}.\bar{b}+\bar{b}.\bar{c}+\bar{c}.\bar{a})+3=0$ then the length of the third vector is**
1) 3   2) 2   3)1   4) 0

10. **If** $\bar{r} = (x+y+2)\bar{i}+(2x-y+3)\bar{j}+(x+2y+7)\bar{k}$ **where** $\bar{r}.\bar{i}$ **= 3,** $\bar{r}.\bar{j}$ **= 5 then** $\bar{r}.\bar{k}$ **=**
1) 4   2) 6   3) 9   4) 8

11. **If** $\bar{a}.\bar{i} = \bar{a}.(\bar{i}+\bar{j}) = \bar{a}.(\bar{i}+\bar{j}+\bar{k})$ then $\bar{a}$ = **(EAM-2002)**
1) $\bar{i}$ 2) $\bar{j}$ 3) $\bar{k}$ 4) $\bar{i}+\bar{j}+\bar{k}$

12. **The vector $\bar{b}$ which is collinear with the vector $\bar{a}=(2,-1,2)$ and satisfies the relation $\bar{a}.\bar{b}=18$ is**
1) $(4,-2,4)$ 2) $(2,1,-1)$
3) $(1,-1,1)$ 4) $(1,1,0)$

13. **If $|\bar{a}|=|\bar{b}|=|\bar{a}-\bar{b}|$, then the angle between $\bar{a}$ and $\bar{b}$ is**
1) $\frac{\pi}{4}$ 2) $\frac{\pi}{2}$ 3) $\frac{\pi}{3}$ 4) $\frac{2\pi}{3}$

14. **If $\bar{a}$ and $\bar{b}$ are two unit vectors and θ be the angle between them, then** $\sin\left(\frac{\theta}{2}\right)$ **=**
1) $|\bar{a}-\bar{b}|$ 2) $|\bar{a}+\bar{b}|$ 3) $\frac{1}{2}|\bar{a}-\bar{b}|$ 4) $\frac{1}{2}|\bar{a}+\bar{b}|$

15. **If a and b are unit vectors and θ is the angle between them ,then a-b will be a unit vector if θ =** **(EAM-1997)**
1) $\frac{\pi}{4}$ 2) $\frac{\pi}{3}$ 3) $\frac{\pi}{6}$ 4) $\frac{\pi}{2}$

16. **If** $\bar{a}=p\bar{i}+3\bar{j}-7\bar{k}, \bar{b}=p\bar{i}-p\bar{j}+4\bar{k}$ **and if the angle between a and b is acute , then the values of p lies in**
1) P<−4 or p>7   2) (-7,4)
3) $P\le -4 or p\ge 7$ 4) [-7,4]

17. **If** $\bar{a}+\bar{b}+\bar{c}$ **= 0,** $|\bar{a}|=3, |\bar{b}|=5$ $and |\bar{c}|=7$ **then the angle between $\bar{a}$ and $\bar{b}$ is**
1) 30°   2) 45°   3) 60°   4) 90°

18. **If $\bar{a},\bar{b},\bar{c}$ are three unit vectors such that $|\bar{a}+\bar{b}+\bar{c}|=1$ and $\bar{a}\perp\bar{b}$. If $\bar{c}$ makes angles α,β with $\bar{a},\bar{b}$ respectively, then $\cos\alpha+\cos\beta$ is equal to**
1) $\frac{3}{2}$ 2) 1   3) -1 4) $\frac{-3}{2}$

19. **Angle between the vectors $\bar{a}=-\bar{i}-2\bar{j}+\bar{k}$** $and\ \bar{b}=x\bar{i}+\bar{j}+(x+1)\bar{k}$
1) Obtuse angle   2) Acute angle
3) Right angle   4) Depends on x

20. **If $\bar{\alpha}$ and $\bar{\beta}$ are non-zero and different vectors such that $|\bar{\alpha}+\bar{\beta}|=|\bar{\beta}-\bar{\alpha}|$ then the angle between $-\bar{\alpha}$ and $\bar{\beta}$ is**
1) $\frac{\pi}{3}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{2}$ 4) $\frac{\pi}{6}$

21. **If the angle between the vectors $(x,3,-7)$ and $(x,-x,4)$ is obtuse, the domain of 'x' is**
1) $(-4,7)$ 2) $[-4,7]$
3) $R-[-4,7]$ 4) $R-(-4,7)$

22. **If the position vectors of A,B and C are respectively $2\bar{i}-\bar{j}+\bar{k}, \bar{i}-3\bar{j}-5\bar{k}$ and $3\bar{i}-4\bar{j}-4\bar{k}$ then** $\cos^2 A=$
1) 0   2) $\frac{6}{41}$ 3) $\frac{35}{41}$ 4) 1

23. **If $\bar{a}=\bar{i}+2\bar{j}-3\bar{k}$ and $\bar{b}=3\bar{i}-\bar{j}+2\bar{k}$ then the angle between the vectors $(\bar{a}+\bar{b})$ and $(\bar{a}-\bar{b})$ is**
1) $\frac{\pi}{4}$ 2) $\frac{\pi}{3}$ 3) $\frac{\pi}{2}$ 4) $\frac{\pi}{6}$

24. **If $\overline{e}_1$ and $\overline{e}_2$ are unit vectors and the vectors $\overline{e}_1 + 2\overline{e}_2$, $5\overline{e}_1 - 4\overline{e}_2$ are at right angles, than the angle between $\overline{e}_1$ and $\overline{e}_2$ is**

1) 30^0 2) 60^0 3) 45^0 4) 75^0

25. **If $\overline{AB}$ =(3,-2,2), $\overline{BC}$ =(-1,0-2) are the adjacent sides of a parallelogram, then the obtuse angle between its diagonals is**

1) $\frac{\pi}{4}$ 2) $\pi/2$ 3) $\pi/3$ 4) $\frac{3\pi}{4}$

26. **If $\overline{a}$ and $\overline{b}$ are unit vectors inclined to x-axis at angle 30^0 and 120^0, then $|\overline{a}+\overline{b}|$ equals**

1) $\sqrt{2/3}$ 2) $\sqrt{2}$ 3) $\sqrt{3}$ 4) 2

27. **If $\overline{AB} = -\overline{i}-2\overline{j}-6\overline{k}$, $\overline{BC} = 2\overline{i}-\overline{j}+\overline{k}$, $\overline{AC} = \overline{i}-3\overline{j}-5\overline{k}$ .Then $\angle B =$**

1) $Cos^{-1}\left(\sqrt{\frac{40}{41}}\right)$ 2) $Cos^{-1}\left(\sqrt{\frac{6}{41}}\right)$

3) $Cos^{-1}\left(\frac{6}{41}\right)$ 4) $Cos^{-1}\left(\frac{62}{63}\right)$

28. **The value of a, for which the points A,B, C with position vectors $2\overline{i} - \overline{j} + \overline{k}, \overline{i} - 3\overline{j} - 5\overline{k}$ and $a\overline{i} - 3\overline{j} + \overline{k}$ respectively are the vertices of a right angled triangle with $\angle C = \frac{\pi}{2}$ are (AIE-2006)**

1) -2 or 1 2) 2 or -1

3) 2 or 1 4) -2 or -1

29. **If $\overline{a}$ =(-1,1,2); $\overline{b}$ =(2,1,-1); $\overline{c}$ =(-2,1,3) then the angle between $2\overline{a}-\overline{c}$ and $\overline{a}+\overline{b}$ is**

1) $\frac{\pi}{4}$ 2) $\frac{\pi}{3}$ 3) $\frac{\pi}{6}$ 4) $\frac{3\pi}{2}$

GEOMETRICAL INTERPRETATION

30. **Given two vectors $\overline{a} = 2\overline{i} - 3\overline{j} + 6\overline{k}$, $\overline{b} = -2\overline{i} + 2\overline{j} - \overline{k}$ and $\lambda = \frac{\text{the projection of } \overline{a} \text{ on } \overline{b}}{\text{the projection of } \overline{b} \text{ on } \overline{a}}$, then the value of λ is**

1) $\frac{3}{7}$ 2) 7 3) 3 4) $\frac{7}{3}$

31. **If the vector OP in XY plane whose magnitude is $\sqrt{3}$ makes an angle 60^0 with Y-axis, the length of the component of the vector in direction of X-axis is**

1) 1 2) $\sqrt{3}$ 3) 1/2 4) 3/2

32. **Let P = (1,0,-1) Q = (-1,2,0) R = (2,0,-3) S=(3,-2,-1), then the length of the component of RS on PQ is**

1) 1/3 2) 2/3 3) 4/3 4) 5/3

33. **If $\overline{a} = 2\overline{i} + \overline{j} + \overline{k}, \overline{b} = \overline{i} + 5\overline{j}, \overline{c} = 4\overline{i} + 4\overline{j} - 2\overline{k}$ then the length of the projection of $\left(3\overline{a} - 2\overline{b}\right)$ in the direction of $\overline{c}$**

1) 3 2) –3 3) 33 4) –33

34. **The angle between $\overline{a}$ and $\overline{b}$ is $\frac{5\pi}{6}$ and the projection of $\overline{a}$ on $\overline{b}$ is $\frac{-6}{\sqrt{3}}$ then $|\overline{a}| =$**

1) 6 2) $\frac{\sqrt{3}}{2}$ 3) 12 4) 4

35. **The component of $a\overline{i} + b\overline{j} + c\overline{k}$ on the Y–axis is**

1) a 2) b 3) c 4) 0

36. **The components of a vector on the co-ordinate axes are 2,1,2. Then the length of the vector is**

(1) 3 (2) 1 (3) 2 (4) 5

37. **If $\overline{a} = 2\overline{i} + 3\overline{j}$ and $\overline{b} = 3\overline{j} + 4\overline{k}$. Then the vector form of the component of $\overline{a}$ along $\overline{b}$ is**

1) $\frac{9\left(3\overline{j}+4\overline{k}\right)}{10\sqrt{3}}$ 2) $\frac{9\left(3\overline{j}+4\overline{k}\right)}{25}$

3) $\frac{9\left(3\bar{j}+4\bar{k}\right)}{\sqrt{13}}$ 4) $3\bar{j}+4\bar{k}$

38. The orthogonal projection of $\bar{a}$ =2$\bar{i}$ +3$\bar{j}$ +3$\bar{k}$ on $\bar{b}$ = $\bar{i}$ -2$\bar{j}$+$\bar{k}$ (where $\bar{i}$. $\bar{j}$. $\bar{k}$ are unit vectors along three mutually perpendicular directions) is (EAM-1996)

1) $\frac{-\bar{i}+2\bar{j}-\bar{k}}{6}$ 2) $\frac{-\bar{i}+2\bar{j}-\bar{k}}{\sqrt{6}}$

3) $\bar{i}-2\bar{j}+\bar{k}$ 4) $-\bar{i}+2\bar{j}-\bar{k}$

39. If $\bar{a}$ = 4$\bar{i}$+6$\bar{j}$ and $\bar{b}$ = 3$\bar{j}$+4$\bar{k}$, then the vector form of the component of a along b is

1) $\frac{18(3\bar{i}+4\bar{k})}{10\sqrt{3}}$ 2) $\frac{18(3\bar{j}+4\bar{k})}{25}$

3) $\frac{18(3\bar{i}+4\bar{k})}{\sqrt{13}}$ 4) $\frac{(3\bar{j}+4\bar{k})}{25}$

40. If $\bar{a}$ =$\bar{i}$+2$\bar{j}$+$\bar{k}$,$\bar{b}$ =2$\bar{j}$+$\bar{k}$-$\bar{i}$, then component of $\bar{a}$ perpendicular to $\bar{b}$ is

1) $\frac{5}{2}\bar{i}+\frac{2}{3}\bar{j}+\frac{1}{2}\bar{k}$ 2) $\frac{5}{3}\bar{i}+2\bar{j}+\frac{1}{3}\bar{k}$

3) $\frac{5\bar{i}+2\bar{j}+\bar{k}}{3}$ 4) $\bar{i}+\bar{j}+\bar{k}$

41. The length of the orthogonal projection of a vector $\bar{i}-2\bar{j}+\bar{k}$ on vector $4\bar{i}-4\bar{j}+7\bar{k}$ is

1) $\frac{19}{6}$ 2) $\frac{19}{7}$

3) $\frac{19\left(4\bar{i}-4\bar{j}+7\bar{k}\right)}{9}$ 4) $\frac{19}{9}$

42. The cartesian equation of the plane perpendicular to vector $3\bar{i}-2\bar{j}-2\bar{k}$ and passing through the point $2\bar{i}+3\bar{j}-\bar{k}$ is

1) 3x+2y+2z=2 2) 3x-2y+2z=2
3) 3x+2y-2z=2 4) 3x-2y-2z=2

43. The perpendicular distance from origin to the plane 3x - 2y - 2z = 2 is

1) $1/\sqrt{17}$ 2) $2/\sqrt{17}$ 3) $3/\sqrt{17}$ 4) $4/\sqrt{17}$

44. The vector equation of the plane which is perpendicular to 2i-3j+k and at a distance of 5 units from the origin is (EAM-1991)

1) $\bar{r}.(2\bar{i}-3\bar{j}+\bar{k})=5\sqrt{14}$ 2) $\bar{r}.(2\bar{i}-3\bar{j}+\bar{k})=5$

3) $\bar{r}.\frac{(2\bar{i}-3\bar{j}+\bar{k})}{\sqrt{14}}$ 4) $\frac{\bar{r}.(2\bar{i}+3\bar{j}+\bar{k})}{\sqrt{14}}$

45. The cartesian equation of the plane passing through the point $(3,-2,1)$ and perpendicular to vector $4\bar{i}+7\bar{j}-4\bar{k}$ is

1) $4x+7y+4z-6=0$ 2) $4x+7y-4z+6=0$
3) $4x-7y-4z-6=0$ 4) $4x-7y+4z+6=0$

46. The angle between planes
$\bar{r}.(2\bar{i}-3\bar{j}+4\bar{k})+11=0$ **and** $\bar{r}.(3\bar{i}-2\bar{j}-3\bar{k})+27=0$ **is**

1) $\pi/6$ 2) $\pi/3$ 3) $\pi/4$ 4) $\pi/2$

47. A particle is acted upon by constant forces $4\bar{i}+\bar{j}-3\bar{k}$ and $3\bar{i}+\bar{j}-\bar{k}$ which displace it from a point $\bar{i}+2\bar{j}+3\bar{k}$ to the point $5\bar{i}+4\bar{j}+\bar{k}$. The work done in standard units by the forces is given by (AIE-2004)

1) 40 2) 30 3) 25 4) 15

48. The force $\bar{F}=3\bar{i}+\bar{j}-\bar{k}$ acts on a particle and it moves from the point $A\left(2\bar{i}-\bar{j}\right)$ to $B\left(2\bar{i}+\bar{j}\right)$. The work done by the force $\bar{F}$ =

1) 1 2) 2 3) 3 4) 4

KEY

01) 2	02\) 2	03) 2	04) 4	05) 4	06) 3
07) 2	08) 2	09) 3	10) 4	11) 1	12) 1
13) 3	14) 3	15) 2	16) 1	17) 3	18) 3
19) 1	20) 3	21) 1	22) 3	23) 3	24) 2
25) 4	26) 2	27. 2	28) 3	29) 3	30) 4
31) 4	32) 3	33) 1	34) 4	35) 2	36) 1
37) 2	38) 1	39) 2	40) 3	41) 4	42. 4
43) 2	44) 1	45) 2	46) 4	47) 1	48) 2

SOLUTIONS

1. $2\left(\bar{A}.\bar{B}+\bar{B}.\bar{C}+\bar{C}.\bar{A}\right)=0$

   $\left|\bar{A}+\bar{B}+\bar{C}\right|^2=9+16+25$

2. $\left|\bar{a}+\bar{b}+\bar{c}\right|=0\Rightarrow 1+4+9+2\left(\bar{a}\bar{b}+\bar{b}.\bar{c}+\bar{c}.\bar{c}\right)=0$

   $\Rightarrow \bar{a}.\bar{b}+\bar{b}.\bar{c}+\bar{c}.\bar{a}=-7$

3. Given $\left|\bar{a}\right|=3,\left|\bar{b}\right|=4$ and $\left|\bar{a}-\bar{b}\right|=5$ we know

   $\left|\bar{a}+\bar{b}\right|^2+\left|\bar{a}-\bar{b}\right|^2=2\left(\left|\bar{a}\right|^2+\left|\bar{b}\right|^2\right)$

   $\left(\bar{a}+b\right)^2+25=2(9+16)$

   $\left|\bar{a}+\bar{b}\right|^2=25$

   $\left|\bar{a}+\bar{b}\right|=5$

4. Given $\left|\bar{a}\right|=1,\left|\bar{b}\right|=2$ $\left|\bar{a}-\bar{b}\right|^2+\left|\bar{a}+2\bar{b}\right|^2=20$

$\left|\bar{a}\right|^2+\left|\bar{b}\right|^2-2\left(\bar{a}.\bar{b}\right)+\left|\bar{a}\right|^2+4\left|\bar{b}\right|^2+4\left(\bar{a}.\bar{b}\right)=20$

   $\Rightarrow 1+4+1+16+2\left(\bar{a}.\bar{b}\right)=20$

   $2\left(\bar{a}.\bar{b}\right)=-2$

   $\bar{a}.\bar{b}=-1$

   We know $\cos\left(\bar{a},\bar{b}\right)=\dfrac{\bar{a}.\bar{b}}{\left|\bar{a}\right|\left|\bar{b}\right|}=\dfrac{-1}{1\times 2}=\dfrac{-1}{2}$

   $\left(\bar{a},\bar{b}\right)=\dfrac{2\pi}{3}$

5. $(\sin\theta)(1)+\cos\theta\left(-\sqrt{3}\right)=0$

   $\Rightarrow \dfrac{\sin\theta}{\cos\theta}=\sqrt{3}\Rightarrow \theta=60^o$

6. Use $\left|\bar{a}\right|^2=\left|\mu^2\right|\left|\bar{b}\right|^2$

7. $\left(\bar{a}+\bar{b}\right).\bar{b}=0$ *and* $\left(\bar{a}+2\bar{b}\right).\bar{a}=0$ *simplify*

8. $\bar{a}.\bar{b}-\bar{a}.\bar{c}-\bar{d}.\bar{b}+\bar{d}.\bar{c}+\bar{b}.\bar{c}-$

   $\bar{b}.\bar{a}-\bar{d}.\bar{c}+\bar{d}.\bar{a}+\ \bar{c}.\bar{a}-\bar{c}.\bar{b}-\bar{d}.\bar{a}+\bar{d}.\bar{b}=0$

9. $\left|\bar{a}+\bar{b}+\bar{c}\right|^2=0\Rightarrow 1+1+\left|\bar{c}\right|^2-3=0\ \Rightarrow\left|\bar{c}\right|=1$

10. $\left.\begin{array}{l}\bar{r}.\bar{i}=3....(1)\\ \bar{r}.\bar{j}=5....(2)\end{array}\right\}$ Solving (1) & (2).

11. Given $\bar{a}.\bar{i}=\bar{a}.\left(\bar{i}+\bar{j}\right)=\bar{a}.\left(\bar{i}+\bar{j}+\bar{k}\right)$

    let $\bar{a}=x\bar{i}+y\bar{j}+z\bar{k}$

    $\bar{a}.\bar{i}=\bar{a}.\left(\bar{i}+\bar{j}\right)$ $\qquad$ $\bar{a}.\left(\bar{i}+\bar{j}\right)=\bar{a}.\left(\bar{i}+\bar{j}+\bar{k}\right)$

    $x=x+y$ $\qquad$ $x+y=x+y+z$

    $y=0$ $\qquad$ $z=0$

    $\bar{a}.\bar{i}=\bar{a}.\left(\bar{i}+\bar{j}+\bar{k}\right)$

    $x=x+y+z$ $\qquad$ $\bar{a}=x$

    $x=1$

12. Given $\bar{b}=\lambda\bar{a}$ $\bar{a}=2\bar{i}-\bar{j}+2\bar{k}$

    $\Rightarrow \bar{b}=\lambda\left(2\bar{i}-\bar{j}+2\bar{k}\right)$ and $\bar{a}.\bar{b}=18$

    $4\lambda+\lambda+4\lambda=18$

    $\lambda=2$

    $\bar{b}=(4,-2,4)$ C

13. $|\bar{a}-\bar{b}|^2=|\bar{a}|^2+|\bar{b}|^2-2|\bar{a}||\bar{b}|\cos\theta$

    $\Rightarrow|\bar{a}|^2=|\bar{a}|^2+|\bar{b}|^2-2|\bar{a}||\bar{b}|\cos\theta$

    $+|\bar{b}|^2=+2|\bar{a}||\bar{b}|\cos\theta$,

    $\cos\theta=\dfrac{1}{2}\Rightarrow\theta=60^0$

14. $\sin\frac{\theta}{2}=\frac{1}{2}\left|\bar{a}-\bar{b}\right|$

15. Given $\left|\bar{a}\right|=\left|\bar{b}\right|=1$ and $\left(\bar{a},\bar{b}\right)=\theta$

$\left|\bar{a}-\bar{b}\right|=1$

$\left|\bar{a}-\bar{b}\right|^2=1$

$\left|\bar{a}\right|^2+\left|\bar{b}\right|^2-2\left(\bar{a}.\bar{b}\right)=1$

$1+1-2\left(\bar{a}.\bar{b}\right)=1$

$\bar{a}.\bar{b}=1/2$

$\left|\bar{a}\right|\left|\bar{b}\right|\cos\left(\bar{a}.\bar{b}\right)=\frac{1}{2}$

$1\times1\cos\theta=1/2$

$\cos\theta=\frac{1}{2}=\cos 60^0$

$\theta=\frac{\pi}{3}$

16. Given $\bar{a}=p\bar{i}+3\bar{j}-7\bar{k},\bar{b}=p\bar{i}-p\bar{j}+4\bar{k}$

and $\bar{a}.\bar{b}>0$

$p^2-3p-28>0$

$\Rightarrow p^2-7p+4p-28>0$

$p\left(p-7\right)+4\left(p-7\right)>0$

$\left(p+4\right)\left(p-7\right)>0\ \bar{a}.\bar{b}=0\ p<-4\left(or\right)p>7$

17. $\bar{a}+\bar{b}+\bar{c}=\bar{0}\qquad\Rightarrow\bar{a}+\bar{b}=-\bar{c}$
S.O.B.S and expand

18. Given $\left|\bar{a}\right|=\left|\bar{b}\right|=\left|\bar{c}\right|=1$ and $\bar{a}.\bar{b}=0$

$\left(\bar{a},\bar{c}\right)=\alpha,\left(\bar{b},\bar{c}\right)=\beta$

$\left|\bar{a}+\bar{b}+\bar{c}\right|^2=1$

$\Rightarrow\left|\bar{a}\right|^2+\left|\bar{b}\right|^2+\left|\bar{c}\right|^2+2\left(\bar{a}.\bar{b}\right)+\bar{b}.\bar{c}+\bar{c}.\bar{a})=1$

$1+1+1+2\left(0+\cos\beta+\cos\alpha\right)=1$

$\cos\alpha+\cos\beta=-1$

19. $\cos\theta=\frac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|}$

20. Here $\bar{\alpha}$ perpendicular $\bar{\beta}\Rightarrow\left(\bar{\alpha},\bar{\beta}\right)=\frac{\pi}{2}$

$\Rightarrow\left(-\bar{\alpha},\bar{\beta}\right)=\pi-\frac{\pi}{2}=\frac{\pi}{2}$

21. $\left(x,3,-7\right).\left(x,-x,4\right)<0$.

22. $\cos A=\frac{\overline{AB}.\overline{AC}}{\left|\overline{AB}\right|\left|\overline{AC}\right|}$.

23. Check $\left(\bar{a}+\bar{b}\right).\left(\bar{a}-\bar{b}\right)=0$ (or) check $\left|\bar{a}\right|=\left|\bar{b}\right|$.

24. $\left(e_1+2e_2\right).\left(5e_1-4e_2\right)=0\Rightarrow5-8+6\cos\theta$

where, $\theta=\left(e_1,e_2\right)\Rightarrow\cos\theta=\frac{1}{2}\Rightarrow\theta=60^0$

25. Given $\overline{AB}=3i-2j+2\bar{k},\ \overline{BC}=-\bar{i}-2\bar{k}$
diagonals of a parallalogran
$\overline{AC}=\overline{AB}+\overline{BC}=2\bar{i}-2\bar{j}$
$\overline{BD}=\overline{BC}-\overline{AB}=-4\bar{i}+2\bar{j}-4\bar{k}$

$$\cos\theta=\frac{\overline{AC}.\overline{BD}}{\left|\overline{AC}\right|\left|\overline{BD}\right|}=\frac{-8-4+0}{\sqrt{8}\sqrt{16+4+16}}=\frac{-12}{6\times2\sqrt{2}}$$

$\cos\theta=\frac{-1}{\sqrt{2}}=\cos\frac{3\pi}{4}$

$\theta=\frac{3\pi}{4}$ Find $\overline{AC}$ & $\overline{BD}$

$\therefore\cos\theta=\frac{\overline{AC}.\overline{BD}}{\left|\overline{AC}\right|\left|\overline{BD}\right|}$

26. Clearly, $\bar{a}$ and $\bar{b}$ are at right angles

$\therefore\left|\bar{a}+\bar{b}\right|^2=\left|\bar{a}\right|^2+\left|\bar{b}\right|^2+2\left|\bar{a}\right|\left|\bar{b}\right|\cos90°$

$\Rightarrow\left|\bar{a}+\bar{b}\right|^2=1+1+0=2\Rightarrow\left|\bar{a}+\bar{b}\right|=\sqrt{2}$

27. $\cos\theta=\frac{\overline{BC}.\overline{BA}}{\left|\overline{BC}\right|\left|\overline{BA}\right|}$

28. Given

$\overline{OA} = 2\bar{i} - \bar{j} + \bar{k},\ \overline{OB} = \bar{i} - 3\bar{j} - 5\bar{k},\ \overline{OC} = a\bar{i} - 3\bar{j} + \bar{k}$

and $\bar{c} = \dfrac{\pi}{2}$

$$\overline{CA}.\overline{CB} = 0$$

$$\left(\overline{OA} - \overline{OC}\right).\left(\overline{OB} - \overline{OC}\right) = 0$$

$$\left((2-a)i + 2j\right).\left((1-a)i - 6\bar{k}\right) = 0$$

$$(2-a)(1-a) + 0 + 0 = 0$$

$$a = 1(or)2$$

29. $2\bar{a} - \bar{c} = (0,1,1), \quad \bar{a} + \bar{b} = (1,2,1)$

then $\cos\theta = \dfrac{2+1}{\sqrt{2}\sqrt{6}} \Rightarrow \cos^{-1}\left(\sqrt{3}/2\right),\ \theta = 30^0$

30. $\lambda = \dfrac{\dfrac{\bar{a}.\bar{b}}{|\bar{b}|}}{\dfrac{\bar{b}.\bar{a}}{|\bar{a}|}} = \dfrac{|\bar{a}|}{|\bar{b}|}$

31. $\sqrt{3}\cos\left(90^0 - 60^0\right)$

32. $\dfrac{\left|\overline{RS}.\overline{PQ}\right|}{\left|\overline{PQ}\right|}$

33. Given $\bar{a} = 2\bar{i} + \bar{j} + \bar{k}, \bar{b} = \bar{i} + 5\bar{j}, \bar{C} = 4\bar{i} + 4\bar{j} - 2\bar{k}$

now

$$3\bar{a} - 2\bar{b} = 3\left(2\bar{i} + \bar{j} + \bar{k}\right) - 2\left(\bar{i} + 5\bar{j}\right) = 4\bar{i} - 7\bar{j} + 3\bar{k}$$

length of projection of $3\bar{a} - 2\bar{b}$ on $\bar{c}$ is

$$= \frac{\left(3\bar{a} - 2\bar{b}\right).\bar{c}}{|\bar{c}|} = \frac{16 - 28 - 6}{\sqrt{16+16+4}} = \frac{-18}{16} = -3.$$

34. $\dfrac{\bar{a}.\bar{b}}{|\bar{b}|} = \dfrac{-6}{\sqrt{3}} \Rightarrow \dfrac{|\bar{a}||\bar{b}|\left(-\dfrac{\sqrt{3}}{2}\right)}{|\bar{b}|} = \dfrac{-6}{\sqrt{3}} \Rightarrow |\bar{a}| = 4.$

35. $\dfrac{\left(a\bar{i} + b\bar{j} + c\bar{k}\right).\bar{j}}{|\bar{j}|} = b.$

36. $\sqrt{a^2 + b^2 + c^2} = \sqrt{4+1+4} = 3$

37. $\dfrac{\left(\bar{a}.\bar{b}\right)\bar{b}}{|\bar{b}|^2}.$

38. Given $\bar{a} = 2\bar{i} + 3\bar{j} + 3\bar{k} \quad \bar{b} = \bar{i} - 2\bar{j} + \bar{k}$

$$\frac{\left(\bar{a}.\bar{b}\right)\bar{b}}{|\bar{b}|^2} = \frac{(2-6+3)\left(\bar{i} - 2\bar{j} + \bar{k}\right)}{\left(\sqrt{1+4+1}\right)^2} = \frac{-1}{6}\left(\bar{i} - 2\bar{j} + \bar{k}\right)$$

39. $\dfrac{\left(\bar{a}.\bar{b}\right)\bar{b}}{|\bar{b}|^2}$

4-0. $\bar{a} - \dfrac{\left(\bar{a}.\bar{b}\right)\bar{b}}{|\bar{b}|^2}$

41. $\dfrac{\left|\bar{a}.\bar{b}\right|}{|\bar{b}|}.$

42. Use $\left(\bar{r} - \bar{a}\right).\bar{b} = 0$

43. Distance $= \dfrac{|d|}{\sqrt{a^2 + b^2 + c^2}}$

44. $\bar{a} = 2i - 3\bar{j} + \bar{k};\ p = 5 \Rightarrow$ plane equation

$$\bar{r}.\frac{\bar{a}}{|\bar{a}|} = p$$

45. $\left(\bar{r} - \bar{a}\right).\bar{b} = 0$

$$\Rightarrow \left[(x-3)\bar{i} + (y+2)\bar{j} + (z-1)\bar{k}\right].$$

$$\left(4\bar{i} + 7\bar{j} - 4\bar{k}\right) = 0.$$

46. $\dfrac{\bar{m}_1.\bar{m}_2}{|\bar{m}_1||\bar{m}_2|} = \cos\theta$

47. $\bar{F} = \bar{F}_1 + \bar{F}_2$ and W.E. $= \bar{F}.\bar{d}$

48. $W = \bar{F}.\overline{AB}$

EXERCISE - II

1. If $\overline{a}$ and $\overline{b}$ are non-collinear unit vectors and $\left|\overline{a}+\overline{b}\right|=\sqrt{3}$ then $\left(2\overline{a}+5\overline{b}\right).\left(3\overline{a}-\overline{b}\right)=$

1) $\frac{15}{4}$ 2) $\frac{15}{2}$ 3) 15 4) 16

2. If $\overline{a},\overline{b}$ and $\overline{c}$ are perpendicular to $\overline{b}+\overline{c},\overline{c}+\overline{a}$ and $\overline{a}+\overline{b}$ respectively and if $\left|\overline{a}+\overline{b}\right|=6$, $\left|\overline{b}+\overline{c}\right|=8$ and $\left|\overline{c}+\overline{a}\right|=10$, then $\left|\overline{a}+\overline{b}+\overline{c}\right|$ is equal to

1) $5\sqrt{2}$ 2) 50 3) $10\sqrt{2}$ 4) 10

3. Let $\overline{u},\overline{v},\overline{w}$ be such that $\left|\overline{u}\right|=1,\left|\overline{v}\right|=2,\left|\overline{w}\right|=3.$ If the projection of $\overline{v}$ along $\overline{u}$ is equal to that of $\overline{w}$ along $\overline{u}$ and $\overline{v},\overline{w}$ are perpendicular to each other, then $\left|\overline{u}-\overline{v}+\overline{w}\right|$ equals

1) 2 2) $\sqrt{7}$ 3) $\sqrt{14}$ 4) 14

4. The length of longer diagonal of the parallelogram constructed on $5\overline{a}+2\overline{b}$ and $\overline{a}-3\overline{b}$ if it is given $\left|\overline{a}\right|=2\sqrt{2};\left|\overline{b}\right|=3,$ and $\left(\overline{a},\overline{b}\right)=\pi/4$ is

1) 15 2) $\sqrt{113}$ 3) $\sqrt{593}$ 4) $\sqrt{395}$

5. If $\overline{b}=4\overline{i}+3\overline{j}$ and c are two vectors perpendicular to each other in the XY plane ,the vector in the same plane having components 1, 2 along b and c respectively is

1) $(-2\overline{i}+11\overline{j})/5$ 2) $(2\overline{i}+11\overline{j})/5$

3) $(-2\overline{i}-11\overline{j})/5$ 4) $(2\overline{i}-11\overline{j})/5$

6. The perpendicular distance of a corner of unit cube from a diagonal not passing through it is

1) $\sqrt{2/3}$ 2) 2/3 3) 1/3 4) 1

7. If A, B are two points on the curve $y=x^2$ in the xoy plane satisfying $\overline{OA}.\overline{i}=1$ and $\overline{OB}.\overline{i}=-2$ then the length of the vector $2\overline{OA}-3\overline{OB}$ is

1) $\sqrt{14}$ 2) $2\sqrt{51}$ 3) $3\sqrt{41}$ 4) $2\sqrt{41}$

8. If a parallelogram is constructed on the vectors $\overline{a}=3\overline{p}-\overline{q},\overline{b}=\overline{p}+3\overline{q}$ and $\left|\overline{p}\right|=\left|\overline{q}\right|=2$ and angle between $\overline{p}$ and $\overline{q}$ is $\frac{\pi}{3}$, then the ratio of the lengths of the sides is

1) $\sqrt{7}:\sqrt{13}$ 2) $\sqrt{6}:\sqrt{2}$

3) $\sqrt{3}:\sqrt{5}$ 4) $1:2$

9. If $\overline{a}=2\overline{m}+\overline{n},\overline{b}=\overline{m}-2\overline{n}$, Angle between the unit vectors m and n is 60^0. a,b are the sides of a parallelogram ,then the lengths of the diagonals are

1) $\sqrt{7},\sqrt{5}$ 2) $\sqrt{13},\sqrt{5}$

3) $\sqrt{7},\sqrt{13}$ 4) $\sqrt{11},\sqrt{13}$

10. If $\overline{a},\overline{b},\overline{c}$ are three mutually perpendicular vectors of equal magnitudes then the vector equally inclined to $\overline{a},\overline{b},\overline{c}$

1) $\frac{\overline{i}+\overline{j}+\overline{k}}{\sqrt{3}}$ 2) $\overline{a}-\overline{b}+\overline{c}$

3) $\frac{\overline{a}-\overline{b}-\overline{c}}{\sqrt{3}}$ 4) $\frac{\overline{a}+\overline{b}+\overline{c}}{\sqrt{3}}$

11. If the position vectors of A,B,C,D are respectively $\left(-6,1,6\right)\left(6,-2,3\right),\left(-2,-3,-1\right)$ *and* $\left(-5,-9,-7\right)$ then

1) $\underline{|BCA}$ is a right angle 2) $\underline{|CDA}$ is a right angle

3) $\underline{|ABD}$ is a right angle 4) $\underline{|ACD}$ is a right angle

12. $|\bar{b}| = 6$, **then** $\bar{b} - 3\bar{c} = \lambda\bar{a}$ **if** $\lambda =$

1) $-9, 3$ 2) $-3, 6$ 3) 6, 3 4) -3, 4

13. If $\bar{a} = -\bar{i}+\bar{j}+\bar{k}$ **and** $\bar{b} = 2\bar{i}+\bar{k}$**, then the vector c satisfying the conditions that (i) it is coplanar with a and b (ii) it is perpendicular to b (iii) a.c =7 is**

1) $-3/2\bar{i}+5/2\bar{j}+3\bar{k}$ 2) $-3\bar{i}+5\bar{j}+6\bar{k}$

3) $-6\bar{i}+\bar{k}$ 4) $-\bar{i}+2\bar{j}+2\bar{k}$

14. If $\bar{a} = 2\bar{i}+\bar{j}+2\bar{k}$, $\bar{b} = 5\bar{i}-3\bar{j}+\bar{k}$**, then the length of the component vector of b perpendicular to a is**

1) $\sqrt{13}$ 2) $\sqrt{26}$ 3) $\sqrt{18}$ 4) $\sqrt{20}$

15. A Parallelogram is constructed with $\bar{a}$ **and** $\bar{b}$ **as adjacent sides such that** $|\bar{a}| = a$ and $|\bar{b}| = b.$ **The vector which coincides with the altitude of the parallelogram and is perpendicular to the vector** $\bar{a}$ **is.**

1) $\bar{b} - \dfrac{(\bar{a}.\bar{b})\bar{a}}{a^2}$ 2) $\bar{a} - \dfrac{(\bar{a}.\bar{b})\bar{b}}{b^2}$

3) $\bar{a} - \dfrac{(\bar{a}.\bar{b})\bar{b}}{a^2}$ 4) $\bar{b} - \dfrac{(\bar{a}.\bar{b})\bar{a}}{b^2}$

16. If $\bar{a}, \bar{b}, \bar{c}$ **are position vectors of the non-collinear points A, B, C respectively, the shortest distance of A from BC is**

1) $\bar{a}.(\bar{b} - \bar{c})$ 2) $\bar{b}.(\bar{c} - \bar{a})$

3) $|\bar{b} - \bar{a}|$ 4) $\sqrt{|\bar{b} - \bar{a}|^2 - \left(\dfrac{(\bar{a} - \bar{b}).(\bar{c} - \bar{b})}{|\bar{c} - \bar{b}|}\right)^2}$

17. If $\bar{a} = 3\bar{i} + \bar{j} + 2\bar{k}$ **and** $\bar{b} = \bar{i} + 2\bar{j} + 3\bar{k}$ **then a unit vector in the direction of the resultant of orthogonal projection of** $\bar{b}$ **on** $\bar{a}$ **and the projection of** $\bar{b}$ **on a line perpendicular to** $\bar{a}$ **is**

1) $\dfrac{\bar{i} + 2\bar{j} + 3\bar{k}}{\sqrt{14}}$ 2) $\dfrac{2\bar{i} + \bar{j} + 3\bar{k}}{\sqrt{14}}$

3) $\dfrac{3\bar{i} + \bar{j} + 2\bar{k}}{\sqrt{14}}$ 4) $\dfrac{\bar{i} + 3\bar{j} + 2\bar{k}}{\sqrt{14}}$

18. The projection of the vector $\bar{i} + \bar{j} + \bar{k}$ **on the line whose vectro equation is** $\bar{r} = (3+t)\bar{i} + (2t-1)\bar{j} + 3t\bar{k}$**, t being the scalar parameter, is**

(1) $\dfrac{1}{\sqrt{14}}$ (2) 6 (3) $\dfrac{6}{\sqrt{14}}$ (4) 10

19. A plane is at a distance of 8 units from the origin and is perpendicular to the vector $2\bar{i} + \bar{j} + 2\bar{k}$ **then the equation of the plane is**

1) $\bar{r}.(2\bar{i} + \bar{j} + 2\bar{k}) = 8$ 2) $\bar{r}.(\bar{i} + 2\bar{j} + 2\bar{k}) = 24$

3) $\bar{r}.(2\bar{i} + 2\bar{j} + \bar{k}) = 24$ 4) $\bar{r}.(2\bar{i} + \bar{j} + 2\bar{k}) = 24$

20. The angle between the planes passing through the points A(0, 0, 0), B(1, 1, 1) , C(3, 2, 1) & the planes passing through A(0, 0, 0), B(1, 1, 1), D(3, 1, 2) is

1) 90^0 2) 45^0 3) 120^0 4) 30^0

21. The point of application of the force (-2,4,7) is displaced from the point (3,-5,1) to the point (5,9,7) .But the force is suddenly halved when the point of application moves half the distance.The work done by the force is

1) 70 2) 70.5 3) 75 4) 75.5

22. If forces of magnitudes 6 and 7 units acting in the directions $\bar{i} - 2\bar{j} + 2\bar{k}$ **and** $2\bar{i} - 3\bar{j} - 6\bar{k}$ **respectively act on a particle which is displaced from the point** $P(2,-1,-3)$ to $Q(5,-1,1)$**, then the work done by the forces is**

1) 4 units 2) -4 units 3) 7 units 4) -7 units

22. A constant force $3\bar{i} + 4\bar{j} - 5\bar{k}$ **acts on a particle at** $\bar{i} + 2\bar{j} + 2\bar{k}$ **and moves it to a point on the z-axis which is 3 units from origin, the work done is**

1) 16 2) -16 3) 14 4) -14

KEY

01) 2	02) 4	03) 3	04) 3	05) 1	06) 4
07) 1	08) 3	09) 4	10) 1	11) 1	12) 1
13) 2	14) 1	15) 4	16. 1	17) 3	18) 4
19) 3	20) 2	21. 1	22) 2		

SOLUTIONS

1. $\left|\bar{a}+\bar{b}\right|=\sqrt{3}\Rightarrow \bar{a}.\bar{b}=\frac{1}{2}$

   Find $\left(2\bar{a}+5\bar{b}\right).\left(3\bar{a}-\bar{b}\right)=\frac{15}{2}$

2. $\left|\bar{a}+\bar{b}\right|^2+\left|\bar{b}+\bar{c}\right|^2+\left|\bar{c}+\bar{a}\right|^2=200$

   $\Rightarrow 2\left(\left|\bar{a}\right|^2+\left|\bar{b}\right|^2+\left|\bar{c}\right|^2\right)=200\left(\because \bar{a}.\bar{b}+\bar{b}.\bar{c}+\bar{c}.\bar{a}=0\right)$

   $\left|\bar{a}+\bar{b}+\bar{c}\right|=\sqrt{100}=10$

3. $\left|\bar{u}-\bar{v}+\bar{w}\right|^2=\left|\bar{u}\right|^2+\left|\bar{v}\right|^2+\left|\bar{w}\right|^2+2\left|\bar{u}.\bar{w}-\bar{u}.\bar{v}-\bar{v}.\bar{w}\right|$

   $=1+4+9+2\left|\bar{u}.\bar{w}-\bar{u}.\bar{v}\right|$

   $\left(\because \bar{v}.\bar{w}=0\right)=14+2\left(\left|\bar{u}\right|\left|\bar{w}\right|\cos\alpha-\left|\bar{u}\right|\left|\bar{v}\right|\cos\beta\right)$

   But, $\left|\bar{v}\right|\cos\beta=\left|\bar{w}\right|\cos\alpha$

   $\therefore \left|\bar{u}-\bar{v}+\bar{w}\right|^2=14\Rightarrow \left|\bar{u}-\bar{v}+\bar{w}\right|=\sqrt{14}$.

4. Longer diagonal $-4\bar{a}-5\bar{b}$

   its magnitude $=\left|-4\bar{a}-5\bar{b}\right|=\sqrt{593}$.

5. Let $\bar{d}=x\bar{i}+y\bar{j}$, and $\bar{c}=-3\bar{i}+4\bar{j}$

   $\frac{\bar{d}.\bar{b}}{\left|\bar{b}\right|}=1\Rightarrow 4x+3y=5,\ \frac{\bar{d}.\bar{c}}{\left|\bar{c}\right|}=2$

   $\Rightarrow -3x+4y=10$

   Solve these two equations..

6. Let $A\left(x_1,y_1\right)\&B\left(x_2,y_2\right)$ lie on the parabola so that $\overline{OA}=\bar{i}+\bar{j}$, $\overline{OB}=-2\bar{i}+4\bar{j}$

7. Find $\left|\bar{a}\right|=\sqrt{9p^2+q^2-6pq\cos 60^o}=2\sqrt{7}$;

   $\left|\bar{b}\right|=\sqrt{p^2+9q^2+6pq\cos 60^o}=2\sqrt{13}$

8. Length of the diagonals are $\left|\bar{a}+\bar{b}\right|,\left|\bar{a}-\bar{b}\right|$

9. Given $\bar{a}.\bar{b}=\bar{b}.\bar{c}=\bar{c}.\bar{a}=0$

   $\left|\bar{a}\right|=\left|\bar{b}\right|=\left|\bar{c}\right|\Rightarrow\left|\bar{a}\right|^2=\left|\bar{b}\right|^2=\left|c\right|^2$

   verification option 4) $\frac{\bar{a}+\bar{b}+\bar{c}}{\sqrt{3}}$

   $\bar{a}.\left(\frac{\bar{a}+\bar{b}+\bar{c}}{\sqrt{3}}\right)=\frac{\left|\bar{a}\right|^2}{\sqrt{3}}$

   $\bar{b}.\left(\frac{\bar{a}+\bar{b}+\bar{c}}{\sqrt{3}}\right)=\frac{\left|\bar{b}\right|^2}{\sqrt{3}}$

   $\bar{c}.\left(\frac{\bar{a}+\bar{b}+\bar{c}}{\sqrt{3}}\right)=\frac{\left|\bar{c}\right|^2}{\sqrt{3}}$

   $\therefore \frac{\bar{a}+\bar{b}+\bar{c}}{\sqrt{3}}$ is equally angle with $\bar{a},\bar{b},\bar{c}$

10. $\therefore \underline{|BCA}=90^0$.

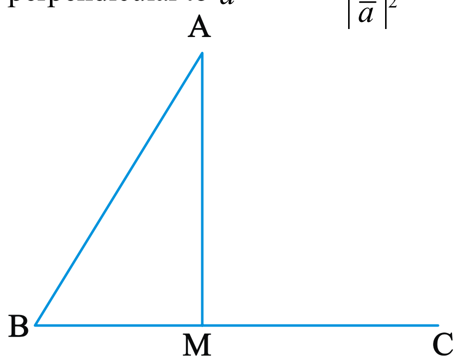
11. Write $\bar{b}=\lambda\bar{a}+3\bar{c}$ squaring. we get

    $\lambda^2\pm 6\lambda-27=0$

12. Take $C=\left(-\frac{3}{2},\frac{5}{2},3\right)$ and verify

13. $\bar{b}-\frac{\left(\bar{a}.\bar{b}\right)\bar{a}}{\left|\bar{a}\right|^2}$

14. Required vector = Component vector of $\bar{b}$ perpendicular to $\bar{a}=\bar{b}-\frac{\left(\bar{b}.\bar{a}\right)\bar{a}}{\left|\bar{a}\right|^2}$

15. 

    $BM=\frac{\overline{BA}.\overline{BC}}{\left|\overline{BC}\right|}$, $AM=\sqrt{AB^2-BM^2}$

16. Required vector $=\dfrac{\bar{b}}{|\bar{b}|}$.

17. Since $\bar{r}=3\bar{i}-\bar{j}+t(\bar{i}+2\bar{j}+3\bar{k})$

So, a vector parallel to the lines is

$\bar{b}=\bar{i}+2\bar{j}+3\bar{k}$

Now, unit vector along the line is

$\dfrac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{I^2+2^2+3^2}}$ i.e $=\dfrac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{14}}$ the projection of $\left(\bar{i}+\bar{j}+\bar{k}\right)$ on lines is

$$\left(\bar{i}+\bar{j}+\bar{k}\right),\frac{\left(\bar{i}+2\bar{j}+3\bar{k}\right)}{\sqrt{14}}=\frac{1+2+3}{\sqrt{14}}=\frac{6}{\sqrt{14}}$$

18. $\bar{r}.\hat{n}=p\Rightarrow\bar{r}.\left(\dfrac{2\bar{i}+\bar{j}+2\bar{k}}{3}\right)=8$

$\Rightarrow\bar{r}.\left(2\bar{i}+\bar{j}+2\bar{k}\right)=24$

19. Plane through ABC is $-x+2y-z=0$

plnae through ABD is $x+y+-2z=0$

$\overline{n_1}=-\bar{i}+2\bar{j}-\bar{k}$, $\overline{n_2}=\bar{i}+\bar{j}-2\bar{k}$

$\cos\theta=\dfrac{\overline{n_1}.\overline{n_2}}{|\overline{n_1}||\overline{n_2}|}=\dfrac{1}{2}\Rightarrow\theta=60^0\ (or)\ 120^0$

20. $F.\dfrac{\bar{d}}{2}+\dfrac{\bar{F}}{2}.\dfrac{\bar{d}}{2}$

21. $\bar{F}=\bar{F}_1+\bar{F}_2=(4,-7,-2),\bar{d}=\overline{AB}=(3,0,4)$

$W.D.=\bar{F}.\bar{d}=4$

22. $W=\bar{F}.\overline{AB}$

$=\left(3\bar{i}+4\bar{j}-5\bar{k}\right)\left(3\bar{k}-(\bar{i}+2\bar{j}+2\bar{k})\right)$

$=\left(3\bar{i}+4\bar{j}-5\bar{k}\right).\left(-\bar{i}-2\bar{j}+\bar{k}\right)$ = -16.

EXERCISE- III

1. The value of a for which the angle between $\bar{a}=2a^2\bar{i}+4a\bar{j}+\bar{k}$ and $\bar{b}=7\bar{i}-2\bar{j}+a\bar{k}$ is obtuse and the angle between $\bar{b}$ and z-axis is acute and less than $\dfrac{\pi}{6}$ is

1) Does not exist   2) Lies in $\left(0,\dfrac{1}{2}\right)$

3) Lies in $(-1,1)$ 4) Lies in $(0,1)$

2. Let $\bar{a}=2\bar{i}+\bar{j}+\bar{k},\bar{b}=\bar{i}+2\bar{j}-\bar{k}$ and a unit vector $\bar{c}$ be coplanar. If $\bar{c}$ is perpendicular to $\bar{a}$, then $\bar{c}$ is

1) $\dfrac{1}{\sqrt{2}}\left(-\bar{j}+\bar{k}\right)$ 2) $\dfrac{1}{\sqrt{3}}\left(-\bar{i}-\bar{j}-\bar{k}\right)$

3) $\dfrac{1}{\sqrt{5}}\left(\bar{i}-2\bar{j}\right)$ 4) $\dfrac{1}{\sqrt{5}}\left(\bar{i}-\bar{j}-\bar{k}\right)$

3. Let $\bar{a}=2\bar{i}-\bar{j}+\bar{k},\bar{b}=\bar{i}+2\bar{j}-\bar{k}$ and $\bar{c}=\bar{i}+\bar{j}-2\bar{k}$ be three vectors. A vector in the plane of $\bar{b}$ and $\bar{c}$ whose projection on $\bar{a}$ is of magnitude $\sqrt{\dfrac{2}{3}}$ is

1) $2\bar{i}-3\bar{j}-3\bar{k}$ 2) $2\bar{i}+3\bar{j}+3\bar{k}$

3) $-2\bar{i}-\bar{j}+5\bar{k}$ 4) $2\bar{i}+\bar{j}+5\bar{k}$

4. If $\sum_{i=1}^{n}\bar{a}_i=\bar{0}$ where $|\bar{a}_i|=1$, for all i, then the value of $\sum\limits_{1\le i<j\le n}\sum\bar{a}_i.\bar{a}_j$ is

1) n^2 2) $-n^2$ 3) n 4) $-\dfrac{n}{2}$

5. The position vector of the foot of the perpendicular from (1,-2,-3) to the line $\bar{r}=\bar{i}+\bar{j}+\lambda\left(2\bar{i}+\bar{j}+\bar{k}\right)$ is

1) $-2\bar{i}-\bar{j}+\bar{k}$ 2) $-\bar{i}-\bar{k}$

3) $\dfrac{\bar{j}+\bar{k}}{2}$ 4) $\bar{i}+\bar{j}+\bar{k}$

6. **The** ΔABC **is defined by the vertices** $A(1,-2,2)B(1,4,0)$ **and** $C(-4,1,1)$**. Let M be the foot of the altitude drawn from the vertex B to side AC. Then** $\overline{BM} =$

1) $\left(\frac{-20}{7},\frac{-30}{7},\frac{10}{7}\right)$ 2) $(-20,-30,10)$

3) $(2,3,-1)$ 4) $(1,2,3)$

7. **If the position vector of a point P is** $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$**, where** $x,y,z \in N$ **and** $\overline{\alpha}$ **is a vector given by** $\overline{\alpha} = \overline{i} + \overline{j} + \overline{k}$**, then the total number of possible positions of point P for which** $\overline{r}.\overline{\alpha} = 10$

1) 36 2) 72 3) 66 4) 100

8. **If** p^{th}, q^{th}, r^{th} **terms of a G.P. are the positive numbers a, b, c then angle between the vectors** $\log a^3\overline{i} + \log b^3\overline{j} + \log c^3\overline{k}$ **and** $(q-r)\overline{i} + (r-p)\overline{j} + (p-q)\overline{k}$ **is**

1) $\frac{\pi}{6}$ 2) $\frac{\pi}{2}$ 3) $\frac{\pi}{3}$ 4) $\sin^{-1}\left(\frac{1}{\sqrt{a^2+b^2+c^2}}\right)$

9. **Let** $\overline{a}, \overline{b}, \overline{c}$ **be vectors of equal magnitude such that the angle between** $\overline{a}$ **and** $\overline{b}$ **is** $\alpha, \overline{b}$ **and** $\overline{c}$ **is** β **and** $\overline{c}$ **and** $\overline{a}$ **is** γ**. Then the minimum value of** $\cos\alpha + \cos\beta + \cos\gamma$ **is**

1) $\frac{1}{2}$ 2) $-\frac{1}{2}$ 3) $\frac{3}{2}$ 4) $-\frac{3}{2}$

10. **Let** $\overline{p}$ **and** $\overline{q}$ **be the position vectors of P and Q repectively with respect to 'O' and** $|\overline{p}| = p, |\overline{q}| = q$**. if R, S, divides PQ Internally and externally in the ratio 2:3 respectively. If OR and OS are perpendicular, then**

1) $9p^2 = 4q^2$ 2) $4p^2 = 9q^2$

3) 9p=4q 4) 4p=9q

11. **The vectors** $3\overline{a} - 5\overline{b}$ **and** $2\overline{a} + \overline{b}$ **are mutually perpendicular and the vectors** $\overline{a} + 4\overline{b}$ **&** $-\overline{a} + \overline{b}$ **are also mutually perpendicular. Then the acute angle between** $\overline{a}$ **and** $\overline{b}$ **is**

1) $\cos^{-1}\left(\frac{19}{5\sqrt{43}}\right)$ 2) $\pi - \cos^{-1}\left(\frac{19}{5\sqrt{43}}\right)$

3) $\cos^{-1}\left(\frac{9}{5\sqrt{43}}\right)$ 4) $\pi - \cos^{-1}\left(\frac{9}{5\sqrt{43}}\right)$

12. **The vectors** $\overline{X}$ **and** $\overline{Y}$ **satisfy the equations** $2\overline{X} + \overline{Y} = \overline{p}, \overline{X} + 2\overline{Y} = \overline{q}$ **where** $\overline{p} = \overline{i} + \overline{j}$ and $\overline{q} = \overline{i} - \overline{j}$ **. If** θ **is the angle between** $\overline{X}$ **and** $\overline{Y}$ **then**

1) $\cos\theta = \frac{4}{5}$ 2) $\sin\theta = \frac{1}{\sqrt{2}}$

3) $\cos\theta = -\frac{4}{5}$ 4) $\cos\theta = -\frac{3}{5}$

13. $A = (2,3,5), B = (-1,3,2)$ **and** $C = (\lambda,5,\mu)$ **are the vertices of a triangle. If the median AM is equally inclined to the coordinates axes, then**

1) $\lambda = 10, \mu = 7$ 2) $\lambda = -10, \mu = 7$

3) $\lambda = 7, \mu = 10$ 4) $\lambda = -7, \mu = -10$

14. **Let** $a = BC, b = CA, c = AB$ **be the sides of the triangle ABC. If G is the centroid of** ΔABC **such that** $\overline{GB}$ **and** $\overline{GC}$ **are inclined at an obtuse angle, then**

1) $5a^2 > b^2 + c^2$ 2) $5c^2 > a^2 + b^2$

3) $5b^2 > a^2 + c^2$ 4) None of these

15. **The position vector of A is** $p\overline{i} + \overline{j}$ **.If A is rotated about O through an angle** $\frac{\pi}{6}$ **in anti clock wise direction. It coincides with B whose position vector** $\overline{i} + q\overline{j}$ **.The value of p, q are**

1) $\sqrt{3}, 3$ 2) $\sqrt{3}, \frac{1}{3}$

3) $\sqrt{3}, \sqrt{3}$ or $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ 4) $\frac{1}{3}, \frac{1}{3}$

16. **In** $\Delta ABC, |\overline{CB}| = a, |\overline{CA}| = b, |\overline{AB}| = c.$ **CD is**

median through the vertex C. Then $\overline{CA}.\overline{CD}$ equals

1) $\frac{1}{4}\left(3a^2+b^2-c^2\right)$ 2) $\frac{1}{4}\left(a^2+3b^2-c^2\right)$

3) $\frac{1}{4}\left(a^2+b^2-3c^2\right)$ 4) $\frac{1}{4}\left(-3a^2+b^2+c^2\right)$

17. If $\overline{a}$ is the position vector of A then the position vector of the foot of the perpendicular from A to the plane $\overline{r}.\overline{b}=\overline{b}.\overline{c}$ is.

1) $\overline{b}+\frac{(\overline{c}-\overline{a})}{|\overline{b}|^2}\overline{b}$ 2) $\overline{a}+\frac{|(\overline{c}-\overline{a}).\overline{b}|\overline{b}}{|\overline{b}|^2}$

3) $\overline{b}+\frac{|(\overline{c}-\overline{b}).\overline{a}|}{|\overline{a}|^2}\overline{b}$ 4) $\overline{a}+\frac{|(\overline{c}-\overline{b}).\overline{a}|}{|\overline{a}|^2}\overline{b}$

18. In triangle ABC if $\overline{AB}=\frac{\overline{u}}{|\overline{u}|}-\frac{\overline{v}}{|\overline{v}|}$ and $\overline{AC}=\frac{2\overline{u}}{|\overline{u}|}$ where $|\overline{u}|\neq|\overline{v}|$ then

1) $1+\sum\cos 2A=0$ 2) $\sum\cos 2A=0$

3) $2+\sum\cos 2A=0$

4) $1+\cos 2A+\cos 2B-\cos 2C=0$

19. Let ABCD be a parallelogram such that $\overline{AB}=\overline{q},\overline{AD}=\overline{p}$ and $\angle BAD$ be an acute angle. If $\overline{r}$ is the vector that coincides with the altitude directed from the vertex B to the side AD, then $\overline{r}$ is given by

1) $\overline{r}=3\overline{q}-\frac{3(\overline{p}.\overline{q})}{(\overline{p}.\overline{p})}\overline{p}$ 2) $\overline{r}=-\overline{q}+\frac{(\overline{p}.\overline{q})}{(\overline{p}.\overline{p})}\overline{p}$

3) $\overline{r}=\overline{q}+\frac{(\overline{p}.\overline{q})}{(\overline{p}.\overline{p})}\overline{p}$ 4) $\overline{r}=-3\overline{q}+\frac{(\overline{p}.\overline{q})}{(\overline{p}.\overline{p})}\overline{p}$

20. In a parallelogram ABCD, $|\overline{AB}|=a,|\overline{AD}|=b,$ and $|\overline{AC}|$=c , then $\overline{DB}.\overline{AB}$ has the value

1) $\frac{3a^2+b^2-c^2}{2}$ 2) $\frac{a^2+3b^2-c^2}{2}$

3) $\frac{a^2-b^2+3c^2}{2}$ 4) $\frac{a^2+3b^2+c^2}{2}$

KEY

01) 1	02) 1	03) 3	04) 4	05) 2	06) 1
07) 1	08) 2	09) 4	10) 1	11) 1	12) 3
13) 3	14) 1	15) 3	16) 2	17) 2	18) 1
19) 2	20) 1				

SOLUTIONS

1. $(2a^2,4a,1).(7,-2,a)<0 \Rightarrow 14a^2-7a<0$

$\Rightarrow a(2a-1)<0 \Rightarrow a(a-1/2)<0$

$\Rightarrow 0<a<\frac{1}{2}$ -------- (1)

$\Rightarrow\frac{(7,-2,a).(0,0,1)}{\sqrt{49+4+a^2}.1}>\frac{\sqrt{3}}{2}$ and $\frac{a}{\sqrt{53+a^2}}>0\Rightarrow a>0....(2)$

$\Rightarrow\frac{a}{\sqrt{53+a^2}}>\frac{\sqrt{3}}{2}\Rightarrow(2a)^2>3(53+a^2)$

$\Rightarrow a^2>159$, $a<-\sqrt{159}\,(or)\,a>\sqrt{159}.......(3)$

$\therefore$ There is no value satisfying $(1)(2)\&(3)$

2. Let $\overline{c}=\alpha\overline{a}+\beta\overline{b}$ and $\overline{c}\overline{a}=0\Rightarrow 1\beta=-2\alpha,\overline{c}=\alpha(\overline{a}-2\overline{b})$. But $|\overline{c}|=1\Rightarrow\alpha=\pm\frac{1}{3\sqrt{2}}$

3. A vector in the plane of $\vec{b}$ and $\vec{c}$ is $\vec{b}+\lambda\vec{c}$, or $(1+\lambda)i+(2+\lambda)j-(1+2\lambda)k$. Its projection on $\overline{a}$ is $\sqrt{\frac{2}{3}}$

$\therefore\frac{2(1+\lambda)-(2+\lambda)-(1+2\lambda)}{\sqrt{6}}=\pm\sqrt{\frac{2}{3}}$

$\therefore\lambda+1=\pm2\Rightarrow\lambda=1,-3$, $\lambda=-3\Rightarrow(3)$

4. $|\overline{a}_1+\overline{a}_2+....+\overline{a}|^2=|\overline{a}_1|^2+|\overline{a}_2|^2+....+|\overline{a}_n|^2+2$

$\sum_{1K,i<j\leq n}\overline{a}_i.\overline{a}_j\Rightarrow 0=n+2(G.E)$

5. $\frac{x-1}{2}=\frac{y-1}{1}=\frac{z}{1}=t \Rightarrow x=2t+1, y=t+1, z=t$

Let, $M=(2t+1, t+1, t)$, $\quad A=(1-2-3)$

D.R'S of $M=(2t, t+3, t+3)$ D.R'S of line $=(2,1,1)$

Pro of D.R'S =0 $\quad \Rightarrow t=-1 \quad \therefore M=(-1,0,-1)$

6. Here $\overline{MB}=$ Component vector of $\overline{AB}$

Perpendicular $\overline{AC}=\overline{AB}-\frac{(\overline{AB}.\overline{AC})\overline{AC}}{AC^2}$ and

hence find $\overline{BM}$

7. $\bar{r}.\bar{\alpha}=10 \Rightarrow x+y+z=10$

No. of solutions $={}^{10-1}C_{3-1}={}^{9}C_2=36$

8. Use $a=t_p=AR^{p-1} \Rightarrow \log a=\log A+(P-1)\log R$ etc.

9. Let $|\bar{a}|=|\bar{b}|=|\bar{c}|=\lambda$

we have, $\bar{a}\,.\,\bar{b}=|\bar{a}||\bar{b}|\cos\alpha=\lambda^2\cos\alpha$

$\bar{b}.\bar{c}=|\bar{b}||\bar{c}|\cos\beta=\lambda^2\cos\beta$

$\bar{c}.\bar{a}=|\bar{c}||\bar{a}|\cos\gamma=\lambda^2\cos\gamma$

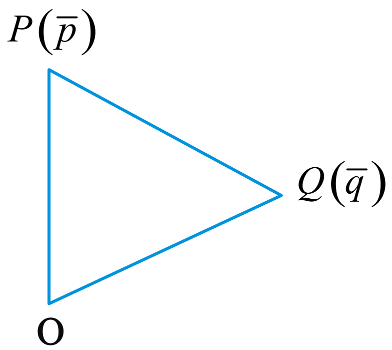
Now, $|\bar{a}+\bar{b}+\bar{c}|^2\geq 0$

$\Rightarrow |\bar{a}|^2+|\bar{b}|+|\bar{c}|^2+2(\bar{a}.\bar{b}+\bar{b}.\bar{c}+\bar{c}.\bar{a})\geq 0$

$\Rightarrow 3\lambda^2+2\lambda^2(\cos\alpha+\cos\beta+\cos\gamma)\geq 0$

$\Rightarrow \cos\alpha+\cos\beta+\cos\gamma\geq -\frac{3}{2}$

10.



$\overline{OR}=\frac{2\bar{q}+3\bar{p}}{5}$

$\overline{OS}=\frac{2\bar{q}-3\bar{p}}{-1} \quad \& \overline{OR}.\overline{OS}=0$

11. $(3\bar{a}-5\bar{b}).(2\bar{a}+\bar{b})=0$

$\Rightarrow 6|\bar{a}|^2-7(\bar{a}.\bar{b})-5|\bar{b}|^2=0$

$\Rightarrow \bar{a}.\bar{b}=\frac{6|\bar{a}|^2-5|\bar{b}|^2}{7}\rightarrow(1)$

$(\bar{a}+4\bar{b}).(\bar{b}-\bar{a})=0$

$\Rightarrow 4|\bar{b}|^2-|\bar{a}|^2-3(\bar{a}.\bar{b})=0$

$\Rightarrow \bar{a}.\bar{b}=\frac{4|\bar{b}|^2-|\bar{a}|^2}{3}\rightarrow(2)$

from (1) & (2) $\frac{6|\bar{a}|^2-5|\bar{b}|^2}{7}=\frac{4|\bar{b}|^2-|\bar{a}|^2}{3}$

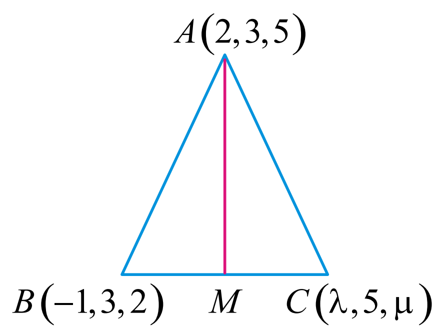
$\Rightarrow 25|\bar{a}|^2=43|\bar{b}|^2 \Rightarrow |\bar{a}|=\sqrt{\frac{43}{25}}|\bar{b}|$

also $\bar{a}.\bar{b}=\frac{|\bar{a}||\bar{b}|^2}{25}$

$\therefore \cos\theta=\frac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|}=\frac{19|\bar{b}|^2}{\sqrt{\frac{43}{25}}|\bar{b}|^2}=\frac{19}{5\sqrt{43}}$

12. $\vec{X}=\frac{1}{3}(\vec{i}+3\vec{j}), \vec{Y}=\frac{1}{3}(\vec{i}-3\vec{j}) \Rightarrow \cos\theta=-\frac{4}{5}$

13.



$M=\left(\frac{\lambda-1}{2},4,\frac{2+\mu}{2}\right)$, $\overline{AM}=\left(\frac{\lambda-5}{2},1,\frac{\mu-8}{2}\right)$

$$\cos\theta = \frac{\overline{AM}.\bar{i}}{|\overline{AM}|} = \frac{\overline{AM}.\bar{j}}{|\overline{AM}|} = \frac{\overline{AM}.\bar{k}}{|\overline{AM}|}$$

$$\Rightarrow \frac{\lambda-5}{2} = 1 = \frac{\mu-8}{2}$$

$$\Rightarrow \lambda-5=2 \ \& \ \mu-8=2 \Rightarrow \lambda=7 \ \& \ \mu=10$$

14. Let $\bar{A}=\bar{0}$; $\overline{AB}=\bar{b}$, $\overline{AC}=\bar{c}$ then

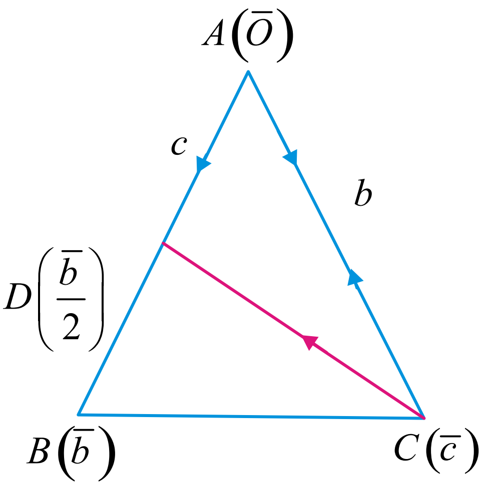
$$\overline{AG} = \frac{\bar{b}+\bar{c}}{3}$$

$$\overline{GB}.\overline{GC}<0 \quad \Rightarrow 5(b.c)-2|b|^2-2|c|^2<0$$

$$5\left(\frac{c^2+b^2-a^2}{2}\right)<2\left(c^2+b^2\right)$$

$$\Rightarrow b^2+c^2<5a^2 \quad \text{Use} \left(\bar{b}.\bar{c}=\frac{c^2+b^2-a^2}{2}\right)$$

15. $|\overline{OA}|=|\overline{OB}|$, $Cos\frac{\pi}{6}=\frac{\overline{OA}.\overline{OB}}{|\overline{OA}|.|\overline{OB}|}$

16. 

Let A be origin & $\overline{AB}=\bar{b},\overline{AC}=\bar{c}$ then

$$|\overline{AB}|=|\bar{b}|=c, |\overline{AC}|=\bar{c}=b$$

$$\overline{CA}=-\bar{c} \ \& \ \overline{CD}=\frac{\bar{b}}{2}-\bar{c}$$

$$\overline{CA}.\overline{CD}=(-\bar{c}).\left(\frac{\bar{b}}{2}-\bar{c}\right)$$

17. If $F=\bar{r}$ then $\overline{AF}$ = Component Vector of $\overline{AC}$

along $\bar{b}$ $\quad \Rightarrow \bar{r}-\bar{a}=\dfrac{|(\bar{c}-\bar{a}).\bar{b}|\bar{b}}{|\bar{b}|^2}$

$$\Rightarrow \bar{r}=\bar{a}+\frac{|(\bar{c}-\bar{a}).\bar{b}|\bar{b}}{|\bar{b}|^2}$$

18. $\overline{AC}=\overline{AB}+\overline{BC}$

$$\overline{BC}=\bar{u}+\bar{v} \Rightarrow \overline{AB}.\overline{BC}=(\bar{u}-\bar{v}).(\bar{u}+\bar{v})=0$$

$$\angle B=\frac{\pi^c}{2} \text{ Hence } \sum\cos A+1=0$$

19. $\overline{AE}$ = vector compenent of $\bar{q}$ on $\bar{p}$

$$\overline{AE}=\frac{(\bar{p}.\bar{q})}{(\bar{p}.\bar{p})}\bar{p} \Rightarrow \bar{q}+\bar{r}=\frac{(\bar{p}.\bar{q})\bar{p}}{(\bar{p}.\bar{p})}$$

20. $\therefore \overline{DB}=\overline{DA}+\overline{AB}$ or $\therefore \overline{DA}=\overline{DB}-\overline{AB}$

$$\therefore (\overline{DA})^2=(\overline{DB})^2+(\overline{AB})^2-2\overline{DB}.\overline{AB}$$

in parallelogram $2\left(a^2+b^2\right)=c^2+DB^2$

$$\therefore (DB)^2=2a^2+2b^2-c^2$$

$$\therefore b^2=2a^2+2b^2-c^2$$

$$\therefore \overline{AB}.\overline{DB}=\frac{3a^2+b^2-c^2}{2}$$

21. Putting $x=1$ in $y=x^2+x+10 \Rightarrow y=12$ and

$\frac{dy}{dx}$ at A is 3. Tangent at A is $y-12=3(x-1)$.

it meets x-axis at B .

$(-3,0)$. $\overline{OB}=-3\bar{i}$, $\overline{AB}=-4\bar{i}-12\bar{j}$

$$\overline{OA}.\overline{AB}=-148$$

JEE MAINS QUESTIONS

1. If the vectors $\overline{p}=(a+1)\overline{i}+a\overline{j}+a\overline{k}, \overline{q}=a\overline{i}+(a+1)\overline{j}+a\overline{k}$ and $\overline{v}=a\overline{i}+a\overline{j}+(a+1)\overline{k}\,(a\in R)$ are coplanar and $3\left(\overline{p}.\overline{q}\right)^2-\lambda\left|\overline{\gamma}\times\overline{p}\right|^2=0$, then the value of λ is __________ [2020]

2. $\overline{a}$ and $\overline{b}$ are unit vectors, then the greatest value of $\sqrt{3}\left|\overline{a}+\overline{b}\right|+\left|\overline{a}-\overline{b}\right|$ is __________ [2020]

3. If $\vec{x}$ and $\vec{y}$ be two non-zero vectors such that $\left|\overline{x}+\overline{y}\right|=\left|x\right|$ and $2\overline{x}+\lambda\overline{y}$ is perpendicular to $\overline{y}$ then the value of λ is [2020]

4. Let the vectors $\overline{a},\overline{b},\overline{c}$ be such that $\left|\overline{a}\right|=2,\left|\overline{b}\right|=4$ and $\left|\overline{c}\right|=4$ if the projection of $\overline{b}$ on $\overline{a}$ is equal to the projection $\overline{c}$ on $\overline{a}$ and $\overline{b}$ is perpendicular to $\overline{c}$, then the value of $\left|\overline{a}+\overline{b}-\overline{c}\right|$ is ------------ [2020]

5. Let $\overline{a},\overline{b}$ and $\overline{c}$ be three unit vectors such that $\left|\overline{a}-\overline{b}\right|^2+\left|\overline{a}-\overline{c}\right|^2=8$ then $\left|\overline{a}+2\overline{b}\right|^2+\left|\overline{a}+2\overline{c}\right|^2$ is equal to ____ [2019]

6. The projection of the line segment joining the points (1,-1,3) and (2,-4,11) on the line joining the points (-1,2,3) and (3,-2,10) is [2020]

7. Let $\overline{a}=2\overline{i}+\lambda_1\overline{j}+3\overline{k}, \overline{b}=4\overline{i}+(3-\lambda_2)\overline{j}+6\overline{k}$ and $\overline{c}=3\overline{i}+6\overline{j}+(\lambda_3-1)\overline{k}$ be three vectors such that $\overline{b}=2\overline{a}$ and $\overline{a}$ is perpendicular to $\overline{c}$ then a possible of $(\lambda_1,\lambda_2,\lambda_3)$ is [2019]

1) $(1,2,1)$ 2) $(1,3,1)$

3) $\left(\frac{-1}{2},4,0\right)$ 4) $\left(\frac{1}{2},4,-2\right)$

8. Let $\overline{\upsilon}$ toe a vector coplanar with the vectors $\overline{a}=2\overline{i}+3\overline{j}-\overline{k}$ and $\overline{b}=\overline{j}+\overline{k}$. If $\overline{\upsilon}$ is perpendicular to $\overline{a}$ and $\overline{\upsilon}.\overline{b}=24$ then $\left|\upsilon\right|^2$ is equal to [2019]

1) 84 2) 336

3) 315 4) 256

KEY

1. 1.00	2. 4.00	3. 1.00	4. 6.00
5. 2.00	6. 8.00	7. 3	8. 2

SOLUTIONS

1. $\begin{vmatrix} a+1 & a & a \\ a & a+1 & a \\ a & a & a+1 \end{vmatrix} = 0$

$$(a+1)\left((a+1)^2-a^2\right)-a\left(a^2+a-a^2\right)+a\left(a^2-a^2-a\right)=0$$

$$\Rightarrow (a+1)(2a+1)-2a^2=0 \Rightarrow 2a^2+3a+1-2a^2=0$$

$$a=-1/3$$

$$\therefore \overline{p}=\frac{1}{3}\left(2\overline{i}-\overline{j}-\overline{k}\right)\overline{q}=\frac{1}{3}\left(-\overline{i}+2\overline{j}-\overline{k}\right),\overline{\gamma}=\frac{1}{3}\left(-\overline{i}-\overline{j}+2\overline{k}\right)$$

$$\overline{p}.\overline{q}=\frac{1}{9}(-2-2+1)=\frac{-1}{3}$$

$$\overline{\gamma}\times\overline{q}=\frac{1}{3}\left(\overline{i}+\overline{j}+\overline{k}\right) \qquad \left|\overline{\gamma}\times\overline{q}\right|=\frac{1}{3}\sqrt{1+1+1}=\frac{1}{\sqrt{3}}$$

$$\therefore \left|\overline{\gamma}\times\overline{q}\right|^2=\frac{1}{3}$$

$$3\left(\overline{p}.\overline{q}\right)^2-\lambda\left|\overline{\gamma}\times\overline{q}\right|^2=0 \Rightarrow 3.\frac{1}{9}-\lambda\frac{1}{3}=0 \Rightarrow \lambda=1$$

2. $\left|\overline{a}+\overline{b}\right|=\sqrt{1+1+2\cos\theta}=2\left|\cos\frac{\theta}{2}\right| \therefore \left|\overline{a}\right|=\left|\overline{b}\right|=1$

similarly $\left|\overline{a}-\overline{b}\right|=2\left|\sin\frac{\theta}{2}\right|$

Now

$$\sqrt{3}\left|\overline{a}+\overline{b}\right|+\left|\overline{a}-\overline{b}\right|=2\left[\sqrt{3}\left|\cos\frac{\theta}{2}\right|+\left|\sin\frac{\theta}{2}\right|\right] \text{min}$$

value $=2\sqrt{\left(\sqrt{3}\right)^2+1}=2\sqrt{3+1}=2\times 2=4$

3. Given $\left|\overline{x}+\overline{y}\right|=|x|$ squaring on both sides we get

$$\left|\overline{x}+\overline{y}\right|^2=\left|\overline{x}\right|^2$$

$$\left|\overline{x}\right|^2+\left|\overline{y}\right|^2+2\overline{x}.\overline{y}=\left|\overline{x}\right|^2 \Rightarrow 2\overline{x}.\overline{y}+\left|\overline{y}\right|^2=0 \text{ ---------}(1)$$

also $2\overline{x}+\lambda\overline{y}$ and $\overline{y}$ are perpendicular

$2\overline{x}.\overline{y}+\lambda\left|\overline{y}\right|^2=0$ ---------(2) comparing (1) and (2) $\lambda=1$

4. Projection of $\overline{b}$ on $\overline{a}$ = projection of $\overline{c}$ on $\overline{a}$

$\overline{a}.\overline{b}=\overline{a}.\overline{c},\ \left|\overline{a}\right|=2,\left|\overline{b}\right|=4,\left|\overline{c}\right|=4$ given $\overline{b}.\overline{c}=0$

now

$$\left|\overline{a}+\overline{b}-\overline{c}\right|^2=\left|\overline{a}\right|^2+\left|\overline{b}\right|^2+\left|\overline{c}\right|^2+2\overline{a}\overline{b}-2\overline{b}\overline{c}-2\overline{a}\overline{c}=4+16+16=36$$

$$\therefore \left|\overline{a}+\overline{b}-\overline{c}\right|=6$$

5. Given $\left|\overline{a}\right|=\left|\overline{b}\right|=\left|\overline{c}\right|=1$ and $\left|\overline{a}-\overline{b}\right|^2+\left|\overline{a}-\overline{c}\right|^2=8$

$$\left|\overline{a}\right|^2+\left|\overline{b}\right|^2-2\overline{a}.\overline{b}+\left|\overline{a}\right|^2+\left|\overline{c}\right|^2-2\overline{a}.\overline{c}=8$$

$$1+1-2\left(\left(\overline{a}.\overline{b}\right)+\overline{a}-\overline{c}\right)+1+1=8$$

$$-2\left(\left(\overline{a}.\overline{b}\right)+\overline{a}.\overline{c}\right)=4$$

$\overline{a}.\overline{b}+\overline{a}-\overline{c}=-2$ now

$$\left|\overline{a}+2\overline{b}\right|^2+\left|\overline{a}+2\overline{c}\right|^2=2\left|\overline{a}\right|^2+4\left|\overline{b}\right|^2+4\left|\overline{c}\right|^2+4\left(\overline{a}.\overline{b}+\overline{a}-\overline{c}\right)$$

$$=2(1)+4(1)+4(1)+4(-2)=2+8-8=2$$

6. Let P(1,-1,3) Q(2,-4,11),R(-1,2,3) and S(3,-2,10)then

$$\overline{PQ} = \overline{OQ} - \overline{OP} = \bar{i} - 3\bar{j} + 8\bar{k}$$

$$\overline{RS} = \overline{OS} - \overline{OR} = 4\bar{i} - 4\bar{j} + 7\bar{k}$$

Projection of $\overline{PQ}$ on $\overline{RS}$ is

$$= \frac{\overline{PQ}.\overline{RS}}{\left|\overline{RS}\right|} = \frac{4+12+56}{\sqrt{16+16+49}} = \frac{72}{9} = 8$$

7. Given $\bar{a}.\bar{c} = 0$ and

$$\bar{b} = 2\bar{a} \Rightarrow 6 + 6\lambda_1 + 3(\lambda_3 - 1) = 0$$

$$\begin{aligned} &3 - \lambda_2 = 2\lambda_1 \\ &\lambda_3 = -2\lambda_1 - 1 \\ &\lambda_2 = 3 - 2\lambda_1 \end{aligned} \quad \text{let } \lambda_1 = 1 \text{ then } \lambda_2 = 1, \lambda_3 = -3$$

Let $\lambda_1 = \frac{-1}{2}$ then $\lambda_2 = 4 \quad \lambda_3 = 0$

let $\lambda_1 = \frac{1}{2}$ *then* $\lambda_2 = 2, \lambda_3 = -2$

$$\therefore (\lambda_1, \lambda_2, \lambda_3) = \left(\frac{-1}{2}, 4, 0\right)$$

8. Let $\bar{\upsilon} = \lambda_1 \bar{a} + \lambda_2 \bar{b}$

$$\bar{\upsilon}.\bar{a} = \lambda_1 \bar{a}.\bar{a} + \lambda_2 \bar{a}.\bar{b}$$

$$O = 14\lambda_1 + 2\lambda_2 \Rightarrow \lambda_2 = -7\lambda_1$$

$$\bar{\upsilon}.\bar{b} = \lambda_1 \bar{a}.\bar{b} + \lambda_2 \bar{b}.\bar{b}$$

$$24 = 2\lambda_1 + 2\lambda_2 \Rightarrow 24 = -12\lambda_1 \Rightarrow \lambda_1 = -2$$

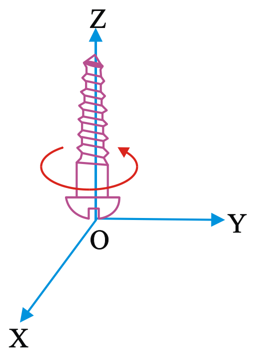
$$\lambda_2 = 14 \Rightarrow \bar{\upsilon} = -4\bar{i} + 8\bar{j} + 16\bar{k}$$

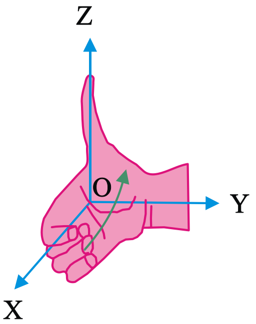
$$|\upsilon|^2 = 336$$

CROSS PRODUCT

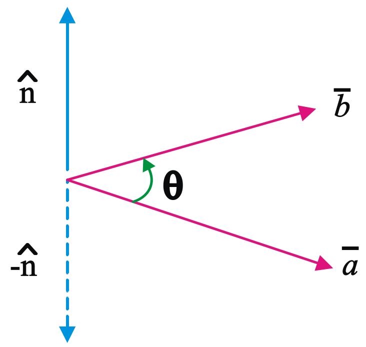
SYNOPSIS

- **Left handed & right handed system (Definition) :**
  Let $\overline{a}, \overline{b}$ and $\overline{c}$ be three non-coplanar vectors. Let O, A ,B and C be points in the space such that no three of them are collinear. Let $\overline{OA} = \overline{a}$, $\overline{OB} = \overline{b}$ and $\overline{OC} = \overline{c}$. Observing from the point C, if the angle of rotation (in the anticlock wise sense) of $\overline{OA}$ to $\overline{OB}$ does not exceed 180° then the vector triad $(\overline{a}, \overline{b}, \overline{c})$ is said to be a Right handed triad or a Right handed System.
- If $(\overline{a}, \overline{b}, \overline{c})$ is not a right handed triad then it is said to be a Left handed triad.
- If $(\overline{a}, \overline{b}, \overline{c})$ is a right handed (left handed) system then the triads $(\overline{b}, \overline{c}, \overline{a})$ and $(\overline{c}, \overline{a}, \overline{b})$ also form right handed (left handed) systems.
- If any two vectors are interchanged then the system changes from R.H.S to L.H.S or L.H.S to R.H.S.
- If any vector of a system is replaced by its additive inverse then the system changes from R.H.S to L.H.S or L.H.S to R.H.S.
- If $(\overline{a}, \overline{b}, \overline{c})$ is a right handed triad and $\overline{a}, \overline{b}, \overline{c}$ are mutually perpendicular to each other then $(\overline{a}, \overline{b}, \overline{c})$ is called an Orthogonal triad.





- **Cross Product (or) Vector Product :**
  If $\overline{a}$ and $\overline{b}$ are two vectors then the cross product or vector product of the vectors represented by $\overline{a} \times \overline{b}$ is defined as,
  $\overline{a} \times \overline{b} = |\overline{a}||\overline{b}|\sin(\overline{a}, \overline{b}).\hat{n}$
  where $\hat{n}$ is unit vector perpendicular to both $\overline{a}$ and $\overline{b}$ such that $\overline{a}, \overline{b}, \hat{n}$ form right handed system



- $\overline{a} \times \overline{b} = \overline{0}$ iff $\overline{a} = \overline{0}$ (or) $\overline{b} = \overline{0}$ (or) $\overline{a}, \overline{b}$ are parallel vectors.
- The vector product of any vector with itself is $\overline{0}$
  Thus, $\overline{a} \times \overline{a} = \overline{0}$, $\overline{b} \times \overline{b} = \overline{0}$ etc...
- If $\overline{a}$, $\overline{b}$ are non-zero and non-parallel vectors then $\overline{a} \times \overline{b}$ is a vector perpendicular to the plane determined by $\overline{a}$ and $\overline{b}$ whose magnitude is $|\overline{a}||\overline{b}|\sin\theta$, where $\theta = (\overline{a}, \overline{b})$
- $|\overline{a} \times \overline{b}| = |\overline{a}||\overline{b}|\sin\theta \le |\overline{a}||\overline{b}|$
- **Properties of cross product of vectors :**
  The cross product of vectors does not obey commutative law. i.e., $\overline{a} \times \overline{b} \ne \overline{b} \times \overline{a}$
  But $\overline{a} \times \overline{b} = -(\overline{b} \times \overline{a})$
- $|\overline{a} \times \overline{b}| = |\overline{b} \times \overline{a}|$

- Let l, m be scalars, then

  i) $(-\bar{a})\times\bar{b}=\bar{a}\times(-\bar{b})=-(\bar{a}\times\bar{b})=\bar{b}\times\bar{a}$

  ii) $(-\bar{a})\times(-\bar{b})=\bar{a}\times\bar{b}$

  iii) $(l\,\bar{a})\times\bar{b}=l(\bar{a}\times\bar{b})=\bar{a}\times(l\,\bar{b})$

  iv) $(l\,\bar{a})\times(m\,\bar{b})=lm(\bar{a}\times\bar{b})$

- **Distributive law :**

  i) $\bar{a}\times(\bar{b}+\bar{c})=\bar{a}\times\bar{b}+\bar{a}\times\bar{c}$

  ii) $(\bar{a}+\bar{b})\times\bar{c}=\bar{a}\times\bar{c}+\bar{b}\times\bar{c}$

- **Vector Products among $\bar{i},\bar{j},\bar{k}$:**

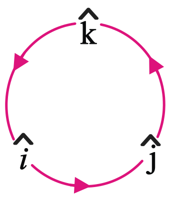
  If $\bar{i},\bar{j},\bar{k}$ are unit Orthogonal triad of vectors then

  $\bar{i}\times\bar{i}=\bar{j}\times\bar{j}=\bar{k}\times\bar{k}=\bar{0}$

  $\bar{i}\times\bar{j}=\bar{k},\ \bar{j}\times\bar{k}=\bar{i},\bar{k}\times\bar{i}=\bar{j},$

  The above result can be easily commited to memory with the help of the following table

$\times$	$\bar{i}$	$\bar{j}$	$\bar{k}$
$\bar{i}$	$\bar{0}$	$\bar{k}$	$-\bar{j}$
$\bar{j}$	$-\bar{k}$	$\bar{0}$	$\bar{i}$
$\bar{k}$	$\bar{j}$	$-\bar{i}$	$\bar{0}$



- **Evaluation of $\bar{a}\times\bar{b}$:**

  If $\bar{a}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k}=(a_1,a_2,a_3)$ and

  $\bar{b}=b_1\bar{i}+b_2\bar{j}+b_3\bar{k}=(b_1,b_2,b_3)$ then

$$\bar{a}\times\bar{b}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

- **Evaluation of $\bar{p}\times\bar{q}$ where base vectors are $\bar{a},\bar{b}\,\&\,\bar{c}$:**

  If $\bar{p}=l_1\bar{a}+l_2\bar{b}+l_3\bar{c}$, $\bar{q}=m_1\bar{a}+m_2\bar{b}+m_3\bar{c}$ are two vectors represented as a linear combination of base vectors $\bar{a},\bar{b},\bar{c},$ then

$$\bar{p}\times\bar{q}=\begin{vmatrix}\bar{b}\times\bar{c} & \bar{c}\times\bar{a} & \bar{a}\times\bar{b}\\ l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\end{vmatrix}$$

- **Angle between two vectors :**

  If $\bar{a}$ and $\bar{b}$ are two non-zero and non-collinear vectors and $(\bar{a},\bar{b})=\theta$ then $\sin\theta=\dfrac{|\bar{a}\times\bar{b}|}{|\bar{a}||\bar{b}|}$

- If $\bar{a}=(a_1,a_2,a_3)$ and $\bar{b}=(b_1,b_2,b_3)$ then

$$\sin\theta=\frac{\sqrt{\sum(a_1b_2-a_2b_1)^2}}{\sqrt{\sum a_1^2}\sqrt{\sum b_1^2}},$$

$$\cos\theta=\frac{\sum a_1b_1}{\sqrt{\sum a_1^2}\sqrt{\sum b_1^2}}$$

- **A Vector perpendicular to the given two vectors :**

  The vectors perpendicular to both $\bar{a}$ and $\bar{b}$ are given by $\lambda(\bar{a}\times\bar{b})$ where $\lambda\in R$

- The unit vectors perpendicular to $\bar{a}$ and $\bar{b}$

$$\pm\frac{(\bar{a}\times\bar{b})}{|\bar{a}\times\bar{b}|}$$

- The vectors perpendicular to $\bar{a}$ and $\bar{b}$ with magnitude m are $\pm\dfrac{m(\bar{a}\times\bar{b})}{|\bar{a}\times\bar{b}|}$

- The unit vectors perpendicular to the plane $\overline{ABC}$ are $\pm\dfrac{(\overline{AB}\times\overline{AC})}{|\overline{AB}\times\overline{AC}|}$

- The unit vectors perpendicular to the plane containing three non-collinear points $\bar{a},\bar{b}$ and $\bar{c}$ are $\pm\dfrac{(\bar{a}\times\bar{b}+\bar{b}\times\bar{c}+\bar{c}\times\bar{a})}{|\bar{a}\times\bar{b}+\bar{b}\times\bar{c}+\bar{c}\times\bar{a}|}$

- The unit vector $\overline{n}$ perpendicular to both $\overline{a}, \overline{b}$ such that $\overline{a},\overline{b},\overline{n}$ form a right handed system is $\overline{n} = \dfrac{\overline{a}\times\overline{b}}{|\overline{a}\times\overline{b}|}$
- The unit vector $\overline{n}$ perpendicular to both $\overline{a}, \overline{b}$ such that $\overline{a},\overline{b},\overline{n}$ form a left handed system is $\overline{n} = -\dfrac{\overline{a}\times\overline{b}}{|\overline{a}\times\overline{b}|}$
- **Vector equation of the line :**
  Vector equation of the line passes through the point $A(\overline{a})$ and parallel to the vector $\overline{b}$ is $(\overline{r}-\overline{a})\times\overline{b} = \overline{0}$, which is called non-parametric form of the line
- Vector equation of the line passes through two points A and B with position vectors $\overline{a}$ and $\overline{b}$ is $(\overline{r}-\overline{a})\times(\overline{b}-\overline{a}) = \overline{0}$, which is called non-parametric form of the line
- **Lagranges Identity :**
  If $\overline{a},\overline{b}$ are two vectors,

$$|\overline{a}\times\overline{b}|^2 = |\overline{a}|^2|\overline{b}|^2 - (\overline{a}.\overline{b})^2 = \begin{vmatrix} \overline{a}.\overline{a} & \overline{a}.\overline{b} \\ \overline{a}.\overline{b} & \overline{b}.\overline{b} \end{vmatrix}$$

- **Geometrical interpretation of cross product :**
  The geometrical interpretation of cross product of two non-zero non collinear vectors $\overline{a},\overline{b}$ is the vector area of a parallelogram whose adjacent sides are $\overline{a},\overline{b}$.
- **Areas :**
  Let D be a plane region bounded by a closed curve C. Let P_1, P_2, P_3 be three points on C (taken in this order).Let $\overline{n}$ be the unit vector perpendicular to the region D such that from the side of $\overline{n}$ the points P_1, P_2 and P_3 are in anticlock sense. If A is the area of the region D, then $A(\overline{n})$ is called the vector area of D. If the points P_1, P_2 and P_3 are in clock sense from the side of $\overline{n}$, then the vector area is $A(-\overline{n})$. In any case the vector area of a plane region D is either $A(\overline{n})$ or $A(-\overline{n})$ so that the area is the magnitude of the vector area.
- The vector area of ΔABC is

$$\frac{1}{2}\left(\overline{AB}\times\overline{AC}\right) = \frac{1}{2}\left(\overline{BC}\times\overline{BA}\right) = \frac{1}{2}\left(\overline{CA}\times\overline{CB}\right)$$

- If ABC is a triangle such that $\overline{AB} = \overline{a}$, $\overline{AC} = \overline{b}$ then
  (i) The vector area of triangle ABC = $\frac{1}{2}(\overline{a}\times\overline{b})$
  (ii) Area of triangle ABC = $\frac{1}{2}|(\overline{a}\times\overline{b})|$
- If $\overline{a},\overline{b},\overline{c}$ are the position vectors of the vertices A, B and C (described in counter clock sense) of ΔABC then
  (i) The Vector area of ΔABC $= \frac{1}{2}(\overline{b}\times\overline{c}+\overline{c}\times\overline{a}+\overline{a}\times\overline{b})$
  (ii) Area of $\Delta ABC = \frac{1}{2}|\overline{b}\times\overline{c}+\overline{c}\times\overline{a}+\overline{a}\times\overline{b}|$
- Let $\overline{a},\overline{b},\overline{c}$ be the position vectors of three points A,B,C then A,B,C are collinear iff $\overline{a}\times\overline{b}+\overline{b}\times\overline{c}+\overline{c}\times\overline{a} = \overline{0}$
- Let $\overline{a},\overline{b},\overline{c}$ be the position vectors of three points A,B,C then $\overline{a}\times\overline{b}+\overline{b}\times\overline{c}+\overline{c}\times\overline{a}$ is vector perpendicular to the plane $\overline{ABC}$
- If ABCD is a parallelogram and $\overline{AB} = \overline{a}$, $\overline{BC} = \overline{b}$ then
  (i) The vector area of ABCD = $(\overline{a}\times\overline{b})$
  (ii) Area of ABCD = $|\overline{a}\times\overline{b}|$
- If ABCD is a parallelogram and $\overline{AC} = \overline{d_1}$, $\overline{BD} = \overline{d_2}$ diagonals of a parallelogram then

(i) The vector area of ABCD = $\frac{1}{2}\left(\overline{d_1}\times\overline{d_2}\right)$ **(or)** $\frac{1}{2}\left(\overline{AC}\times\overline{BD}\right)$. (ii) Area = $\frac{1}{2}\left|\left(\overline{d_1}\times\overline{d_2}\right)\right|$

- Area of quadrilateral $ABCD=\frac{1}{2}\left|\overline{AC}\times\overline{BD}\right|$ square units where $\overline{AC}$ and $\overline{BD}$ are diagonals.
- **Physical Applications of cross product :** Let O be the point of reference (origin) and $\overline{OP}=\overline{r}$ be the position vector of a point P on the line of action of a force $\overline{F}$. Then the moment of the force $\overline{F}$ about O is given by $\overline{M}=\overline{r}\times\overline{F}$. $\overline{M}$ is also called vector moment
- The vector equation of a line passing through the point A ($\overline{a}$) and perpendicular to the vectors $\overline{b}$, $\overline{c}$ is $\overline{r}=\overline{a}+t\left(\overline{b}\times\overline{c}\right)$ where 't' is a scalar.
- The length of the projection of $\overline{b}$ on a vector perpendicular to $\overline{a}$ in the plane generated by $\overline{a},\overline{b}$ is $\frac{\left|\overline{a}\times\overline{b}\right|}{\left|\overline{a}\right|}$
- **Moment of a force :**
(Torque or Vector moment): Let O be the point of reference (orgin) and $\overline{OP}=\overline{r}$ be the position vector of a point P on the line of action of a force 'F'. Then the moment of the force 'F' about 'O' is given by $\overline{r}\times\overline{F}$
- If $\overline{a}=a_1\overline{l}+a_2\overline{m}+a_3\overline{n},\overline{b}=b_1\overline{l}+b_2\overline{m}+b_3\overline{n}$. Where $\overline{l},\overline{m},\overline{n}$ form a right handed system of non-coplanar vectors, then

$$\overline{a}\times\overline{b}=\begin{vmatrix}\overline{m}\times\overline{n} & \overline{n}\times\overline{l} & \overline{l}\times\overline{m}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

- **Some useful Results:** For any vector $\overline{a}$, $\left|\overline{a}\times\overline{i}\right|^2+\left|\overline{a}\times\overline{j}\right|^2+\left|\overline{a}\times\overline{k}\right|^2=2\left|\overline{a}\right|^2$
- $\overline{a},\overline{b},\overline{a}\times\overline{b}$ form a right handed system.
- $\overline{a},\ \overline{b},\ \overline{b}\times\overline{a}$ form a left handed system.
- If $\overline{a},\overline{b},\overline{c}$ are mutually perpendicular vectors then $\overline{a}\times\overline{b},\ \overline{b}\times\overline{c},\ \overline{c}\times\overline{a}$ are also mutually perpendicular vectors.
- If $\overline{a},\ \overline{b},\ \overline{c}$ are in right handed system then $\overline{a}\times\overline{b},\ \overline{b}\times\overline{c},\ \overline{c}\times\overline{a}$ are also in right handed system.

Eg : 1

If $13\overline{a}=3\overline{i}+4\overline{j}+12\overline{k}$, $13\overline{b}=4\overline{i}-12\overline{j}+3\overline{k}$, $13\overline{c}=12\overline{i}+3\overline{j}-4\overline{k}$ **then the value of** $\overline{a}\times\overline{b}$ **in terms of** $\overline{c}$ **is.**

Sol. $13\overline{a}\times13\overline{b}=\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ 3 & 4 & 12\\ 4 & -12 & 3\end{vmatrix}$

$=156\overline{i}+39\overline{j}-52\overline{k}$

$=13\left(12\overline{i}+3\overline{j}-4\overline{k}\right)=169\overline{c}$, $\therefore\left(\overline{a}\times\overline{b}\right)=\overline{c}$

Eg : 2

If $\left|\overline{a}\right|=4,\left|\overline{b}\right|=2$ **the angle between** $\overline{a}$ **and** $\overline{b}$ **is** $\pi/6$**. Then** $\left|\left(\overline{a}\times\overline{b}\right)\right|^2$ **is** **(AIE-2002)**

Sol. $\left|\left(\overline{a}\times\overline{b}\right)\right|^2=\left|\overline{a}\right|^2\left|\overline{b}\right|^2\sin^2\theta$ = (16)(4)(1/4) = 16

Eg : 3

Find the unit vector perpendicular to the plane determined by the vectors

$\overline{a}=4\overline{i}+3\overline{j}-\overline{k}$ & $\overline{b}=2\overline{i}-6\overline{j}-3\overline{k}$

Sol. $\overline{a}\times\overline{b}=\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ 4 & 3 & -1\\ 2 & -6 & -3\end{vmatrix}=5\left(-3\overline{i}+2\overline{j}-6\overline{k}\right)$

unit vector perpendicular to the plane determined by the vectors $\overline{a}$ *and* $\overline{b}$

is $\pm\frac{\overline{a}\times\overline{b}}{\left|\overline{a}\times\overline{b}\right|}=\pm\left(\frac{-3\overline{i}+2\overline{j}-6\overline{k}}{7}\right)$

Eg : 4

A force $\bar{F} = 2\bar{i} + \bar{j} - \bar{k}$ **acts at point A whose position vectors is** $2\bar{i} - \bar{j}$ **. Find the moment of force** $\bar{F}$ **about the origin.**

Sol. Given, $\bar{F} = 2\bar{i} + \bar{j} - \bar{k}$, $\overline{OA} = 2\bar{i} - \bar{j}$

Now $\overline{OA} \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & -1 & 0 \\ 2 & 1 & -1 \end{vmatrix} = \bar{i} + 2\bar{j} + 4\bar{k}$

Magnitude of moment of $\bar{F}$ about O

$= \left|\overline{OA} \times \bar{F}\right| = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{21}$

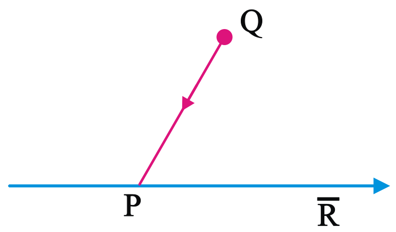
Eg : 5

Forces $2\bar{i} + \bar{j}, 2\bar{i} - 3\bar{j} + 6\bar{k}$ **and** $-\bar{i} + 2\bar{j} - \bar{k}$ **act at a point P, with position vector** $4\bar{i} - 3\bar{j} - \bar{k}$ **. Find the vector moment of the resultant of these forces about the point Q whose position vector is** $6\bar{i} + \bar{j} - 3\bar{k}$

Sol. Let $\bar{F}_1 = 2\bar{i} + \bar{j}$, $\bar{F}_2 = 2\bar{i} - 3\bar{j} + 6\bar{k}$

$\bar{F}_3 = -\bar{i} + 2\bar{j} - \bar{k}$

Thier resultant $\bar{R} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 = 3\bar{i} + 5\bar{k}$



Also $\overline{QP} = \overline{OP} - \overline{OQ}$

$= \left(4\bar{i} - 3\bar{j} - \bar{k}\right) - \left(6\bar{i} + \bar{j} - 3\bar{k}\right)$

$= -2\bar{i} - 4\bar{j} + 2\bar{k}$

Vector moment of $\bar{R}$ about the point Q

$\overline{QP} \times \bar{R} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -2 & -4 & 2 \\ 3 & 0 & 5 \end{vmatrix}$

$= (-20-0)\bar{i} - (-10-6)\bar{j} + (0+12)\bar{k}$

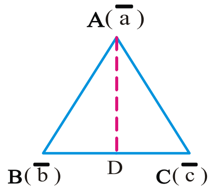
$= -20\bar{i} + 16\bar{j} + 12\bar{k}$

✈ The perpendicular distance from a point 'P' to the line joining the points A,B is $\dfrac{\left|\overline{AP} \times \overline{AB}\right|}{\left|\overline{AB}\right|}$

✈ The perpendicular distance from $A(\bar{a})$ to the line through $B(\bar{b})$ and $C(\bar{c})$ is $\dfrac{\left|\overline{BC} \times \overline{BA}\right|}{\left|\overline{BC}\right|}$

Proof: Perpendilar distance from A to BC=AD

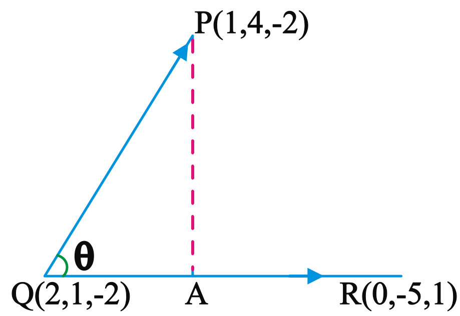
$= \dfrac{2\Delta}{(base)} = \dfrac{2.\frac{1}{2}\left|\left(\overline{BC} \times \overline{BA}\right)\right|}{\left|\overline{BC}\right|} = \dfrac{\left|\left(\overline{BC} \times \overline{BA}\right)\right|}{\left|\overline{BC}\right|}$



Eg : 6

If Q = (2,1,-2) and R = (0,-5,1). Find the perpendicular distance from P(1,4,-2) to QR.

Sol.



The perpendicular distance from P to $QR = \dfrac{\left|\overline{QP} \times \overline{QR}\right|}{\left|\overline{QR}\right|}$

$\overline{QP} = -\bar{i} + 3\bar{j}, \overline{QR} = -2\bar{i} - 6\bar{j} + 3\bar{k}$

$\therefore \overline{QP} \times \overline{QR} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -1 & 3 & 0 \\ -2 & -6 & 3 \end{vmatrix} = 9\bar{i} + 3\bar{j} + 12\bar{k}$

$\left|\overline{QP} \times \overline{QR}\right| = \sqrt{81 + 9 + 144} = 3\sqrt{26}$

$\therefore PA = \dfrac{3\sqrt{26}}{\sqrt{49}} = \dfrac{3}{7}\sqrt{26}$

EXERCISE - I

1. If $|\bar{a}|=2, |\bar{b}|=4, (\bar{a}, \bar{b})=\frac{\pi}{6}$ then $|\bar{a}\times\bar{b}|^2=$
   1) 16 2) 2 3) 775 4) 36

2. If $\bar{a}=2\bar{i}-\bar{j}+\bar{k}, \bar{b}=3\bar{i}+4\bar{j}-\bar{k}$ then $|\bar{a}\times\bar{b}|=$
   1) 9 2) $3\sqrt{10}$ 3) $\sqrt{155}$ 4) $5\sqrt{5}$

3. If $(2\bar{i}+6\bar{j}+27\bar{k})\times(\bar{i}+\lambda\bar{j}+\mu\bar{k})=\bar{0}$ then values of λ, μ are
   1) 3, 27 2) $3, \frac{27}{2}$ 3) $\frac{27}{2}, 3$ 4) $3, \frac{9}{2}$

4. If $|\bar{a}|=1, |\bar{b}|=2, (\bar{a},\bar{b})=\frac{2\pi}{3}$ then $\{(\bar{a}+3\bar{b})\times(3\bar{a}-\bar{b})\}^2=$
   1) 425 2) 375 3) 325 4) 300

5. If $\bar{a}=\bar{i}-3\bar{j}+2\bar{k}, \bar{b}=2\bar{i}+\bar{j}-\bar{k}$ then the length of the component vector of $\bar{a}\times\bar{b}$ along $5\bar{i}-\bar{k}$ is.
   1) $\sqrt{\frac{1}{13}}$ 2) $\sqrt{\frac{2}{13}}$ 3) $\sqrt{\frac{3}{13}}$ 4) $\sqrt{\frac{4}{13}}$

6. If $\bar{a}+\bar{b}+\bar{c}=\bar{0}$ then $\bar{a}\times\bar{b}=$
   1) $\bar{c}\times\bar{b}$ 2) $\bar{b}\times\bar{c}$ 3) $\bar{a}\times\bar{c}$ 4) $2\bar{b}\times\bar{c}$

7. If $\bar{a}=\bar{i}+2\bar{j}+3\bar{k}, \bar{b}=-\bar{i}+2\bar{j}+\bar{k}, \bar{c}=3\bar{i}+\bar{j}$ and $\bar{d}$ is normal to both $\bar{a}$ and $\bar{b}$, then $(\bar{c}, \bar{d})$
   1) $\cos^{-1}\left(\frac{4}{\sqrt{30}}\right)$ 2) $\sin^{-1}\left(\frac{4}{\sqrt{30}}\right)$
   3) $\cos^{-1}\left(\frac{2}{\sqrt{30}}\right)$ 4) $\sin^{-1}\left(\frac{2}{\sqrt{30}}\right)$

8. $|\bar{p}|=2, |\bar{q}|=3$ then $\frac{|\bar{p}\times\bar{q}|}{\sin(\bar{p}, \bar{q})}=$
   1) 6 2) 3/2 3) 2/3 4) 1

9. The value of $\bar{i}.\bar{i}+|\bar{i}\times\bar{j}|+\bar{j}.\bar{j}+|\bar{j}\times\bar{k}|+\bar{k}.\bar{k}+|\bar{k}\times\bar{i}|=$
   1) 0 2) 2 3) 4 4) 6

10. The value of $|\bar{i}\times\bar{j}+\bar{j}\times\bar{k}+\bar{k}\times\bar{i}|=$
    1) 0 2) 1 3) $\sqrt{3}$ 4) $\sqrt{5}$

11. $\bar{c}$ is a unit Vector orthogonal to $\bar{a}, \bar{b}$ and $\bar{a}, \bar{b}, \bar{c}$ are in R.H.S $\bar{a}=\bar{i}+\bar{j}+\bar{k}, \bar{b}=2\bar{j}+2\bar{k}$ then $\bar{c}=$
    1) $\frac{\bar{i}+\bar{j}}{\sqrt{2}}$ 2) $\frac{\bar{j}+\bar{k}}{\sqrt{2}}$ 3) $\frac{\bar{i}-\bar{k}}{\sqrt{2}}$ 4) $\frac{\bar{k}-\bar{j}}{\sqrt{2}}$

12. $\bar{a},\bar{b}$ are two vectors such that $|\bar{a}|=3$, $|\bar{b}|=\frac{\sqrt{2}}{3}$. If $\bar{a}\times\bar{b}$ is unit vector then $(\bar{a},\bar{b})=$
    1) $\frac{\pi}{6}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{3}$ 4) $\frac{\pi}{2}$

13. If $|\bar{a}|=2, |\bar{b}|=7$ and $\bar{a}\times\bar{b}=3\bar{i}+2\bar{j}+6\bar{k}$, then $(\bar{a}, \bar{b})=$
    1) 30^0 2) 60^0 3) 45^0 4) 75^0

14. If $\bar{a}=2\bar{i}+2\bar{j}+\bar{k}, \bar{b}=5\bar{i}+\bar{j}+2\bar{k}$ then $|\bar{a}\times\bar{b}|^2+(\bar{a}.\bar{b})^2=$
    1) 270 2) 120 3) 170 4) 110

15. $\frac{|\bar{a}\times\bar{b}|^2+(\bar{a}.\bar{b})^2}{2a^2b^2}$, where $|\bar{a}|=a, |\bar{b}|=b$ is
    1) 1 2) $\frac{1}{2}$ 3) 2 4) $\frac{1}{4}$

16. $\bar{a}=\bar{i}+\bar{j}, \bar{b}=2\bar{i}-\bar{k}$ & $\bar{r}\times\bar{a}=\bar{b}\times\bar{a}$, $\bar{r}\times\bar{b}=\bar{a}\times\bar{b}$ the $\bar{r}=$
    1) $-\bar{i}+\bar{j}+\bar{k}$ 2) $3\bar{i}-\bar{j}+\bar{k}$
    3) $3\bar{i}+\bar{j}-\bar{k}$ 4) $\bar{i}-\bar{j}-\bar{k}$

17. The area of the triangle formed by the points whose position vectors are $-3\bar{i}+\bar{j}$, $5\bar{i}+2\bar{j}+\bar{k}$ and $\bar{i}-2\bar{j}+3\bar{k}$ is
    1) $\sqrt{23}$ sq. units 2) $\sqrt{21}$ sq. units
    3) $\sqrt{305}$ sq. units 4) $\sqrt{33}$ sq. units

18. The area of the triangle formed by the points A (2,3,4), B (3,4,2) and C (4,2,3) is.

1) $3\sqrt{3}$ 2) $\frac{3\sqrt{3}}{2}$ 3) $\frac{\sqrt{3}}{2}$ 4) $\frac{5\sqrt{3}}{2}$

19. If the adjacent sides of a parallelogram are $\bar{i}+2\bar{j}+3\bar{k}$, and $-3\bar{i}-2\bar{j}+\bar{k}$ then the area of the parallelogram is.

1) $6\sqrt{5}$ 2) $7\sqrt{5}$ 3) $8\sqrt{5}$ 4) $5\sqrt{7}$

20. If $ABCD$ is a quadrilateral such that $\overline{AB}=\bar{i}+2\bar{j}, \overline{AD}=\bar{j}+2\bar{k}$ and $\overline{AC}=2(\bar{i}+2\bar{j})+3(\bar{j}+2\bar{k})$. then area of the quadrilateral ABCD is

1) $\frac{5\sqrt{21}}{2}$ 2) $\frac{3\sqrt{21}}{2}$ 3) $\frac{\sqrt{21}}{2}$ 4) $\frac{7}{2}$

21. If the Vectors $3\bar{i}+\bar{j}-2\bar{k}$, $\bar{i}-3\bar{j}+4\bar{k}$ are diagonals of a quadrilateral then the Vector area is

1) $\bar{i}+7\bar{j}-5\bar{k}$ 2) $\bar{i}-7\bar{j}+5\bar{k}$

3) $-\bar{i}+2\bar{j}+5\bar{k}$ 4) $-\bar{i}-7\bar{j}-5\bar{k}$

22. The area of the parallelogram constructed on the Vectors $\bar{a}=\bar{p}+2\bar{q}$ and $\bar{b}=2\bar{p}+\bar{q}$ as sides, where $\bar{p}, \bar{q}$ are unit Vectors forming an angle of 60^0 in square units is

1) $\frac{3}{2}$ 2) $\frac{3\sqrt{3}}{2}$ 3) $\frac{3\sqrt{3}}{4}$ 4) $\frac{1}{2}$

23. If $\overline{OA}=\bar{a}$, $\overline{OB}=10\bar{a}+2\bar{b}$ and $\overline{OC}=\bar{b}$, where A and C are non collinear points. Let p denote the area of the Quadrillateral OABC and q denote the area of a parallelogram with $\overline{OA}$ and $\overline{OC}$ as adjacent sides. The p/q=

1) 4 2) 6 3) 8 4) 10

24. If $\bar{a}=2\bar{i}-3\bar{j}+\bar{k}, \bar{b}=-\bar{i}+\bar{k}, \bar{c}=2\bar{j}-\bar{k}$, then the area of the parallelogram is having diagonals $\bar{a}+\bar{b}$ and $\bar{b}+\bar{c}$ (in sq.units) is

1) $\sqrt{21}$ 2) $\frac{\sqrt{21}}{2}$ 3) $\sqrt{19}$ 4) $\frac{\sqrt{19}}{2}$

25. Area of rectangle having vertices A, B, C and D with P.V's $-\bar{i}+\frac{1}{2}\bar{j}+4\bar{k}$, $\bar{i}+\frac{1}{2}\bar{j}+4\bar{k}$, $\bar{i}-\frac{1}{2}\bar{j}+4\bar{k}$ and $-\bar{i}-\frac{1}{2}\bar{j}+4\bar{k}$ respectively is

1) $\frac{1}{2}$ 2) 1 3) 2 4) 4

26. If $\bar{a}, \bar{b}, \bar{c}$ are the vertices of a triangle ABC then $|\bar{a}\times\bar{b}+\bar{b}\times\bar{c}+\bar{c}\times\bar{a}|=$

1) Area of the triangle ΔABC

2) Two times Area of the triangle ΔABC

3) Three times Area of the triangle ΔABC

4) Four times Area of the triangle ΔABC

27. If $\bar{a}$ and $\bar{b}$ are such that $|\bar{a}|=3, |\bar{b}|=2$, $(\bar{a},\bar{b})=\pi/3$ then the area of the triangle with adjacent sides $\bar{a}+2\bar{b}$ and $2\bar{a}+\bar{b}$ in sq. units is

1) $3\sqrt{3}$ 2) $9\sqrt{3}$ 3) $\frac{9\sqrt{3}}{2}$ 4) $\frac{9}{2}$

28. If $\bar{a}\times\bar{i}=\bar{j}$ then $\bar{a}.\bar{i}=$

1) any scalar 2) 0 3) 1 4) 2

29. If $\bar{a},\bar{b},\bar{c}$ are non-coplanar vectors such that $\bar{b}\times\bar{c}=\bar{a}$, $\bar{c}\times\bar{a}=\bar{b}$ and $\bar{a}\times\bar{b}=\bar{c}$, then $|\bar{a}+\bar{b}+\bar{c}|=$

1) 1 2) 2 3) 3 4) $\sqrt{3}$

30. If $\bar{r}\times\bar{a}=\bar{b}\times\bar{a}, \bar{r}\times\bar{b}=\bar{a}\times\bar{b}$, $\bar{a}\neq 0, \bar{b}\neq 0, \bar{b}\neq\lambda\bar{a}$, $\bar{a}$ is not perpendicular to $\bar{b}$ then $\bar{r}$ =

1) $\bar{a}-\bar{b}$ 2) $\bar{a}+\bar{b}$ 3) $\bar{a}\times\bar{b}+\bar{a}$ 4) $\bar{a}\times\bar{b}+\bar{b}$

31. If $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$ then $(\overline{r} \times \overline{i}).(\overline{r} \times \overline{j}) + xy$

1) 0 2) 1 3) $\overline{r}$ 4) $|\overline{r}|$

32. A(1,2,5), B(5,7,9), and C(3,2,-1), are given three points. A unit Vector normal to the plane of the triangle ABC.

1) $\dfrac{15\overline{i}+16\overline{j}-5\overline{k}}{\sqrt{506}}$ 2) $\dfrac{-15\overline{i}+16\overline{j}-5\overline{k}}{\sqrt{506}}$

3) $\dfrac{-15\overline{i}+16\overline{j}+5\overline{k}}{\sqrt{506}}$ 4) $\dfrac{\overline{i}+\overline{j}+\overline{k}}{\sqrt{3}}$

33. A vector of length $\sqrt{7}$ which is perpendicular to $2\overline{j}-\overline{k}$ and $-\overline{i}+2\overline{j}-3\overline{k}$ and makes obtuse angle with y-aixs is

1) $\dfrac{1}{\sqrt{5}}\left(4\overline{i}-\overline{j}+\sqrt{18}\overline{k}\right)$ 2) $\dfrac{1}{\sqrt{3}}\left(4\overline{i}-\overline{j}-2\overline{k}\right)$

3) $\dfrac{1}{\sqrt{3}}\left(-4\overline{i}+\overline{j}+2\overline{k}\right)$ 4) $\dfrac{1}{\sqrt{3}}\left(-4\overline{i}-\overline{j}+2\overline{k}\right)$

34. Given $\overline{a}=\overline{i}+\overline{j}-\overline{k}$, $\overline{b}=-\overline{i}+2\overline{j}+\overline{k}$ and $\overline{c}=-\overline{i}+2\overline{j}-\overline{k}$. A unit vector perpendicular to both $\overline{a}+\overline{b}$ & $\overline{b}+\overline{c}$ is

1) $\dfrac{2\overline{i}+\overline{j}+\overline{k}}{\sqrt{6}}$ 2) $\overline{j}$ 3) $\overline{k}$ 4) $\dfrac{\overline{i}+\overline{j}+\overline{k}}{\sqrt{3}}$

35. The unit vector normal to the plane containing $\overline{a} = \overline{i} - \overline{j} - \overline{k}$ and $\overline{b} = \overline{i} + \overline{j} + \overline{k}$ is

1) $\overline{j} - \overline{k}$ 2) $\dfrac{\overline{j}-\overline{k}}{\sqrt{2}}$ 3) $\dfrac{\overline{j}+\overline{k}}{\sqrt{2}}$ 4) $\dfrac{\overline{i}+\overline{j}}{\sqrt{2}}$

36. $|\overline{a}|=|\overline{b}|=2, \overline{p}=\overline{a}+\overline{b}, \overline{q}=\overline{a}-\overline{b}$, if $|\overline{p}\times\overline{q}| = 2\left[K-\left(\overline{a}.\overline{b}\right)^2\right]^{1/2}$ then K=

1) 16 2) 8 3) 4 4) 1

37. If $\overline{a} = 2\overline{i} + 2\overline{j} + 3\overline{k}$, $\overline{b} = -\overline{i} + 2\overline{j} + \overline{k}$ and $\overline{c} = 3\overline{i} + \overline{j} + \dfrac{17}{6}\overline{k}$ then $\overline{a} + t\overline{b}$ is parallel to $\overline{c}$ if t is equalto

1) $-\dfrac{4}{7}$ 2) 6 3) -3 4) 2

38. The perpendicular distance of the point (6,-4,4) on to the line joining the points A (2,1,2), B (3,-1,4) is.

1) 1 2) 2 3) 3 4) 4

39. If $\overline{AB}=\overline{b}$ and $\overline{AC}=\overline{c}$ then the length of the perpendicular from A to the line BC is

1) $\left|\dfrac{\overline{b}\times\overline{c}}{\overline{b}+\overline{c}}\right|$ 2) $\left|\dfrac{\overline{b}\times\overline{c}}{\overline{b}-\overline{c}}\right|$ 3) $\dfrac{1}{2}\left|\dfrac{\overline{b}\times\overline{c}}{\overline{b}-\overline{c}}\right|$ 4) $2\left|\dfrac{\overline{b}\times\overline{c}}{\overline{b}-\overline{c}}\right|$

40. If the projection of Vector $\overline{OA}$ on unit Vector $\overline{OB}$ equals twice the area of ΔOAB in magnitude, then $\angle AOB$ in radian is

1) 0 2) $\dfrac{\pi}{2}$ 3) $\dfrac{\pi}{4}$ 4) π

41. Let $\overline{a}=\overline{i}+\overline{j}, \overline{b}=2\overline{i}-\overline{k}$. Then the point of intersection of the lines $\overline{r}\times\overline{a}=\overline{b}\times\overline{a}$ and $\overline{r}\times\overline{b}=\overline{a}\times\overline{b}$ is

1) $3\overline{i}+\overline{j}-\overline{k}$ 2) $3\overline{i}-\overline{j}-\overline{k}$

3) $3\overline{i}-3\overline{j}-\overline{k}$ 4) $3\overline{i}+3\overline{j}+\overline{k}$

42. Point of intersection of the lines $\overline{r}\times\overline{a}=\overline{b}\times\overline{a}$ and $\overline{r}\times\overline{b}=\overline{a}\times\overline{b}$ is

1) $\overline{a}$ 2) $\overline{b}-\overline{a}$ 3) $\overline{a}-\overline{b}$ 4) $\overline{a}+\overline{b}$

43. The moment about the point $\overline{i}+2\overline{j}+3\overline{k}$ of a force represented by $\overline{i}+\overline{j}+\overline{k}$ acting through the point $2\overline{i}+3\overline{j}+\overline{k}$ is

1) $3\overline{i}-3\overline{j}$ 2) $-3\overline{i}+2\overline{j}+\overline{k}$

3) $\overline{i}-3\overline{j}$ 4) $\overline{i}-3\overline{j}+\overline{k}$

44. The moment of a force $\overline{i}+\overline{j}+\overline{k}$ acting through the point $A=-2\overline{i}+3\overline{j}+\overline{k}$ about the point $B=\overline{i}+2\overline{j}+3\overline{k}$ is

1) $3\overline{i}+\overline{j}+4\overline{k}$ 2) $3\overline{i}-\overline{j}-4\overline{k}$

3) $3\overline{i}+\overline{j}-4\overline{k}$ 4) $3\overline{i}+\overline{j}+4\overline{k}$

45. The torque about the point $3\overline{i}-\overline{j}+3\overline{k}$ of a force $4\overline{i}+2\overline{j}+\overline{k}$ through the point $5\overline{i}+2\overline{j}+4\overline{k}$, is

1) $\bar{i}+2\bar{j}-8\bar{k}$ 2) $\bar{i}+2\bar{j}+8\bar{k}$

3) $\bar{i}-2\bar{j}-8\bar{k}$ 4) $-\bar{i}-2\bar{j}-8\bar{k}$

KEY

01) 1	02) 3	03) 2	04) 4	05) 2	06) 2
07) 1	08) 1	09) 4	10) 3	11) 4	12) 2
13) 1	14) 1	15) 2	16) 3	17) 3	18) 2
19) 1	20) 1	21) 4	22) 2	23) 2	24) 2
25) 3	26) 2	27) 3	28) 1	29) 4	30) 2
31) 1	32) 2	33) 2	34) 3	35) 2	36) 1
37) 1	38) 3	39) 2	40) 3	41) 1	42) 4
43) 1	44) 3	45) 1			

SOLUTIONS

1. $\left|\bar{a}\times\bar{b}\right|^2=|\bar{a}|^2\left|\bar{b}\right|^2\sin^2\left(\bar{a},\bar{b}\right)$

2. $\bar{a}\times\bar{b}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 2 & -1 & 1\\ 3 & 4 & -1\end{vmatrix}$

3. $\left(6\mu-27\lambda\right)\bar{i}+(27-2\mu)\bar{j}\quad+\left(2\lambda-6\right)\bar{k}=\bar{0}$

$\Rightarrow\lambda=3,\ \mu=\dfrac{27}{2}$

4. **Given** $\left|\bar{a}\right|=1,\left|\bar{b}\right|=2,\left(\bar{a},\bar{b}\right)=\dfrac{2\pi}{3}$

$$\left\{\left(\bar{a}+3\bar{b}\right)\times\left(3\bar{a}-\bar{b}\right)\right\}^2=\left(3\left(\bar{a}\times\bar{a}\right)-\bar{a}\times\bar{b}+9\left(\bar{a}\times\bar{b}\right)-3\left(\bar{b}\times\bar{b}\right)\right)^2=\left(0-10\left(\bar{a}\times\bar{b}\right)\right)^2$$

$$=100\left|\bar{a}\times\bar{b}\right|^2=100\left|\bar{a}\right|^2|\bar{b}|^2\sin^2\left(\bar{a},\bar{b}\right)$$

$$=100\times1\times4\sin^2(120)$$

5. **given** $\bar{a}=\bar{i}-3\bar{j}+2\bar{k},\ \bar{b}=2\bar{i}+\bar{j}-\bar{k}$

$$\bar{a}\times\bar{b}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 1 & -3 & 2\\ 2 & 1 & -1\end{vmatrix}=\bar{i}(3-2)-\bar{j}(-1-4)+\bar{k}(1+6)=\bar{i}+5\bar{j}+7\bar{k}$$

length of component of $\bar{a}\times\bar{b}$ **along** $5\bar{i}-\bar{k}$

$$=\frac{\left|\left(\bar{a}\times\bar{b}\right).\left(5\bar{i}-\bar{k}\right)\right|}{\left|5\bar{i}-\bar{k}\right|}=\frac{\left|\left(\bar{i}+5\bar{j}+7\bar{k}\right).\left(5\bar{i}-\bar{k}\right)\right|}{\left|5\bar{i}-\bar{k}\right|}$$

$$=\frac{|5+0-7|}{\sqrt{26}}=\frac{2}{\sqrt{26}}=\sqrt{\frac{2}{13}}$$

6.

$$=100\times4\times\frac{3}{4}=300$$

7. $\bar{d}=\bar{a}\times\bar{b}$, $\cos\theta=\dfrac{\bar{d}.\bar{c}}{\left|\bar{d}\right|\left|\bar{c}\right|}$

8. $\dfrac{\bar{p}\times\bar{q}}{\sin\left(\bar{p}.\bar{q}\right)}=\dfrac{|\bar{p}||\bar{q}|\sin\left(\bar{p}.\bar{q}\right)}{\sin\left(\bar{p}.\bar{q}\right)}=|\bar{p}||\bar{q}|=6$

9. $1+1+1+1+1+1=6$

10. $\left|\bar{k}+\bar{i}+\bar{j}\right|=\sqrt{3}$

11. $\bar{c}=\dfrac{\bar{a}\times\bar{b}}{\left|\bar{a}\times\bar{b}\right|}$

12. $\left|\bar{a}\times\bar{b}\right|=1\ \Rightarrow\left|\bar{a}\right|\left|\bar{b}\right|\sin\left(\bar{a},\bar{b}\right)=1$

13. given $\left|\bar{a}\right|=2,\left|\bar{b}\right|=7$ and $\bar{a}\times\bar{b}=3\bar{i}+2\bar{j}+6\bar{k}$

$$\sin\left(\bar{a},\bar{b}\right)=\frac{\left|\bar{a}\times\bar{b}\right|}{\left|\bar{a}\right|\left|\bar{b}\right|}=\frac{\sqrt{9+4+36}}{2\times7}=\frac{7}{2\times7}=\frac{1}{2}$$

$$\left(\bar{a},\bar{b}\right)=30^0=\frac{\pi}{6}$$

14. $\left(\bar{a}\times\bar{b}\right)^2+\left(\bar{a}\,.\bar{b}\right)^2=|\bar{a}|^2\left|\bar{b}\right|^2$

15. $\left(\bar{a}\times\bar{b}\right)^2+\left(\bar{a}\,.\bar{b}\right)^2=|\bar{a}|^2\left|\bar{b}\right|^2$

16. $\bar{a}=\bar{i}+\bar{j},\bar{b}=2\bar{i}-\bar{k}$ and $\bar{\gamma}\times\bar{a}=\bar{b}\times\bar{a},\bar{\gamma}\times\bar{b}=\bar{a}\times\bar{b}$

$$\gamma\times\bar{a}=-\left(\bar{a}\times\bar{b}\right),\bar{\gamma}\times\bar{b}=\bar{a}\times\bar{b}$$

$$\bar{\gamma}\times\bar{a}=-\left(\bar{\gamma}\times\bar{b}\right)$$

$$\bar{\gamma}\times\left(\bar{a}\times\bar{b}\right)=0$$

$$\gamma=\bar{a}+\bar{b}=3\bar{i}+\bar{j}-\bar{k}$$

17. Given $\overline{OA}=-3\bar{i}+\bar{j},\overline{OB}=5\bar{i}+2\bar{j}+\bar{k}$

$$\overline{OC}=\bar{i}-2\bar{j}+3\bar{k}$$

$$\overline{AB}=\overline{OB}-\overline{OA}=8\bar{i}+\bar{j}+\bar{k}$$

$\overline{AC} = \overline{OC} - \overline{OA} = 4\bar{i} - 3\bar{j} + 3\bar{k}$

area of $\Delta ABC = \frac{1}{2}\left|\overline{AB} \times \overline{AC}\right|$

$$\overline{AB} \times \overline{AC} = \begin{vmatrix} i & j & k \\ 8 & 1 & 1 \\ 4 & -3 & 3 \end{vmatrix} = \bar{i}(6) = \bar{j}(20) + \bar{k}(-28)$$

$= \frac{1}{2}\sqrt{36 + 400 + 784}$

$= \frac{1}{2}\sqrt{1220} = \sqrt{305}$

18. $\frac{1}{2}\left|\overline{AB} \times \overline{AC}\right|$

19. $\left|\bar{a} \times \bar{b}\right|$

20. $\frac{1}{2}\left|\overline{AC} \times \overline{BD}\right|$

21. $\frac{1}{2}\left(\bar{a} \times \bar{b}\right)$

22. $\left|\bar{a} \times \bar{b}\right| = \left|-3\left(\bar{p} \times \bar{q}\right)\right|$

$= 3\left|\bar{p}\right|\left|\bar{q}\right|\sin\left(\bar{p}, \bar{q}\right)$

23. $p = \frac{1}{2}\left|\overline{OB} \times \overline{AC}\right| = \frac{1}{2}\left|\left(10\bar{a} + 2\bar{b}\right) \times \left(\bar{b} - \bar{a}\right)\right|$

$= 6\left|\bar{a} \times \bar{b}\right|$

$q = \left|\overline{OA} \times \overline{OC}\right| = \left|\bar{a} \times \bar{b}\right|$, $\frac{p}{q} = 6$

24. $\bar{x} = \bar{a} + \bar{b} = \bar{i} - 3\bar{j} + 2\bar{k}$

$\bar{y} = \bar{b} + \bar{c} = -\bar{i} + 2\bar{j}$, $\frac{1}{2}\left|\bar{x} \times \bar{y}\right|$

25. Area = $\left|\overline{AB} \times \overline{AD}\right|$

26. $\Delta = \frac{1}{2}\left|\overline{AB} \times \overline{AC}\right|$, $\Delta = \frac{1}{2}\left|\left(\bar{b} - \bar{a}\right) \times \left(\bar{c} - \bar{b}\right)\right|$

$\Delta = \frac{1}{2}\left|\bar{a} \times \bar{b} + \bar{b} \times \bar{c} + \bar{c} \times \bar{a}\right|$

$2\Delta = \left|\bar{a} \times \bar{b} + \bar{b} \times \bar{c} + \bar{c} \times \bar{a}\right|$

27. $\frac{1}{2}\left|\left(\bar{a} + 2\bar{b}\right) \times \left(2\bar{a} + \bar{b}\right)\right|$

$\frac{1}{2}\left|-3\left(\bar{a} \times \bar{b}\right)\right| = \frac{3}{2}\left|\bar{a} \times \bar{b}\right|$

28. given $\bar{a} \times \bar{i} = \bar{j}$

let $\bar{a} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\bar{a} = \bar{i}\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x & y & z \\ 1 & 0 & 0 \end{vmatrix} = i(0) - j(0 - z) + \bar{k}(0 - y) = z\bar{j} - \bar{k}y$$

$\bar{a} + i = \bar{j}$

$z\bar{j} - \bar{k}y = j$

$z = 1, y = 0$

$\bar{a} = x\bar{i} + \bar{k}$

now $\bar{a}.\mathrm{i} = \left(x\bar{i} + \bar{k}\right).i = x$ **anyscalar**

29. We have,

$\bar{a} \times \bar{b} = \bar{c},\ \bar{b} \times \bar{c} = \bar{a},\ and\ \bar{c} \times \bar{a} = \bar{b}$

$\Rightarrow \bar{a} \perp \bar{b} \perp \bar{c}\ \ and\ \Rightarrow \left|\bar{a}\right| = \left|\bar{b}\right| = \left|\bar{c}\right| = 1$

$\Rightarrow \left|\bar{a}\right|^2 + \left|\bar{b}\right|^2 + \left|\bar{c}\right|^2 = 3\ \left[\because \bar{a}.\bar{b} = \bar{b}.\bar{c} = \bar{c}.\bar{a} = 0\right]$

$\Rightarrow \left|\bar{a} + \bar{b} + \bar{c}\right|^2 = 3\ \Rightarrow \left|\bar{a} + \bar{b} + \bar{c}\right| = \sqrt{3}$

30. $\bar{r} \times \bar{a} = -\left(\bar{a} \times \bar{b}\right), \bar{r} \times \bar{b} = \bar{a} \times \bar{b}$

$\bar{r} \times \bar{a} = -\bar{r} \times \bar{b}$, $\bar{r} \times \left(\bar{a} + \bar{b}\right) = \bar{0}$

$\bar{r} \parallel \left(\bar{a} + \bar{b}\right)$, $\quad \bar{r} = \lambda\left(\bar{a} + \bar{b}\right)$, $\lambda = 1$

31. Find $\bar{r} \times \bar{i}$ and $\bar{r} \times \bar{j}$

32. Apply formula $\left(\pm \frac{\overline{AB} \times \overline{AC}}{\left|\overline{AB} \times \overline{AC}\right|}\right)$

33. **Le** $\bar{a} = 2\bar{j} - \bar{k}$ $\quad \bar{b} = -\bar{i} + 2\bar{j} - 3\bar{k}$ **required vector is** $\sqrt{7}\frac{\left(\bar{a} \times \bar{b}\right)}{\left|\bar{a} \times \bar{b}\right|}$ **coefficient of j<0**

$$\bar{a}\times\bar{b}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 0 & 2 & -1\\ -1 & 2 & -3\end{vmatrix}=i(-4)-\bar{j}(0-1)+\bar{k}(0+2)$$

$$=-4\bar{i}+\bar{j}+2\bar{k}=4\bar{i}-\bar{j}-2\bar{k}$$

$$|\bar{a}\times\bar{b}|=\sqrt{16+1+4}=\sqrt{21}=\sqrt{3}\sqrt{7}$$

required vector $=\dfrac{4i-j+2\bar{k}}{\sqrt{3}}$

34. $\bar{x}=\bar{a}+\bar{b}, \bar{y}=\bar{b}+\bar{c}$ unit vector $=\pm\dfrac{(\bar{x}\times\bar{y})}{|\bar{x}\times\bar{y}|}$

35. $\pm\dfrac{(\bar{a}\times\bar{b})}{(\bar{a}\times\bar{b})}$

36. Put $|\bar{p}\times\bar{q}|=2\sqrt{2^4-(\bar{p}.\bar{q})^2}$

37. **Given**

$$\bar{a}=2\bar{i}+2\bar{j}+3\bar{k}, \bar{b}=-\bar{i}+2\bar{j}+\bar{k}, \bar{c}=3\bar{i}+\bar{j}+\frac{17}{6}\bar{k}$$

Now

$$\bar{a}\times t\bar{b}=2\bar{i}+2\bar{j}+3\bar{k}+t(-\bar{i}+2\bar{j}+\bar{k})=i(2-t)+j(2+2t)+\bar{k}(3+t)$$

given $\bar{a}+t\bar{b}$ **is parallel to** $\bar{c}$

$$\frac{2-t}{3}=\frac{2+2t}{1}=\frac{3+t}{\frac{17}{6}}$$

$$\Rightarrow\frac{2-t}{3}=2+2t$$

$$2-t=6+6t$$

$$-4=7t\Rightarrow t=\frac{-4}{7}$$

38. $C(6,-4,4), A(2,1,2), B(3,-1,4)$

$$\frac{|\overline{CA}\times\overline{AB}|}{|\overline{AB}|}$$

39. $\dfrac{|\overline{AB}\times\overline{BC}|}{\overline{BC}}=\dfrac{|\bar{b}\times(\overline{BA}+\overline{AC})}{|\overline{BC}|}=\dfrac{|\bar{b}\times(-\bar{b}+\bar{c})|}{|\bar{c}-\bar{b}|}=\dfrac{|\bar{b}\times\bar{c}|}{|\bar{b}-\bar{c}|}$

40. $\overline{OA}.\overline{OB}=2\frac{1}{2}|\overline{OA}\times\overline{OB}|$

$$|\overline{OA}||\overline{OB}|\cos(\overline{OA},\overline{OB})=|\overline{OA}||\overline{OB}|\sin(\overline{OA},\overline{OB})$$

$$(\overline{OA},\overline{OB})=\frac{\pi}{4}$$

41. The equation of the two lines are

$(\bar{r}-\bar{b})\times\bar{a}=\bar{0}$ *and* $(\bar{r}-\bar{a})\times\bar{b}=\bar{0}$

$\therefore(\bar{r}-\bar{b})$ is parallel to $\bar{a}$ and $(\bar{r}-\bar{a})$ is parallel to $\bar{b}$, $\therefore\bar{r}-\bar{b}+p\bar{a},\bar{r}=\bar{a}+q\bar{b}$

For their of intersection we have identical values of $\bar{r}$. $\therefore p=q=1$

Hence , $\therefore\bar{r}=\bar{a}+\bar{b}$

42. The equation of the two lines are

$(\bar{r}-\bar{b})\times\bar{a}=0$ *and* $(\bar{r}-\bar{a})\times\bar{b}=0$

$\therefore(\bar{r}-\bar{b})$ is parallel to $\bar{a}$

and $(\bar{r}-\bar{a})$ is parallel to $\bar{b}$

$\therefore\bar{r}=\bar{b}+p\bar{a},\bar{r}=\bar{a}+q\bar{b}$

For their of intersection we have identical values of $\bar{r}$. ,

$\therefore p=q=1$ Hence , $\therefore\bar{r}=\bar{a}+\bar{b}$

43. Here , $\bar{r}=(2\bar{i}+3\bar{j}+\bar{k})-(\bar{i}+2\bar{j}+3\bar{k})$

$$\bar{F}=\bar{i}+\bar{j}+\bar{k}$$

$$\bar{r}\times\bar{F}=(\bar{i}+\bar{j}-2\bar{k})\times(\bar{i}+\bar{j}+\bar{k})$$

$$=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 1 & 1 & -2\\ 1 & 1 & 1\end{vmatrix}=3\bar{i}-3\bar{j}$$

44. $\overline{M}=\overline{BA}\times\bar{F}$

45. We have, $\overline{F} = 4\overline{i} + 2\overline{j} + \overline{k}$

$\overline{OP} = 2\overline{i} + 3\overline{j} + \overline{k} = \overline{r}$

EXERCISE - II

1. If $|\overline{a}|=2, |\overline{b}| = 4$ **then** $\dfrac{|\overline{a}\times\overline{b}|^2}{1-\cos^2(\overline{a}, \overline{b})} =$

1) 8 2) 2 3) 64 4) 32

2. If $\overline{a}=2\overline{i}+\overline{j}+3\overline{k}, \overline{b}=p\overline{i}+\overline{j}+q\overline{k}$ **and** $\overline{b}\times\overline{a}=\overline{0}$ **then**

1) (p, q) = (2, 3) 2) (p, q) = (-2, -3)
3) (p, q) = (1, 2) 4) (p, q) = (-1, -2)

3. Three vectors $\overline{a},\overline{b},\overline{c}$ **are such that** $\overline{a}\times\overline{b} = 3(\overline{a}\times\overline{c})$**. Also** $|\overline{a}|=|\overline{b}|=1, |\overline{c}|=\frac{1}{3}$**. If the angle between** $\overline{b}$ **and** $\overline{c}$ **is** 60^0**, then**

1) $\overline{b} = 3\overline{c}+\overline{a}$ 2) $\overline{b} = 3\overline{a}-\overline{c}$
3) $\overline{b} = 3\overline{c}+2\overline{a}$ 4) $\overline{b} = 3\overline{c}-2\overline{a}$

4. Let $\overline{a},\overline{b},\overline{c}$ **be unit vectors such that** $\overline{a}.\overline{b}=\overline{a}.\overline{c}=0$ **and the angle between** $\overline{b}$ **and** $\overline{c}$ **is** $\frac{\pi}{6}$ **if** $\overline{a}=n(\overline{b}\times\overline{c})$**, then value of n is**

1) ± 1 2) ± 2 3) $\pm\sqrt{3}$ 4) 0

5. Let $\overline{a}=2\overline{i}+\overline{k}, \overline{b}=\overline{i}+\overline{j}+\overline{k}$ **and** $\overline{c}=4\overline{i}-3\overline{j}+7\overline{k}$ **be three vectors. The vector which satisfies** $\overline{r}\times\overline{b}=\overline{c}\times\overline{b}$ **and** $\overline{r}.\overline{a}=0$ **is**

1) $\overline{i}+8\overline{j}+2\overline{k}$ 2) $-\overline{i}-8\overline{j}+2\overline{k}$
3) $-\overline{i}-8\overline{j}-2\overline{k}$ 4) $\overline{i}+8\overline{j}-2\overline{k}$

6. If a vector $\overline{r}$ **satisfies the equation** $\overline{r}\times(\overline{i}+2\overline{j}+\overline{k})=\overline{i}-\overline{k}$**, then** $\overline{r}$ **is equal to**

1) $\overline{i}+3\overline{j}+\overline{k}$ 2) $3\overline{i}-7\overline{j}-3\overline{k}$
3) $\overline{k}+t(\overline{i}+2\overline{j}+\overline{k})$ where t is any scalar
4) $2\overline{i}+(t+3)\overline{j}-5\overline{k}$ where t is any scalar

7. $\overline{a},\overline{b}$ **are such that** $|\overline{a}|=\sqrt{3}$ **,** $|\overline{b}|=2$ **and** $(\overline{a},\overline{b})=\frac{\pi}{3}$**. Then the area of the triangle with adjacent sides** $\overline{a}+2\overline{b}$ **and** $2\overline{a}+\overline{b}$ **is**

1) $5\sqrt{3}$ 2) 15 3) 9/2 4) 15/2

8. ABCD is a quadrilateral with $\overline{AB}=\overline{a}, \overline{AD}=\overline{b}, \overline{AC}=2\overline{a}+3\overline{b}$**. If the area of parallelogram ABCD is p times the area of the parallelogram with AB, AD as adjacent sides, then p is equal to**

1) 5 2) 5/2 3) 1 4) 1/2

9. If $\overline{a}\times\overline{b}=\overline{c}\times\overline{d}, \overline{a}\times\overline{c}=\overline{b}\times\overline{d}$ **then**

1) $\overline{a}-\overline{d}$ is parallel to $\overline{b}-\overline{c}$
2) $\overline{a}-\overline{b}$ is parallel to $\overline{c}-\overline{d}$
3) $\overline{a}-\overline{c}$ is parallel to $\overline{b}-\overline{d}$
4) $\overline{a}+\overline{b}$ is parallel to $\overline{c}+\overline{d}$

10. If $\overline{a},\overline{b},\overline{c}$ **be three vectors such that** $|\overline{a}+\overline{b}+\overline{c}|=1, \overline{c}=\lambda(\overline{a}\times\overline{b})$ **and** $|\overline{a}|=\frac{1}{\sqrt{2}}$, $|\overline{b}|=\frac{1}{\sqrt{3}}, |\overline{c}|=\frac{1}{\sqrt{6}}$**, then the angle between** $\overline{a}$ **and** $\overline{b}$ **is**

1) $\frac{\pi}{6}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{3}$ 4) $\frac{\pi}{2}$

11. $\overline{a}+2\overline{b}+3\overline{c}=\overline{0}$ **and** $\overline{a}\times\overline{b}+\overline{b}\times\overline{c}+\overline{c}\times\overline{a}=\ell(\overline{b}\times\overline{c})$**, then** $\ell =$

1) 2 2) 4 3) 6 4) 8

12. Let $\overline{a},\overline{b},\overline{c}$ **be three vectors satisfying** $\overline{a}\times\overline{b}=2\overline{a}\times\overline{c}$**,** $|\overline{a}|=|\overline{c}|=1, \overline{b}=4$ **and** $|\overline{b}\times\overline{c}|=\sqrt{15}$**. If** $\overline{b}-2\overline{c}=\lambda\overline{a}$ **then** λ **is**

1) 1 2) -1 3) 2 4) ± 4

13. If α, β are roots of the equation $x^2+2x+5=0$ and $\bar{a}=(\alpha+\beta)\bar{i}+\alpha\beta\bar{j}$,

$\bar{b}=\alpha\beta\bar{i}+(\alpha+\beta)\bar{j}+(\alpha^2+\beta^2)\bar{k}$ then

$\bar{a}\times\bar{b}=$

1) $\bar{i}+12\bar{j}+12\bar{k}$ 2) $-30\bar{i}+12\bar{j}-5\bar{k}$

3) $-30\bar{i}-12\bar{j}-21\bar{k}$ 4) $\bar{i}-12\bar{j}+29\bar{k}$

14. If $\bar{x}$ and $\bar{y}$ are two non-collinear vectors and ABC is a triangle with sides a, b, c satisfying $(20a-15b)\bar{x}+(15b-12c)\bar{y}+(12c-20a)$ $(\bar{x}\times\bar{y})=\bar{0}$**, then the triangle ABC is**

1) an acute angle triangle
2) an obtuse angle triangle
3) a right angle triangle
4) an isosceles triangle

15. If A (1, 2, 3), B (2, 3, 1), C (3, 1, 2) then the length of the altitude through C is

1) 3 2) $3\sqrt{3}$ 3) $3\sqrt{2}$ 4) $3/\sqrt{2}$

16. If $\bar{a}=\bar{i}+\bar{j}, \bar{b}=2\bar{j}-\bar{k}$ & $\bar{r}\times\bar{a}=\bar{b}\times\bar{a}$, $\bar{r}\times\bar{b}=\bar{a}\times\bar{b}$, **then** $\frac{\bar{r}}{|\bar{r}|}=$

1) $\frac{1}{\sqrt{11}}(\bar{i}-3\bar{j}+\bar{k})$ 2) $\frac{1}{\sqrt{13}}(\bar{i}-\bar{j}+\bar{k})$

3) $\frac{1}{\sqrt{11}}(\bar{i}+3\bar{j}-\bar{k})$ 4) $\frac{1}{\sqrt{13}}(\bar{i}+\bar{j}-\bar{k})$

17. The vector equation of the line passing through the point $\bar{i}-2\bar{j}+\bar{k}$ and perpendicular to the vectors $2\bar{i}-3\bar{j}-\bar{k}$, $\bar{i}+4\bar{j}-2\bar{k}$ **is**

1) $\bar{r}=(\bar{i}-2\bar{j}+\bar{k})+t(\bar{i}-7\bar{j}+\bar{k})$

2) $\bar{r}=(\bar{i}-2\bar{j}+\bar{k})+t(3\bar{i}+\bar{j}-3\bar{k})$

3) $\bar{r}=(\bar{i}-2\bar{j}+\bar{k})+t(10\bar{i}+3\bar{j}+11\bar{k})$

4) $\bar{r}=\bar{i}$

18. If the position vectors of the three points A,B,C, are $\bar{i}+\bar{j}+\bar{k}$, $2\bar{i}+3\bar{j}-4\bar{k}$ and $7\bar{i}+4\bar{j}+9\bar{k}$, then the unit vector perpendicular to the plane of the triangle ABC is

1) $(31\bar{i}-38\bar{j}-9\bar{k})/2486$

2) $(31\bar{i}-38\bar{j}+9\bar{k})/\sqrt{2486}$

3) $(31\bar{i}-38\bar{j}-9\bar{k})/\sqrt{2486}$

4) $(31\bar{i}+38\bar{j}+9\bar{k})/2486$

19. A unit vector making an obtuse angle with x-axis and perpendicular to the plane containing the points

$\bar{i}+2\bar{j}+3\bar{k}, 2\bar{i}+3\bar{j}+4\bar{k}$ **and** $\bar{i}+5\bar{j}+7\bar{k}$ **also makes an obtuse angle with**

1) y-axis 2) z-axis
3) both y and z axes 4) both x and y axes

20. A force $\bar{F}=2\bar{i}-\lambda\bar{j}+5\bar{k}$ is applied at the point A(1,2,5). If its moment about the point (-1,-2,3) is $16\bar{i}-6\bar{j}+2\lambda\bar{k}$ then $\lambda=$

1) -2 2) -1 3) 0 4) 2

KEY

01) 3	02) 1	03) 1	04) 2	05) 2	06) 1
07) 3	08) 2	09) 1	10) 4	11) 3	12) 4
13) 3	14) 3	15) 4	16)3	17) 3	18) 2
19) 2	20) 1				

SOLUTIONS

1. $\frac{|\bar{a}|^2|\bar{b}|^2\sin^2(\bar{a},\bar{b})}{\sin^2(\bar{a},\bar{b})}=|a|^2|b|^2$

2. $\bar{b}\,||\,\bar{a}$, $\frac{p}{2}=\frac{1}{1}=\frac{q}{3}$, $p=2, q=3$

3. $\bar{a}\times\bar{b}=3(\bar{a}\times\bar{c}) \quad \Rightarrow \bar{a}\times(\bar{b}-3\bar{c})=\bar{0}$

$\Rightarrow \bar{a}\,||\,(\bar{b}-3\bar{c})=\bar{0} \quad \Rightarrow \bar{a}\,||\,\bar{b}-3\bar{c}$

$\Rightarrow \bar{b}-3\bar{c}=\lambda\bar{a} \quad \Rightarrow |\bar{b}-3\bar{c}|^2=\lambda^2|\bar{a}|^2$

$\Rightarrow |\overline{b}|^2 + 9|\overline{c}|^2 - 6(\overline{b}\,.\,\overline{c}) = \lambda^2 |\overline{a}|^2$

$\Rightarrow\ 2 - 6 \times 1 \times \frac{1}{3} \cos 60^0 = \lambda^2$

4. Given $\overline{a}.\overline{b} = 0 \Rightarrow \overline{a}$ is perpendicular to $\overline{b}$.

$\overline{a}\,.\,\overline{c} = 0 \Rightarrow \overline{a}$ is perpendicular to $\overline{c}$

$\therefore \overline{a}$ is perpendicular to the plane of $\overline{b}$ and $\overline{c}$. Also $\overline{a}$ is a unit vector.

$\therefore\ \overline{a} = \pm \frac{\overline{b} \times \overline{c}}{|\overline{b} \times \overline{c}|}$ .... (1)

But $|\overline{b} \times \overline{c}| = |\overline{b}||\overline{c}| \sin \frac{\pi}{6} = 1.1.\frac{1}{2}$.

$\therefore$ from (1) we have $\overline{a} = \pm 2(\overline{b} \times \overline{c})\ \therefore n = \pm 2$.

5. **Given** $\overline{a} = 2\overline{i} + \overline{k}, \overline{b} = \overline{i} + \overline{j} + \overline{k}$ **and** $\overline{c} = 4\overline{i} - 3\overline{j} + 7\overline{k}$

$2(4+\lambda) + 0 + (\lambda + 7) = 0$

$3\lambda + 15 = 0$

$\lambda = -5$

$\therefore \overline{\gamma} = -\overline{i} - 8\overline{j} + 2\overline{k}$

6. Let $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$

$\overline{\gamma} \times \overline{b} = \overline{c} \times \overline{b}$ **and** $\overline{\gamma}.\overline{a} = 0$

$(\overline{\gamma} - \overline{c}) \times \overline{b} = 0$

let $\overline{\gamma} = x\overline{i} + y\overline{j} + z\overline{k}$

$\overline{\gamma} - \overline{c} = \lambda \overline{b}$

$\overline{\gamma} = \overline{c} + \lambda \overline{b}$

$x\overline{i} + y\overline{j} + z\overline{k} = \overline{i}(4+\lambda) + \overline{j}(\lambda - 3) + \overline{k}(\lambda + 3)$

$x = 4 + \lambda, y = \lambda - 3, z = \lambda + 3$

$\overline{\gamma} = (4+\lambda)\overline{i} + (\lambda - 2)\overline{j} + (\lambda + 7)\overline{k}, \overline{a} = 2\overline{i} + \overline{k}$

$\overline{\gamma}.\overline{a} = 0$

$\therefore (x\overline{i} + y\overline{j} + z\overline{k}) \times (\overline{i} + 2\overline{j} + \overline{k}) = \overline{i} - \overline{k}$

$\Rightarrow \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ x & y & z \\ 1 & 2 & 1 \end{vmatrix} = \overline{i} - \overline{k}$

$\Rightarrow \overline{i}(y - 2z) - \overline{j}(x - z) + \overline{k}(2x - y) = \overline{i} - \overline{k}$

On comparing $z = x,\ y = 2x + 1$

$\therefore \overline{r} = x\overline{i} + (2x+1)\overline{j} + x\overline{k}$ ...(1), For x = 1 x = 3

$\overline{r} = \overline{i} + 3\overline{j} + \overline{k}$ *and* $\overline{r} = 3\overline{i} + 7\overline{j} + 3\overline{k}$

Also (1) $\Rightarrow \overline{r} = \overline{j} + x(\overline{i} + 2\overline{j} + \overline{k})$ or

$\overline{r} = \overline{j} + t(\overline{i} + 2\overline{j} + \overline{k})$ where t is scalar.

7. $\frac{1}{2}|(\overline{a} + 2\overline{b}) \times (2\overline{a} + \overline{b})|$

$\frac{1}{2}|-3(\overline{a} \times \overline{b})| = \frac{3}{2}|\overline{a} \times \overline{b}| = \frac{3}{2}|\overline{a}||\overline{b}| \sin \frac{\pi}{3}$

8. $\Rightarrow \overline{c}\,.\,\overline{a} = \overline{c}\,.\,\overline{b} = 0$, $|\overline{a} + \overline{b} + \overline{c}|^2 = 1$

$\Rightarrow \frac{1}{2} + \frac{1}{3} + \frac{1}{6} + 2\{|\overline{a}||\overline{b}| \cos \theta\} = 1$

$\Rightarrow \cos \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

10.

. **Given** $\overline{AC} = 2\overline{a} + 3\overline{b}, \overline{AB} = \overline{a}, \overline{AD} = \overline{b}$ **and**

$\frac{1}{2}|\overline{AC} \times \overline{BD}| = P|\overline{AB} \times \overline{AD}|$

$\frac{1}{2}|(2\overline{a} + 3\overline{b}) \times (\overline{b} - \overline{a})| = P|\overline{a} \times \overline{b}| \Rightarrow \frac{1}{2}|2(\overline{a} \times \overline{b}) + 3(\overline{a} \times \overline{b})| = P|\overline{a} \times \overline{b}|$

$\frac{5}{2}|\overline{a} \times \overline{b}| = P|\overline{a} \times \overline{b}|$

$P = \frac{5}{2}$

9. $\left(\overline{a}-\overline{d}\right)\times\left(\overline{b}-\overline{c}\right)=\overline{0}$, $\quad \left(\overline{a}-\overline{d}\right)\parallel\left(\overline{b}-\overline{c}\right)$

11. $p\overline{a}+q\overline{b}+r\overline{c}=0$

$$\Rightarrow \overline{a}\times\overline{b}+\overline{b}\times\overline{c}+\overline{c}\times\overline{a}=\left(\frac{p+q+r}{p}\right)\left(\overline{b}\times\overline{c}\right)$$

12. If angle between $\overline{b}\,\&\,\overline{c}$ is α and $\left|\overline{b}\times\overline{c}\right|=\sqrt{15}$

$$\left|\overline{b}\right|\left|\overline{c}\right|\sin\alpha=\sqrt{15}\Rightarrow\sin\alpha=\frac{\sqrt{15}}{4}\Rightarrow\cos\alpha=\frac{1}{4}$$

$\because\ \overline{a}\times\overline{b}=2\overline{a}\times\overline{c}\ \Rightarrow\overline{a}\times\left(\overline{b}-2\overline{c}\right)=\overline{0}$

$\Rightarrow \overline{b}-2\overline{c}=\lambda\overline{a}\quad\Rightarrow\left|\overline{b}-2\overline{c}\right|=\left|\lambda\overline{a}\right|$

13. $\alpha+\beta=-2, \alpha\beta=5$

$\alpha^2+\beta^2=\left(\alpha+\beta\right)^2-2\alpha\beta$

14. Since $\overline{x},\overline{y}\ \overline{x}\times\overline{y}$ are non-coplanar vectors.

$=20a-15b=0, 15b-12c=0, 12c-20a=0$

$4a=3b, 5b=4c$ and $3c=5a$

$$\Rightarrow\frac{a}{3}=\frac{b}{4}=\frac{c}{5}=\left(\lambda\, say\right)$$

$\Rightarrow a=3\lambda, b=4\lambda, c=5\lambda$, $c^2=a^2+b^2$

15. $\dfrac{\left|\overline{CA}\times\overline{CB}\right|}{\left|\overline{AB}\right|}$

16. Given $\left(\overline{r}\times\overline{a}\right)+\left(\overline{r}\times\overline{b}\right)=\left(\overline{b}\times\overline{a}\right)+\left(\overline{a}\times\overline{b}\right)$

(by adding) $=\left(\overline{b}\times\overline{a}\right)-\left(\overline{b}\times\overline{a}\right)=\overline{0}$

$\Rightarrow\overline{r}\times\left(\overline{a}+\overline{b}\right)=\overline{0}\Rightarrow\overline{r}$ parallel to $\left(\overline{a}+\overline{b}\right)$

17. $\overline{a}=\overline{i}-2\overline{j}+\overline{k}$, $\quad\overline{b}=2\overline{i}-3\overline{j}-\overline{k}$,

$\overline{c}=\overline{i}+4\overline{j}-2\overline{k}$, $\quad\overline{r}=\overline{a}+t\left(\overline{b}\times\overline{c}\right)$

18. $\pm\dfrac{\overline{AB}\times\overline{AC}}{\left|\overline{AB}\times\overline{AC}\right|}$

19. Let the given points be A, B and C respectively.

The unit vectors perpendicular to the plane containing A, B and C are given by

$$\pm\frac{\overline{AB}\times\overline{AC}}{\left|\overline{AB}\times\overline{AC}\right|}=\pm\frac{1}{\sqrt{26}}\left(-\overline{i}+4\overline{j}-3\overline{k}\right)$$

Here coefficient of $\overline{k}<0$

$\therefore$ It makes obtuse angle with z-axis.

20. **Given** $\overline{F}=2\overline{i}-\lambda\overline{j}+5\overline{k}, \overline{OA}=\overline{i}+2\overline{j}+5\overline{k}$

$\overline{F}=3\overline{i}+4\overline{j}+5\overline{k}$

$\overline{r}=\overline{OP}-\overline{OQ}=-2\overline{i}-4\overline{j}+\overline{k}$, $\overline{M}=\overline{r}\times\overline{F}$

$\overline{OB}=-\overline{i}-2\overline{j}+3\overline{k}$

$\overline{BA}=\overline{OA}-\overline{OB}=2\overline{i}+4\vec{j}+2\overline{k}$

$$\overline{BA}\times\overline{F}=\begin{vmatrix} i & j & k \\ 2 & 4 & 2 \\ 2 & -\lambda & 5 \end{vmatrix}=16\overline{i}-6\overline{j}+2\lambda\overline{k}$$

$i\left(20+2\lambda\right)-6\overline{j}+\overline{k}\left(-2\lambda-8\right)=16\overline{i}-6\overline{j}+2\lambda\overline{k}$

$20+2\lambda=16$

$2\lambda=-4$

$\lambda=-2$

EXERCISE - III

1. **Given** $\left|\bar{a}\right| = \left|\bar{b}\right| = 1$ **and** $\left|\bar{a}+\bar{b}\right| = \sqrt{3}$. **If** $\bar{c}$ **be a vector such that** $\bar{c}-\bar{a}-2\bar{b} = 3\left(\bar{a}\times\bar{b}\right)$, **then** $\bar{c}\,.\,\bar{b}$ **is equal to**

1) $-\frac{1}{2}$ 2) $\frac{1}{2}$ 3) $\frac{3}{2}$ 4) $\frac{5}{2}$

2. $\overline{A} = 2\overline{i} + \overline{k}, \overline{B} = \overline{i} + \overline{j} + \overline{k}$ **and** $\overline{C} = 4\overline{i} - 3\overline{j} + 7\overline{k}$ **,then the Vector** $\overline{R}$ **satisfying the conditions** $\overline{R} \times \overline{B} = \overline{C} \times \overline{B}$ **and** $\overline{R}.\overline{A} = 0$

1) (-1, -8, 2) 2) (1,-8,-2)
3) (-2,8,4) 4) (1,0,-2)

3. **If** $\bar{a}$ **and** $\bar{b}$ **are vectors such that** $\left|\bar{a}+\bar{b}\right| = \sqrt{29}$ **and** $\bar{a}\times\left(2\bar{i}+3\bar{j}+4\bar{k}\right) = \left(2\bar{i}+3\bar{j}+4\bar{k}\right)\times\bar{b}$ **then possible value of** $\left(\bar{a}+\bar{b}\right).\left(-7\bar{i}+2\bar{j}+3\bar{k}\right)$ is

1) 0 2) 3 3) 4 4) 8

4. **If** $\bar{a}$ **and** $\bar{b}$ **are unit vectors and** $\bar{c}$ **satisfies** $2(\bar{a}\times\bar{b}) + \bar{c} = \bar{b}\times\bar{c}$ **then the maximum value of** $\left|\left(\bar{a}\times\bar{c}\right).\bar{b}\right|$ **is**

5. **Let** $\bar{a}(2,1,-1), \bar{b}(1,2,1), \bar{c}(2,-1,3)$ **and** $\bar{d}(3,-1,2)$ **be four vectors. The projection of the vector** $\bar{a}+\bar{c}$ **on the vector** $(\bar{b}-\bar{d})\times\bar{c}$ **is**

1) $\sqrt{2}$ 2) $\sqrt{3}$ 3) $\sqrt{6}$ 4) $\sqrt{7}$

6. **Let** $\bar{a}, \bar{b}$ **be two non collinear unit vectors. If** $\bar{\alpha} = \bar{a} - \left(\bar{a}\,.\,\bar{b}\right)\bar{b},\ \bar{\beta} = \bar{a}\times\bar{b}$ **then**

1) $|\bar{\alpha}| = |\bar{\beta}|$ 2) $|\bar{\alpha}|^2 = |\bar{\beta}|$ 3) $|\bar{\beta}|^2 = |\bar{\alpha}|$ 4) $|\bar{\alpha}| = 2|\bar{\beta}|$

7. **Consider the parallopiped with sides** $\bar{a} = 3\bar{i} + 2\bar{j} + \bar{k}, \bar{b} = \bar{i} + \bar{j} + 2\bar{k}$ **and** $\bar{c} = \bar{i} + 3\bar{j} + 3\bar{k}$**, then angle between** $\bar{a}$ **and the plane containing the face determined by** $\bar{b}$ **and** $\bar{c}$ **is**

1) $\sin^{-1}\frac{1}{3}$ 2) $\cos^{-1}\frac{9}{14}$ 3) $\sin^{-1}\frac{9}{14}$ 4) $\sin^{-1}\frac{2}{3}$

KEY

01) 4 02) 1 03) 3 04) 3 05) 3

06) 1 07) 3

SOLUTIONS

1. $\left|\bar{a}+\bar{b}\right| = \sqrt{3},\ \left|\bar{a}+\bar{b}\right|^2 = 3 \Rightarrow \bar{a}\,.\,\bar{b} = \frac{1}{2}$

Now, $\bar{c}-\bar{a}-2\bar{b} = 3(\bar{a}\times\bar{b})$

$\Rightarrow (\bar{c}-\bar{a}-2\bar{b})\,.\,\bar{b} = 3\left\{(\bar{a}\times\bar{b})\,.\,\bar{b}\right\}$

$\Rightarrow \bar{c}\,.\,\bar{b} - \bar{a}\,.\,\bar{b} - 2(\bar{b}\,.\,\bar{b}) = 0$

$\Rightarrow \bar{c}\,.\,\bar{b} - \frac{1}{2} - 2\times 1 = 0 \Rightarrow \bar{c}\,.\,\bar{b} = \frac{5}{2}$

2. $\left(\bar{R}-\bar{C}\right)\times\bar{B} = \bar{0},\ \bar{R}-\bar{C} \parallel \bar{B},\ \bar{R} = \bar{C} + \lambda\bar{B}$

$\bar{R}.\bar{A} = 0 \Rightarrow \lambda = \frac{-\bar{C}.\bar{A}}{\bar{A}.\bar{B}}$

3. $\left(\bar{a}+\bar{b}\right)\times\left(2\bar{i}+3\bar{j}+4\bar{k}\right) = \bar{0}$

$\Rightarrow \bar{a}+\bar{b} = \lambda\left(2\bar{i}+3\bar{j}+4\bar{k}\right)$

$\Rightarrow \sqrt{29} = \pm\lambda\sqrt{29} \Rightarrow \lambda = \pm 1$

4. $2(\bar{a}\times\bar{b})+\bar{c}=\bar{b}\times\bar{c}$

Let $\bar{c}=\alpha_1\,\bar{a}+\alpha_2\bar{b}+\alpha_3(\bar{a}\times\bar{b})$

$\therefore\ 2(\bar{a}\times\bar{b})+\alpha_1\,\bar{a}+\alpha_2\bar{b}+\alpha_3(\bar{a}\times\bar{b})=$

$\bar{b}\times\left\{\alpha_1\bar{a}+\alpha_2\bar{b}+\alpha_3\left(\bar{a}\times\bar{b}\right)\right\}$

$\Rightarrow(\bar{a}\times\bar{b})+\alpha_1\,\bar{a}+\alpha_2\bar{b}+\alpha_3(\bar{a}\times\bar{b})=$

$-\alpha_1\,(\bar{a}\times\bar{b})+\alpha_3(\bar{a}-(\bar{a}.\bar{b})\bar{b})$

$\Rightarrow 2+\alpha_3=-\alpha_1;\ \alpha_1=\alpha_3,\ \alpha_2=-(\bar{a}.\bar{b})\alpha_3$

$\therefore\alpha_1=\alpha_3=-1,\alpha_2=\bar{a}.\bar{b}$

$\therefore\left|(\bar{a}\times\bar{b}).\bar{c}\right|=\left|(\bar{a}\times\bar{b}).(-\bar{a}+(\bar{a}.\bar{b})\bar{b}-\bar{a}\times\bar{b})\right|$

$=\left|0+0-\left|\bar{a}\times\bar{b}\right|^2\right|\quad=\left|\bar{a}\right|^2\left|\bar{b}\right|^2\sin^2\theta$

max.value = 1 , $(\because\sin^2\theta\le 1)$

5. We have $\bar{b}-\bar{d}=-2\bar{i}+3\bar{j}-\bar{k}$ and

$\bar{c}=2\bar{i}-\bar{j}+3\bar{k}$

$\therefore\ \bar{p}=(\bar{b}-\bar{d})\times\bar{c}=4(2\bar{i}+\bar{j}-\bar{k})$

$\therefore\left|\bar{p}\right|=4\sqrt{6}$

If $\bar{q}=\bar{a}+\bar{c}$, then $\bar{q}\ .\ \bar{p}=24$

So, the projection of $\bar{q}\ on\ \bar{p}$ is

$$\frac{\bar{p}\ .\ \bar{q}}{\left|\bar{p}\right|}=\frac{24}{4\sqrt{6}}=\sqrt{6}$$

6. $\left|\bar{\beta}^2\right|=\left|\bar{a}\times\bar{b}\right|^2=\left|\bar{a}\right|^2\left|\bar{b}\right|^2-\left|\bar{a}\ .\ \bar{b}\right|^2$

$=1-\cos^2\theta=\sin^2\theta$

$\left|\bar{\alpha}\right|^2=\left|\bar{a}\right|^2+\left(\bar{a}\ .\ \bar{b}\right)^2\left(\bar{b}\right)^2-2\left(\bar{a}\ .\ \bar{b}\right)^2$

$=1+\cos^2\theta-2\cos^2\theta=1-\cos^2\theta=\sin^2\theta$

$\left|\bar{\alpha}\right|=\left|\bar{\beta}\right|$

7. $\bar{b}\times\bar{c}=3\bar{i}-\bar{j}+2\bar{k}$. Let θ be angle between $\bar{a}$ and plane containing $\bar{b}$ and $\bar{c}$ then

$$\sin\theta=\left|\frac{\bar{a}.\left(\bar{b}\times\bar{c}\right)}{\left|\bar{a}\right|\left|\bar{b}\times\bar{c}\right|}\right|=\frac{9}{14}$$

JEE MAINS QUESTIONS

1. Let $\bar{a}, \bar{b}$ and $\bar{c}$ be three unit vectors such that $\bar{a}+\bar{b}+\bar{c}=0$ If $\lambda=\bar{a}.\bar{b}+\bar{b}.\bar{c}+\bar{c}.\bar{a}$ and $\bar{d}=\bar{a}\times\bar{b}+\bar{b}+\bar{c}+\bar{c}\times\bar{a}$ then the ordered pair $(\lambda, \bar{d})$ is equal to [2020]

1) $\left(\frac{3}{2}, 3(\bar{a}\times\bar{c})\right)$ 2) $\left(\frac{-3}{2}, 3(\bar{c}\times\bar{b})\right)$

3) $\left(\frac{3}{2}, 3(\bar{b}\times\bar{c})\right)$ 4) $\left(\frac{-3}{2}, 3(\bar{a}\times\bar{b})\right)$

2. Let $\bar{a}=\bar{i}-2\bar{j}+\bar{k}$ and $\bar{b}=\bar{i}-\bar{j}+\bar{k}$ be two vectors. If $\bar{c}$ is a vector such that $\bar{b}\times\bar{c}=\bar{b}\times\bar{a}$ and $\bar{c}.\bar{a}=0$ then $\bar{c}.\bar{b}=$__________________ [2020]

1) $\frac{-3}{2}$ 2) $\frac{1}{2}$

3) $\frac{-1}{2}$ 4) -1

3. Let $\bar{a}=\bar{i}+2\bar{j}+4\bar{k}, \bar{b}=\bar{i}+\lambda\bar{j}+4\bar{k}$ and $\bar{c}=2\bar{i}+4\bar{j}+(\lambda^2-1)\bar{k}$ be coplanar vectors, then the non-zero vector $\bar{a}\times\bar{c}$ is [2019]

1) $-10\bar{i}-5\bar{j}$ 2) $-10\bar{i}+5\bar{j}$

3) $-14\bar{i}+5\bar{j}$ 4) $-14\bar{i}-5\bar{j}$

4. Let $\bar{a}=\bar{i}-\bar{j}, \bar{b}=\bar{i}+\bar{j}+\bar{k}$ and $\bar{c}$ be a vector such that $\bar{a}\times\bar{c}+\bar{b}=\bar{0}$ and $\bar{a}.\bar{c}=4$ then $|\bar{c}|^2$ is equal to [2019]

1) $\frac{19}{2}$ 2) 8 3) $\frac{17}{2}$ 4) 9

5. Let $\bar{a}=\bar{i}+\bar{j}+\bar{k}, \bar{c}=\bar{j}-\bar{k}$ and a vector $\bar{b}$ be such that $\bar{a}\times\bar{b}=\bar{c}$ and $\bar{a}.\bar{b}=3$ then $|\bar{b}|=$ [2020]

1) $\frac{11}{3}$ 2) $\frac{11}{\sqrt{3}}$ 3) $\sqrt{\frac{11}{3}}$ 4) $\frac{\sqrt{11}}{3}$

KEY

1) 4 2) 3 3) 2 4) 1

5) 3

SOLUTIONS

1. Given $\left|\bar{a}+\bar{b}+\bar{c}\right|=0 \Rightarrow \left|\bar{a}+\bar{b}+\bar{c}\right|^2=0$

$$\left|\bar{a}\right|^2+\left|\bar{b}\right|^2+\left|\bar{c}\right|^2+2\left(\bar{a}.\bar{b}+\bar{b}.\bar{c}+\bar{c}.\bar{a}\right)=0$$

$$3+2\left(\bar{a}.\bar{b}+\bar{b}.\bar{c}+\bar{c}.\bar{a}\right)=0$$

$$\Rightarrow \bar{a}.\bar{b}+\bar{b}.\bar{c}+\bar{c}.\bar{a}=\frac{-3}{2}\Rightarrow \lambda=\frac{-3}{2}$$

$$\bar{d}=\bar{a}\times\bar{b}+\bar{b}\times\left(-\bar{a}-\bar{b}\right)+\left(-\bar{a}-\bar{b}\right)\times\bar{a}$$

$$\left(\therefore \bar{c}=-\bar{a}-\bar{b}\right)=\bar{a}\times\bar{b}+\bar{a}\times\bar{b}-0-0+\bar{a}\times\bar{b}$$

$$\bar{d}=3\left(\bar{a}\times\bar{b}\right)$$

2. Given

$$\bar{a}=\bar{i}-2\bar{j}+\bar{k},\bar{b}=\bar{i}-\bar{j}+\bar{k} \text{ and } \bar{b}\times\bar{c}=\bar{b}\times\bar{a} \text{ and } \bar{c}.\bar{a}=0$$

$$\bar{a}\times\left(\bar{b}\times\bar{c}\right)=\bar{a}\times\left(\bar{b}\times\bar{a}\right)\Rightarrow\left(\bar{a}.\bar{c}\right)\bar{b}-\left(\bar{a}.\bar{b}\right)\bar{c}=\left(\bar{a}.\bar{a}\right)\bar{b}-\left(\bar{a}.\bar{b}\right)\bar{a}$$

$$0-(1+2+1)\bar{c}=(6)\left(\bar{i}-\bar{j}+\bar{k}\right)-(1+2+1)\left(\bar{i}-2\bar{j}+\bar{k}\right)\Rightarrow-4\bar{c}=2\bar{i}+2\bar{j}+2\bar{k}$$

$$\bar{c}=\frac{-1}{2}\left(\bar{i}+\bar{j}+\bar{k}\right) \text{ now } \bar{b}.\bar{c}=\frac{-1}{2}(1=1+1)=\frac{-1}{2}$$

3. Given

$$\bar{a}=\bar{i}+2\bar{j}+4\bar{k},\bar{b}=\bar{i}+\lambda\bar{j}+4\bar{k},\bar{c}=2\bar{i}+4\bar{j}+\left(\lambda^2-1\right)\bar{k}$$

e r a

coplanar

$$\begin{vmatrix}1 & 2 & 4\\ 1 & \lambda & 4\\ 2 & 4 & \lambda^2-1\end{vmatrix}=0$$

$$\Rightarrow 1\left(\lambda^3-\lambda-16\right)-2\left(\lambda^2-1-8\right)+4\left(4-2\lambda\right)=0$$

$$\Rightarrow \lambda^3-2\lambda^2-9\lambda+18=0\Rightarrow\lambda=2,3,-3$$

if $\lambda=\pm3$ then $\bar{a}$ is parallel $\bar{c}$ $\therefore \lambda=2$

$$\bar{a}\times\bar{b}=\begin{vmatrix}i & j & k\\ 1 & 2 & 4\\ 2 & 4 & 3\end{vmatrix}=i(6-16)-j(3-8)+\bar{k}(4-4)=-10\bar{i}+5\bar{j}$$

4.Given $\bar{a}=\bar{i}-\bar{j}$ $\bar{b}=\bar{i}+\bar{j}+\bar{k}$ and $\bar{a}\times\bar{c}+\bar{b}=0$

$\bar{a}.\bar{c}=4$

$$\left|\bar{a}\right|\left|\bar{c}\right|\cos\theta=4\Rightarrow\cos\theta=\frac{2\sqrt{2}}{\left|\bar{c}\right|}\Rightarrow\left|\bar{a}\times\bar{c}\right|=\left|\bar{b}\right|$$

$$\left|\bar{a}\right|\left|\bar{c}\right|\sin\theta=\left|\bar{b}\right|$$

$$\sqrt{2}\left|\bar{c}\right|\sin\theta=\sqrt{3}\quad S.O.B.S$$

$$2\left|\bar{c}\right|^2\sin^2\theta=3$$

5. Given $\bar{a}=\bar{i}+\bar{j}+\bar{k}$ $\bar{c}=\bar{j}-\bar{k},\bar{a}.\bar{b}=3$,

$\bar{a}\times\bar{b}=\bar{c}$

$$\Rightarrow\bar{a}\times\left(\bar{a}\times\bar{b}\right)=\bar{a}\times\bar{c}$$

$$\left(\bar{a}.\bar{b}\right)\bar{a}-\left(\bar{a}.\bar{a}\right)\bar{b}=\bar{a}\times\bar{c}$$

$$3\left(\bar{i}+\bar{j}+\bar{k}\right)-3\bar{b}=-2\bar{i}+\bar{j}+\bar{k}\Rightarrow5\bar{i}+2\bar{j}+2\bar{k}=3\bar{b}$$

$$\bar{b}=\frac{1}{3}\left(5\bar{i}+2\bar{j}+2\bar{k}\right)\Rightarrow\left|\bar{b}\right|=\frac{1}{3}\sqrt{33}=\sqrt{\frac{11}{3}}$$

SCALAR TRIPLE PRODUCT

SYNOPSIS

- The dot Product of the vector $\bar{a}\times\bar{b}$ with the vector $\bar{c}$ is a scalar triple product of the three vectors $\bar{a},\bar{b},\bar{c}$ and it is written as $(\bar{a}\times\bar{b}).\bar{c}$
  It is a scalar quantity. We write it as $[\bar{a}\,\bar{b}\,\bar{c}]$ and read as box of $\bar{a},\bar{b},\bar{c}$.
- $(\bar{a}\times\bar{b}).\bar{c}=\bar{a}.(\bar{b}\times\bar{c})$ i.e., dot and cross can be interchanged in a scalar triple product.
- $[\bar{a}\ \ \bar{b}\ \ \bar{c}]=[\bar{b}\ \ \bar{c}\ \ \bar{a}]=[\bar{c}\ \ \bar{a}\ \ \bar{b}]=$
  $-[\bar{b}\ \ \bar{a}\ \ \bar{c}]=-[\bar{c}\ \ \bar{b}\ \ \bar{a}]=-[\bar{a}\ \ \bar{c}\ \ \bar{b}]$
- If $\bar{i},\bar{j},\bar{k}$ is orthogonal unit vector triad then
  i) $[\bar{i}\ \bar{j}\ \bar{k}]=[\bar{j}\ \bar{k}\ \bar{i}]=[\bar{k}\ \bar{i}\ \bar{j}]=1$
  ii) $[\bar{j}\ \bar{i}\ \bar{k}]=[\bar{k}\ \bar{j}\ \bar{i}]=[\bar{i}\ \bar{k}\ \bar{j}]=-1$
- If $\bar{a},\bar{b},\bar{c}$ are three mutually perpendicular vectors then $[\bar{a}\,\bar{b}\,\bar{c}]=\pm|\bar{a}||\bar{b}||\bar{c}|$
- If $\bar{a}=(a_1,a_2,a_3)=a_1\bar{i}+a_2\bar{j}+a_3\bar{k}$
  $\bar{b}=(b_1,b_2,b_3)=b_1\bar{i}+b_2\bar{j}+b_3\bar{k}$
  $\bar{c}=(c_1,c_2,c_3)=c_1\bar{i}+c_2\bar{j}+c_3\bar{k}$, then
  $$[\bar{a}\ \ \bar{b}\ \ \bar{c}]=\begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$
- $[\bar{a}\ \bar{b}\ \bar{c}]=0$ if $\bar{a}=\bar{b}$ or $\bar{b}=\bar{c}$ or $\bar{c}=\bar{a}$ or at least one vector is null vector, or atleast two of the three vectors are collinear, or $\bar{a},\bar{b},\bar{c}$ are coplanar.
- The four points A,B,C,D are coplanar $\Leftrightarrow[\overline{AB}\ \overline{AC}\ \overline{AD}]=0$.
- $\bar{a},\bar{b},\bar{c}$ is a vector triad in a right handed system, $\Leftrightarrow\ [\bar{a}\ \bar{b}\ \bar{c}]>0$
- $\bar{a},\bar{b},\bar{c}$ is a vector triad in a left handed system, $\Leftrightarrow\ [\bar{a}\ \bar{b}\ \bar{c}]<0$.
- If $[\bar{a}\,\bar{b}\,\bar{c}]\neq 0$ then vectors $\bar{a},\bar{b},\bar{c}$ are non-coplanar
- If $\bar{a},\bar{b},\bar{c}$ are coplanar then
  i) $[\bar{a}\times\bar{b}\ \ \bar{b}\times\bar{c}\ \ \bar{c}\times\bar{a}]=0$
  ii) $\bar{a}\times\bar{b}\ \ \bar{b}\times\bar{c}\ \ \bar{c}\times\bar{a}$ are coplanar
- If $\bar{a}+\bar{b}+\bar{c}=\bar{0}$ then $\bar{a},\bar{b},\bar{c}$ are coplanar
- $\bar{a},\bar{b},\bar{c}$ are three vectors then $\bar{a}-\bar{b},\bar{b}-\bar{c},\bar{c}-\bar{a}$ are coplanar
- **Volume :**
  The volume of the parallelopiped
  i) with $\bar{a},\bar{b},\bar{c}$ as coterminus edges is $|[\bar{a}\ \bar{b}\ \bar{c}]|$ cubic units
  ii) with A,B,C,D as vertices of coterminus edges is $|[\overline{AB}\ \overline{AC}\ \overline{AD}]|$ cubic units.
- The volume of tetrahedron
  i) with $\bar{a},\bar{b},\bar{c}$ as Coterminus edges is $\frac{1}{6}|[\bar{a}\ \ \bar{b}\ \ \bar{c}]|$ cubic units.
  ii) with A,B,C,D as vertices is $\frac{1}{6}|[\overline{AB}\ \ \overline{AC}\ \ \overline{AD}]|$ cubic units.
- The centroid of tetrahedron divides the line joining any vertex to the centroid of its opposite face in the ratio 3:1

- The volume of the triangular prism whose adjacent sides are represented by the vectors $\bar{a}, \bar{b}$ and $\bar{c}$ is $\frac{1}{2}\left|\left[\bar{a} \quad \bar{b} \quad \bar{c}\right]\right|$ cubic units.
- If $\bar{a}, \bar{b}, \bar{c}$ are three vectors, l, m, n are three real numbers, then $\left[l\bar{a} \ m\bar{b} \ n\bar{c}\right] = lmn\left[\bar{a} \ \bar{b} \ \bar{c}\right]$.
- For any three vectors $\bar{a}, \bar{b}$ and $\bar{c}$ and λ scalar
$$\left[\lambda\bar{a} \ \bar{b} \ \bar{c}\right] = \left[\bar{a} \ \lambda\bar{b} \ \bar{c}\right] = \left[\bar{a} \ \bar{b} \ \lambda\bar{c}\right] = \lambda\left[\bar{a} \ \bar{b} \bar{c}\right]$$
- If $\bar{l}, \bar{m}, \bar{n}$ are three non-coplanar vectors and $\bar{a} = a_1\bar{l} + a_2\bar{m} + a_3\bar{n}$, $\bar{b} = b_1\bar{l} + b_2\bar{m} + b_3\bar{n}$, $\bar{c} = c_1\bar{l} + c_2\bar{m} + c_3\bar{n}$, then
$$\left[\bar{a} \ \bar{b} \ \bar{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\left[\bar{l} \ \bar{m} \ \bar{n}\right]$$
- If $\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$ are coplanar points then
$\left[\bar{a} \ \bar{b} \ \bar{c}\right] = \left[\bar{b} \ \bar{c} \ \bar{d}\right] + \left[\bar{c} \ \bar{a} \ \bar{d}\right] + \left[\bar{a} \ \bar{b} \ \bar{d}\right]$
- i) $\left[\bar{a}+\bar{b} \ \ \bar{b}+\bar{c} \ \ \bar{c}+\bar{a}\right] = 2\left[\bar{a} \ \bar{b} \ \bar{c}\right]$
ii) $\left[\bar{a}-\bar{b} \ \ \bar{b}-\bar{c} \ \ \bar{c}-\bar{a}\right] = 0$
- 1) $\left[\bar{a} \ \bar{b} \ \bar{c}\right]\left[\bar{l} \ \bar{m} \ \bar{n}\right] = \begin{vmatrix} \bar{a}.\bar{l} & \bar{b}.\bar{l} & \bar{c}.\bar{l} \\ \bar{a}.\bar{m} & \bar{b}.\bar{m} & \bar{c}.\bar{m} \\ \bar{a}.\bar{n} & \bar{b}.\bar{n} & \bar{c}.\bar{n} \end{vmatrix}$

2) $\left[\bar{a} \bar{b} \bar{c}\right]^2 = \begin{vmatrix} \bar{a}.\bar{a} & \bar{a}.\bar{b} & \bar{a}.\bar{c} \\ \bar{b}.\bar{a} & \bar{b}.\bar{b} & \bar{b}.\bar{c} \\ \bar{c}.\bar{a} & \bar{c}.\bar{b} & \bar{c}.\bar{c} \end{vmatrix}$

- **Vector equation of a plane :**

i) Vector equation of a plane passing through three non-collinear points having position vectors $\bar{a}, \bar{b}$ and $\bar{c}$ is $\left[\bar{r}-\bar{a} \ \ \bar{b}-\bar{a} \ \ \bar{c}-\bar{a}\right] = 0$ or
$\bar{r}.(\bar{b}\times\bar{c} + \bar{c}\times\bar{a} + \bar{a}\times\bar{b}) = \left[\bar{a} \ \bar{b} \ \bar{c}\right]$
$\left[\bar{r} \bar{b} \bar{c}\right] + \left[\bar{r} \bar{c} \bar{a}\right] + \left[\bar{r} \bar{a} \bar{b}\right] = \left[\bar{a} \bar{b} \bar{c}\right]$

ii) A unit vector perpendicular to the plane containing three non-collinear points $\bar{a}, \bar{b}, \bar{c}$ is $\pm\dfrac{(\bar{a}\times\bar{b}) + (\bar{b}\times\bar{c}) + (\bar{c}\times\bar{a})}{\left|\bar{a}\times\bar{b} + \bar{b}\times\bar{c} + \bar{c}\times\bar{a}\right|}$

iii) The length of the perpendicular from the origin to the plane containing three non-collinear points $\bar{a}, \bar{b}, \bar{c}$ is $\dfrac{\left|\left[\bar{a} \bar{b} \bar{c}\right]\right|}{\left|\bar{a}\times\bar{b} + \bar{b}\times\bar{c} + \bar{c}\times\bar{a}\right|}$

- Vector equation of a plane passing through a given point with position vector $\bar{a}$ and parallel to $\bar{b}, \bar{c}$ is $[\bar{r}-\bar{a} \ \bar{b} \ \bar{c}] = 0$ (or) $\left[\bar{r} \ \bar{b} \ \bar{c}\right] = \left[\bar{a} \ \bar{b} \ \bar{c}\right]$
- Vector equation of a plane passing through the points $\bar{a}, \bar{b}$ and parallel to $\bar{c}$ is $\left[\bar{r}-\bar{a} \ \ \bar{b}-\bar{a} \ \ \bar{c}\right] = 0$
- Vector equation of a plane passing through origin and the points $\bar{b}, \bar{c}$ is $\left[\bar{r} \ \bar{b} \ \bar{c}\right] = 0$.
- $\left[\bar{a} \ \bar{b} \ \lambda\bar{c}+\mu\bar{d}\right] = \lambda\left[\bar{a} \ \bar{b} \ \bar{c}\right] + \mu\left[\bar{a} \ \bar{b} \ \bar{d}\right]$ where λ & μ are scalars.
- **Three nonparallel planes :**
$\bar{r}.\bar{a} = p_1, \bar{r}.\bar{b} = p_2, \bar{r}.\bar{c} = p_3$ represents three planes in normal form then

i) The planes intersect at the point
$$\bar{r} = \frac{p_1\left(\bar{b}\times\bar{c}\right) + p_2\left(\bar{c}\times\bar{a}\right) + p_3\left(\bar{a}\times\bar{b}\right)}{\left[\bar{a} \ \bar{b} \ \bar{c}\right]}, \text{ where}$$
$\left[\bar{a} \bar{b} \bar{c}\right] \neq 0$.

ii) The planes intersect along a line if $\left[\bar{a} \bar{b} \bar{c}\right] = 0$, $p_1\left(\bar{b}\times\bar{c}\right) + p_2\left(\bar{c}\times\bar{a}\right) + p_3\left(\bar{a}\times\bar{b}\right) = \bar{0}$,

iii) The planes form a triangular prism if $\left[\bar{a} \bar{b} \ \bar{c}\right] = 0, p_1\left(\bar{b}\times\bar{c}\right) + p_2\left(\bar{c}\times\bar{a}\right) + p_3\left(\bar{a}\times\bar{b}\right) \neq \bar{0}$

- **Skew Lines :** If two straight lines in space do not intersect and are also not parallel, then the two lines are called Skew lines.
- In other words, the two skew lines are not coplanar.
- l and m are two skew lines. If P is a point on l and Q is a point on m such that $PQ \perp l$ and $PQ \perp m$, then PQ is called the shortest distance and $\overleftrightarrow{PQ}$ is called the shortest distance line between the skew lines l, m.

✈ The lines $\overline{r}=\overline{a}+s\overline{b}$ and $\overline{r}=\overline{c}+t\overline{d}$ intersect each other $\Leftrightarrow \left[\overline{a}-\overline{c}\ \ \overline{b}\ \ \overline{d}\right]=0$

✈ The shortest distance between the skew lines $\overline{r}=\overline{a}+s\overline{b}$ and $\overline{r}=\overline{c}+t\overline{d}$ is

$$\left|\frac{\left[\overline{a}-\overline{c}\ \ \overline{b}\ \ \overline{d}\right]}{\left|\overline{b}\times\overline{d}\right|}\right| \text{ or } \left|\left(\overline{a}-\overline{c}\right).\frac{\overline{b}\times\overline{d}}{\left|\overline{b}\times\overline{d}\right|}\right|.$$

✈ Let the position vectors of A,B,C,D are $\overline{a},\overline{b},\overline{c},\overline{d}$ .Then the shortest distance between two lines AB,CD is $\dfrac{\left|\left[\overline{AC}\ \ \overline{AB}\ \ \overline{CD}\right]\right|}{\left|\overline{AB}\times\overline{CD}\right|}$

✈ **Reciprocal system of vectors :**

i) If $\overline{a},\overline{b},\overline{c}$ be any three non-coplanar vectors such that $\left[\overline{a}\ \overline{b}\ \overline{c}\right]\neq 0$, then the three vectors $\overline{a^1},\overline{b^1},\overline{c^1}$ defined by ,

$$\overline{a^1}=\frac{\overline{b}\times\overline{c}}{\left[\overline{a}\ \ \overline{b}\ \ \overline{c}\right]},\overline{b^1}=\frac{\overline{c}\times\overline{a}}{\left[\overline{a}\ \ \overline{b}\ \ \overline{c}\right]},\overline{c^1}=\frac{\overline{a}\times\overline{b}}{\left[\overline{a}\ \ \overline{b}\ \ \overline{c}\right]}$$

are called the reciprocal system of vectors to the given vectors $\overline{a},\overline{b},\overline{c}$ respectively.

ii) $\overline{a}.\overline{a^1}=\overline{a}.\dfrac{\overline{b}\times\overline{c}}{\left[\overline{a}\ \ \overline{b}\ \ \overline{c}\right]}=\dfrac{\overline{a}.\left(\overline{b}\times\overline{c}\right)}{\left[\overline{a}\ \ \overline{b}\ \ \overline{c}\right]}=1$ also

$\overline{b}.\overline{b^1}=\overline{c}.\overline{c^1}=1\cdot$

iii) $\overline{a}.\overline{b^1}=\overline{a}.\overline{c^1}=\overline{b}.\overline{a^1}=\overline{b}.\overline{c^1}=\overline{c}.\overline{a^1}=\overline{c}.\overline{b^1}=0$

iv) $\left[\overline{a^1}\ \ \overline{b^1}\ \ \overline{c^1}\right]=\dfrac{1}{\left[\overline{a}\ \ \overline{b}\ \ \overline{c}\right]}$

Eg: 1

Let a,b,c be distinct non-negative numbers. If the vectors $a\overline{i}+a\overline{j}+c\overline{k}$, $\overline{i}+\overline{k}$ and $c\overline{i}+c\overline{j}+b\overline{k}$ lie in a plane then c is the

Sol: Since the vectors are lie in a plane.

$$\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix}=0 \Rightarrow c^2=ab$$

c is the G.M. of ab.

Eg: 2

The volume (in cubic unit) of the tetrahedron with edges $\overline{i}+\overline{j}+\overline{k},\overline{i}-\overline{j}+\overline{k}\ and\ \overline{i}+2\overline{j}-\overline{k}$ is

[EAM-2007]

Sol : We know that, volume of tetrahedron

$$=\frac{1}{6}\left[\overline{a}\ \ \overline{b}\ \ \overline{c}\right]=\frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}=\frac{2}{3}$$

Eg: 3

If the volume of parallelopiped with coterminus edges $4\overline{i}+5\overline{j}+\overline{k},-\overline{j}+\overline{k}$ and $3\overline{i}+9\overline{j}+p\overline{k}$ is 34 cubic units, then p=

[Eam-2006]

Sol : Coterminus edges of a parallelopiped are $4\overline{i}+5\overline{j}+\overline{k},-\overline{j}+\overline{k}$ and $3\overline{i}+9\overline{j}+p\overline{k}$

Volume of parallelopiped = 34

$$\Rightarrow\begin{vmatrix} 4 & 5 & 1 \\ 0 & -1 & 1 \\ 3 & 9 & p \end{vmatrix}=34$$

$$\Rightarrow 4(-p-9)-5(-3)+1(3)=34$$

$$\therefore p=-13$$

Eg: 5

If $\overline{i}-2\overline{j}$, $3\overline{j}+\overline{k}$ and $\lambda\overline{i}+3\overline{j}$ are coplanar, then λ is equal to **[Eam-2006]**

Sol: Given $\overline{i}-2\overline{j}$, $3\overline{j}+\overline{k}$ and $\lambda\overline{i}+3\overline{j}$ are coplanar,

then $\begin{vmatrix} 1 & -2 & 0 \\ 0 & 3 & 1 \\ \lambda & 3 & 0 \end{vmatrix}=0$

$$\Rightarrow -3+2(-\lambda)=0 \qquad \Rightarrow \lambda=-\frac{3}{2}$$

Eg: 5

Let $\overline{a}=\overline{j}-\overline{k}$ and $\overline{c}=\overline{i}-\overline{j}-\overline{k}$ Then, the vector $\overline{b}$ satisfying $\overline{a}\times\overline{b}+\overline{c}=\overline{0}$ and $\overline{a}.\overline{b}=3$, is **(AIE-2010)**

Sol : We have, $\overline{a}\times\overline{b}+\overline{c}=\overline{0}$

$\Rightarrow \overline{a}\times\left(\overline{a}\times\overline{b}\right)+\overline{a}\times\overline{c}=\overline{0}$

$\Rightarrow \left(\overline{a}.\overline{b}\right)\overline{a}-\left(\overline{a}.\overline{a}\right)\overline{b}+\overline{a}\times\overline{c}=\overline{0}$

$\Rightarrow 3\overline{a}-2\overline{b}+\overline{a}\times\overline{c}=\overline{0}\ \Rightarrow 2\overline{b}=3\overline{a}+\overline{a}\times\overline{c}$

$\Rightarrow 2\overline{b}=3\overline{j}-3\overline{k}-2\overline{i}-\overline{j}-\overline{k}=-2\overline{i}+2\overline{j}-4\overline{k}$

$\Rightarrow \overline{b}=-\overline{i}+\overline{j}-2\overline{k}$

Eg: 6

Let $\overline{a}=\overline{i}-\overline{j},\ \overline{b}=\overline{j}-\overline{k},\ \overline{c}=\overline{k}-\overline{i}$ **. If** $\overline{d}$ **is a unit vector such that** $\overline{a}.\overline{d}=0=\left[\overline{b}\ \overline{c}\ \overline{d}\right]$**, then** $\overline{d}$ **is (are)**

Sol : Let $\overline{d}=d_1\overline{i}+d_2\overline{j}+d_3\overline{k}$

$\overline{a}.\overline{d}=0\Rightarrow d_1-d_2=0 \quad \Rightarrow d_1=d_2$ ______ (1)

$\left|\overline{d}\right|=1\Rightarrow d_1^2+d_2^2+d_3^2=1$ _______ (2)

$\left[\overline{b}\ \overline{c}\ \overline{d}\right]=0$

$$\Rightarrow\begin{vmatrix}0 & 1 & -1\\ -1 & 0 & 1\\ d_1 & d_2 & d_3\end{vmatrix}=0$$

$\Rightarrow -1\left(d_3-d_1\right)-1\left(-d_2\right)=0\Rightarrow 2d_1+d_3=0$ __ (3)

Using (1) (2) and (3)

$$d_1=d_2=\pm\frac{1}{\sqrt{6}},\quad d_3=\mp\frac{2}{\sqrt{6}}$$

$$\therefore \overline{d}=\pm\frac{1}{\sqrt{6}}\left(\overline{i}+\overline{j}-2\overline{k}\right)$$

EXERCISE - I

1. $\left[\overline{i}\ \overline{j}\ \overline{k}\right]+\left[\overline{j}\ \overline{k}\ \overline{i}\right]+\left[\overline{k}\ \overline{i}\ \overline{j}\right]+\left[\overline{i}\ \overline{k}\ \overline{j}\right]+\left[\overline{j}\ \overline{i}\ \overline{k}\right]+\left[\overline{k}\ \overline{j}\ \overline{i}\right]$

1) 1 2) 2 3) 6 4) 0

2. If $\overline{a}=\overline{i}-2\overline{j}-3\overline{k},\overline{b}=2\overline{i}+\overline{j}-\overline{k}$, $\overline{c}=\overline{i}+3\overline{j}-2\overline{k}$ **then** $\overline{a}.\left(\overline{b}\times\overline{c}\right)$

1) 10 2) -10 3) -20 4) 20

3. $\left(2\overline{i}-3\overline{j}+\overline{k}\right).\left(\overline{i}-\overline{j}+2\overline{k}\right)\times\left(2\overline{i}+\overline{j}+\overline{k}\right)$

1) -14 2) 14 3) -12 4) 12

4. $\left(\overline{a}+2\overline{b}-\overline{c}\right).\left(\overline{a}-\overline{b}\right)\times\left(\overline{a}-\overline{b}-\overline{c}\right)=$

1) $-\left[\overline{a}\ \overline{b}\ \overline{c}\right]$ 2) $2\left[\overline{a}\ \overline{b}\ \overline{c}\right]$ 3) $3\left[\overline{a}\ \overline{b}\ \overline{c}\right]$ 4) 0

5. If $\overline{a}=2\overline{i}-\overline{j},\ \overline{b}=4\overline{j}+\overline{k},\overline{c}=3\overline{k}$ **then** $\left(2\overline{a}+\overline{b}+\overline{c}\right).\left(-\overline{b}+2\overline{c}\right)\times\overline{c}$

1) 48 2) 28 3) -28 4) -48

6. If $\left[\overline{a}+2\overline{b}\ \ 2\overline{b}+\overline{c}\ \ 5\overline{c}+\overline{a}\right]=k\left[\overline{a}\,\overline{b}\,\overline{c}\right]$ **then k =**

1) 2 2) 4 3) 8 4) 12

7. If $\left[\overline{i}+4\overline{j}+6\overline{k}\ \ 2\overline{i}+a\overline{j}+3\overline{k}\ \ \overline{i}+2\overline{j}-3\overline{k}\right]=18$ **then a =**

1) -4 2) 4 3) ±4 4) 1

8. If $\overline{u},\overline{v},\overline{w}$ **are three non coplanar vectors then** $\left(\overline{u}+\overline{v}-\overline{w}\right).\left\{\left(\overline{u}-\overline{v}\right)\times\left(\overline{v}-\overline{w}\right)\right\}=$

1) $\overline{0}$ 2) $\overline{u}.\left(\overline{v}\times\overline{w}\right)$

3) $\overline{u}.\left(\overline{w}\times\overline{v}\right)$ 4) $3\overline{u}.\left(\overline{v}\times\overline{w}\right)$

9. $\overline{a},\overline{b}\ and\ \overline{c}$ **are mutually perpendicular unit vectors then** $\left[\overline{a}\,\overline{b}\,\overline{c}\right]=$

1) 0 2) -1 3) 1 4) ±1

10. If $\left[\overline{a}\ \overline{b}\ \overline{c}\right]=1$ **then**

$$\frac{\overline{a}.\left(\overline{b}\times\overline{c}\right)}{\left(\overline{c}\times\overline{a}\right).\overline{b}}+\frac{\overline{b}.\left(\overline{c}\times\overline{a}\right)}{\left(\overline{a}\times\overline{b}\right).\overline{c}}+\frac{\overline{c}\left(\overline{a}\times\overline{b}\right)}{\left(\overline{b}\times\overline{c}\right).\overline{a}}$$

1) 3 2) 1 3) -1 4) 0

11. If $\left[\overline{a}\,\overline{b}\,\overline{c}\right]\neq0$ **then** $\dfrac{\left[\overline{a}+\overline{b}\ \ \overline{b}+\overline{c}\ \ \overline{c}+\overline{a}\right]}{\left[\overline{a}\ \overline{b}\ \overline{c}\right]}=$

1) 1 2) -1 3) 2 4) -3

12. If $\overline{a}=2\overline{i}+3\overline{j},\ \overline{b}=\overline{i}+\overline{j}+\overline{k}$, $\overline{c}=3\overline{i}+\lambda\overline{j}+2\overline{k}$ **and** $\left[\overline{a}\ \overline{b}\ \overline{c}\right]=-1$ **then** $\lambda=$

1) 4 2) 3 3) 2 4) 1

13. **The volume of the parallelopiped having coterminus edges $\bar{i}+\bar{j}+\bar{k}$, $\bar{i}-\bar{j}$, $\bar{i}+2\bar{j}-\bar{k}$ is**
1) 2 2) 3 3) 5 4) 7

14. **If $\bar{a}=2\bar{i}+3\bar{j}$, $\bar{b}=\bar{i}+\bar{j}+\bar{k}$, $\bar{c}=\lambda\bar{i}+4\bar{j}+2\bar{k}$ are coterminus edges of a parallelopiped of volume 2 cubic units then λ is**
1) 1 2) 2 3) 3 4) 4

15. **The volume of parallelopiped with vectors $\bar{a}+2\bar{b}-\bar{c}$, $\bar{a}-\bar{b}$, $\bar{a}-\bar{b}-\bar{c}$ as coterminus edges is $k\left[\bar{a}\ \bar{b}\ \bar{c}\right]$ then $|k|$ is**
1) -2 2) 2 3) -3 4) 3

16. **If $\left[\bar{a}\ \bar{b}\ \bar{c}\right]=-4$ then the volume of the parallelopiped with coterminus edges $\bar{a}+2\bar{b}$, $2\bar{b}+\bar{c}$, $3\bar{c}+\bar{a}$ is (in cu units)**
1) 32 2) -32 3) 8 4) 12

17. **The volume of tetrahedron with edges $\bar{i}\times\bar{j}\quad \bar{j}\times\bar{k}\quad \bar{k}\times\bar{i}$**
1) 1 2) $1/6$ 3) 3 4) $1/3$

18. **Volume of the tetrahedron with vertices at $(0,0,0),(1,0,0),(0,1,0)$ and $(0,0,1)$ is ____ (cu units)**
1) $\frac{1}{6}$ 2) $\frac{1}{4}$ 3) $\frac{1}{3}$ 4) $\frac{1}{2}$

19. **If the volume of the tetrahedron with edges $2\bar{i}+\bar{j}+\bar{k}$, $\bar{i}+a\bar{j}+\bar{k}$ and $\bar{i}+2\bar{j}-\bar{k}$ is one cubic unit then $a=$ _____**
1) 1 2) -2 3) 2 4) -1

20. **If $\bar{a},\bar{b},\bar{c}$ are vectors such that $\left[\bar{a}\ \bar{b}\ \bar{c}\right]=4$ then $\left[\bar{a}\times\bar{b}\ \ \bar{b}\times\bar{c}\ \ \bar{c}\times\bar{a}\right]$ is __**
1) 16 2) 64 3) 4 4) 8

21. **The vector equation of the plane passing through the points $(1,-2,5),(0,-5,-1)$ and $(-3,5,0)$ is ____**
1) $\left[\bar{r}-\left(\bar{i}-2\bar{j}+5\bar{k}\right)\ \ -5\bar{i}-\bar{k}\ \ -3\bar{i}+5\bar{j}\right]=0$
2) $\left[\bar{r}\ \ -5\bar{i}-\bar{k}\ \ -3\bar{i}+5\bar{j}\right]=0$
3) $\left[\bar{r}-\left(\bar{i}-2\bar{j}+5\bar{k}\right)\ \ -\bar{i}-3\bar{j}-6\bar{k}\ \ -4\bar{i}+7\bar{j}+5\bar{k}\right]=0$
4) $\left[\bar{r}-\bar{i}-3\bar{j}-6\bar{k}-4\bar{i}+7\bar{j}-5\bar{k}\right]=0$

22. **The vector equation of the plane passing through the points $\bar{i}-2\bar{j}+\bar{k}$, $3\bar{k}-2\bar{j}$ and parallel to the vector $2\bar{i}+\bar{j}+\bar{k}$ is ____**
1) $\left[\bar{r}-\left(\bar{i}-2\bar{j}+\bar{k}\right)\ \ 3\bar{k}-2\bar{j}\ \ 2\bar{i}+\bar{j}+\bar{k}\right]=0$
2) $\left[\bar{r}-\left(\bar{i}-2\bar{j}+\bar{k}\right)\ \ -\bar{i}+2\bar{k}\ \ 2\bar{i}+\bar{j}+\bar{k}\right]=0$
3) $\left[\bar{r}-\left(\bar{i}-2\bar{j}+\bar{k}\right)\ \ 2\bar{k}-2\bar{j}\ \ \bar{i}+3\bar{j}+\bar{k}\right]=0$
4) $\left[\bar{r}-\left(\bar{i}-2\bar{j}+\bar{k}\right)\ \ -\bar{i}+2\bar{k}\ \ \bar{i}+3\bar{j}+\bar{k}\right]=0$

23. **The vector equation of the plane passing through $\bar{i}+\bar{j}+\bar{k}$ and parallel to the vectors $2\bar{i}+3\bar{j}-\bar{k}$, $\bar{i}+2\bar{j}+3\bar{k}$ is ____**
1) $\left[\bar{r}-\left(\bar{i}+\bar{j}+\bar{k}\right)\ \ 2\bar{i}+3\bar{j}-\bar{k}\ \ \bar{i}+2\bar{j}+3\bar{k}\right]=0$
2) $\left[\bar{r}\ \ 2\bar{i}+3\bar{j}-\bar{k}\ \ \bar{i}+2\bar{j}+5\bar{k}\right]=0$
3) $\left[\bar{r}-\left(\bar{i}+\bar{j}+\bar{k}\right)\ \ \bar{i}+2\bar{j}-2\bar{k}\ \ \bar{j}+2\bar{k}\right]=0$
4) $\left[\bar{r}\ \ \bar{i}+2\bar{j}-2\bar{k}\ \ \bar{j}+2\bar{k}\right]=0$

24. **The equation of the plane passing through the point with position vector $\bar{a}$ and perpendicular to $\bar{b}$ is ____ [EAM- 2019]**
1) $\bar{r}.\left(\bar{a}\times\bar{b}\right)=0$ 2) $\bar{r}=\bar{a}\times\bar{b}$
3) $\bar{r}=\bar{b}\times\bar{a}$ 4) $\left(\bar{r}-\bar{a}\right)\bar{b}=0$

25. **The equation of the plane passing through the points $A=(2,3,-1)$, $B=(4,5,2)$, $C=(3,6,5)$ is**
1) $3x+9y+4z+25=0$
2) $3x-9y+4z+25=0$
3) $3x-9y+4z-25=0$
4) $3x-9y-4z-25=0$

26. **The vector equation of the plane containing the line $\bar{r}=\bar{a}+s\bar{b}$ and parallel to the line $\bar{r}=\bar{c}+t\bar{d}$ is**

1) $\left[\overline{r}-\overline{a}\quad \overline{b}\quad \overline{d}\right]=0$ 2) $\left[\overline{r}-\overline{b}\quad \overline{c}\quad \overline{d}\right]=0$

3) $\left[\overline{r}-\overline{d}\quad \overline{a}\quad \overline{b}\right]=0$ 4) $\left[\overline{r}-\overline{c}\quad \overline{a}\quad \overline{d}\right]=0$

27. The distance between the plane whose equation is $\overline{r}.\left(2\overline{i}+\overline{j}-3\overline{k}\right)=5$ and the line whose equation is $\overline{r}=\overline{i}+\lambda\left(2\overline{i}+5\overline{j}+3\overline{k}\right)$ is

1) $\frac{3}{\sqrt{14}}$ 2) $\frac{5}{\sqrt{14}}$ 3) 5 4) 0

28. The shortest distance between the lines whose equations are $\overline{r}=t\left(\overline{i}+\overline{j}+\overline{k}\right)$,

$\overline{r}=\overline{k}+s\left(\overline{i}-2\overline{j}+3\overline{k}\right)$ **is**

1) 3 2) $\frac{3}{\sqrt{38}}$ 3) $\sqrt{\frac{3}{14}}$ 4) $\frac{2}{\sqrt{13}}$

29. The shortest distance between the lines

$\overline{r}=\overline{i}+2\overline{j}+3\overline{k}+s\left(2\overline{i}+3\overline{j}+4\overline{k}\right)$ **and**

$\overline{r}=\left(2\overline{i}+4\overline{j}+5\overline{k}\right)+t\left(3\overline{i}+4\overline{j}+5\overline{k}\right)$ **is___**

1) $\frac{1}{6}$ 2) $\frac{1}{\sqrt{6}}$ 3) $\frac{1}{3}$ 4) $\frac{1}{\sqrt{3}}$

30-. The lines $\overline{r}=\overline{a}+t\overline{b}$, $\overline{r}=\overline{c}+s\overline{d}$ are coplanar if

1) $\left(\overline{a}-\overline{b}\right).\overline{c}\times\overline{d}=0$ 2) $\left(\overline{a}-\overline{c}\right).\overline{b}\times\overline{d}=0$

3) $\left(\overline{b}-\overline{c}\right).\overline{a}\times\overline{d}=0$ 4) $\left(\overline{b}-\overline{d}\right).\overline{a}\times\overline{c}=0$

31. If $\overline{a}',\overline{b}',\overline{c}'$ represents the reciprocal system of vectors of $\overline{a},\overline{b},\overline{c}$ then $\overline{a}\,\overline{a}'+\overline{b}\,\overline{b}'+\overline{c}\,\overline{c}'=$

1) 0 2) 1 3) 2 4) 3

KEY

01) 4	02) 3	03) 3	04) 3	05) 4	06) 4
07) 2	08) 2	09) 4	10) 1	11) 3	12) 1
13) 3	14) 4	15) 4	16) 2	17) 2	
18) 1	1)9 2	20) 1	21) 3	22) 2	23) 1
24) 4	25) 2	26) 1	27) 1	28) 2	29) 2
30) 2	31) 4				

SOLUTIONS

1. $\left[\overline{i}\,\overline{j}\,\overline{k}\right]=\left[\overline{j}\,\overline{k}\,\overline{i}\right]=\left[\overline{k}\,\overline{i}\,\overline{j}\right]=1$

$\left[\overline{i}\,\overline{k}\,\overline{j}\right]=\left[\overline{j}\,\overline{i}\,\overline{k}\right]=\left[\overline{k}\,\overline{j}\,\overline{i}\right]=-1$

2. $\left[\overline{a}\,\overline{b}\,\overline{c}\right]=\begin{vmatrix}1 & -2 & -3\\ 2 & 1 & -1\\ 1 & 3 & -2\end{vmatrix}$

3. $\begin{vmatrix}2 & -3 & 1\\ 1 & -1 & 2\\ 2 & 1 & 1\end{vmatrix}$

4. $\begin{vmatrix}1 & 2 & -1\\ 1 & -1 & 0\\ 1 & -1 & -1\end{vmatrix}\left[\overline{a}\,\overline{b}\,\overline{c}\right]$

5. $\begin{vmatrix}2 & 1 & 1\\ 0 & -1 & 2\\ 0 & 0 & 1\end{vmatrix}\left[\overline{a}\,\overline{b}\,\overline{c}\right]$

6. $\begin{vmatrix}1 & 2 & 0\\ 0 & 2 & 1\\ 1 & 0 & 5\end{vmatrix}\left[\overline{a}\,\overline{b}\,\overline{c}\right]$

7. $\begin{vmatrix}1 & 4 & 6\\ 2 & a & 3\\ 1 & 2 & -3\end{vmatrix}=18$

8. $\begin{vmatrix}1 & 1 & -1\\ 1 & -1 & 0\\ 0 & 1 & -1\end{vmatrix}\left[\overline{u}\ \overline{v}\ \overline{w}\right]$

9. $\left[\overline{a}\ \overline{b}\ \overline{c}\right]=\pm\left|\overline{a}\right|\left|\overline{b}\right|\left|\overline{c}\right|$

10. given $\left[\bar{a}\,\bar{b}\,\bar{c}\right]=1=\left[\bar{b}\,\bar{c}\,\bar{a}\right]=\left[\bar{c}\,\bar{a}\,\bar{b}\right]$

$$\frac{\bar{a}.\left(\bar{b}\times\bar{c}\right)}{\left(\bar{c}\times\bar{a}\right).\bar{b}}+\frac{\bar{b}.\left(\bar{c}\times\bar{a}\right)}{\left(\bar{a}\times\bar{b}\right).\bar{c}}+\frac{\bar{c}.\left(\bar{a}\times\bar{b}\right)}{\left(\bar{b}\times\bar{c}\right).\bar{a}}$$

$$=\frac{\left[\overline{abc}\right]}{\left[\overline{cab}\right]}\times\frac{\left[\overline{bca}\right]}{\left[\overline{abc}\right]}\times\frac{\left[\overline{cab}\right]}{\left[\overline{bca}\right]}=1+1+1=3$$

11. $\dfrac{2\left[\bar{a}\ \bar{b}\ \bar{c}\right]}{\left[\bar{a}\ \bar{b}\ \bar{c}\right]}$

12. $\begin{vmatrix}2 & 3 & 0\\ 1 & 1 & 1\\ 3 & \lambda & 2\end{vmatrix}=-1$

13. $V=\left|\left[\bar{a}\ \bar{b}\ \bar{c}\right]\right|$

14. $V=\left|\left[\bar{a}\ \bar{b}\ \bar{c}\right]\right|=2$

15. $k\left[\bar{a}\ \bar{b}\ \bar{c}\right]=\begin{vmatrix}1 & 2 & -1\\ 1 & -1 & 0\\ 1 & -1 & 0\end{vmatrix}\left[\bar{a}\ \bar{b}\ \bar{c}\right]$

16. $\begin{vmatrix}1 & 2 & 0\\ 0 & 2 & 1\\ 1 & 0 & 3\end{vmatrix}\left[\bar{a}\ \bar{b}\ \bar{c}\right]$

17. given edges $\begin{array}{l}\bar{a}=\bar{i}\times\bar{j}=\bar{k}\\ \bar{b}=\bar{j}\times\bar{k}=\bar{i}\\ \bar{c}=\bar{k}\times i=-j\end{array}$

volume of tetra hedron

$$=\frac{1}{6}\left[\overline{abc}\right]=\frac{1}{6}\left[\overline{ki}-\bar{j}\right]=\frac{1}{6}\left(k\times i\right)-\bar{j}=\frac{1}{6}\left|-1\right|=\frac{1}{6}$$

18. $V=\frac{1}{6}\left[\overline{OA}\ \overline{OB}\ \ \overline{OC}\ \right]$ *cubic units*

19. Given volume of the tetrahedron =1

$$\frac{1}{6}\left[\overline{abb}\right]=1$$

$$\begin{bmatrix}2 & 1 & 1\\ 1 & a & 1\\ 1 & 2 & -1\end{bmatrix}=6$$

$$\Rightarrow 2(-a-2)-1(-2)+1(2-a)=6$$
$$-2a-4+2+2-a=6$$
$$-3a=6$$
$$a=-2$$

20. $\left[\bar{a}\ \bar{b}\ \bar{c}\right]^2$

21. $V=\left[\bar{r}-\bar{a}\quad \bar{b}-\bar{a}\quad \bar{c}-\bar{a}\right]=0$

22. $V=\left[\bar{r}-\bar{a}\quad \bar{b}-\bar{a}\quad \bar{c}\right]=0$

23. $V=\left[\bar{r}-\bar{a}\quad \bar{b}\quad \bar{c}\right]=0$

24. $\bar{r}.\bar{a}=\bar{r}.\bar{b}$

25. the equation of plane passing though the three points $\bar{a},\bar{b},\bar{c}$ is $\left[\overline{AP}\ \overline{AB}\ \overline{AC}\right]=0$

$$\Rightarrow\begin{vmatrix}x-2 & y-3 & z+1\\ 2 & 2 & 3\\ 1 & 3 & 6\end{vmatrix}=0$$

$$(x-2)(3)-(y-3)(9)+(z+1)(4)=0$$
$$3x-6-9y+27+4z+4=0$$
$$3x-9y+4z+25=0$$

26. $\left[\bar{r}-\bar{a}\quad \bar{b}\quad \bar{d}\right]=0$

27. $D=\dfrac{\left|ax_1+by_1+cz_1+d\right|}{\sqrt{a^2+b^2+c^2}}$

28. $S.D.=\dfrac{\left|\left[\bar{a}-\bar{c}\ \bar{b}\ \bar{d}\right]\right|}{\left|\bar{b}\times\bar{d}\right|}$

29. Given lines

$$\bar{\gamma}=\bar{i}+2\bar{j}+3\bar{k}+\delta\left(2\bar{i}+3\bar{j}+4\bar{k}\right)-1$$
$$\bar{\gamma}=\left(2\bar{i}+4\bar{j}+5\bar{k}\right)+t\left(3\bar{i}+4\bar{j}+5\bar{k}\right)-2$$

comparing with

$\bar{\gamma} = \bar{a} + \delta\bar{b}$

$\bar{\gamma} = \bar{c} + t\bar{d}$

$\bar{a} = \bar{i} + 2\bar{j} + 3\bar{k}$ $\bar{b} = 2\bar{i} + 3\bar{j} + 4\bar{k}$

$\bar{c} = 2\bar{i} + 4\bar{j} + 5\bar{k}$ $\bar{d} = 3\bar{i} + 4\bar{j} + 5\bar{k}$

$$\left[\bar{a}-\bar{c}\ \bar{b}\ \bar{d}\right] = \begin{vmatrix} -1 & -2 & -2 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = -1(-1)+2(-2)-2(8-9) = 1-4+2 = -1$$

$$\bar{b}\times\bar{d} = \begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \bar{i}(-1) - \bar{j}(-2) + \bar{k}(-1)$$

$\left|\bar{b}\times\bar{d}\right| = \sqrt{1+4+1} = \sqrt{16}$

shortest distance between the lines 1,2 is

$$\frac{\left|\left[\bar{a}-\bar{c}\,\bar{b}\,\bar{d}\right]\right|}{\left|\bar{b}\times\bar{d}\right|} = \frac{1}{\sqrt{16}}$$

30. $\left[\bar{a}-\bar{c}\ \bar{b}\ \bar{d}\right] = (\bar{a}-\bar{c}).\bar{b}\times\bar{d} = 0$

31. $\bar{a}.\bar{a}^1 = \bar{b}.\bar{b}^1 = \bar{c}.\bar{c}^1 = 1$

EXERCISE - II

1. If $\bar{a},\bar{b},\bar{c}$ form a left handed orthogonal system and $\bar{a}.\bar{a} = 4, \bar{b}.\bar{b} = 9, \bar{c}.\bar{c} = 16$ then $\left[\bar{a}\ \bar{b}\ \bar{c}\right] =$

1) 24 2) -24 3) 12 4) -12

2. For any three non-zero vectors $\bar{a},\bar{b},\bar{c}$, $\left|\bar{a}\times\bar{b}.\bar{c}\right| = |\bar{a}||\bar{b}||\bar{c}|$ hold if and only if

1) $\bar{a}.\bar{b} = \bar{b}.\bar{c} = \bar{c}.\bar{a} = 0$ 2) $\bar{a}.\bar{b} + \bar{b}.\bar{c} + \bar{c}.\bar{a} = 0$

3) $\bar{a} + \bar{b} + \bar{c} = \bar{0}$ 4) $\left[\bar{a}\ \bar{b}\ \bar{c}\right] = 0$

3. If $\bar{a}$ is a perpendicular to $\bar{b}$ and $\bar{c}$, $|\bar{a}| = 2, |\bar{b}| = 3, |\bar{c}| = 4$ and the angle between $\bar{b}$ and $\bar{c}$ is $\frac{2\pi}{3}$, then $\left|\left[\bar{a}\ \bar{b}\ \bar{c}\right]\right| =$

1) 24 2) 12 3) $12\sqrt{3}$ 4) $24\sqrt{3}$

4. If $\bar{a},\bar{b},\bar{c}$ are three non-zero and non-null vectors and $\bar{r}$ is any vector in space, then $\left[\bar{b}\ \bar{c}\ \bar{r}\right]\bar{a} + \left[\bar{c}\ \bar{a}\ \bar{r}\right]\bar{b} + \left[\bar{a}\ \bar{b}\ \bar{r}\right]\bar{c}$ is equal to

1) $2\left[\bar{a}\ \bar{b}\ \bar{c}\right]\bar{r}$ 2) $3\left[\bar{a}\ \bar{b}\ \bar{c}\right]\bar{r}$

3) $\left[\bar{a}\ \bar{b}\ \bar{c}\right]\bar{r}$ 4) $5\left[\bar{a}\ \bar{b}\ \bar{c}\right]\bar{r}$

5. If $\bar{a} = 2\bar{i} - \bar{j} + \bar{k}$, $\bar{b} = \bar{i} + 2\bar{j} - 3\bar{k}$, $\bar{c} = 3\bar{i} + \mu\bar{j} + 5\bar{k}$ are coplanar then μ is a root of the equation

1) $x^2 + 3x = 4$ 2) $x^2 + 2x = 6$

3) $x^2 + 3x = 6$ 4) $x + 5 = 0$

6. $\bar{a} = \bar{i} - \bar{k}, \bar{b} = x\bar{i} + \bar{j} + (1-x)\bar{k}$ and $\bar{c} = y\bar{i} + \lambda\bar{j} + (1+x-y)\bar{k}$ then $\left[\bar{a}\ \bar{b}\ \bar{c}\right]$ depends on

1) neither x nor y 2) both x and y

3) only x 4) only y

7. Let $\bar{a}$ be a unit vector $\bar{b} = 2\bar{i} + \bar{j} - \bar{k}$ and $\bar{c} = \bar{i} + 3\bar{k}$ then maximum value of $\left[\bar{a}\ \bar{b}\ \bar{c}\right]$ is

1) -1 2) $\sqrt{10}+\sqrt{6}$ 3) $\sqrt{10}-\sqrt{6}$ 4) $\sqrt{59}$

8. If $x(\bar{a}\times\bar{b}) + y(\bar{b}\times\bar{c}) + z(\bar{c}\times\bar{a}) = \bar{r}$ and $\left[\bar{a}\ \bar{b}\ \bar{c}\right] = \frac{1}{8}$ then $x+y+z =$

1) $\bar{r}.(\bar{a}+\bar{b}+\bar{c})$ 2) $4\left[\bar{r}.(\bar{a}+\bar{b}+\bar{c})\right]$

3) $8\left[\bar{r}.(\bar{a}+\bar{b}+\bar{c})\right]$ 4) 0

9. $\begin{vmatrix} \bar{a}.\bar{a} & \bar{a}.\bar{b} & \bar{a}.\bar{c} \\ \bar{b}.\bar{a} & \bar{b}.\bar{b} & \bar{b}.\bar{c} \\ \bar{c}.\bar{a} & \bar{c}.\bar{b} & \bar{c}.\bar{c} \end{vmatrix} = 0$ **then the vectors $\bar{a}+\bar{b}$, $\bar{b}+\bar{c}$, $\bar{c}+\bar{a}$ are**

1) Coplanar 2) Non coplanar

3) Unit vectors 4) Mutually perpendicular

10. The volume of the parallelopiped with coterminus edges $l\bar{i} - 5\bar{k}, \bar{i} - \bar{j} + m\bar{k}$ and $3\bar{i} - 5\bar{j}$ is 8. Then 'l' and 'm' are related as

1) $3lm+2=0$ 2) $lm+2=0$
3) $3lm-2=0$ 4) $5lm+2=0$

11. If $\bar{a},\bar{b},\bar{c}$ are non coplanar vectors, then $\begin{vmatrix} \bar{a}.\bar{a} & \bar{a}.\bar{b} & \bar{a}.\bar{c} \\ \bar{b}.\bar{a} & \bar{b}.\bar{b} & \bar{b}.\bar{c} \\ \bar{c}.\bar{a} & \bar{c}.\bar{b} & \bar{c}.\bar{c} \end{vmatrix} =$

1) $[\bar{a}\ \bar{b}\ \bar{c}]$ 2) $[\bar{a}\ \bar{b}\ \bar{c}]^2$ 3) 1 4) 0

12. The lines $\bar{r}=\bar{i}+\bar{j}-\bar{k}+s(3\bar{i}-\bar{j})$ **and** $\bar{r}=4\bar{i}-\bar{k}+t(2\bar{i}+3\bar{k})$

1) intersect 2) do not intersect
3) are skew lines 4) cannot be determined

13. The shortest distance between the lines $\bar{r}=3\bar{i}+5\bar{j}+7\bar{k}+\lambda(\bar{i}+2\bar{j}+\bar{k})$
$\bar{r}=-\bar{i}-\bar{j}+\bar{k}+\mu(7\bar{i}-6\bar{j}+\bar{k})$ **is**

1) $\frac{16}{5\sqrt{5}}$ 2) $\frac{26}{5\sqrt{5}}$ 3) $\frac{46}{5\sqrt{5}}$ 4) $\frac{36}{5\sqrt{5}}$

14. The lines $\bar{r}=\bar{i}-\bar{j}+\bar{k}+s(\bar{i}+2\bar{j}-3\bar{k})$ **and** $\bar{r}=(\bar{i}-2\bar{j}+3\bar{k})+t(-\bar{i}+\bar{j}+2\bar{k})$

1) Intersect 2) Do not intersect
3) skew lines 4) Cannot be determine

15. The shortest distance between the lines through the points $(2,3,1),(4,5,2)$ and parallel to the vectors $(3,4,2),(4,5,3)$ respectively is

1) $\frac{\sqrt{6}}{7}$ 2) $\frac{1}{\sqrt{6}}$ 3) $\frac{2}{\sqrt{3}}$ 4) 9

16. Let $\bar{p}=\frac{\bar{b}\times\bar{c}}{[\bar{a}\ \bar{b}\ \bar{c}]},\bar{q}=\frac{\bar{c}\times\bar{a}}{[\bar{a}\ \bar{b}\ \bar{c}]},\bar{r}=\frac{\bar{a}\times\bar{b}}{[\bar{a}\ \bar{b}\ \bar{c}]}$ **and $\bar{a},\bar{b},\bar{c}$ being any three non-coplanar vectors then**

$\bar{p}.(\bar{a}+\bar{b})+\bar{q}.(\bar{b}+\bar{c})+\bar{r}.(\bar{c}+\bar{a})=$

1) -3 2) 0 3) 3 4) -2

17. If $\overline{a^1},\overline{b^1},\overline{c^1}$ is the reciprocal system of vector triad of $\bar{a},\bar{b}$ and $\bar{c}$, then $\bar{a}.\overline{b^1}+\bar{b}.\overline{c^1}+\bar{c}.\overline{a^1}=$

1) 0 2) 1 3)2 4) 3

18. If $\bar{p}\ \bar{q}\ \bar{r}$ is reciprocal system of vector triad $\bar{a},\bar{b}$ and $\bar{c}$ then $[\bar{a}\ \bar{b}\ \bar{c}][\bar{p}\ \bar{q}\ \bar{r}]=$

1) 0 2) 1 3) 2 4) 3

19. Let $c_1=(1,0,0),c_2=(1,1,0),c_3=(1,1,1)$**, then the reciprocal of** $c_1=$

1) $\bar{i}+\bar{j}$ 2) $\bar{i}-\bar{j}$ 3) $\bar{j}-\bar{k}$ 4) $\bar{k}-\bar{i}$

KEY

01) 2	02) 1	03) 3	04) 3	05) 1	06) 3
07) 4	08) 3	09) 1	10) 4	11) 2	12) 1
13) 2	14) 3	15) 2	16) 3	17) 1	18) 2
19) 2					

SOLUTIONS

1. $|[\bar{a}\ \bar{b}\ \bar{c}]|=|\bar{a}\times\bar{b}.\bar{c}|=-|\bar{a}||\bar{b}||\bar{c}|=-2\times3\times4=-24$

2. $\bar{a},\bar{b},\bar{c}$ are non zero vectors
$|\bar{a}\times\bar{b}.\bar{c}|=|\bar{a}||\bar{b}||\bar{c}|$
$\Leftrightarrow \bar{a}.\bar{b}=\bar{b}.\bar{c}=\bar{c}.\bar{a}=0$ (conceptual)

3. $[\bar{a}\ \bar{b}\ \bar{c}]=|\bar{a}||\bar{b}||\bar{c}|(\cos 0)\left(\sin\frac{2\pi}{3}\right)=12\sqrt{3}$

4. let $\bar{r}=x\bar{a}+y\bar{b}+z\bar{c}-1$
$(\bar{b}\times\bar{c}).\bar{r}=(\bar{b}\times\bar{c}).(x\bar{a}+y\bar{b}+z\bar{c})$
$[\bar{b}\ \bar{c}\ \bar{r}]=x[\bar{b}\bar{c}\bar{a}]+0+0$
$x=\frac{[\bar{b}\bar{c}\bar{r}]}{[\bar{a}\bar{b}\bar{c}]}$

similarly $y=\frac{[\bar{c}\bar{a}\bar{r}]}{[\bar{a}\bar{b}\bar{c}]},z=\frac{[\bar{a}\bar{b}\bar{r}]}{[\bar{a}\bar{b}\bar{c}]}$

substitute the value of x,y,z in -1

$\bar{r}=\frac{[\bar{b}\bar{c}\bar{r}]\bar{a}+[\bar{c}\bar{a}\bar{r}]\bar{b}+[\bar{a}\bar{b}\bar{r}]\bar{r}}{[\bar{a}\bar{b}\bar{c}]}$

5. $\left[\bar{a}\ \bar{b}\ \bar{c}\right]=0 \Rightarrow \mu=-4$

6. $\begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1-x-y \end{vmatrix}=0 \Rightarrow 2x=1 \Rightarrow x=\dfrac{1}{2}$

7. $V=\left|\left(\bar{a}\times\bar{b}\right).\bar{c}\right|=\left|\bar{a}\times\bar{b}\right|\left|\bar{c}\right|=\left(\sqrt{59}\right)(1)=\sqrt{59}$

8. $\bar{r}.\bar{a}=y\left[\bar{a}\ \bar{b}\ \bar{c}\right],\quad \bar{r}.\bar{b}=z\left[\bar{a}\ \bar{b}\ \bar{c}\right]$

$\bar{r}.\bar{c}=x\left[\bar{a}\ \bar{b}\ \bar{c}\right]$

$\bar{r}.\left[\bar{a}+\bar{b}+\bar{c}\right]=\left[x+y+z\right].\left[\bar{a}\ \bar{b}\ \bar{c}\right]$

$x+y+z=8\left[\bar{r}.\left(\bar{a}+\bar{b}+\bar{c}\right)\right]$

9. $\left[\bar{a}\ \bar{b}\ \bar{c}\right]^2=0 \quad \Rightarrow\left[\bar{a}\ \bar{b}\ \bar{c}\right]=0$

$\left[\bar{a}+\bar{b}\quad \bar{b}+\bar{c}\quad \bar{c}+\bar{a}\right]=2\left[\bar{a}\ \bar{b}\ \bar{c}\right]=0$

10. $\begin{vmatrix} l & 0 & -5 \\ 1 & -1 & m \\ 3 & -5 & 0 \end{vmatrix}=+8$

11. $\left[\bar{a}\,\bar{b}\,\bar{c}\right]\left[\bar{a}\,\bar{b}\,\bar{c}\right]=\left[\bar{a}\,\bar{b}\,\bar{c}\right]^2$

12. $\left[\bar{a}-\bar{c}\ \bar{b}\ \bar{d}\right]=0 \Rightarrow$ lines are intersect

13. $\text{SD}=\dfrac{\left|\left[\bar{a}-\bar{c}\quad \bar{b}\ \ \bar{d}\right]\right|}{\left|\bar{b}\times\bar{d}\right|}$

14. $\left[\bar{a}-\bar{c}\ \ \bar{b}\ \ \bar{d}\right]\neq 0 \Rightarrow$ skew lines

15. $\dfrac{\left|\left[\bar{a}-\bar{c}\quad \bar{b}\ \ \bar{d}\right]\right|}{\left|\bar{b}\times\bar{d}\right|}$

16.

$$\left(\bar{a}+\bar{b}\right).\frac{\left(\bar{b}\times\bar{c}\right)}{\left[\bar{a}\bar{b}\bar{c}\right]}+\left(\bar{b}+\bar{c}\right).\frac{\left(\bar{c}\times\bar{a}\right)}{\left[\overline{abc}\right]}+\left(\bar{c}+\bar{a}\right).\frac{\left(\bar{a}+\bar{b}\right)}{\left[\overline{abc}\right]}$$

$$=\frac{\bar{a}\left(\bar{b}\times\bar{c}\right)+0}{\left[\bar{a}\bar{b}\bar{c}\right]}+\frac{\bar{b}.\left(\bar{c}\times\bar{a}\right)+0}{\left[\bar{a}\bar{b}\bar{c}\right]}+\frac{\bar{c}\left(\bar{a}\times\bar{b}\right)+0}{\left[\bar{a}\bar{b}\bar{c}\right]}$$

$$=\frac{\left[\bar{a}\bar{b}\bar{c}\right]}{\left[\bar{a}\bar{b}\bar{c}\right]}+\frac{\left[\bar{b}\bar{c}\bar{a}\right]}{\left[\bar{a}\bar{b}\bar{c}\right]}+\frac{\left[\bar{c}\bar{a}\bar{b}\right]}{\left[\bar{a}\bar{b}\bar{c}\right]}$$

$$\left(\bar{a}+\bar{b}\right).\frac{\left(\bar{b}\times\bar{c}\right)}{\left[\bar{a}\ \bar{b}\ \bar{c}\right]}+\left(\bar{b}+\bar{c}\right).\frac{\left(\bar{c}\times\bar{a}\right)}{\left[\bar{a}\ \bar{b}\ \bar{c}\right]}+\left(\bar{c}+\bar{a}\right).\frac{\left(\bar{a}\times\bar{b}\right)}{\left[\bar{a}\ \bar{b}\ \bar{c}\right]}$$

$=1+1+1=3$

17. $\bar{a}.\dfrac{\left(\bar{c}\times\bar{a}\right)}{\left[\bar{a}\ \bar{b}\ \bar{c}\right]}+\bar{b}.\dfrac{\left(\bar{a}\times\bar{b}\right)}{\left[\bar{a}\ \bar{b}\ \bar{c}\right]}+\bar{c}.\dfrac{\left(\bar{b}\times\bar{c}\right)}{\left[\bar{a}\ \bar{b}\ \bar{c}\right]}=0+0+0=0$

18. $\left[\bar{p}\ \bar{q}\ \bar{r}\right]=\dfrac{1}{\left[\bar{a}\ \bar{b}\ \bar{c}\right]}$

19. Given $c_1=\left(1,0,0\right),c_2=\left(1,1,0\right),c_3=\left(1,1,1\right)$

reciprocol of $c_1=\dfrac{c_2\times c_3}{\left[c_1c_2c_3\right]}$

$$\left[c_1c_2c_3\right]=\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}=1\left(1-0\right)-0\left(1-0\right)+0\left(1-1\right)=1$$

$$c_2\times c_3=\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}=\bar{i}\left(1-0\right)-\bar{j}\left(1-0\right)+\bar{k}\left(1-1\right)$$

reciprocal of $c_1=\dfrac{\bar{i}-\bar{j}}{1}=\bar{i}-\bar{j}$

EXERCISE - III

1. If $4\overline{a}+5\overline{b}+9\overline{c}=\overline{0}$

then $(\overline{a}\times\overline{b})\times[(\overline{b}\times\overline{c})\times(\overline{c}\times\overline{a})]=$

1) A vector perpendicular to the plane of $\overline{a},\overline{b}$ & $\overline{c}$

2) $4\overline{a}+5\overline{b}+9\overline{c}$ 3) $\overline{0}$ 4) $[\overline{a}\,\overline{b}\,\overline{c}]$

2. A tetrahedron of volume V=5 has three of its vertices at the points $A(2,1,-1), B(3,0,1)$ **and** $C(2,-1,3)$**. The fourth vertex D lies on the y-axis. Then D is**

1) $(0,8,0)$ 2) $(0,-7,0)$

3) $(0,8,0)$ or $(0,-7,0)$ 4) $(0,7,0)$

3. If $\overline{r}$ **is a unit vector such that**

$\overline{r}=x(\overline{b}\times\overline{c})+y(\overline{c}\times\overline{a})+z(\overline{a}\times\overline{b})$**, then**

$|(\overline{r}.\overline{a})(\overline{b}\times\overline{c})+(\overline{r}.\overline{b})(\overline{c}\times\overline{a})+(\overline{r}.\overline{c})(\overline{a}\times\overline{b})|=$

1) $[\overline{a}\ \overline{b}\ \overline{c}]$ 2) 1 3) $|[\overline{a}\ \overline{b}\ \overline{c}]|$ 4) 0

4. If $\overline{a}$ *and* $\overline{b}$ **are two mutually perpendicular unit vectors and the vectors**

$x\overline{a}+x\overline{b}+z(\overline{a}\times\overline{b}), \overline{a}+(\overline{a}\times\overline{b})$ *and*

$z\overline{a}+z\overline{b}+y(\overline{a}\times\overline{b})$ **lie in a plane, then z is**

1) A.M. of x and y 2) G.M. of x and y

3) H,M, of x and y 4) Equal to zero

5. If $\overline{a},\overline{b},\overline{c}$ **are three non-coplanar, non-zero vectors, then**

$(\overline{a}.\overline{a})(\overline{b}\times\overline{c})+(\overline{a}.\overline{b})(\overline{c}\times\overline{a})+(\overline{a}.\overline{c})(\overline{a}\times\overline{b})=$

1) $[\overline{a}\ \overline{b}\ \overline{c}]\overline{c}$ 2) $[\overline{b}\ \overline{c}\ \overline{a}]\overline{a}$

3) $[\overline{c}\ \overline{a}\ \overline{b}]\overline{b}$ 4) $\overline{0}$

6. Let $\overline{a},\overline{b},\overline{c}$ **be the position vectors of the points A,B,C respectively and** α,β **and** γ **be the inclinations between** $\overline{b}\,\overline{c};\overline{a},\overline{b}$ **and** $\overline{a},\overline{c}$ **if the volume of the tetrahedron OABC is V then**

1) $V^2=\dfrac{a^2b^2c^2}{36}\begin{vmatrix}1 & \cos\beta & \cos\gamma\\ \cos\beta & 1 & \cos\alpha\\ \cos\gamma & \cos\alpha & 1\end{vmatrix}$

2) $V^2=\dfrac{a^2b^2c^2}{6}\begin{vmatrix}1 & \cos\beta & \cos\gamma\\ \cos\beta & 1 & \cos\alpha\\ \cos\gamma & \cos\alpha & 1\end{vmatrix}$

3) $V^2=\dfrac{a^2b^2c^2}{36}\begin{vmatrix}0 & \cos\beta & \cos\gamma\\ \cos\beta & 0 & \cos\alpha\\ \cos\gamma & \cos\alpha & 0\end{vmatrix}$

4) $V^2=\dfrac{a^2b^2c^2}{36}\begin{vmatrix}1 & \sin\beta & \sin\gamma\\ \sin\beta & 1 & \sin\alpha\\ \sin\gamma & \sin\alpha & 1\end{vmatrix}$

7. If $\overline{u},\overline{v},\overline{w}$ **are non co-planar vectors and p,q are real numbers then the equality**

$[3\overline{u}\ p\overline{v}\ p\overline{w}]-[p\overline{v}\ \overline{w}\,q\overline{u}]-[2\overline{w}\,q\overline{v}\ q\overline{u}]=0$

holds for

1) exactly two values of (p, q)
2) more than two but not all values of (p, q)
3) all values of (p,q)
4) exactly one value of (p,q)

8. If $\overline{a},\overline{b},\overline{c}$ **are non co-planar vectors and** λ **is a real number then**

$[\lambda(\overline{a}+\overline{b})\ \lambda^2\overline{b}\ \lambda\overline{c}]=[\overline{a}\ \overline{b}+\overline{c}\ \overline{b}]$ **for**

1) exactly two values of λ
2) exactly three values of λ
3) no value of λ
4) exactly one value of λ

KEY

01) 3 02) 3 03) 3 04) 2 05) 2 06) 1
07) 4 08) 3

SOLUTIONS

1. $4\bar{a}+5\bar{b}+9\bar{c}=0 \Rightarrow$ vectors $\bar{a},\bar{b}$ and $\bar{c}$ are coplanar $\Rightarrow \bar{b}\times\bar{c}$ and $\bar{c}\times\bar{a}$ are collinear

$\Rightarrow (\bar{b}\times\bar{c})\times(\bar{c}\times\bar{a})=0$

Hence (C) is the correct answer

2. $D=(0,y,0)$, $\frac{1}{6}\begin{vmatrix}1 & -1 & 2\\ 0 & -2 & 4\\ -2 & y-1 & 1\end{vmatrix}=\pm 5$

3. $\bar{r}=x(\bar{b}\times\bar{c})+y(\bar{c}\times\bar{a})+z(\bar{a}\times\bar{b})$

$\therefore \bar{r}\,.\,\bar{a}=x[\bar{a}\;\bar{b}\;\bar{c}],\ \bar{r}\,.\,\bar{b}=y[\bar{a}\;\bar{b}\;\bar{c}]$ and

$\bar{r}\,.\,\bar{c}=z[\bar{a}\;\bar{b}\;\bar{c}]$

substituting the values of x, y, z in (i), we get

$[\bar{a}\bar{b}\bar{c}]\bar{r}=(\bar{r}\,.\bar{a})(\bar{b}\times\bar{c})+(\bar{r}.\bar{b})(\bar{c}\times\bar{a})+(\bar{r}.\bar{c})(\bar{a}\times\bar{b})$

$|(\bar{r}.\bar{a})(\bar{b}\times\bar{c})+(\bar{r}.\bar{b})(\bar{c}\times\bar{a})+(\bar{r}.\bar{c})(\bar{a}\times\bar{b})|-|[\bar{a}\bar{b}\bar{c}]\bar{r}|$

$|[\bar{a}\;\bar{b}\;\bar{c}]|\,|\bar{r}|=|[\bar{a}\;\bar{b}\;\bar{c}]|\ [\because |\bar{r}|=1]$

4. $\begin{vmatrix}x & x & z\\ 1 & 0 & 1\\ z & z & y\end{vmatrix}[\bar{a}\,\bar{b}\;\bar{a}\times\bar{b}\,]=0$, $\therefore z^2=xy$

5. Since $\bar{a},\bar{b},\bar{c}$ are non-coplanar vectors.
Therefore the vectors
$\bar{a}\times\bar{b},\ \bar{b}\times\bar{c},\ \bar{c}\times\bar{a}$ are non-coplanar

$\bar{a}=x(b\times\bar{c})+y(\bar{c}\times\bar{a})+z(\bar{a}\times\bar{b})$

Taking dot products successively with $\bar{a},\bar{b},\bar{c}$

$$x=\frac{\bar{a}\,.\,\bar{a}}{[\bar{a}\;\bar{b}\;\bar{c}]},\ y=\frac{\bar{a}\,.\,\bar{b}}{[\bar{a}\;\bar{b}\;\bar{c}]},\ z=\frac{\bar{a}\,.\,\bar{c}}{[\bar{a}\;\bar{b}\;\bar{c}]}$$

Substituting these values in (i), we obtain that the given expression is equal to $[\bar{a}\;\bar{b}\;\bar{c}]\bar{a}$.

6. We have $\bar{a},\bar{b},\bar{c}$ be the position vectors of the points A,B,C with respect to $0.\alpha,\beta,\gamma$ be the angles between $\bar{b},\bar{c};\bar{a},\bar{b};\bar{a},\bar{c}$ .OABC is the tetrahedron. Let $\overline{OA}=\bar{a}=x_1\bar{i}+y_1\bar{j}+z_1\bar{k}$

$\overline{OB}=\bar{b}=x_2\bar{i}+y_2\bar{j}+z_2\bar{k}$

$\overline{OC}=\bar{c}=x_3\bar{i}+y_3\bar{j}+z_3\bar{k}$

V = volume of tetrahedron $=\frac{1}{6}[\overline{OA},\overline{OB},\overline{OC}]$

$$V=\frac{1}{6}\begin{vmatrix}x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\end{vmatrix}$$

$$V^2=\frac{1}{36}\begin{vmatrix}x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\end{vmatrix}\begin{vmatrix}x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\end{vmatrix}$$

$$=\frac{1}{36}\begin{vmatrix}\sum x_1^2 & \sum x_1x_2 & \sum x_1x_3\\ \sum x_1x_2 & \sum x_2^2 & \sum x_2x_3\\ \sum x_3x_1 & \sum x_2x_3 & \sum x_3^2\end{vmatrix}$$

$$=\frac{1}{36}\begin{vmatrix}a^2 & ab\cos\beta & ac\cos\gamma\\ ab\cos\beta & b^2 & bc\cos\alpha\\ ca\cos\gamma & bc\cos\alpha & c^2\end{vmatrix}$$

$$V^2=\frac{a^2b^2c^2}{36}\begin{vmatrix}1 & \cos\beta & \cos\gamma\\ \cos\beta & 1 & \cos\alpha\\ \cos\gamma & \cos\alpha & 1\end{vmatrix}$$

7. $[3\bar{u}\;p\bar{v}\;p\bar{w}]-[p\bar{v}\;\bar{w}q\bar{u}]-[2\bar{w}q\bar{v}\;q\bar{u}]=0$

$\Rightarrow 3p^2[\bar{u}\;\bar{v}\;\bar{w}]-pq[\bar{u}\;\bar{v}\;\bar{w}]-2q^2[\bar{u}\;\bar{v}\;\bar{w}]$

$\Rightarrow (3p^2-pq+2q^2)[\bar{u}\;\bar{v}\;\bar{w}]=0$

$\Rightarrow 3p^2-pq+2q^2=0\ (\because[\bar{u}\;\bar{v}\;\bar{w}]\neq 0)$

$\Rightarrow p=0,\ q=0$

8. $[\lambda(\bar{a}+\bar{b})\;\lambda^2\bar{b}\;\lambda\bar{c}]=[\bar{a}\;\bar{b}+\bar{c}\;\bar{b}]$

$\Rightarrow \lambda^4[\bar{a}\;\bar{b}\;\bar{c}]=[\bar{a}\;\bar{c}\;\bar{b}]\ \Rightarrow \lambda^4=-1$

which is not possible for any real λ

VECTOR TRIPLE PRODUCT & PRODUCT OF FOUR VECTORS

SYNOPSIS

- **Vector triple Product :**
  The vector product of $\overline{a}\times\overline{b}$ and $\overline{c}$ is a vector triple Product of three vectors $\overline{a},\overline{b}$ and $\overline{c}$. It is denoted by $\left(\overline{a}\times\overline{b}\right)\times\overline{c}$
- $(\overline{a}\times\overline{b})\times\overline{c}=(\overline{a}.\overline{c})\overline{b}-(\overline{b}.\overline{c})\overline{a}$. This is a vector in the plane of $\overline{a}$ and $\overline{b}$.
- $\overline{a}\times\left(\overline{b}\times\overline{c}\right)=\left(\overline{a}.\overline{c}\right)\overline{b}-\left(\overline{a}.\overline{b}\right)\overline{c}$. This is a vector in the plane of $\overline{b},\overline{c}$
- $(\overline{a}\times\overline{b})\times\overline{c}=-\overline{c}\times(\overline{a}\times\overline{b})$
- $\overline{a},\overline{b},\overline{c}$ are non-zero vectors and $(\overline{a}\times\overline{b})\times\overline{c}=\overline{a}\times(\overline{b}\times\overline{c})\Leftrightarrow\overline{a}\,\&\,\overline{c}$ are collinear (Parallel) (or) $(\overline{a}\times\overline{c})\times\overline{b}=\overline{0}$
- Vector triple product is not associative. If $\overline{a},\overline{b},\overline{c}$ are non-zero, non-orthogonal vectors., then $(\overline{a}\times\overline{b})\times\overline{c}\neq\overline{a}\times(\overline{b}\times\overline{c})$.
- $\overline{a}\times(\overline{b}\times\overline{c})+\overline{b}\times\left(\overline{c}\times\overline{a}\right)+\overline{c}\times(\overline{a}\times\overline{b})=\overline{0}$
  $\Rightarrow\overline{a}\times(\overline{b}\times\overline{c}),\overline{b}\times(\overline{c}\times\overline{a}),\overline{c}\times(\overline{a}\times\overline{b})$ are coplanar
- $\overline{i}\times\left(\overline{j}\times\overline{k}\right)+\overline{j}\times\left(\overline{k}\times\overline{i}\right)+\overline{k}\times\left(\overline{i}\times\overline{j}\right)=\overline{0}$
- $\overline{i}\times\left(\overline{a}\times\overline{i}\right)+\overline{j}\times\left(\overline{a}\times\overline{i}\right)+\overline{k}\times\left(\overline{a}\times\overline{k}\right)=2\overline{a}$
  where $\overline{a}$ is any vector
- $\left|\overline{a}\times(\overline{b}\times\overline{c})+\overline{b}\times(\overline{c}\times\overline{a})\right|=\left|(\overline{a}\times\overline{b})\times\overline{c}\right|$
- $[\overline{a}\times\overline{b}\ \ \overline{b}\times\overline{c}\ \ \overline{c}\times\overline{a}]=[\overline{a}\,\overline{b}\,\overline{c}]^2$
- **Scalar Product of Four Vectors :**
  $(\overline{a}\times\overline{b}).(\overline{c}\times\overline{d})$ is a scalar product of four vectors. It is a dot product of the vectors $\overline{a}\times\overline{b}$ and $\overline{c}\times\overline{d}$.
- $(\overline{a}\times\overline{b}).(\overline{c}\times\overline{d})=\left(\overline{a}.\overline{c}\right)(\overline{b}.\overline{d})-\left(\overline{a}.\overline{d}\right)(\overline{b}.\overline{c})$
  $=\begin{vmatrix}\overline{a}.\overline{c} & \overline{a}.\overline{d}\\ \overline{b}.\overline{c} & \overline{b}.\overline{d}\end{vmatrix}$
- $(\overline{a}\times\overline{c}).\left(\overline{b}\times\overline{d}\right)=\begin{vmatrix}\overline{a}.\overline{b} & \overline{a}.\overline{d}\\ \overline{c}.\overline{b} & \overline{c}.\overline{d}\end{vmatrix}$
- $\left(\overline{a}\times\overline{b}\right).\left(\overline{a}\times\overline{b}\right)=\left|\overline{a}\times\overline{b}\right|^2=\left|\overline{a}\right|^2\left|\overline{b}\right|^2-\left(\overline{a}.\overline{b}\right)^2$
- **Vector Product of Four Vectors :**
  $\left(\overline{a}\times\overline{b}\right)\times(\overline{c}\times\overline{d})$ is a vector product of four vectors.
- $\left(\overline{a}\times\overline{b}\right)\times\left(\overline{c}\times\overline{d}\right)=\left[\overline{a}\,\overline{b}\,\overline{d}\right]\overline{c}-\left[\overline{a}\,\overline{b}\,\overline{c}\right]\overline{d}$
  $=\left[\overline{c}\,\overline{d}\,\overline{a}\right]\overline{b}-\left[\overline{c}\,\overline{d}\,\overline{b}\right]\overline{a}$
- $\left[\overline{a}\,\overline{b}\,\overline{c}\right]\overline{d}=\left[\overline{b}\,\overline{c}\,\overline{d}\right]\overline{a}+\left[\overline{c}\,\overline{a}\,\overline{d}\right]\overline{b}+\left[\overline{a}\,\overline{b}\,\overline{d}\right]\overline{c}$
- $\left[\overline{a}\times\overline{b}\ \overline{b}\times\overline{c}\ \overline{c}\times\overline{a}\right]=\left[\overline{a}\,\overline{b}\,\overline{c}\right]^2$
- If $\overline{a},\overline{b},\overline{c}$ are non coplanar vectors,
  i.e.,$\left[\overline{a}\,\overline{b}\,\overline{c}\right]\neq 0$ then any vector $\overline{r}$ in space can be expressed as a linear combination of $\overline{a},\overline{b},\overline{c}$
  i.e., $\overline{r}=\dfrac{\left[\overline{r}\,\overline{b}\,\overline{c}\right]}{\left[\overline{a}\,\overline{b}\,\overline{c}\right]}\overline{a}+\dfrac{\left[\overline{r}\,\overline{c}\,\overline{a}\right]}{\left[\overline{a}\,\overline{b}\,\overline{c}\right]}\overline{b}+\dfrac{\left[\overline{r}\,\overline{a}\,\overline{b}\right]}{\left[\overline{a}\,\overline{b}\,\overline{c}\right]}\overline{c}$
  i.e., in the form $\overline{r}=x\overline{a}+y\overline{b}+z\overline{c}$
- If $\overline{a},\overline{b},\overline{c}$ and $\overline{d}$ are coplanar then $\left(\overline{a}\times\overline{b}\right)\times\left(\overline{c}\times\overline{d}\right)=\overline{0}$
- If $\overline{a},\overline{b},\overline{c}$ and $\overline{d}$ are parallel vectors (or) collinear vectors, then $(\overline{a}\times\overline{b})\times\left(\overline{c}\times\overline{d}\right)=\overline{0}$

✈ To find the direction of a line with greatest slope :

Let π_1, π_2 be two planes intersecting in a line l_1 then the line of greatest slope in π_1 is the line lying in the plane π_1 and perpendicular to the line l_1.

Note : Let $\bar{a}, \bar{b}$ be the vectors along the normals to the planes π_1 and π_2 respectively then the vector $\bar{a} \times (\bar{a} \times \bar{b})$ will be along the line of greatest slope in π_1.

Eg : 1

Let $\bar{a} = 2\bar{i} + \bar{j} + \bar{k}, \bar{b} = \bar{i} + 2\bar{j} - \bar{k}$ and a unit vector $\bar{c}$ be coplanar. If $\bar{c}$ is perpendicular to $\bar{a}$, then $\bar{c}$ is equal to

Sol: Required unit vector is $\pm \dfrac{\bar{a} \times (\bar{a} \times \bar{b})}{|\bar{a} \times (\bar{a} \times \bar{b})|}$

$$\bar{a} \times (\bar{a} \times \bar{b}) = (\bar{a}.\bar{b})\bar{a} - (\bar{a}.\bar{a})\bar{b} = -9\bar{j} + 9\bar{k}$$

$$\therefore \bar{c} = \pm \frac{1}{\sqrt{2}}(-\bar{j} + \bar{k})$$

Eg : 2

Let $\bar{a} = \bar{i} + \bar{j}$ and $\bar{b} = 2\bar{i} - \bar{k}$ then point of intersection of the line

$\bar{r} \times \bar{a} = \bar{b} \times \bar{a}$ and $\bar{r} \times \bar{b} = \bar{a} \times \bar{b}$ **is**

Sol: We have $\bar{r} \times \bar{a} = \bar{b} \times \bar{a} \Rightarrow (\bar{r} - \bar{b}) \times \bar{a} = \bar{0}$

$$\Rightarrow \bar{r} - \bar{b} \parallel \bar{a} \Rightarrow \bar{r} - \bar{b} = \lambda \bar{a}$$

$$\Rightarrow \bar{r} = \bar{b} + \lambda \bar{a}$$

Similarly, the equation of the line $\bar{r} \times \bar{b} = \bar{a} \times \bar{b}$ can be written as $\bar{r} = \bar{a} + \mu \bar{b}$

For the point of intersection of the above two lines, we have $\bar{a} + \mu \bar{b} = \bar{b} + \lambda \bar{a} \Rightarrow \lambda = \mu = 1$

$$\therefore \bar{r} = \bar{a} + \bar{b} = 3\bar{i} + \bar{j} - \bar{k}$$

Eg : 3

$(\bar{b} \times \bar{c}) \times (\bar{c} \times \bar{a})$ **is equal to**

Sol : $(\bar{b} \times \bar{c}) \times (\bar{c} \times \bar{a}) = \{(\bar{b} \times \bar{c}).\bar{a}\}\bar{c} - \{(\bar{b} \times \bar{c}).\bar{c}\}\bar{a}$

$$= [\mathrm{a\ b\ c}]\bar{c} - [\mathrm{b\ c\ c}]\bar{a} = [\mathrm{a\ b\ c}]\bar{c}$$

EXERCISE - I

1. **If** $\bar{a} = \bar{i} + \bar{j} - \bar{k}, \bar{b} = \bar{i} - \bar{j} + \bar{k}$, $\bar{c} = \bar{i} - \bar{j} - \bar{k}$ **then** $\bar{a} \times (\bar{b} \times \bar{c}) =$

1) $\bar{i} - \bar{j} + \bar{k}$ 2) $2\bar{i} - 2\bar{j}$
3) $3\bar{i} - \bar{j} + \bar{k}$ 4) $2\bar{i} + 2\bar{j} - \bar{k}$

2. **If** $\bar{a} = \bar{i} - 2\bar{j} - 3\bar{k}, \bar{b} = 2\bar{i} + \bar{j} - \bar{k}$ **and** $\bar{c} = \bar{i} + 3\bar{j} - 2\bar{k}$, **and** $\bar{a} \times (\bar{b} \times \bar{c}) = p\bar{i} + q\bar{j} + r\bar{k}$ **,** **then** $p + q + r =$

1) -4 2) 4 3) 2 4) -2

3. $\bar{i} \times (\bar{j} \times \bar{k}) + \bar{j} \times (\bar{k} \times \bar{i}) + \bar{k} \times (\bar{i} \times \bar{j}) =$

1) $\bar{i}$ 2) $\bar{j}$ 3) $\bar{k}$ 4) Null vector

4. $\bar{a} = 2\bar{i} + 3\bar{j} - 4\bar{k}, \bar{b} = \bar{i} + \bar{j} + \bar{k}$, $\bar{c} = 4\bar{i} + 2\bar{j} + 3\bar{k}$ **then** $|\bar{a} \times (\bar{b} \times \bar{c})| =$ **(EAM-2000)**

1) $\sqrt{10}$ 2) 1 3) 2 4) $\sqrt{5}$

5. **If** $\bar{a} = \bar{i} + \bar{j} + \bar{k}, \bar{b} = \bar{i} - \bar{j} + \bar{k}, \bar{c} = 2\bar{i} + 3\bar{j} - \bar{k}$**,** **then** $(\bar{a} \times \bar{b}) \times \bar{c} =$

1) $2\bar{i} - 6\bar{j} + 2\bar{k}$ 2) $6\bar{i} - 2\bar{j} + 6\bar{k}$
3) $-6\bar{i} + 2\bar{j} - 6\bar{k}$ 4) $6\bar{i} + 2\bar{j} + 6\bar{k}$

6. **If** $\bar{a} = \bar{i} + \bar{j} + \bar{k}, \bar{b} = \bar{i} + \bar{j}, \bar{c} = \bar{i}$ **and** $(\bar{a} \times \bar{b}) \times \bar{c} = \lambda \bar{a} + \mu \bar{b}$, **then** $\lambda + \mu =$

1) 1 2) 0 3) -1 4) 2

7. $\bar{a} \times (\bar{b} \times \bar{c}) + \bar{b} \times (\bar{c} \times \bar{a}) + \bar{c} \times (\bar{a} \times \bar{b}) =$

1) 0 2) $\bar{0}$ 3) 1 4) $(\bar{a} \times \bar{b}).\bar{c}$

8. $(\bar{a} \times \bar{b}) \times \bar{c} = \bar{a} \times (\bar{b} \times \bar{c})$ **if and only if**

1) $(\bar{a} \times \bar{c}) \times \bar{b} = \bar{0}$ 2) $\bar{a} \times (\bar{c} \times \bar{b}) = \bar{0}$
3) $\bar{c} \times (\bar{b} \times \bar{a}) = \bar{0}$ 4) $[\bar{a}\,\bar{b}\,\bar{c}] = 1$

7. $\bar{i}\times(\bar{a}\times\bar{i})+\bar{j}\times(\bar{a}\times\bar{j})+\bar{k}\times(\bar{a}\times\bar{k})=$

1) $3\bar{a}$ 2) $2\bar{a}$ 3) $\bar{a}$ 4) $\bar{0}$

6. **The vector $(\bar{a}\times\bar{b})\times\bar{c}$ is perpendicular to**

1) $\bar{c}$ 2) $\bar{a}\times\bar{b}$ 3) both 1 and 2

4) $\bar{b},\bar{c}$

12. **If $\bar{a},\bar{b}$ are two unit vectors such that $|\bar{a}\times\bar{b}|=2$ then the value of $[\bar{a}\ \bar{b}\ \bar{a}\times\bar{b}]$ is**

1) 1 2) 2 3) 4 4) 0

8. $\bar{a}=2\bar{i}+3\bar{j}-\bar{k},\bar{b}=-\bar{i}+2\bar{j}-4\bar{k}$, $\bar{c}=\bar{i}+\bar{j}+\bar{k}$, $\bar{d}=\bar{i}-\bar{j}-\bar{k}$ **then** $(\bar{a}\times\bar{b}).(\bar{c}\times\bar{d})=$ ___

1) 4 2) 24 3) 36 4) 4

9. **If** $(\bar{a}\times\bar{b})(\bar{c}\times\bar{b})+(\bar{a}.\bar{b})(\bar{b}.\bar{c})=\lambda(\bar{a}.\bar{c})$ **then** $\lambda=$

1) $|\bar{a}|^2$ 2) $|\bar{b}|^2$ 3) $|\bar{c}|^2$ 4) 0

10. $(\bar{a}\times\bar{i})(\bar{b}\times\bar{i})+(\bar{a}\times\bar{j})(\bar{b}\times\bar{j})+(\bar{a}\times\bar{k})(\bar{b}\times\bar{k})=$ **[EAM-2018]**

1) $\bar{a}.\bar{b}$ 2) $3(\bar{a}.\bar{b})$ 3) 0 4) $2(\bar{a}.\bar{b})$

9. $(\bar{a}\times\bar{b}).(\bar{c}\times\bar{d})=(\bar{a}.\bar{c})(\bar{b}.\bar{d})+K(\bar{a}.\bar{d})(\bar{b}.\bar{c})$ **then the value of K is**

1) 1 2) 0 3) -2 4) -1

10. $(\bar{b}\times\bar{c}).(\bar{a}\times\bar{d})+(\bar{c}\times\bar{a}).(\bar{b}\times\bar{d})+(\bar{a}\times\bar{b}).(\bar{c}\times\bar{d})=$

1) 0 2) 1 3) 2 4) -1

12. **If $\bar{a}$, $\bar{b}$ lie in a plane normal to the plane containing $\bar{c}$ and $\bar{d}$ then** $|(\bar{a}\times\bar{b}).(\bar{c}\times\bar{d})|=$

1) 4 2) 1 3) 0 4) 3

13. **If** $\bar{a}=2\bar{i}+\bar{j}+3\bar{k},\bar{b}=3\bar{i}+2\bar{j}+\bar{k}$, $\bar{c}=\bar{i}-\bar{j}-4\bar{k},\bar{d}=\bar{i}+2\bar{j}-\bar{k}$ **then** $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})=$

1) $24(\bar{i}+\bar{j}-2\bar{k})$ 2) $24(\bar{i}-\bar{j}-\bar{k})$

3) $12(2\bar{i}+\bar{j}-3\bar{k})$ 4) $12(\bar{i}-2\bar{j}+3\bar{k})$

14 **If four vectors $\bar{a},\bar{b},\bar{c},\bar{d}$ are coplanar, then** $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})=$

1) $[\bar{a}\,\bar{b}\,\bar{c}]\bar{d}$ 1) $[\bar{b}\,\bar{c}\,\bar{d}]\bar{a}$

3) $[\bar{c}\ \bar{d}\ \bar{a}]\bar{b}$ 4) Null vector

15. **If** $(\bar{b}\times\bar{c})\times(\bar{c}\times\bar{a})=3\bar{c}$ **then** $[\bar{b}\times\bar{c}\quad \bar{c}\times\bar{a}\quad \bar{a}\times\bar{b}]=$

1) 2 2) 7 3) 9 4) 11

14. $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})=l\bar{c}+m\bar{d}$ **then m is**

1) $[\bar{a}\ \bar{b}\ \bar{c}\ \bar{d}]$ 2) $[\bar{c}\ \bar{b}\ \bar{d}]$

3) $[\bar{b}\ \bar{c}\ \bar{d}]$ 4) $-[\bar{a}\ \bar{b}\ \bar{c}]$

15. **If** $\alpha\bar{a}+\beta\bar{b}+\gamma\bar{c}=\bar{0}$ **then** $(\bar{a}\times\bar{b})\times\{(\bar{b}\times\bar{c})\times(\bar{c}\times\bar{a})\}=$

1) $\bar{0}$

2) A vector perpendicular to the plane of $\bar{a},\bar{b},\bar{c}$

3) A scalar quantity 4) $2[\bar{a}\ \bar{b}\ \bar{c}]$

KEY

01) 2	02) 1	03) 4	04) 4	05) 2	06) 2
07) 2	08) 1	09) 2	10) 3	11) 3	12) 4
13) 2	14) 4	15) 4	16) 1	17) 3	18) 1
19) 4	20) 3	21) 4	22) 1		

SOLUTIONS

1. $(\bar{a}.\bar{c})\bar{b}-(\bar{a}.\bar{b})\bar{c}$

2. $\bar{a}=\bar{i}-2\bar{j}-3\bar{k},\bar{b}=2\bar{i}+2\bar{j}-\bar{k},\bar{c}=\bar{i}+3\bar{j}-2\bar{k}$

$$\bar{a}\times(\bar{b}\times\bar{c})=p\bar{i}+q\bar{j}+r\bar{k}$$

$$\Rightarrow(\bar{a}.\bar{c})\bar{b}-(\bar{a}.\bar{b})\bar{c}=p\bar{i}+g\bar{j}+r\bar{k}$$

$$\Rightarrow(1-6+6)(2\bar{i}+\bar{j}-\bar{k})-(2-2+3)(\bar{i}+3\bar{j}-2\bar{k})=p\bar{i}+q\bar{j}+r\bar{k}$$

$$\Rightarrow 2i+\bar{j}-\bar{k}-3i-9\bar{j}+6\bar{k}=p\bar{i}+q\bar{j}+r\bar{k}$$

$$\Rightarrow -\bar{i}-8\bar{j}+j\bar{k}=p\bar{i}+qj+r\bar{k}$$

comparing p = 1, q= -8, r = 5

now p+q+r =-1-8+5=-4

$\Rightarrow (1-6+6)(2\bar{i}+\bar{j}-\bar{k})-(2-2+3)(\bar{i}+3\bar{j}-2\bar{k})=p\bar{i}+q\bar{j}+r\bar{k}$

3. $\bar{a}\times(\bar{b}\times\bar{c})=(\bar{a}.\bar{c})\bar{b}-(\bar{a}.\bar{b})\bar{c}$

4. $\left|(\bar{a}.\bar{c})\bar{b}-(\bar{a}.\bar{b})\bar{c}\right|$

   and $\left|a_1\bar{i}+a_2\bar{j}+a_3\bar{k}\right|=\sqrt{a_1^2+a_2^2+a_3^2}$

5. $(\bar{a}.\bar{c})\bar{b}-(\bar{c}.\bar{b})\bar{a}$

6. $(\bar{a}\times\bar{c})\times\bar{b}=\lambda\bar{a}+\mu\bar{b}$

   $\Rightarrow(\bar{a}.\bar{c})\bar{b}-(\bar{b}.\bar{c})\bar{a}=\lambda\bar{a}+\mu\bar{b}$

   $\Rightarrow\lambda=-(\bar{b}.\bar{c}),\ \mu=(\bar{c}.\bar{a})$

7. $\sum\left(\bar{a}\times(\bar{b}\times\bar{c})\right)=\sum\left((\bar{a}.\bar{c})\bar{b}-(\bar{a}.\bar{b})\bar{c}\right)=\bar{0}$

8. $(\bar{a}\times\bar{b})\times\bar{c}=\bar{a}\times(\bar{b}\times\bar{c})$

   $\Rightarrow(\bar{c}.\bar{a})\bar{b}-(\bar{c}.\bar{b})\bar{a}=(\bar{a}.\bar{c})\bar{b}-(\bar{a}.\bar{b})\bar{c}$

   $\Rightarrow(\bar{a}.\bar{b})\bar{c}-(\bar{c}.\bar{b})\bar{a}=0\quad\Rightarrow(\bar{a}\times\bar{c})\times\bar{b}=0$

9. $\sum\bar{i}\times(\bar{a}\times\bar{i})=\sum\left((\bar{i}.\bar{i})\bar{a}-(\bar{i}.\bar{a})\bar{i}\right)$

   $=\sum\left(\bar{a}-(\bar{a}.\bar{i})\bar{i}\right)=3\bar{a}-\bar{a}=2\bar{a}$

10. Cross product of any two vectors is perpendicular to both the vectors

11. $\left[\bar{a}\ \ \bar{b}\ \ \bar{a}\times\bar{b}\right]=(\bar{a}\times\bar{b}).(\bar{a}\times\bar{b})=\left|\bar{a}\times\bar{b}\right|^2$

12. Given $\bar{a}=2\bar{i}+3\bar{j}-\bar{k},\bar{b}=-\bar{i}+2\bar{j}-4\bar{k},\bar{c}=\bar{i}+\bar{j}+\bar{k},\bar{d}=\bar{i}-\bar{j}-\bar{k}$

$$(\bar{a}\times\bar{b}).(\bar{c}\times\bar{d})=\begin{vmatrix}\bar{a}.\bar{c} & \bar{a}.\bar{d}\\ \bar{b}.\bar{c} & \bar{b}.\bar{d}\end{vmatrix}=\begin{vmatrix}4 & 0\\ -3 & 1\end{vmatrix}=4-0=4$$

13. $(\bar{a}.\bar{c})(\bar{b}.\bar{b})-(\bar{a}.\bar{b})(\bar{b}.\bar{c})+(\bar{a}.\bar{b})(\bar{b}.\bar{c})=\lambda(\bar{a}.\bar{c})$

    $\Rightarrow(\bar{a}.\bar{c})\left|\bar{b}\right|^2=\lambda(\bar{a}.\bar{c})\qquad\therefore\lambda=\left|\bar{b}\right|^2$

14. Use scalar product of four vectors formula

15. $(\bar{a}.\bar{c})(\bar{b}.\bar{d})-(\bar{a}.\bar{d})(\bar{b}.\bar{c})=(\bar{a}.\bar{c})(\bar{b}.\bar{d})+K(\bar{a}.\bar{d})(\bar{b}.\bar{c})$

    $\Rightarrow K=-1$

16. $(\bar{b}\times\bar{c}).(\bar{a}\times\bar{d})+(\bar{c}\times\bar{a}).(\bar{b}\times\bar{d})+(\bar{a}\times\bar{b}).(\bar{c}\times\bar{d})$

$=(\bar{b}\bar{a})(\bar{c}\bar{d})-(\bar{b}\bar{d})(\bar{a}\bar{c})+(\bar{c}\bar{b})(\bar{a}\bar{d})-(\bar{c}\bar{d})(\bar{a}\bar{b})+(\bar{a}\bar{c})(\bar{b}\bar{d})-(\bar{a}\bar{d})(\bar{b}\bar{c})=0$

17. $(\bar{a}\times\bar{b})\perp(\bar{c}\times\bar{d})\Rightarrow(\bar{a}\times\bar{b}).(\bar{c}\times\bar{d})=0$

18. Given

$$\left[\bar{a}\bar{b}\bar{d}\right]=\begin{vmatrix}2 & 1 & 3\\ 3 & 2 & 1\\ 1 & 2 & -1\end{vmatrix}$$

$=$

$=2(-2-2)-1(-3-1)+3(6-2)=-8+4+12=8$

$$\left[\bar{a}\bar{b}\bar{c}\right]=\begin{vmatrix}2 & 1 & 3\\ 3 & 2 & 1\\ 1 & -1 & -1\end{vmatrix}=2(-7)-1(-13)+3(-5)=-14+13-15=-16$$

Now $(\bar{a}\times\bar{b})\times(\bar{c}\times\bar{d})=\left[\bar{a}\bar{b}\bar{d}\right]\bar{c}-\left[\bar{a}\bar{b}\bar{c}\right]\bar{d}$

$=8(\bar{i}-\bar{j}-4\bar{k})+16(\bar{i}+2\bar{j}-\bar{k})=24\bar{i}+24\bar{j}-48\bar{k}=24(\bar{i}+\bar{j}-2\bar{k})$

19. $\left[\bar{a}\ \bar{b}\ \bar{d}\right]\bar{c}-\left[\bar{a}\ \bar{b}\ \bar{c}\right]\bar{d}=\bar{0}-\bar{0}=\bar{0}$

20. $\left[\bar{b}\ \ \bar{c}\ \ \bar{a}\right]\bar{c}=3\bar{c}\quad\Rightarrow\left[\bar{a}\ \bar{b}\ \bar{c}\right]=3$

    Required value $\left[\bar{a}\ \bar{b}\ \bar{c}\right]^2=9$

21. $\Rightarrow\left[\bar{a}\ \bar{b}\ \bar{d}\right]\bar{c}-\left[\bar{a}\ \bar{b}\ \bar{c}\right]\bar{d}=l\bar{c}+m\bar{d}$

    $\therefore m=-\left[\bar{a}\ \bar{b}\ \bar{c}\right]$

22. The vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ are coplanar

    $\Rightarrow\bar{b}\times\bar{c}\ and\ \bar{c}\times\bar{a}$ are parallel

    $\Rightarrow(\bar{b}\times\bar{c})\times(\bar{c}\times\bar{a})=\bar{0}\quad\therefore(\bar{a}\times\bar{b})\times\bar{0}=\bar{0}$

EXERCISE - II

1. $\bar{i}\times\left[\left(\bar{a}\times\bar{b}\right)\times\bar{i}\right]+\bar{j}\times\left[\left(\bar{a}\times\bar{b}\right)\times\bar{j}\right]+\bar{k}\times\left[\left(\bar{a}\times\bar{b}\right)\times\bar{k}\right]=$

   1) $\bar{0}$ 2) $\left(\bar{a}\cdot\bar{b}\right)\bar{b}$ 3) $\bar{b}$ 4) $2\left(\bar{a}\times\bar{b}\right)$

2. $\bar{a},\bar{b},\bar{c}$ **are three unit vectors such that** $\bar{a}\times(\bar{b}\times\bar{c})=\frac{1}{2}\bar{b}$**, then** $(\bar{a},\bar{b})=$**,** $(\bar{a},\bar{c})=$ ($\bar{b},\bar{c}$ are non-collinear)

   1) $90^0,60^0$ 2) $60^0,90^0$ 3) $30^0,60^0$ 4) $45^0,30^0$

3. **If** $\bar{a},\bar{b},\bar{c}$ **are non-coplanar unit vectors such that** $\bar{a}\times(\bar{b}\times\bar{c})=\frac{\bar{b}+\bar{c}}{\sqrt{2}}$**, then the angle between** $\bar{a}$ **and** $\bar{b}$ **is**

   1) $\frac{3\pi}{4}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{2}$ 4) π

4. $\bar{a}\times\left(\bar{b}\times\bar{c}\right)$ **is parallel to** $\bar{b}$ **, then**

   1) $\bar{a}\parallel\bar{c}$ 2) $\bar{b}\parallel\bar{c}$ 3) $\bar{a}\parallel\bar{b}$

   4) $\bar{a},\bar{b}\ \&\ \bar{c}$ are parallel to each other

5. **If** $\bar{a}$ *and* $\bar{b}$ **are unit vectors then the vectors** $\left(\bar{a}+\bar{b}\right)\times\left(\bar{a}\times\bar{b}\right)$ **is parallel to the vector**

   1) $\bar{a}-\bar{b}$ 2) $\bar{a}+\bar{b}$ 3) $2\bar{a}-\bar{b}$ 4) $2\bar{a}+\bar{b}$

6. $\bar{b},\bar{c}$ **are unit vectors and** $|\bar{a}|=7$**.** $\bar{a}\times\left(\bar{b}\times\bar{c}\right)+\bar{b}\times\left(\bar{c}\times\bar{a}\right)=\frac{1}{2}\bar{a}$ **then the angle between** $\bar{a}$ **and** $\bar{c}$ **is**

   1) $\frac{\pi}{3}$ 2) $\frac{\pi}{6}$ 3) $\frac{\pi}{2}$ 4) $\frac{\pi}{4}$

7. **If** $\bar{a},\bar{b},\bar{c}$ **are unit vectors and** $\bar{b},\bar{c}$ **are non collinear vectors satisfying.** $\left(\bar{a},\bar{b}\right)=\alpha,\left(\bar{a},\bar{c}\right)=\beta$ **and** $\bar{a}\times\left(\bar{b}\times\bar{c}\right)=\frac{\bar{b}+\bar{c}}{2}$ **then** $\cos\left(\alpha+\beta\right)=$

   1) 0 2) 1 3) -1 4) $\frac{1}{2}$

8. **If** $\bar{a},\bar{b},\bar{c}$ **are any three vectors such that** $\left(\bar{a}+\bar{b}\right).\bar{c}=\left(\bar{a}-\bar{b}\right).\bar{c}=0$ **then** $\left(\bar{a}\times\bar{b}\right)\times\bar{c}=$

   1) $\bar{0}$ 2) $\bar{a}$ 3) $\bar{b}$ 4) $\bar{c}$

9. $\sum\left(\left(\bar{a}\times\bar{i}\right)\times\bar{j}\right)^2=$

   1) $\bar{a}^2$ 2) $2\bar{a}^2$ 3) $3\bar{a}^2$ 4) $4\bar{a}^2$

10. **If** $\bar{a}=\bar{i}+\bar{j}+\bar{k},\ \bar{b}=\bar{i}+\bar{j}-\bar{k}$**,** $\bar{c}=\bar{i}-\bar{j}+\bar{k},\bar{d}=\bar{i}-\bar{j}-\bar{k}$ **then** $\left|\left(\bar{a}\times\bar{b}\right)\times\left(\bar{c}\times\bar{d}\right)\right|=$

    1) 8 2) 1 3) 2 4) 3

11. $\bar{a}=2\bar{i}+\bar{j}-2\bar{k}$ **and** $\bar{b}=\bar{i}+\bar{j}$ **if** $\bar{c}$ **is a vector such that** $\bar{a}\cdot\bar{c}=|\bar{c}|,|\bar{c}-\bar{a}|=2\sqrt{2}$ **and the angle between** $\bar{a}\times\bar{b}$ **and** $\bar{c}$ **is** 30^0**, then** $\left|\left(\bar{a}\times\bar{b}\right)\times\bar{c}\right|=$

    1) 2/3 2) 3/2 3) 2 4) 3

12. **If** $\bar{a}=2\bar{i}+\bar{j}+\bar{k},\ \bar{b}=\bar{i}+2\bar{j}+2\bar{k}$**,** $\bar{c}=\bar{i}+\bar{j}+2\bar{k}$ **and** $\bar{a}\times\left(\bar{b}\times\bar{c}\right)=$ $(1+\alpha)\bar{i}+\beta(1+\alpha)\bar{j}+\gamma(1+\alpha)(1+\beta)\bar{k}$ **then**

    1) $\alpha=-2,\beta=-4,\gamma=\frac{-2}{3}$ 2) $\alpha=2,\beta=-4,\gamma=\frac{2}{3}$

    3) $\alpha=-2,\beta=4,\gamma=\frac{2}{3}$ 4) $\alpha=2,\beta=4,\gamma=\frac{-2}{3}$

13. **If** $\bar{a}$ **is a unit vector then** $\bar{a}\times\left\{\bar{a}\times\left(\bar{a}\times\bar{b}\right)\right\}=$

    1) $\bar{a}\times\bar{b}$ 2) $\bar{b}\times\bar{a}$ 3) $\left(\bar{a}\times\bar{b}\right)$ 4) $2\left(\bar{b}\times\bar{a}\right)$

14. **If** $\bar{a},\bar{b},\bar{c}$ **are three vectors of magnitude** $\sqrt{3},1,2$ **such that** $\bar{a}\times\left(\bar{a}\times\bar{c}\right)+3\bar{b}=\bar{0}$ **and** θ **is the angle between** $\bar{a}$ **and** $\bar{c}$ **then** $\cos^2\theta=$

    1) $\frac{3}{4}$ 2) $\frac{1}{2}$ 3) $\frac{1}{4}$ 4) $\frac{1}{5}$

15. **If** $\bar{a}\times\left(\bar{a}\times\bar{b}\right)=\bar{b}\times\left(\bar{b}\times\bar{c}\right)$ **and** $\left(\bar{a}.\bar{b}\right)\neq0$**, then** $\left[\bar{a}\ \bar{b}\ \bar{c}\right]=$

    1) 1 2) 2 3) 3 4) 0

16. If $\bar{a},\bar{b},\bar{c}$ are non-zero, non-coplanar vectors then $\{\bar{a}\times(\bar{b}+\bar{c})\}\times\{\bar{b}\times(\bar{c}-\bar{a})\}$ is collinear with the vector

1) $\bar{a}+\bar{b}+\bar{c}$ 2) $\bar{a}-\bar{b}+\bar{c}$

3) $\bar{a}+\bar{b}-\bar{c}$ 4) $\bar{a}-\bar{b}-\bar{c}$

17. $\left[\bar{a}\times\bar{b}\ \ \bar{a}\times\bar{c}\ \ \bar{d}\right]=$ **[EAM-2020]**

1) $(\bar{a}.\bar{b})[\bar{a}\ \bar{c}\ \bar{d}]$ 2) $(\bar{a}.\bar{d})[\bar{a}\ \bar{b}\ \bar{c}]$

3) $(\bar{b}.\bar{c})[\bar{a}\ \bar{b}\ \bar{d}]$ 4) $(\bar{a}.\bar{c})[\bar{b}\ \bar{c}\ \bar{d}]$

KEY

01) 4	02) 1	03) 1	04) 2	05) 1	06) 3
07) 3	08) 1	09) 1	10) 1	11) 2	12) 1
13) 2	14) 1	15) 4	16) 4	17) 2	

SOLUTIONS

1.

$$\bar{i}\times\left[(\bar{a}\times\bar{b})\times i\right]+\bar{j}\times\left[(\bar{a}\times\bar{b})\times j\right]+\bar{k}\times\left[(\bar{a}\times\bar{b})\times\bar{k}\right]$$

let $\bar{a}\times\bar{b}=\bar{p}=a_1\bar{i}+a_2 j+a_3\bar{k}$

$$=i\times(\bar{p}\times i)+j\times(p\times j)+\bar{k}\times(\bar{p}\times\bar{k})$$

$$=(i.i)\bar{p}-(i.\bar{p})i+(j.\bar{j})p-(j.\bar{p})j+(\bar{k}.\bar{k})\bar{p}-(\bar{k}.\bar{p})\bar{k}$$

$$=3\bar{p}-(a_1 i+a_2 j+a_3\bar{k})$$

$$=3\bar{p}-\bar{p}=2\bar{p}=2(\bar{a}\times\bar{b})$$

2. $\bar{a}\times(\bar{b}\times\bar{c})=\frac{1}{2}\bar{b}$

$$\Rightarrow(\bar{a}.\bar{c})\bar{b}-(\bar{a}.\bar{b})\bar{c}=\frac{1}{2}\bar{b}-0(\bar{c})$$

$$\Rightarrow(\bar{a}.\bar{c})=\frac{1}{2},\ \bar{a}.\bar{b}=0\quad\Rightarrow|\bar{a}||\bar{c}|\cos(\bar{a},\bar{c})=\frac{1}{2}$$

$$\Rightarrow\cos(\bar{a},\bar{c})=\frac{1}{2}\quad\therefore(\bar{a},\bar{c})=\frac{\pi}{3}$$

$$\bar{a}.\bar{b}=0\Rightarrow(\bar{a},\bar{b})=\frac{\pi}{2}$$

3. $(\bar{a}.\bar{c})\bar{b}-(\bar{a}.\bar{b})\bar{c}=\frac{1}{\sqrt{2}}\bar{b}+\frac{1}{\sqrt{2}}\bar{c}$

$$\Rightarrow\bar{a}.\bar{c}=\frac{1}{\sqrt{2}},\ \bar{a}.\bar{b}=\frac{-1}{\sqrt{2}}$$

$$\Rightarrow(\bar{a},\bar{c})=\frac{\pi}{4},\quad(\bar{a},\bar{b})=\frac{3\pi}{4}$$

4. Given $\bar{a}\times(\bar{b}\times\bar{c})\times\bar{b}=0$

$$\Rightarrow\left((\bar{a}.\bar{c})\bar{b}-(\bar{a}.\bar{b})\bar{c}\right)\times\bar{b}=0$$

$$\Rightarrow(\bar{a}.\bar{c})(\bar{b}\times\bar{b})-(\bar{a}.\bar{b})(\bar{c}\times\bar{b})=0$$

$$0+(\bar{a}.\bar{b})(\bar{b}\times\bar{c})=0$$

$$\bar{b}\times\bar{c}=0$$

$\bar{b}$ is parallel to $\bar{c}$

5. $(\bar{a}+\bar{b})\times(\bar{a}\times\bar{b})$

$$=\left((\bar{a}+\bar{b}).\bar{b}\right)\bar{a}-\left((\bar{a}+\bar{b}).\bar{a}\right)\bar{b}$$

$$=(\bar{a}.\bar{b})\bar{a}+(\bar{b}.\bar{b}).\bar{a}-(\bar{a}.\bar{a})\bar{b}-(\bar{b}.\bar{a})\bar{b}$$

$$=(\bar{a}.\bar{b})\bar{a}+\bar{a}-\bar{b}-(\bar{a}.\bar{b})\bar{b}$$

$$=\{(\bar{a}.\bar{b})+1\}(\bar{a}-\bar{b})$$

6. $\Rightarrow(\bar{a}.\bar{c})\bar{b}-(\bar{a}.\bar{b})\bar{c}+(\bar{a}.\bar{b})\bar{c}-(\bar{b}.\bar{c})\bar{a}=\frac{1}{2}\bar{a}$

$$\Rightarrow(\bar{a}.\bar{c})\bar{b}-\left((\bar{b}.\bar{c})+\frac{1}{2}\right)\bar{a}=\bar{0}$$

$\Rightarrow \bar{a}.\bar{c} = 0,\ \bar{b}.\bar{c} = -\frac{1}{2}$

7. $(\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c} - \frac{\bar{b}}{2} - \frac{\bar{b}}{2} = \bar{0}$

$\Rightarrow \bar{a}.\bar{c} - \frac{1}{2} = 0,\ \bar{a}.\bar{b} + \frac{1}{2} = 0$

$\Rightarrow \bar{a}.\bar{c} = \frac{1}{2},\ \bar{a}.\bar{b} = -\frac{1}{2}$

$\cos\beta = \frac{1}{2},\ \cos\alpha = -\frac{1}{2} \ \Rightarrow \beta = 60^0,\ \alpha = 120^0$

8. $\bar{a}.\bar{c} = \bar{b}.\bar{c} = 0$

$(\bar{a} \times \bar{b}) \times \bar{c} = (\bar{c} \times \bar{a})\bar{b} = -(\bar{c}.\bar{b})\bar{a} = \bar{0}$

9. $\sum\left[(\bar{j}.\bar{a})\bar{i} - (\bar{j}.\bar{i})\bar{a}\right]^2$

$= \sum\left[(\bar{j}.\bar{a})\bar{i}\right]^2 = a_2^2 + a_3^2 + a_1^2 = \bar{a}^2$

10. $\left[\bar{a}\ \bar{b}\ \bar{d}\right]\bar{c} - \left[\bar{a}\ \bar{b}\ \bar{c}\right]\bar{d}$

11. $|\bar{a}| = 3,\ |\bar{a} \times \bar{b}| = |2\bar{i} + 2\bar{j} + \bar{k}| = 3$

$|\bar{c} - \bar{a}| = 2\sqrt{2}$

$\Rightarrow |\bar{c} - \bar{a}|^2 = 8 \quad \Rightarrow |\bar{c}|^2 + |\bar{a}|^2 - 2(\bar{a}.\bar{c}) = 8$

$\Rightarrow c^2 + 9 - 2c = 8 \quad \because \bar{a}.\bar{c} = |\bar{c}|$

$\Rightarrow c^2 - 2c + 1 = 0 \ \Rightarrow (c-1)^2 = 0$

$\therefore c = 1 \quad \therefore |\bar{c}| = 1$

$|(\bar{a} \times \bar{b}) \times \bar{c}| = |\bar{a} \times \bar{b}||\bar{c}|\sin(30) = 3 \times 1 \times \frac{1}{2} = \frac{3}{2}$

12.

Given $\bar{a} = 2\bar{i} + \bar{j} + \bar{k}, \bar{b} = \bar{i} + 2\bar{j} + 2\bar{k}, \bar{c} = \bar{i} + \bar{j} + 2\bar{k}$

$\bar{a} \times (\bar{b} \times \bar{c}) = (1+\alpha)\bar{i} + \beta(1+\alpha)\bar{j} + r(1+\alpha)(1+\beta)\bar{k}$

$(\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b})\bar{c} = (1+\alpha)\bar{i} + \beta(1+\alpha)\bar{j} + \gamma(1+\alpha)(1+\beta)\bar{k}$

$5(\bar{i} + 2\bar{j} + 2\bar{k}) - 6(\bar{i} + \bar{j} + 2\bar{k})$

$= (1+\alpha)\bar{i} + \beta(1+\alpha)\bar{j} + \gamma(1+\alpha)(1+\beta)\bar{k}$

$-\bar{i} + 4\bar{j} - 2\bar{k} = (1+\alpha)\bar{i} + \beta(1+\alpha)j + \gamma(1+\alpha)(1+\beta)\bar{k}$

$1 + \alpha = -1$

$\alpha = -2$

$\beta(1+\alpha) = 4$

$\beta(1-2) = 4$

$\beta = -4$

$\gamma(1+\alpha)(1+\beta) = -2$

$\gamma(-1)(-3) = -2$

$\gamma = \frac{-2}{3}$

13. $\Rightarrow \bar{a} \times \{(\bar{a}.\bar{b})\bar{a} - (\bar{a}.\bar{a})\bar{b}\}$

$= (\bar{a}.\bar{b})(\bar{a} \times \bar{a}) - (\bar{a}.\bar{a})(\bar{a} \times \bar{b}) \ = \bar{b} \times \bar{a}$

14. $(\bar{a}.\bar{c})\bar{a} - (\bar{a}.\bar{a})\bar{c} + 3\bar{b} = \bar{0}$

$\Rightarrow 2\sqrt{3}\cos\theta\bar{a} - 3\bar{c} + 3\bar{b} = \bar{0}$

$\Rightarrow 2\cos\theta\bar{a} - \sqrt{3}\bar{c} + \sqrt{3}\bar{b} = \bar{0}$

$\Rightarrow |2\cos\theta\bar{a} - \sqrt{3}\bar{c}|^2 = (-\sqrt{3}\bar{b})^2$

$\Rightarrow 12\cos^2\theta + 9 - 24\cos^2\theta = 0 \ \Rightarrow \cos^2\theta = \frac{3}{4}$

15. Given $\bar{a} \times (\bar{a} \times \bar{b}) = \bar{b} \times (\bar{b} \times \bar{c})$ and $(\bar{a}.\bar{b}) \neq 0$

$(\bar{a}.\bar{b})\bar{a} - (\bar{a}.\bar{a})\bar{b} = (\bar{b}.\bar{c})\bar{b} - (\bar{b}.\bar{b})\bar{c}$

$\bar{b} = \bar{c}$

Now $[\bar{a}\bar{b}\bar{c}] = [\bar{a}\bar{b}\bar{b}] = 0$

16. $\{(\bar{a} \times \bar{b}) - (\bar{c} \times \bar{a})\} \times \{(\bar{b} \times \bar{a}) + (\bar{a} \times \bar{b})\}$

$= (\bar{a} \times \bar{b}) \times (\bar{b} \times \bar{a}) + (\bar{a} \times \bar{b}) \times (\bar{a} \times \bar{b})$

$+ (\bar{b} \times \bar{a}) \times (\bar{c} \times \bar{a}) + (\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{a})$

17. $\left[\bar{a} \times \bar{b}\ \ \bar{a} \times \bar{c}\ \ \bar{d}\right] = (\bar{a} \times \bar{b}).\{(\bar{a} \times \bar{c}) \times \bar{d}\}$

$= (\bar{a} \times \bar{b}).\{(\bar{a}.\bar{d})\bar{c} - (\bar{c}.\bar{d})\bar{a}\}$

$= (\bar{a}.\bar{d}).\left[\bar{a}\ \ \bar{b}\ \ \bar{c}\right] - 0$

EXERCISE - III

1. Let $\overline{a} = \overline{j} - \overline{k}, \overline{c} = \overline{i} - \overline{j} - \overline{k}$ then vector $\overline{b}$ satisfying $\overline{a} \times \overline{b} + \overline{c} = \overline{0}$ and $\overline{a}.\overline{b} = 3$ is

1) $-\overline{i} + \overline{j} - 2\overline{k}$ 2) $2\overline{i} - \overline{j} + 2\overline{k}$

3) $\overline{i} - \overline{j} - 2\overline{k}$ 4) $\overline{i} + \overline{j} - 2\overline{k}$

2. Let $\overline{a}$ be a unit vector and $\overline{b}$ be a non-zero vector not parallel to $\overline{a}$. If two sides of a triangle are represented by the vectors $\sqrt{3}\left(\overline{a} \times \overline{b}\right)$ and $\overline{b} - \left(\overline{a}.\overline{b}\right)\overline{a}$ then the angles of the triangle are

1) $90^0, 60^0, 30^0$ 2) $45^0, 45^0, 90^0$

3) $60^0, 60^0, 60^0$ 4) $75^0, 45^0, 60^0$

3. Unit vectors $\overline{a}, \overline{b}, \overline{c}$ are coplanar. A unit vector $\overline{d}$ is perpendicular to them. If $\left(\overline{a} \times \overline{b}\right) \times \left(\overline{c} \times \overline{d}\right) = \frac{1}{6}\overline{i} - \frac{1}{3}\overline{j} + \frac{1}{3}\overline{k}$ and the angle between $\overline{a}$ and $\overline{b}$ is 30^0 then $\overline{c} =$

1) $\frac{\overline{i} - 2\overline{j} + 2\overline{k}}{3}$ 2) $\frac{2\overline{i} + \overline{j} - \overline{k}}{3}$

3) $\frac{-\overline{i} + 2\overline{j} + 3\overline{k}}{3}$ 4) $\frac{-\overline{i} + 2\overline{j} + \overline{k}}{3}$

4. Let $\overline{a} = \overline{i} - \overline{j}, \overline{b} = \overline{j} - \overline{k}, \overline{c} = \overline{k} - \overline{i}$. If $\overline{d}$ is a unit vector such that $\overline{a}.\overline{d} = 0 = \left[\overline{b}\ \overline{c}\ \overline{d}\right]$, then $\overline{d}$ is equal to

1) $\pm\left(\frac{\overline{i} + \overline{j} - 2\overline{k}}{\sqrt{6}}\right)$ 2) $\pm\left(\frac{\overline{i} + \overline{j} - \overline{k}}{\sqrt{3}}\right)$

3) $\pm\left(\frac{\overline{i} + \overline{j} + \overline{k}}{\sqrt{3}}\right)$ 4) $\pm\overline{k}$

5. In a regular tetrahedron the angle between any two faces is

1) $\sin^{-1}\frac{1}{3}$ 2) $\cos^{-1}\frac{1}{3}$ 3) $\tan^{-1}\frac{1}{\sqrt{3}}$ 4) $\cos^{-1}\frac{1}{\sqrt{3}}$

KEY

01) 1 02) 1 03) 1 04) 1
05) 2

SOLUTIONS

1. $\overline{a} \times \overline{b} + \overline{c} = \overline{0} \Rightarrow \overline{a} \times \left(\overline{a} \times \overline{b}\right) + \left(\overline{a} \times \overline{c}\right) = \overline{0}$

$\left(\overline{a}.\overline{b}\right)\overline{a} - \left(\overline{a}.\overline{a}\right)\overline{b} + \left(\overline{a} \times \overline{c}\right) = \overline{0}$

$3\overline{a} - 2\overline{b} + \overline{a} \times \overline{c} = \overline{0}$

$\Rightarrow 2\overline{b} = 3\overline{a} + \overline{a} \times \overline{c} = \overline{0}$

$\therefore \overline{b} = -\overline{i} + \overline{j} - 2\overline{k}$

2. Two sides are $\sqrt{3}\left(\overline{a} \times \overline{b}\right), \left(\overline{a} \times \overline{b}\right) \times \overline{a}$.

Observe that $\sqrt{3}\left(\overline{a} \times \overline{b}\right).\left\{\left(\overline{a} \times \overline{b}\right) \times \overline{a}\right\} = 0$

$\Rightarrow$ angle between these two sides is 90^0.

Lengths of these two sides are in the ratio $\sqrt{3}:1$. So the remaining angles are $60^0, 30^0$.

3. $\overline{a}, \overline{b}, \overline{c}$ are coplanar $\Rightarrow \left[\overline{a}\ \overline{b}\ \overline{c}\right] = 0$.

$\left[\overline{a}\ \overline{b}\ \overline{d}\right]\overline{c} - \left[\overline{a}\ \overline{b}\ \overline{c}\right]\overline{d} = \frac{1}{6}\overline{i} - \frac{1}{3}\overline{j} + \frac{1}{3}\overline{k}$

$\Rightarrow \left[\left(\overline{a} \times \overline{b}\right).\overline{d}\right]\overline{c} = \frac{1}{6}\overline{i} - \frac{1}{3}\overline{j} + \frac{1}{3}\overline{k}$

4. $\overline{d}$ is a vector perpendicular to $\overline{a}$ and coplanar with $\overline{b}$ and $\overline{c}$. Hence $\overline{d}$ is a vector collinear with $\overline{a} \times \left(\overline{b} \times \overline{c}\right)$ $\Rightarrow \overline{d} = \pm\frac{\overline{a} \times \left(\overline{b} \times \overline{c}\right)}{\left|\overline{a} \times \left(\overline{b} \times \overline{c}\right)\right|}$

5. Let OABC be the regular Tetrahedron and $\bar{a}, \bar{b}, \bar{c}$ be the position vectors of A, B, C.

OA = OB = OC = AB = BC = CA

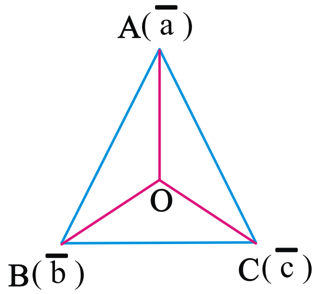
$\Rightarrow |\bar{a}| = |\bar{b}| = |\bar{c}| = |\bar{b} - \bar{a}| = |\bar{c} - \bar{b}| = |\bar{a} - \bar{b}|$

Let θ be the angle between the faces OBC and OCA. $\Rightarrow \cos\theta = \left|\dfrac{(\bar{b} \times \bar{c}).(\bar{c} \times \bar{a})}{|\bar{b} \times \bar{c}||\bar{c} \times \bar{a}|}\right|$ ...... (1)

Observe that $\bar{a}.\bar{b} = \bar{b}.\bar{c} = \bar{c}.\bar{a}$

$= |\bar{a}||\bar{b}|\cos\dfrac{\pi}{3} = \dfrac{1}{2}|\bar{a}|^2$

$|\bar{a} \times \bar{b}| = |\bar{b} \times \bar{c}| = |\bar{c} \times \bar{a}| = |\bar{a}||\bar{a}|\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}|\bar{a}|^2$



JEE MAINS QUESTIONS

1. Let the volume of a parallelopiped whose coterminous edges are given by $\bar{u} = \bar{i} + \bar{j} + \lambda\bar{k}, \bar{v} = \bar{i} + \bar{j} + 3\bar{k}$ and $\bar{w} = 2\bar{i} + \bar{j} + \bar{k}$ be 1cu.unit. if θ be the angle between the edges $\bar{u}$ and $\bar{w}$ then $\cos\theta$ can be [2019]

1) $\dfrac{7}{6\sqrt{6}}$ 2) $\dfrac{7}{6\sqrt{3}}$ 3) $\dfrac{5}{7}$ 4) $\dfrac{5}{3\sqrt{3}}$

2.If $\bar{a} = 2\bar{i} + \bar{j} + 2\bar{k}$, then the value of

$\left|\bar{i} \times (\bar{a} \times \bar{i})\right|^2 + \left|\bar{j} \times (\bar{a} \times \bar{j})\right|^2 + \left|\bar{k} \times (\bar{a} \times \bar{k})\right|^2$ is equal to --------- [2020]

3.Let x_0 be the point of local maximum of $f(x) = \bar{a}.(\bar{b} \times \bar{c})$ where

$\bar{a} = x\bar{i} - 2\bar{j} + 3\bar{k}, \bar{b} = -2\bar{i} + x\bar{j} - \bar{k}$ and

$\bar{c} = 7\bar{i} - 2\bar{j} + x\bar{k}$ then the value of $\bar{a}.\bar{b} + \bar{b} - \bar{c} + \bar{c} - \bar{a}$ at $x = x_0$ is [2020]

1) -4 2)-30 3)14 4)-22

KEY

1) 2 2) 18 3)

SOLUTIONS

1. iven $\overline{\upsilon}=\overline{i}+\overline{j}+\lambda\overline{k}$ $\overline{v}=\overline{i}+\overline{j}+3\overline{k}$ and $\overline{w}=2\overline{i}+\overline{j}+\overline{k}$ volume of parallelopiped $=\left[\overline{\upsilon}\,\overline{v}\,\overline{\omega}\right]$

$$\Rightarrow\begin{vmatrix}1&1&\lambda\\1&1&3\\2&1&1\end{vmatrix}=\pm1\Rightarrow-\lambda+3=\pm1$$

$\lambda=2(or)\lambda=4$ for $\lambda=2$

$$\cos\theta=\frac{2+1+2}{\sqrt{6}\sqrt{6}}=\frac{5}{6}\text{ for }\qquad\lambda=4$$

$$\cos\theta=\frac{2+1+4}{\sqrt{6}\sqrt{18}}=\frac{7}{6\sqrt{3}}$$

2.

$$\overline{i}\times\left(\overline{a}\times i\right)=\left(\overline{i}.\overline{i}\right)\overline{a}-\left(\overline{i}.\overline{a}\right)\overline{i}=\left(2\overline{i}+\overline{j}+2\overline{k}\right)-2\overline{i}=\overline{j}+2\overline{k}$$

similary

$$\overline{j}\times\left(\overline{a}\times\overline{j}\right)=2\overline{i}+2\overline{k}$$

$$\overline{k}\times\left(\overline{a}\times\overline{k}\right)=2\overline{i}+\overline{j}$$

$$\left|\overline{j}+2\overline{k}\right|^2+\left|2\overline{i}+2\overline{k}\right|^2+\left|2\overline{i}+\overline{j}\right|^2$$

$$=\left(\sqrt{1+4}\right)^2+\left(\sqrt{4+4}\right)^2+\left(\sqrt{4+1}\right)^2=5+8+5=18$$

3. It is given that

$$f(x)=\overline{a}.\left(\overline{b}\times\overline{c}\right)=\begin{vmatrix}x&-2&3\\-2&x&-1\\7&-2&x\end{vmatrix}=x^3-27x+26$$

$$f(x)=x^3-27x+26$$

$$f^1(x)=3x^2-27$$

for critical point $f^1(x)=0$

$$3x^2-27=0\Rightarrow x=-3,3$$

$$f^{11}(x)=6x$$

$$f^{11}(x)\,at\;x=-3=-18<$$

$f(x)$ is maximum at $x_0=-3$

then

$$\overline{a}\overline{b}+\overline{b}\overline{c}+\overline{c}\overline{a}=-2x-2x-3-14-2x-x+7x+4+3x$$

$$=3x-13$$

ADVANCED LEVEL QUESTIONS

SINGLE ANSWER TYPE QUESTIONS

1. the shortest distance between two opposite edges regular tetrahedron of side k is

A) $\frac{k}{2}$ B) $\frac{k}{3}$ C) $\frac{k}{\sqrt{2}}$ D) $\frac{k}{\sqrt{3}}$

2. The shortest distance between the lines $\vec{r} = 3\vec{i} - 15\vec{j} + 9\vec{k} + \lambda\left(2\vec{i} - 7\vec{j} + 5\vec{k}\right)$ and $\vec{r} = \left(-\vec{i} + \vec{j} + 9\vec{k}\right) + \mu\left(2\vec{i} + \vec{j} - 3\vec{k}\right)$ is

A) $\sqrt{34}$ B) $\sqrt{3}$ C) $4\sqrt{3}$ D) $2\sqrt{3}$

3. If $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$, $\vec{c} = \hat{i} + \hat{j} + 2\hat{k}$ and $(1+\alpha)\hat{i} + \beta(1+\alpha)\hat{j} + \gamma(1+\alpha)(1+\beta)\hat{k} = \vec{a}\times\left(\vec{b}\times\vec{c}\right)$ then α, β and γ are

A) $-2, -4, -\frac{2}{3}$ B) $2, -4, \frac{2}{3}$

C) $-2, 4, \frac{2}{3}$ D) $2, 4, -\frac{2}{3}$

4. The reflection of the line $\vec{r} = \vec{a} + t\vec{b}$ in the plane $\vec{r}.\vec{n} = q$

A) $\vec{r} = \vec{a} + \left(\frac{\vec{q} - \vec{a}.\vec{n}}{|\vec{n}|^2}\right)\vec{n} + \vec{b}$

B) $\vec{r} = \vec{a} + 2\left(\frac{q - \vec{a}.\vec{n}}{|\vec{n}|^2}\right)\vec{n} + t\left(\vec{b} - 2\frac{\vec{b}.\vec{n}}{|\vec{n}|^2}\vec{n}\right)$

C) $\vec{r} = \vec{a} + \vec{b} - \frac{2\left(\vec{b}.\vec{n}\right)}{|\vec{n}|}\vec{n}$ D) None

5. The distance of the point $P\left(\vec{a}\right)$ from the plane $\vec{r}.\vec{n} = q$ measured parallel to the line $\vec{r} = \vec{b} + t\vec{c}$

A) $\left|\frac{q - \vec{a}.\vec{n}}{|\vec{c}.\vec{n}|}\right||\vec{c}|$ B) $\left(\frac{q^2\vec{a}.\vec{n}}{|\vec{n}|}\right)|\vec{c}|$

C) $\frac{\vec{q} - \vec{a}.\vec{n}}{|\vec{c}|}$ D) $\vec{q} + \frac{|\vec{b} - \vec{c}|}{|\vec{n}|}$

6. The three planes $\vec{r}.\vec{n}_1 = p_1$, $\vec{r}.\vec{n}_2 = p_2$, $\vec{r}.\vec{n}_3 = p_3$ have a common line of intersection then

$p_1\left(\vec{n}_2 \times \vec{n}_3\right) + p_2\left(\vec{n}_3 \times \vec{n}_1\right) + p_3\left(\vec{n}_1 \times \vec{n}_2\right) =$

A) $\vec{n}_1 + \vec{n}_2 + \vec{n}_3$ B) $\vec{0}$

C) $\vec{n}_1 - \vec{n}_2 + \vec{n}_3$ D) $\vec{n}_1 + 2\vec{n}_2 + 3\vec{n}_3$

7. Vectors $\vec{a}, \vec{b}$ are non-zero, non-collinear vectors such that $|\vec{a}| = 2$, $\vec{a}.\vec{b} = 1$ and angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$, if $\vec{r}$ is any vector satisfying $\vec{r}.\vec{a} = 2$, $\vec{r}.\vec{b} = 8$, $\left(\vec{r} + 2\vec{a} - 10\vec{b}\right).\left(\vec{a}\times\vec{b}\right) = 4\sqrt{3}$ also $\left(\vec{r} + 2\vec{a} - 10\vec{b}\right) = \lambda\left(\vec{a}\times\vec{b}\right)$ then $\lambda =$

A) $\frac{4}{\sqrt{3}}$ B) $\frac{2}{\sqrt{3}}$ C) $\sqrt{3}$ D) $\frac{1}{\sqrt{3}}$

8. Let a, b, c be distinct non-negative numbers. If the vectors $a\hat{i} + a\hat{j} + c\hat{k}$, $\hat{i} + \hat{k}$ and $c\hat{i} + c\hat{j} + b\hat{k}$ lie in a plane then c is

[IIT JEE 1993]

A) the Arithmetic Mean of a and b

B) the Geometric Mean of a and b

C) the Harmonic Mean of a and b

D) equal to zero.

9. **Let** $\vec{a}=\hat{i}-\hat{j}, \vec{b}=\hat{j}-\hat{k}, \vec{c}=\hat{k}-\hat{i}$. **If** $\vec{d}$ **is a unit vector such that** $\vec{a}.\vec{d}=0=[\vec{b}\,\vec{c}\,\vec{d}]$ **then** $\vec{d}$ **equals to** **[IIT JEE 1995]**

A) $\pm\frac{\hat{i}+\hat{j}-2\hat{k}}{\sqrt{6}}$ B) $\pm\frac{\hat{i}+\hat{j}-\hat{k}}{\sqrt{3}}$

C) $\pm\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ D) $\pm\hat{k}$

10. **If** $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=4\hat{i}+3\hat{j}+4\hat{k}$ **and** $\vec{c}=\hat{i}+\alpha\hat{j}+\beta\hat{k}$ **are linearly dependent vectors and** $|\vec{c}|=\sqrt{3}$ **then** **[IIT JEE 2018]**

A) $\alpha=1,\beta=-1$ B) $\alpha=1,\beta=\pm1$

C) $\alpha=-1,\beta=\pm1$ D) $\alpha=\pm1,\beta=1$

11. **For three vectors** $\vec{u},\vec{v},\vec{w}$ **which of the following expressions is not equal to any of the remaining three?** **[IIT JEE 1998]**

A) $\vec{u}.(\vec{v}\times\vec{w})$ B) $(\vec{v}\times\vec{w}).\vec{u}$

C) $\vec{v}.(\vec{u}\times\vec{w})$ D) $(\vec{u}\times\vec{v}).\vec{w}$

12. **Let** $\vec{a}=2\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}$. **If** $\vec{c}$ **is a vector such that** $\vec{a}.\vec{c}=|\vec{c}|, |\vec{c}-\vec{a}|=2\sqrt{2}$ **and the angle between** $(\vec{a}\times\vec{b})$ **and** $\vec{c}$ **is 30° then** $|(\vec{a}\times\vec{b})\times\vec{c}|$ **is equal to** **[IIT JEE 1999]**

A) $\frac{2}{3}$ B) $\frac{3}{2}$ C) 2 D) 3

13. **Let** $\vec{a}=2\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$ **and a unit vector** $\vec{c}$ **be coplanar. If** $\vec{c}$ **is perpendicular to** $\vec{a}$ **then** $\vec{c}$ **is equal to** **[IIT JEE 1999]**

A) $\frac{1}{\sqrt{2}}(-\hat{j}+\hat{k})$ B) $\frac{1}{\sqrt{3}}(-\hat{i}-\hat{j}-\hat{k})$

C) $\frac{1}{\sqrt{5}}(\hat{i}-2\hat{j})$ D) $\frac{1}{\sqrt{5}}(\hat{i}-\hat{j}-\hat{k})$

14. **If** $\vec{a},\vec{b}$ **and** $\vec{c}$ **are unit coplanar vectors, then the scalar triple product** $[2\vec{a}-\vec{b},2\vec{b}-\vec{c},2\vec{c}-\vec{a}]$ **is equal to** **[IIT JEE 2020]**

A) 0 B) 1 C) $-\sqrt{3}$ D) $\sqrt{3}$

15. **Let** $\vec{a}=\hat{i}-\hat{k}$, $\vec{b}=x\hat{i}+\hat{j}+(1-x)\hat{k}$ **and** $\vec{c}=y\hat{i}+x\hat{j}+(1+x-y)\hat{k}$ **then** $[\vec{a}\ \vec{b}\ \vec{c}]$ **depends on** **[IIT JEE 2001]**

A) only x B) only y

C) neither x nor y D) both x and y

16. **Let** $\vec{V}=2\vec{i}+\vec{j}-\vec{k}$ and $\vec{W}=\vec{i}+3\vec{k}$. **If** $\vec{U}$ **is a unit vector then the maximum value of the scalar triple product** $[\vec{U}\vec{V}\vec{W}]$ **is** **[IIT JEE 2002]**

A) - 1 B) $\sqrt{10}+\sqrt{6}$

C) $\sqrt{59}$ D) $\sqrt{60}$

17. **The value of 'a' so that the volume of parallelopiped formed by** $\hat{i}+a\hat{j}+\hat{k}, \hat{j}+a\hat{k}$ **and** $a\hat{i}+\hat{k}$ **become minimum, is** **[IIT JEE 2003]**

A) -3 B) 3 C) $\frac{1}{\sqrt{3}}$ D) $\sqrt{3}$

18. **The unit vector which is orthogonal to the vector** $3\hat{i}+2\hat{j}+6\hat{k}$ **and is coplanar with the vectors** $2\hat{i}+\hat{j}+\hat{k}$ **and** $\hat{i}-\hat{j}+\hat{k}$ **is** **[IIT JEE 2004]**

A) $\frac{2\hat{i}-6\hat{j}+\hat{k}}{\sqrt{41}}$ B) $\frac{2\hat{i}-3\hat{j}}{\sqrt{13}}$

C) $\frac{3\hat{j}-\hat{k}}{\sqrt{10}}$ D) $\frac{4\hat{i}+3\hat{j}-3\hat{k}}{\sqrt{34}}$

19. **Let** $\vec{a}=\hat{i}+2\hat{j}+\hat{k}, \vec{b}=\hat{i}-\hat{j}+\hat{k}, \vec{c}=\hat{i}+\hat{j}-\hat{k}$. **A vector coplanar to** $\vec{a}$ **and** $\vec{b}$ **has a projection along** $\vec{c}$ **of magnitude** $\frac{1}{\sqrt{3}}$ **then the vector is** **[IIT JEE 2006]**

A) $4\hat{i}-\hat{j}+4\hat{k}$ B) $4\hat{i}+\hat{j}-4\hat{k}$

C) $2\hat{i}+\hat{j}+\hat{k}$ D) none of these

20. **The number of distinct real values of** λ**, for which the vectors** $-\lambda^2\hat{i}+\hat{j}+\hat{k}, \hat{i}-\lambda^2\hat{j}+\hat{k}$ **and** $\hat{i}+\hat{j}-\lambda^2\hat{k}$ **are coplanar is** **[IIT JEE 2006]**

A) 0 B) 1 C) 2 D) 3

21. **If** $\vec{a},\vec{b},\vec{c}$ **are non-coplanar unit vectors such that** $\vec{a}\times(\vec{b}\times\vec{c})=\frac{(\vec{b}+\vec{c})}{\sqrt{2}}$ **then the angle between** $\vec{a}$ **and** $\vec{b}$ **is** **[IIT JEE 1995]**

A) $\frac{3\pi}{4}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{2}$ D) π

22. **If $\vec{a},\vec{b}$ and $\vec{c}$ are three non-coplanar vectors then $\left(\vec{a}+\vec{b}+\vec{c}\right).[(\vec{a}+\vec{b})\times(\vec{a}+\vec{c})]$ equals**

[IIT JEE 1995]

A) 0 B) $[\vec{a}\ \vec{b}\ \vec{c}]$

(C) $2.[\vec{a}\ \vec{b}\ \vec{c}]$ (D) $-[\vec{a}\ \vec{b}\ \vec{c}]$

23. **Let $\vec{p},\vec{q},\vec{r}$ be three mutually perpendicular vectors of the same magnitude. If a vector $\vec{x}$ satisfies the equation $\vec{p}\times\left((\vec{x}-\vec{q})\times\vec{p}\right)+\vec{q}\times\left((\vec{x}-\vec{r})\times\vec{q}\right)+\vec{r}\times\left((\vec{x}-\vec{p})\times\vec{r}\right)=\vec{0}$ then $\vec{x}$ is given by** **[IIT JEE 1997]**

(A) $\frac{1}{2}(\vec{p}+\vec{q}-2\vec{r})$ (B) $\frac{1}{2}(\vec{p}+\vec{q}+\vec{r})$

(C) $\frac{1}{3}(\vec{p}+\vec{q}+\vec{r})$ (D) $\frac{1}{3}(2\vec{p}+\vec{q}-\vec{r})$

24. **Let the vectors $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ be such that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=0$. Let P_1 and P_2 be planes determined by the pairs of vectors $\vec{a},\vec{b}$ and $\vec{c},\vec{d}$ respectively then the angle between P_1 and P_2 is** **[IIT JEE 2000]**

(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

KEY

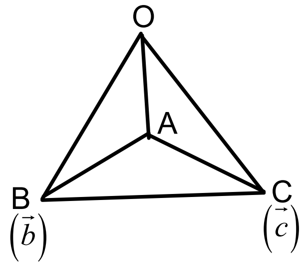
1) C	2) C	3) A	4) B	5) B	6) B
7) A	8) A	9) C	10) D	11) B	12) D
13) C	14) B	15) A	16) C	17) B	18) C
19) C	20) C	21) B	22) A	23) B	24) A

SOLUTIONS

1.Consider $\overrightarrow{OB}=\vec{k}$

its opposite edge $\overrightarrow{AC}=\vec{c}-\vec{a}$

we have to find shortest distance between $\overrightarrow{OA}$ and $\overrightarrow{AC}$



equation $\overrightarrow{OA}$ is $\vec{r}=1-\vec{b}$

equation $\overrightarrow{AC}$ is $\vec{r}=\left(1-\vec{s}\right)\vec{a}+s\vec{c}$

$$\vec{r}=\vec{a}+s\left(\vec{c}-\vec{a}\right)$$

S.D projection $\vec{a}$ on $\vec{b}\times\left(\vec{c}-\vec{a}\right)$

$$=\frac{\vec{a}\times\left(\vec{b}\times\vec{c}-\vec{b}\times\vec{a}\right)}{\left|\vec{b}\times\vec{c}-\vec{b}\times\vec{a}\right|}=\frac{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}{\vec{b}\times\left(\vec{c}-\vec{a}\right)}$$

$$=\frac{\frac{k^3}{\sqrt{2}}}{\left|\vec{b}\right|\left|\vec{c}-\vec{a}\right|\sin 90}$$

$$=\frac{k^3}{\sqrt{2}k^2}=\left(\frac{k}{\sqrt{2}}\right)$$

2.Using the formula

$$S.D=\left|\frac{\left(\vec{a}_2-\vec{a}_1\right).\left(\vec{b}_1\times\vec{b}_2\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right|$$

$$\vec{a}_2-\vec{a}_1=4\vec{i}-16\vec{j}$$

$$\vec{b}_1\times\vec{b}_2=16\vec{i}+16\vec{j}+16\vec{k}$$

$$\frac{\vec{b}_1\times\vec{b}_2}{\left|\vec{b}_1\times\vec{b}_2\right|}=\frac{1}{\sqrt{3}}\left(\vec{i}+\vec{j}+\vec{k}\right)$$

Hence S.D= $\frac{1}{\sqrt{3}}|4-16| = \frac{12}{\sqrt{3}} = 4\sqrt{3}$

3. $\vec{a}\times\left(\vec{b}\times\vec{c}\right) = (\vec{a}.\vec{c})\vec{b} - \left(\vec{a}.\vec{b}\right)\vec{c}$

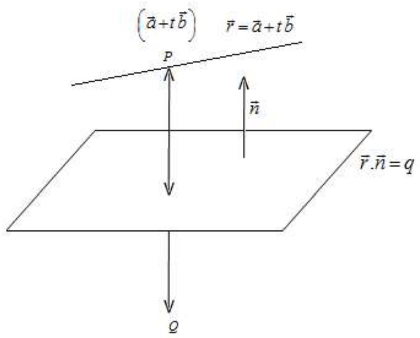
$= 5\left(\hat{i}+2\hat{j}+2\hat{k}\right) - 6\left(\hat{i}+\hat{j}+2\hat{k}\right)$

$\Rightarrow (1+\alpha)\hat{i} + \beta(1+\alpha)\hat{j} + \gamma(1+\alpha)(1+\beta)\hat{k}$

$= -\hat{i} + 4\hat{j} - 2\hat{k}$

$\Rightarrow 1+\alpha = -1, \beta = -4$ and $\gamma(-1)(-3) = -2$

$\Rightarrow \gamma = -\frac{2}{3}$

4. 

Let $P\left(\vec{a}+t\vec{b}\right)$ be any point on the line

Equation of the line $\overleftrightarrow{PQ}$ is

$\vec{r} = \left(\vec{a}+t\vec{b}\right) + \lambda\vec{n}$

Let P.V. of Q is

$\vec{a}+t\vec{b}+\lambda\vec{n}$

Mid $(PQ) = \left(\frac{\vec{a}+t\vec{b}+\vec{a}+t\vec{b}+\lambda\vec{n}}{2}\right)$

$= \vec{a}+t\vec{b}+\frac{\lambda}{2}\vec{n}$ lies on $\vec{r}.\vec{n} = q$

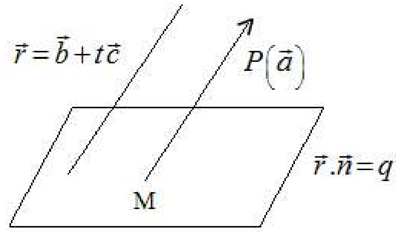
$\left(\vec{a}+t\vec{b}+\frac{\lambda}{2}\vec{n}\right).\vec{n} = q$

$\vec{a}.\vec{n} + t\vec{b}.\vec{n} + \frac{\lambda}{2}|\vec{n}|^2 = q$

$\lambda = \frac{2\left(q-\left(\vec{a}+t\vec{b}\right)\right).\vec{n}}{|\vec{n}|}$

$\therefore$ P.V of $Q = \vec{a}+t\vec{b}+\frac{2\left(q-\left(\vec{a}+t\vec{b}\right).\vec{n}\right)}{|\vec{n}|}\vec{n}$

$\vec{a} + \frac{2(q-\vec{a}.\vec{n})}{|\vec{n}|^2}\vec{n} + t\left(\vec{b} - 2\frac{\left(\vec{b}.\vec{n}\right)}{|n|^2}\vec{n}\right)$

5. 

Any point on the line $\overleftrightarrow{PM}$ is

$\vec{r} = \vec{a}+t\vec{c}$

Suppose it represents M

$M\left(\vec{a}+t\vec{c}\right)$ lies on $\vec{r}.\vec{n} = q$

$\left(\vec{a}+t\vec{c}\right).\vec{n} = q$

$\vec{a}.\vec{n} + t\left(\vec{c}.\vec{n}\right) = q$

$t = \frac{\vec{q}-\vec{a}.\vec{n}}{\vec{c}.\vec{n}}$

$\left|\overleftrightarrow{MP}\right| = \left|\vec{a} - \left(\vec{a}+t\vec{c}\right)\right|$

$= |t\ \ \vec{c}|$

$= |t|\,|\vec{c}|$

$= \left(\frac{(\vec{q}-\vec{a}.\vec{n})}{\vec{c}.\vec{n}}\right)|\vec{c}|$

6. Equation of plane passing through intersating two planes

$\vec{r}.\overrightarrow{n_1} + \overrightarrow{n_2} + \lambda\overrightarrow{n_2} = p_1 + \lambda p_2$. Where λ is a parameter

This plane and $\vec{r}.\overrightarrow{n_3} = p_3$ are identical for same value of λ.

$\overrightarrow{n_1} + \lambda\overrightarrow{n_2} = kn_3$

$p_1 + \lambda p_2 = kp_3$

$\Rightarrow \overrightarrow{n_1}\times\overrightarrow{n_3} + \lambda\overrightarrow{n_2}\times\overrightarrow{n_3} = 0$

$\overrightarrow{n_1}\times\overrightarrow{n_2} = k\,\overrightarrow{n_3}\times\overrightarrow{n_2}$

$\therefore (p_1+\lambda p_2)\left(\overrightarrow{n_2}\times\overrightarrow{n_3}\right) = kp_3\left(\overrightarrow{n_2}\times\overrightarrow{n_3}\right) = p\overrightarrow{n_2}\times\overrightarrow{n_1}$

$\therefore \left(\overrightarrow{n_2}\times\overrightarrow{n_3}\right) + p_2\lambda\left(\overrightarrow{n_2}\times\overrightarrow{n_3}\right) + p_3\overrightarrow{n_1}\times\overrightarrow{n_2} = 0$

$$\Rightarrow p_1\left(\overrightarrow{n_2}\times\overrightarrow{n_3}\right)+p_2\left(\overrightarrow{n_3}+\overrightarrow{n_1}\right)+p_3\left(\overrightarrow{n_1}\times\overrightarrow{n_2}\right)=0$$

7. $\vec{r}=\alpha\vec{a}+\beta\vec{b}+\gamma\left(\vec{a}\times\vec{b}\right)$

$$\left.\begin{array}{l}\vec{r}.\vec{a}=2\Rightarrow 4\alpha+\beta=2\\ \vec{r}.\vec{b}=8\Rightarrow \alpha+\beta=8\end{array}\right\}\Rightarrow \begin{array}{l}\alpha=-2\\ \beta=10\end{array}$$

$\left(\vec{r}+2\vec{a}-10\vec{b}\right)=\lambda\left(\vec{a}\times\vec{b}\right)$

$\left(\vec{r}+2\vec{a}-10\vec{b}\right).\left(\vec{a}\times\vec{b}\right)=\lambda\left|\vec{a}\times\vec{b}\right|^2$

$\Rightarrow \lambda=\dfrac{4}{\sqrt{3}}$

8. $\begin{vmatrix}a & a & c\\ 1 & 0 & 1\\ c & c & b\end{vmatrix}=0\Rightarrow\begin{vmatrix}a & a & c-a\\ 1 & 0 & 0\\ c & c & b-c\end{vmatrix}=0$

$C_3\rightarrow C_3-C_1$

Expanding along R_2 we get

$\Rightarrow c^2-ac-ab+ac=0\Rightarrow c^2=ab$

$\Rightarrow a,c,b$ are in G.P

$\therefore$ c is the G.M of a and b

9. (A) Let $\vec{d}=x\hat{i}+y\hat{j}+z\hat{k}$

Where $x^2+y^2+z^2=1$ ....(i)

($\vec{d}$ being unit vector)

$\therefore \quad \vec{a}.\vec{d}=0$

$\Rightarrow x-y=0\Rightarrow x=y$ ....(ii)

$\Rightarrow\begin{vmatrix}0 & 1 & -1\\ -1 & 0 & 1\\ x & y & z\end{vmatrix}$

$\Rightarrow x+y+z=0$

$\Rightarrow 2x+z=0$ ....(using (ii)

$\Rightarrow z=-2x$ ....(iii)

From (i) ,(II) (III)

$x^2+x^2+4x^2=1$

$x=\pm\dfrac{1}{\sqrt{6}}$

10. $\left[\vec{a}\vec{b}\vec{c}\right]=0$ & $1+\alpha^2+\beta^2=3$

11. $\left[\vec{u},\vec{v},\vec{w}\right]=-\left[\vec{v}\,\vec{u}\,\vec{w}\right]=\left[\vec{v}\,\vec{w}\,\vec{u}\right]$

12. $\left|\left(\vec{a}\times\vec{b}\right)\times\vec{c}\right|=\left|\vec{a}\times\vec{b}\right|\left|\vec{c}\right|\sin 30^0$

$=\dfrac{1}{2}\left|\vec{a}\times\vec{b}\right|\left|\vec{c}\right|$ .....(i)

We have $\vec{a}=2\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}$

$\Rightarrow \vec{a}\times\vec{b}=2\hat{i}-2\hat{j}+\hat{k}$

$\Rightarrow\left|\vec{a}\times\vec{b}\right|=\sqrt{9}=3$

Also given $\left|\vec{c}-\vec{a}\right|=2\sqrt{2}$

$\Rightarrow\left|\vec{c}-\vec{a}\right|^2=8$

Substituting values of $\left|\vec{a}\times\vec{b}\right|$ and $\left|\vec{c}\right|$ in (i) we get

$\left|\left(\vec{a}\times\vec{b}\right)\times\vec{c}\right|=\dfrac{1}{2}\times3\times1=\dfrac{3}{2}$

13. (a) As c is coplanar with a and b we take

$\vec{c}=a\vec{a}+b\vec{b}$ ....(i)

Where a, b are scalars

As $\vec{c}$ is perpendicular to $\vec{a}$, $\vec{c}.\vec{a}=0$

$\therefore$ From (i) we get , $0=a\left(\vec{a}.\vec{a}\right)+b\left(\vec{b}.\vec{a}\right)$

$\Rightarrow 0=a(6)+b(2+2-1)$

$\Rightarrow b=-2a$

Thus,

$\vec{c}=a\left(\vec{a}-2\vec{b}\right)=a\left(-3j+3k\right)=3a\left(-j+k\right)$

$\Rightarrow\left|c\right|^2=9a^2(1+1)=18a^2\Rightarrow 1=18a^2$

$\Rightarrow a=\pm\dfrac{1}{3\sqrt{2}}$

$\therefore c=\pm\dfrac{1}{\sqrt{2}}(-j+k)$

14. $\vec{a},\vec{b},\vec{c}$ are unit coplanar vectors ,

$2\vec{a}-\vec{b},2\vec{b}-\vec{c}$ and $2\vec{c}-\vec{a}$ are also coplanar vectors, being linear combination of $\vec{a},\vec{b}$ and $\vec{c}$

Thus $\left[2\vec{a}-\vec{b}\;2\vec{b}-\vec{c}\;2\vec{c}-\vec{a}\right]=0$

15. (C) $\vec{a}=\hat{i}-\hat{k},\vec{b}=x\hat{i}+\hat{j}+(1-x)\hat{k}$

$\vec{c}=y\hat{i}+xj+(1+x-y)\hat{k}$

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix}$$

$= 1\left(1+x-y-x+x^2\right)-1\left(x^2-y\right)=1$

$\therefore$ depends neither on x nor on y.

16. Given that $\vec{v} = 2\hat{i}+\hat{j}-\hat{k}$ and $\vec{w} = \hat{i}+3\hat{k}$ and $\vec{u}$ is a unit vector

$\therefore |\vec{u}| = 1$

Now $[\vec{u}\,\vec{v}\,\vec{w}] = \vec{u}.(\vec{v}\times\vec{w})$

$\vec{u}.\left(2\hat{i}+\hat{j}-\hat{k}\right)\times\left(\hat{i}+3\hat{k}\right)$

$= \vec{u}.\left(3\hat{i}-7\hat{j}-\hat{k}\right) = \sqrt{3^2+7^2+1^2}\cos\theta$

which is max. when cos $\theta = 1$

$\therefore$ max. value of $[\vec{u}\,\vec{v}\ \vec{w}] = \sqrt{59}$

17. Vol. of parallelopiped formed by $\vec{u} = \hat{i}+a\hat{j}+\hat{k}, v = \hat{j}+a\hat{k}, w = a\hat{i}+\hat{k}$ is

$$V = [\vec{u}\,\vec{v}\,\vec{w}] = \begin{vmatrix} 1 & a & 1 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix}$$

$= 1(1-0)-a\left(0-a^2\right)+1(0-a)\ = 1+a^3-a$

For V to be $\min \frac{dV}{da} = 0 \Rightarrow 3a^2-1=0$

$\Rightarrow a = \pm\frac{1}{\sqrt{3}}$

But $a>0 \Rightarrow a = \frac{1}{\sqrt{3}}$

18. (C) Any vector coplanar to $\vec{a}$ and $\vec{b}$ can be written as $\vec{r} = a+\lambda\vec{b}$

$\vec{r} = (1+2\lambda)\hat{i}+(-1+\lambda)\hat{j}+(1+\lambda)\hat{k}$

since $\vec{r}$ is orthogonal to $5\hat{i}+2\hat{j}+6\hat{k}$

$\Rightarrow 5(1+2\lambda)+2(-1+\lambda)+6(1+\lambda)=0$

$\Rightarrow 9+18\lambda = 0 \Rightarrow \lambda = -\frac{1}{2} \quad \therefore \vec{r}$ is $3\hat{j}-\hat{k}$

$\therefore \vec{r}$ is a unit vector

$\therefore \hat{r} = \frac{3j-k}{\sqrt{10}}$

19. A vector in the plane of $\vec{a}$ and $\vec{b}$ is

$\vec{u} = \vec{a}+\lambda\vec{b} = (1+\lambda)\hat{i}+(2-\lambda)\hat{j}+(1+\lambda)\hat{k}$

Projection of $\vec{u}$ on $c = \frac{1}{\sqrt{3}} \Rightarrow \frac{\vec{u}.\vec{c}}{|\vec{c}|} = \frac{1}{\sqrt{3}}$

$\Rightarrow \vec{u}.\vec{c} = 1$

$\Rightarrow |1+\lambda+2-\lambda-1-\lambda| = 1$

$\Rightarrow |2-\lambda| = 1 \Rightarrow \lambda = 1$ or 3

$\Rightarrow \vec{u} = 2\hat{i}+\hat{j}+2\hat{k}$ or $4\hat{i}-\hat{j}+4\hat{k}$

20. $$\begin{vmatrix} -\lambda^2 & 1 & 1 \\ 1 & -\lambda^2 & 1 \\ 1 & 1 & -\lambda^2 \end{vmatrix} = 0$$

$\Rightarrow \lambda^6-3\lambda^2-2=0$

$\Rightarrow \left(1+\lambda^2\right)^2\left(\lambda^2-2\right)=0 \Rightarrow \lambda = \pm\sqrt{2}$

21. We have $\vec{a}\times\left(\vec{b}\times\vec{c}\right) = \frac{\vec{b}+\vec{c}}{\sqrt{2}}$

$\Rightarrow (\vec{a}.\vec{c})\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c} = \frac{\vec{b}}{\sqrt{2}}+\frac{\vec{c}}{\sqrt{2}}$

$\Rightarrow \left(\vec{a}.\vec{c}-\frac{1}{\sqrt{2}}\right)\vec{b}-\left(\vec{a}.\vec{b}+\frac{1}{\sqrt{2}}\right)\vec{c} = 0$

$\Rightarrow$ then $\vec{a}.\vec{c}-\frac{1}{\sqrt{2}} = 0$ amd $\vec{a}.\vec{b}+\frac{1}{\sqrt{2}} = 0$

$\therefore \vec{a}.\vec{b} = -\frac{1}{\sqrt{2}} \Rightarrow \left(\vec{a},\vec{b}\right) = \frac{3\pi}{4}$

22. $\left(\vec{a}+\vec{b}+\vec{c}\right).[\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}+\vec{c}\right)]$

$= \left(\vec{a}+\vec{b}+\vec{c}\right).[\vec{a}\times\vec{a}+\vec{a}\times\vec{c}+\vec{b}\times\vec{a}+\vec{b}\times\vec{c}]$

$= \left\{\vec{a}.\left(\vec{a}.\times\vec{c}\right)+\vec{a}.\left(\vec{b}\times\vec{a}\right)+\vec{a}.\left(\vec{b}\times\vec{c}\right)\right\}$

$+\left\{\vec{b}.\left(\vec{a}\times\vec{c}\right)+\vec{b}.\left(\vec{b}\times\vec{a}\right)+\vec{b}.\left(\vec{b}\times\vec{c}\right)\right\}$

$+\left\{\vec{c}.\left(\vec{a}\times\vec{c}\right)+\vec{c}.\left(\vec{b}\times\vec{a}\right)+\vec{c}.\left(\vec{b}\times\vec{c}\right)\right\}$

$= \left[\vec{a}\ \vec{b}\ \vec{c}\right] - \left[\vec{a}\ \vec{b}\ \vec{c}\right] - \left[\vec{a}\ \vec{b}\ \vec{c}\right] = -\left[\vec{a}\ \vec{b}\ \vec{c}\right]$

23. (B) As p,q and r are three mutually perpendicular vectors of same magnitude , so let us consider

$\vec{p} = a\hat{i}, \vec{q} = a\hat{j}, \vec{r} = a\hat{k}$

Also let $\vec{x} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

Given that $\vec{x}$ satisfies that equation

$\vec{p} \times \left[(x - q) \times \vec{p}\right] + \vec{q} \times \left[(\vec{x} - \vec{r}) \times \vec{q}\right]$

$+\vec{r} \times \left[(\vec{x} - \vec{p}) \times \vec{r}\right] = 0$ ....(i)

$\Rightarrow 3\vec{a}^2\vec{x} - \vec{a}^2\vec{x} - \vec{a}^2\left(\vec{p} + \vec{q} + \vec{r}\right) = 0$

$\Rightarrow \vec{x} = \frac{1}{2}\left(\vec{p} + \vec{q} + \vec{r}\right)$

24. If θ is the angle between P_1 and P_2

$$\left.\begin{array}{l} N_1 = \vec{a} \times \vec{b} \\ N_2 = \vec{c} \times \vec{d} \end{array}\right\} N_1 \times N_2 = 0$$

then $|N_1| \times |N_2| \sin\theta = 0$

or $\sin\theta = 0 \Rightarrow \theta = 0$

MULTIPLE ANSWER TYPE QUESTIONS

1. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-coplanar vectors and $\vec{d}$ be a non-zero vector perpendicular to $\left(\vec{a} + \vec{b} + \vec{c}\right)$. Now $\vec{d} = \left(\vec{a} \times \vec{b}\right)\sin x + \left(\vec{b} \times \vec{c}\right)\cos y + 2\left(\vec{c} \times \vec{a}\right)$ then

A) $\frac{\vec{d}.(\vec{a} + \vec{c})}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = 2$ B) $\frac{\vec{d}.(\vec{a} + \vec{c})}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = -2$

C) minimum value of $x^2 + y^2$ is $\frac{\pi^2}{4}$

D) minimum value of $x^2 + y^2$ is $\frac{5\pi^2}{4}$

2. The scalars l and m such that $l\vec{a} + m\vec{b} = \vec{c}$, where $\vec{a}, \vec{b}$ and $\vec{c}$ are given vectors, are equal to

A) $l = \frac{\left(\vec{c} \times \vec{b}\right).\left(\vec{a} \times \vec{b}\right)}{\left(\vec{a} \times \vec{b}\right)^2}$

B) $l = \frac{\left(\vec{c} \times \vec{a}\right).\left(\vec{b} \times \vec{a}\right)}{\left|\vec{b} \times \vec{a}\right|}$

C) $m = \frac{\left(\vec{c} \times \vec{a}\right).\left(\vec{b} \times \vec{a}\right)}{\left(\vec{b} \times \vec{a}\right)^2}$

D) $m = \frac{\left(\vec{c} \times \vec{a}\right).\left(\vec{b} \times \vec{a}\right)}{\left|\vec{b} \times \vec{a}\right|}$

3. A particle in equilibrium is subjected to four forces $\overrightarrow{F_1} = -10\hat{k}, \overrightarrow{F_2} = u\left[\frac{4}{13}\hat{i} - \frac{12}{13}\hat{j} + \frac{3}{13}\hat{k}\right]$,

$\overrightarrow{F_3} = V\left[-\frac{4}{13}\hat{i} - \frac{12}{13}\hat{j} + \frac{3}{13}\hat{k}\right],$

$\overrightarrow{F_4} = W\left[\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right]$ **,then**

A) $U = \frac{65}{3}(1 - 3\cot\theta)$

B) $V = \frac{65}{3}(1 + 3\cot\theta)$

C) $W = 40\operatorname{cosec}\theta$

D) $U + V = \frac{130}{3}$

4. Let $\bar{x}, \bar{y}$ and $\bar{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\frac{\pi}{3}$. If $\bar{a}$ is a non-zero vector perpendicular to $\bar{x}$ and $\bar{y} \times \bar{z}$ and $\bar{b}$ is a non zero vector perpendicular to $\bar{y}$ and $\bar{z} \times \bar{x}$, then (IIT-2014)

A) $\bar{b} = (\bar{b}.\bar{z})(\bar{z} - \bar{x})$ B) $\bar{a} = (\bar{a}.\bar{y})(\bar{y} - \bar{z})$

C) $\bar{a}.\bar{b} = -(\bar{a}.\bar{y})(\bar{b}.\bar{z})$ D) $\bar{a} = (\bar{a}.\bar{y})(\bar{z} - \bar{y})$

5. Let $\vec{A}$ be vector parallel to line of intersection of planes P_1 and P_2 through origin. P_1 is parallel to the vectors $2\hat{j} + 3\hat{k}$ and $4\hat{j} - 3\hat{k}$ and P_2 is parallel to $\hat{j} - \hat{k}$ and $3\hat{i} + 3\hat{j}$, then the angle between vector $\vec{A}$ and $2\hat{i} + \hat{j} - 2\hat{k}$ is

[IIT JEE 2016]

A) $\frac{\pi}{2}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{6}$ D) $\frac{3\pi}{4}$

6. Let $\vec{a}$ and $\vec{b}$ be two non-collinear unit vectors. If $\vec{u} = \vec{a} - (\vec{a}.\vec{b})\vec{b}$ and $\vec{v} = \vec{a} \times \vec{b}$ then $|\vec{v}|$ is

[IIT JEE 2017]

A) $|\vec{u}|$ B) $|\vec{u}| + |\vec{u}.\vec{a}|$

C) $|\vec{u}| + |\vec{u}.\vec{b}|$ D) $|\vec{u}| + \vec{u}.(\vec{a} + \vec{b})$

7. Which of the following expressions are meaningful [IIT JEE 1998]

A) $\vec{u}.(\vec{v} \times \vec{w})$ B) $(\vec{u}.\vec{v}).\vec{w}$

C) $(\vec{u}.\vec{v})\vec{w}$ D) $\vec{u} \times (\vec{v}.\vec{w})$

8. Let $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}, \vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = \hat{i} + \hat{j} - 2\hat{k}$ three vectors. A vector in the plane of $\vec{b}$ and $\vec{c}$, whose projection on $\vec{a}$ is of magnitude $\sqrt{\frac{2}{3}}$ is [IIT JEE 1993]

A) $2\hat{i} + 3\hat{j} - 3\hat{k}$ B) $2\hat{i} + 3\hat{j} + 3\hat{k}$

C) $-2\hat{i} - \hat{j} + 5\hat{k}$ D) $2\hat{i} + \hat{j} + 5\hat{k}$

KEY

1) BD	2) AC	3) ABCD	4) ABC
5) BD	6) AC	7) AC	8) AC

SOLUTIONS

1. $\vec{d}.\vec{a} = \left[\vec{a}\ \vec{b}\ \vec{c}\right]\cos y = -\vec{d}.\left(\vec{b}+\vec{c}\right)$

$\Rightarrow \cos y = \dfrac{\vec{d}.\left(\vec{b}+\vec{c}\right)}{\left[\vec{a}.\vec{b}.\vec{c}\right]}$

Similarly,

$\sin x = -\dfrac{\vec{d}.\left(\vec{a}+\vec{b}\right)}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}$ and $\dfrac{\vec{d}.\left(\vec{a}+\vec{c}\right)}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = -2$

$\therefore \sin x + \cos y + 2 = 0$

$\Rightarrow \sin x + \cos y = -2$

$\Rightarrow \sin x = -1, \cos y = -1$

Since we want the minimum value of $x^2 + y^2$ is $\dfrac{5\pi^2}{4}$

2. Here $\left(l\vec{a} + m\vec{b}\right) \times \vec{b} = \vec{c} \times \vec{b} \Rightarrow l\vec{a} \times \vec{b} = \vec{c} \times \vec{b}$

$\Rightarrow l\left(\vec{a}\times\vec{b}\right)^2 = \left(\vec{c}\times\vec{b}\right).\left(\vec{a}\times\vec{b}\right)$

$\Rightarrow l = \dfrac{\left(\vec{c}\times\vec{b}\right).\left(\vec{a}\times\vec{b}\right)}{\left(\vec{a}\times\vec{b}\right)^2}$

Similarly, $m = \dfrac{\left(\vec{c}\times\vec{a}\right).\left(\vec{b}\times\vec{a}\right)}{\left(\vec{b}\times\vec{a}\right)^2}$

3. As forces are in equilibrium we have $\vec{F_1} + \vec{F_2} + \vec{F_3} + \vec{F_4} = \vec{0}$

equating the coefficients of $\hat{i}, \hat{j}, \hat{k}$ and solve the equations

4. $\bar{x}.\bar{y} = \bar{y}.\bar{z} = \bar{z}.\bar{x} = \sqrt{2} \times \sqrt{2} \times \dfrac{1}{2} = 1.$

Let $\bar{a} = \lambda\left(\bar{x} \times \left(\bar{y} \times \bar{z}\right)\right) = \lambda\left(\left(\bar{x}.\bar{z}\right)\bar{y} - \left(\bar{x}.\bar{y}\right)\bar{z}\right)$

$= \lambda\left(\bar{y} - \bar{z}\right)$

$\bar{a}.\bar{y} = \lambda$

$\therefore \bar{a} = \left(\bar{a}.\bar{y}\right)\left(\bar{y} - \bar{z}\right)$

Similarly $\bar{b} = \left(\bar{b}.\bar{z}\right)\left(\bar{z} - \bar{x}\right)$

$\bar{a}.\bar{b} = -\left(\bar{a}.\bar{y}\right)\left(\bar{b} - \bar{z}\right)$

5. Let vector $\overline{AO}$ be parallel to line of intersection of planes P_1 and P_2 through origin.

i.e.$\left[(2j + 3k) \times (4j - 3k)\right] \times \left[(j - k) \times (3i + 3j)\right]$

6. $\vec{u}.\vec{a} = |\vec{a}|^2 - |\vec{a}.\vec{b}|^2$

$|\vec{v}|^2 = |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta = |\vec{a}|^2 |\vec{b}|^2 - \left(\vec{a}.\vec{b}\right)^2$

$|\vec{u}|^2 = |\vec{a}|^2 |\vec{b}|^2 + \left(\vec{a}.\vec{b}\right)|\vec{b}|^2 - 2\left(\vec{a}.\vec{b}\right)^2$

$= |\vec{a}|^2 + |\vec{b}|^2 - \left(\vec{a}.\vec{b}\right)^2$

7. Clearly (B), (D) are meaningless

8. $\because 2\hat{i} + 3\hat{j} - 3\hat{k}$ & $-2\hat{i} - \hat{j} + 5\hat{k}$ satisfies both conditions.

COMPREHENSION TYPE QUESTIONS

Passage - 1

Let ABC be a triangle, AD, BE, CF be the angular bisectors of its interior angles. These bisectors are concurrent at a point I called incentre of the triangle. we know that from geometry that $\frac{BD}{DC}=\frac{AB}{AC}$ **. If** $BC=\alpha$, $CA=\beta$ **and** $AB=\gamma$ **and with reference to the same origin. Let** $\vec{a},\vec{b},\vec{c}$ **be position vectors of A,B,C respectively, then**

1. The positon vector of I must be

(A) $\frac{\vec{a}+\vec{b}+\vec{c}}{3}$ (B) $\frac{\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}}{3}$

(C) $\frac{\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}}{\alpha+\beta+\gamma}$ (D) $\frac{\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}}{2}$

2. If 'r' is the perpendicular distance of I from the side BC, then $\overrightarrow{IB}.\overrightarrow{IC}$ **must be**

a) $r^2 \cos ec\left(\frac{B}{2}\right)\cos ec\left(\frac{C}{2}\right)$

b) $r^2 \cos ec\left(\frac{B}{2}\right)\cos ec\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)$

c) $-r^2 \cos ec\left(\frac{B}{2}\right)\cos ec\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)$

d) $-r^2 \cos ec\left(\frac{A}{2}\right)\cos ec\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$

3. If 'r' is the perpendicular distance of I from the side BC, then $\left|\overrightarrow{IB}\times\overrightarrow{IC}\right|$ **must be**

a) $r^2 \cos ec\left(\frac{B}{2}\right)\cos ec\left(\frac{C}{2}\right)$

b) $r^2 \cos ec\left(\frac{B}{2}\right)\cos ec\left(\frac{C}{2}\right)\cos\left(\frac{A}{2}\right)$

c) $r^2 \cos ec\left(\frac{B}{2}\right)\cos ec\left(\frac{C}{2}\right)\cos ec\left(\frac{A}{2}\right)$

d) $r^2 \cos ec\left(\frac{B}{2}\right)\cos ec\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)$

Passage - 2

If $\vec{a}$, $\vec{b}$, $\vec{c}$ **be non coplanar unit vectors equally inclined to one another at an angle** θ **and if** $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}=p\vec{a}+q\vec{b}+r\vec{c}$ **then**

4. p = ...

(A) $\frac{1}{\sqrt{2\cos\theta}}$ (B) $\frac{1}{\sqrt{1+2\cos\theta}}$

(C) $\frac{1}{\sqrt{1-\cos\theta}}$ (D) $\frac{1}{\sqrt{1-2\cos\theta}}$

5. r = ...

(A) $\frac{1}{\sqrt{2\cos\theta}}$ (B) $\frac{1}{\sqrt{2\sin\theta}}$

(C) $\frac{1}{\sqrt{1+2\cos\theta}}$ (D) $\frac{1}{\sqrt{1-2\cos\theta}}$

6. q = ...

(A) $\frac{-2\cos\theta}{\sqrt{1+2\cos\theta}}$ (B) $\frac{-2\sin\theta}{\sqrt{1+2\cos\theta}}$

(C) $\frac{-2\cos\theta}{\sqrt{1+\cos\theta}}$ (D) $\frac{1}{\sqrt{1+2\cos\theta}}$

KEY

1. C	2. C	3.B
4. B	5. C	6.A

SOLUTIONS

1. $\frac{BD}{DC}=\frac{AB}{AC}=\frac{\gamma}{\beta}$

position vector of $D=\frac{\gamma\vec{c}+\beta\vec{b}}{\gamma+\beta}$

Now $BD=\frac{\alpha}{\gamma+\beta}.\gamma$

In $\Delta BDA, BI$ is bisector of ΔBDA also

$$\frac{ID}{IA}=\frac{BD}{AB}=\frac{\frac{\gamma\alpha}{\gamma+\beta}}{\gamma}=\frac{\alpha}{\gamma+\beta}$$

$$I=\frac{\alpha\vec{a}+\frac{(\gamma+\beta)(\gamma\vec{c}+\beta\vec{b})}{\gamma+\beta}}{\alpha+\beta+\gamma}=\frac{\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}}{\alpha+\beta+\gamma}$$

2. $\overrightarrow{IB}.\overrightarrow{IC}=\left|\overrightarrow{IB}\right|\left|\overrightarrow{IC}\right| \cos\left(180^0-\left(\frac{B}{2}+\frac{C}{2}\right)\right)$

$$=r\cos ec\left(\frac{B}{2}\right)r\cos ec\left(\frac{C}{2}\right)\left(-\cos\left(\frac{B+C}{2}\right)\right)$$

$$=-r^2\cos ec\left(\frac{B}{2}\right)r\cos ec\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)$$

3. $\left|\overrightarrow{IB}\times\overrightarrow{IC}\right|=r^2\cos ec\left(\frac{B}{2}\right)\cos ec\left(\frac{C}{2}\right)\cos\left(\frac{A}{2}\right)$

4, 5, 6.

$\vec{a}\times\vec{b}+\vec{b}\times\vec{c}=p\vec{a}+q\vec{b}+r\vec{c}$ ... (1)

taking dot product with $\vec{a}, \vec{b}, \vec{c}$ respectively in (1)

$\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]=p+q\cos\theta+r\cos\theta$ ... (2)

$0=p\cos\theta+q+r\cos\theta$ ... (3)

$\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]=p\cos\theta+q\cos\theta+r$ ... (4)

$(2)+(3)+(4)$ gives :

$$p+q+r=\frac{2\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]}{2\cos\theta+1} \quad \text{... (5)}$$

$(2)-(5)\times\cos\theta$ gives

$$p=\frac{\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]}{(2\cos\theta+1)(1-\cos\theta)}=r$$

$$q=\frac{-2\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]\cos\theta}{(2\cos\theta+1)(1-\cos\theta)}$$

But $\left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right]=\begin{vmatrix}1&\cos\theta&\cos\theta\\\cos\theta&1&\cos\theta\\\cos\theta&\cos\theta&1\end{vmatrix}$

$$=(1+2\sin\theta)(1-\sin\theta)^2$$

$$p=r=\frac{1}{\sqrt{1+2\cos\theta}},\ q=\frac{-2\cos\theta}{\sqrt{1+2\cos\theta}}$$

MATRIX MATCHING TYPE QUESTIONS

1.**Column I**

(A) The line $\vec{r} = (\hat{i}+\hat{j})+t(\hat{i}-\hat{k})$ where t is scalar passes through the point

(B) The line $\vec{r} = (\hat{i}+\hat{j})+t(\hat{i}-\hat{k})$ where t is scalar and the plane $\vec{r}.(2\hat{i}+\hat{j}+\hat{k}) = 2$ intersect at the point

(C) The point on the line $\vec{r} = (\hat{i}+\hat{j})+t(\hat{i}-\hat{k})$ where t is scalar, which is at a distance of 3 units from the point having position vector $\hat{i}$ is/are

(D) The volume of the parallelopiped having adjacent sides $\hat{i}+\hat{k}, 2\hat{i}+\hat{j}+\hat{k}$ and $\vec{c}$ is 4 cubic units then $\vec{c}$ may be

Column II

(p) $-\hat{i}-\hat{j}+2\hat{k}$

(q) $-\hat{i}+\hat{j}+2\hat{k}$

(r) $-2\hat{i}+\hat{j}+3\hat{k}$

(s) $\hat{j}+\hat{k}$

(t) $3\hat{i}+\hat{j}-2\hat{k}$

2.**COLUMN-I**

(A) The distance of the point (1,0,1) from the line $\bar{r} = \left(\bar{i}+2\bar{j}+4\bar{k}\right)+\lambda\left(\bar{i}-\bar{k}\right)$ is

(B) If A denote the image of origin in the line $\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-1}{1}$ then OA=

(C) If $\bar{a}, \bar{b}$ and $\vec{c}$ are three non- coplanar uni-modular vectors, each inclined with other at an angle 30^0, then volume of tetrahedron whose edges are $\bar{a}, \bar{b}$ and $\bar{c}$ is

(D) If $\bar{a}, \bar{b}$ and $\bar{c}$ are unit vectros such that $\bar{a}.\bar{b} = 0, \left(\bar{a}-\bar{c}\right).\left(\bar{b}+\bar{c}\right) = 0$ and $\bar{c} = \lambda\bar{a}+\mu\bar{b}+\omega\left(\bar{a}\times\bar{b}\right)$ where λ, μ and ω are scalars then $2\mu(\mu+1)+\omega^2 =$

COLUMN-II

(p) $\frac{1}{2}\sqrt{3\sqrt{3}-5}$

(q) 0

(r) $\sqrt{\frac{17}{2}}$

(s) $\sqrt{\frac{106}{9}}$

KEY

01) (A)–(q,r,s,t); (B)–(s); (C)–(q,t); (D)–(q,t)

02) A-r, B-s, C-p, D-q

SOLUTIONS

1. (A) The point (1+t , 1, - t) where t is real parameter, always lie on the given line.
(B) Solving the two equations we have
$$(1+t)2+1-t=2$$
$$\Rightarrow t=-1$$
So ; $\vec{r}=\hat{j}+\hat{k}$
(C) Any point on the line is $(1+t,1-t)$
$$\sqrt{(1+t-1)^2+1+t^2}=3$$
$$\Rightarrow t=\pm 2$$
So, the point is $3\vec{i}+\vec{j}-2\vec{k}$ and
$$-\vec{i}+\vec{j}+2\vec{k}$$
(D) Let $\vec{C}=x\hat{i}+y\hat{j}+z\hat{k}$
$$\left\|\begin{matrix} x & y & z \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{matrix}\right\|=4$$
$$\Rightarrow |(y-z)+1(x-2y)|=4$$
$$\Rightarrow |x-y-z|=4$$

2. (A) Find the foot of perpendicular of P(1,0,1) on $\frac{x-1}{1}=\frac{y-2}{0}=\frac{z-4}{-1}$ then find distance
(B) same as (A)
(C) $\bar{a}.\bar{a}=\bar{b}.\bar{b}=\bar{c}.\bar{c}=1$ and
$$\bar{a}.\bar{b}=\bar{b}.\bar{c}=\bar{c}.\bar{a}=\frac{\sqrt{3}}{2}$$
$$V=\frac{1}{6}\left[\bar{a}\ \bar{b}\ \bar{c}\right]$$
$$V^2=\frac{1}{36}\left[\bar{a}\ \bar{b}\ \bar{c}\right]^2$$
$$=\frac{1}{36}\begin{vmatrix} \bar{a}.\bar{a} & \bar{a}.\bar{b} & \bar{a}.\bar{c} \\ \bar{b}.\bar{a} & \bar{b}.\bar{b} & \bar{b}.\bar{c} \\ \bar{c}.\bar{a} & \bar{c}.\bar{b} & \bar{c}.\bar{c} \end{vmatrix}$$

(D) $\left(\bar{a}-\bar{c}\right).\left(\bar{b}+\bar{c}\right)=0$
$$\bar{a}.\bar{b}+\bar{c}.\left(\bar{a}-\bar{b}\right)=\left|\bar{c}\right|^2$$
$$\left(\bar{a}-\bar{b}\right).\left(\lambda\bar{a}+\mu\bar{b}+\omega\left(\bar{a}\,\bar{b}\,\bar{c}\right)\right)=1$$
$$\Rightarrow \lambda-\mu=1$$
$$\Rightarrow \lambda=\mu+1$$
$$\Rightarrow (\mu+1)^2+\mu^2+\omega^2=1$$
$$\left[\bar{a}\ \bar{b}\ \bar{c}\right]=0 \ \ \& \ \ \bar{a}\times\bar{b}\neq\bar{o}$$
$$\left(\bar{c}\times\bar{b}\right).\left(\bar{a}\times\bar{b}\right)=\begin{vmatrix} \bar{c}.\ \bar{a} & \bar{c}.\ \bar{b} \\ \bar{b}.\ \bar{a} & \bar{b}.\ \bar{b} \end{vmatrix}$$
$$\left(\bar{a}\times\bar{c}\right).\left(\bar{a}\times\bar{b}\right)=\begin{vmatrix} \bar{a}.\ \bar{a} & \bar{a}.\ \bar{b} \\ \bar{c}.\ \bar{a} & \bar{c}.\ \bar{b} \end{vmatrix}$$

MATRICES & DETERMINANTS

SYNOPSIS

- **Definition :** A rectangular arrangement of numbers (which may be real or complex numbers) in rows and columns, is called a matrix. This arrangement is enclosed by open ( ) or closed [ ] brackets. The numbers are called the elements of the matrix or entries of the matrix.
- **Order of Matrix :** A matrix having 'm' rows and 'n' columns is called a matrix of order '$m \times n$' or simply '$m \times n$' matrix (read as m by n matrix). A matrix A of order '$m \times n$' is usually written in the following manner

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & \dots & a_{1j} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & \dots & a_{2j} & \dots & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & a_{i3} & \dots & \dots & a_{ij} & \dots & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & \dots & a_{mj} & \dots & \dots & a_{mn} \end{bmatrix}_{m\times n}$$

In a compact form the above matrix is represented by

$A = \left[a_{ij}\right]_{m\times n}, 1 \le i \le m, 1 \le j \le n \,(or)\, A = \left[a_{ij}\right]$

The numbers $a_{11}, a_{12}, \ldots\ldots$, etc. are known as the elements of the matrix A. The element a_{ij} in the matrix A belongs to i^{th} row and j^{th} column.

Note: A matrix of order '$m \times n$' contains mn elements. Every row of such a matrix contains n elements and every column contains m elements.

Eg : Order of matrix $\begin{bmatrix} 3 & -1 & 5 \\ 6 & 2 & -7 \end{bmatrix}$ is 2×3

- **Types of matrices :**

Row matrix: A matrix is said to be a row matrix if it has only one row and any number of columns.

Eg : $[5\ 0\ 3]$ is a row matrix of order 1×3

- **Column matrix :** A matrix is said to be a column matirx if it has only one column and any number of rows.

Eg : $\begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$ is a column matrix of order 3×1

- **Singleton matrix :** If in a matrix there is only one element then it is called singleton matrix.

Thus, $A = \left[a_{ij}\right]_{m\times n}$ is a singleton matrix,

if $m = n = 1$

Eg : $[2], [3], [a], [-3]$ are singleton matrices.

- **Null matrix (or) Zero matrix :** If in a matrix all the elements are zeros then it is called a zero matrix and it is generally denoted by O. Thus $A = \left[a_{ij}\right]_{m\times n}$ is a zero matrix if

$a_{ij} = 0 \,\forall$ i and j.

Eg : $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are all zero matrices, but of different orders.

- **Rectangular matrix :** In a matrix if the number of rows is not equal to the number of columns then the matrix is called a rectangular matrix.

Eg : $\begin{bmatrix} 3 & -2 \\ 5 & 0 \\ 7 & 1 \end{bmatrix}_{3\times 2}$

- **Square matrix :** If number of rows and number of columns in a matrix are equal, then it is called a square matrix.

Eg : $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3 × 3.

- **Principal diagonal of a square matrix:** In a square matrix the diagonal from first element of the first row to the last element of the last row is called the principal diagonal of the square matrix.

Eg : $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

then a_{11}, a_{22}, a_{33} constitutes diagonal of A

- **Diagonal matrix :** In a square matrix if all the elements outside the principal diagonal are zeros, the elements of principal diagonal may or may not be zero, then the matrix is said to be diagonal matrix.

Eg : $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a diagonal matrix of order 3×3, which can be denoted by diag $[2,0,4]$

Note: If A = diag($d_1, d_2, d_3 ... d_n$) then

$A^n = diag\left(d_1{}^n, d_2{}^n, d_3{}^n .....d_n{}^n\right)$

- **Identity matrix or Unit matrix :**

A square matrix in which each element in the principal diagonal is '1' and rest are all zeros is called an identity matrix or unit matrix. Thus, the square matrix $A = \left[a_{ij}\right]$ is an identity matrix, if $a_{ij} = \begin{cases} 1, if \ i = j \\ 0, if \ i \neq j \end{cases}$

We denote the identity matrix of order n by I_n.

Eg : $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 2 and 3 respectively.

- **Scalar matrix :** A square matrix whose all non diagonal elements are zeros and diagonal elements are equal is called a scalar matrix.

Thus, if $A = \left[a_{ij}\right]$ is a square matrix and $a_{ij} = \begin{cases} \alpha, \text{if } i = j \\ 0, \text{ if } i \neq j \end{cases}$, then A is a scalar matrix.

Eg : $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Unit matrix and null square matrices are also scalar matrices.

- **Triangular matrix :** A square matrix $\left[a_{ij}\right]$ is said to be triangular matrix if each element above or below the principal diagonal is zero. It is of two types

Upper triangular matrix : A square matrix $\left[a_{ij}\right]$ is called the upper triangular matrix, if $a_{ij} = 0$ when $i > j$.

Eg : $\begin{bmatrix} 3 & 1 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & 6 \end{bmatrix}$ is an upper triangular matrix of order 3 × 3.

Lower triangular matrix: A square matrix $\left[a_{ij}\right]$ is called the lower triangular matrix, if $a_{ij} = 0$ when $i < j$.

Eg : $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 2 \end{bmatrix}$ is a lower triangular matrix of order 3 × 3

- **Trace of a matrix :** The sum of diagonal elements of a square matrix A is called the trace of matrix A, which is denoted by tr (A). Trace is also called as spur.

  i.e., tr $(A)=\sum_{i=1}^{n} a_{ii}=a_{11}+a_{22}+.....a_{nn}$

Properties of trace of a matrix :

Let $A=\left[a_{ij}\right]_{n\times n}$ and $B=\left[b_{ij}\right]_{n\times n}$ and λ be a scalar

i) $tr(\lambda A)=\lambda tr(A)$

ii) $tr(A\pm B)=tr(A)\pm tr(B)$

iii) $tr(AB)=tr(BA)$

iv) $tr(I_n)=n$ v) $tr(AB)\neq tr(A).tr(B)$

vi) If A,B,C are square matrices of order n, then Tr (ABC) = Tr (BCA) = Tr (CAB)

- **Equality of matrices :** Two matrices A and B are said to be equal, if

i) A and B are of the same type (order)

ii) The corresponding elements of A and B are equal.

Eg : $A=\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 0 & 1 \\ -1 & -3 & 5 \end{bmatrix}$ are equal iff a = 2, b = 0, c = 1, d = -1, e = -3, f = 5

- **Addition of matrices :** If A and B are two matrices of the same type, then their sum denoted by A+B is defined to be the matrix of the same type and is obtained by adding the corresponding elements of A and B.

  i.e., If $A=\left[a_{ij}\right]_{m\times n}$ and $B=\left[b_{ij}\right]_{m\times n}$ then

  $A+B=[C_{ij}]_{m\times n}$ where $C_{ij}=a_{ij}+b_{ij}$

- **Properties :**

i) A+B = B+A (commutative law)

ii) (A+B)+C = A+(B+C) (Associative law)

iii) A+O=O+A = A where O is zero matrix which is called additive identity.

iv) $A+(-A)=O=(-A)+A$, where $(-A)$ is obtained by changing the sign of every element of A, which is additive inverse of the matrix A.

- **Difference of two matrices :** If A and B are two matrices of the same type then A-B is defined as A+(-B)

  Note: If two matrices A and B are of different orders, we can not define A+B and A-B.

- **Multiplication of a matrix by a scalar:**

  Let $A=\left[a_{ij}\right]_{m\times n}$ be a matrix and k be a number, then the matrix which is obtained by multiplying every element of A by k is called scalar multiplication of A by k and it is denoted by kA.

  Thus, if $A=\left[a_{ij}\right]_{m\times n}$,

  then $kA=Ak=\left[ka_{ij}\right]_{m\times n}$.

- **Properties of scalar multiplication :** If A, B are matrices of the same order and λ,μ are any two scalars then

i) $\lambda(A+B)=\lambda A+\lambda B$

ii) $(\lambda+\mu)A=\lambda A+\mu A$

iii) $\lambda(\mu A)=(\lambda\mu A)=\mu(\lambda A)$

iv) $(-\lambda A)=-(\lambda A)=\lambda(-A)$

- **Multiplication of matrices :** Two matrices A and B are conformable for the product AB if the number of columns in A is same as the number of rows in B thus, if

  $A=\left[a_{ik}\right]_{m\times p}$, $B=\left[b_{kj}\right]_{p\times n}$ then

  $AB=\left[C_{ij}\right]_{m\times n}$ where $C_{ij}=\sum_{k=1}^{p} a_{ik}b_{kj}$

- **Properties of matrix multiplication :** If A,B and C are three matrices such that their product is defined, then

i) $AB\neq BA$,(Generally not commutative)

ii) $(AB)C=A(BC)$=ABC, (Associative law)

iii) $IA=A=AI$, where I is identitiy matrix for matrix multiplication.

iv) $A(B+C)=AB+AC$, (Distributive law)

v) (A + B).C = AC + BC, (Distributive law)

vi) If $AB = O$, then either A or B need not be equal to O.

vii) If AB = AC then B need not be equal to C even if $A \neq O$.

✈ Remember that if A and B are two square matrices of the same order, then

i) $(A+B)^2 = A^2 + B^2 + AB + BA$

ii) $(A-B)^2 = A^2 + B^2 - AB - BA$

iii) $A^m A^n = A^{m+n}$ iv) $(A^m)^n = A^{mn} = (A^n)^m$

v) $I^n = I \ \forall n \in N$

vi) If a square matrix, which is commutative with every square matrix of the same order for multiplication then it is necessarily a scalar matrix.

✈ **Transpose of a matrix :** The matrix obtained from a given matrix A by changing its rows into columns or columns into rows is called transpose of matrix A and is denoted by A^T *or* $A^{'}$

From the definition it is obvious that if order of A is $m \times n$, then order of A^T is $n \times m$

✈ **Properties of transpose of a matrix :**

Let A and B be two matrices then,

$(i)\left(A^T\right)^T = A$

ii) $(A \pm B)^T = A^T \pm B^T$, A and B being of the same order

iii) $(kA)^T = kA^T$, k be any scalar

iv) $(AB)^T = B^T A^T$, A and B being conformable for the product AB

v) $(A_1 A_2 A_3 .... A_{n-1} A_n)^T = A_n^T A_{n-1}^T .... A_3^T A_2^T A_1^T$

vi) A=B $\Leftrightarrow A^T = B^T$

✈ **Symmetric matrix:** A square matrix $A = [a_{ij}]$ is called symmetric matrix if $a_{ij} = a_{ji}$ $\forall\ i, j\ i.e., A^T = A$

Eg : $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is a symmetric matrix

✈ **Skew-Symmetric matrix :** A square matrix $A = [a_{ij}]$ is called skew -symmetric matrix if $a_{ij} = -a_{ji}\ \forall$ i, j i.e., $A^T = -A$.

Eg : $\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$

Note: i) All principal diagonal elements of a skew-symmetric matrix are always zeros

ii) Trace of a skew-symmetric matrix is zero

✈ **Properties of symmetric and Skew-symmetric matrices :**

i) If A is a square matrix, then $A + A^T, AA^T, A^T A$ are symmetric matrices

ii) I f A is square matrix then $A - A^T$ is a skew-symmetric matrix.

iii) If A is a symmetric matrix, then $-A, KA, A^T, A^n, A^{-1}, B^T AB$ are also symmetric matrices, where $n \in N$, $K \in R$ and B is a square matrix of order that of A

iv) If A is a skew-symmetric matrix, then

a) A^{2n} is symmetric matrix for $n \in N$

b) A^{2n+1} is a skew-symmetric matrix for $n \in N$

c) kA is also skew-symmetric matrix, where $k \in R$

v) If A, B are two symmetric matrices, then

a) $A \pm B, AB + BA$ are also symmetric matrices,

b) AB-BA is a skew-symmetric matrix,

c) AB is a symmetric matrix, when AB=BA

vi) If A, B two skew-symmetric matrices, then

a) $A \pm B$, AB-BA are skew-symmetric matrices

b) AB+BA is a symmetric matrix.

vii) a) If A is a skew-symmetric matrix and B is a square matrix of order that of A then $B^T AB$ is also skew-symmetric matrix.

b) If A is a skew-symmetric matrix and C is a column matrix, then C^T AC is a zero matrix.

viii) Every square matrix A can be uniquely expressed as sum of a symmetric and skew-symmetric matrices of same order

ix) If A,B are symmetric matrices of same order and $X = AB + BA$, $y = AB - BA$ then $XY + YX = 0$

i.e., $A = \left[\frac{1}{2}\left(A + A^T\right)\right] + \left[\frac{1}{2}\left(A - A^T\right)\right]$.

Note: If a matrix A is both symmetric and skew-symmetric then A is null matrix.

✈ **Orthogonal matrix:** A square matrix A is called orthogonal if $AA^T = I = A^T A$ *i.e.,* $A^{-1} = A^T$

✈ Properties of Orthogonal matrix :

i) Every orthogonal matrix is non -singular.

ii) Every orthogonal matrix is invertible.

iii) If A is orthogonal, then A^T *and* A^{-1} are also orthogonal.

iv) If A and B are orthogonal matrices of same order then AB and BA are also orthogonal.

v) The sum of the squares of elements of any row or column of an orthogonal matrix is 1.

vi) The sum of the products of the corresponding elements of any two rows or columns is 0

vii) If A is an orthogonal matrix and B = AP (P-non-singular matrix) then PB^{-1} is also orthogonal matrix.

✈ Special types of matrices :

i) Idempotent matrix: A square matrix A is called an idempotent matrix if $A^2 = A$

Eg : A= $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is an idempotent matrix.

Properties :

i) If A, B are two idempotent matrices and AB = BA = O then A+B is idempotent matrix.

ii) If A is idempotent matrix then I-A is also idempotent

iii) Every non-singular idempotent matrix is always unit matrix

iv) If AB=A , BA=B then A and B are idempotent matrices and $A^n + B^n = A + B$

ii) Involutory matrix: A square matrix A is called an involutory matrix if $A^2 = I$

Eg : $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is an involutory matrix.

iii) Nilpotent matrix: A square matrix A is called a nilpotent matrix if there exists atleast one $p \in N$ such that $A^p = O$, where the least value p is called index of the nilpotent matrix A

Eg : A= $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is a nilpotent matrix of index 3.

Note: Trace of a nilpotent matrix is zero.

iv) Periodic matrix: A matrix A is called a periodic matrix if $A^{k+1} = A$ where k is a positive integer. The least value of k is said to be period of A

v) Conjugate of a matrix: The matrix obtained from any given matrix A containing complex numbers as its elements, on replacing its elements by the corresponding conjugate complex numbers is called conjugate of A and is denoted by $\overline{A}$.

Eg : A = $\begin{bmatrix} 1+2i & 2-3i & 3+4i \\ 4-5i & 5+6i & 6-7i \\ 8 & 7+8i & 7 \end{bmatrix}$ then

$\overline{A} = \begin{bmatrix} 1-2i & 2+3i & 3-4i \\ 4+5i & 5-6i & 6+7i \\ 8 & 7-8i & 7 \end{bmatrix}$

vi) Transposed conjugate of a matrix: The transpose of the conjugate of a matrix A is called transposed conjugate of A and is denoted by A^{θ} or A^*. The conjugate of the transpose of A is the same as the transpose of the conjugate of A

i.e., $\left(\overline{A^T}\right) = \left(\overline{A}\right)^T = A^{\theta}$.

Eg : $A=\begin{bmatrix}1+2i & 2-3i & 3+4i\\ 4-5i & 5+6i & 6-7i\\ 8 & 7+8i & 7\end{bmatrix}$

then $A^{\theta}=\begin{bmatrix}1-2i & 4+5i & 8\\ 2+3i & 5-6i & 7-8i\\ 3-4i & 6+7i & 7\end{bmatrix}$

vii) Hermitian matrix: A square matrix 'A' is said to be Hermitian matrix if $A^{\theta}=A$

Eg : $\begin{bmatrix}3 & 3-4i & 5+2i\\ 3+4i & 5 & -2+i\\ 5-2i & -2-i & 2\end{bmatrix}$ is Hermitian matrix.

Note: All the principal diagonal elements of herimitian matrix are real.

viii) Skew Hermitian matrix : A square matrix A is said to be a skew-Hermitian if

$A^{\theta}=-A$

Eg : $\begin{bmatrix}3i & -3+2i & -1-i\\ 3+2i & -2i & -2-4i\\ 1-i & 2-4i & 0\end{bmatrix}$

is a skew-hermitian matrix.

Note: All the principal diagonal elements of skew-hermitian matrix are either zero or purely imaginary.

ix) Unitary matrix: A square matrix A is said to be unitary, if $A^{\theta}A=I=AA^{\theta}$

The determinant of unitary matrix is one and it is non-singular.

✈ Determinant of a square matrix :

The determinant of the square matrix

$A=\begin{bmatrix}a_1 & b_1\\ a_2 & b_2\end{bmatrix}$ is the unique

number $a_1b_2-a_2b_1$ and is denoted by

$$\text{Det } A=|A|=\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}=a_1b_2-a_2b_1$$

✈ Minor of an element in a square matrix:

Let $A=[a_{ij}]$ be a square matrix. The minor of an element a_{ij} in A is the determinant of the square matrix that remains after deleting the i^{th} row and j^{th} column of A. It is denoted by M_{ij}.

✈ Cofactor of an element of a square matrix :

The cofactor of an element in the i^{th} row and the j^{th} column of a matrix is defined as its minor multiplied by $(-1)^{i+j}$. If A_{ij} is the cofactor of a_{ij}, then $A_{ij}=(-1)^{i+j}M_{ij}$

Let $A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$

i) Minor of a_{23} is $(M_{23})=\begin{vmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{vmatrix}$

$=a_{11}a_{32}-a_{31}a_{12}$

ii) Cofactor of a_{31} is

$A_{31}=(-1)^{3+1}M_{31}=+\begin{vmatrix}a_{12} & a_{13}\\ a_{22} & a_{23}\end{vmatrix}$

iii) The cofactors of $a_{11},a_{12},a_{13}.....$ are denoted by $A_{11},A_{12},A_{13},.......$

iv) For the matrix $\begin{bmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{bmatrix}$, the cofactor

matrix is given by $\begin{bmatrix}A_1 & B_1 & C_1\\ A_2 & B_2 & C_2\\ A_3 & B_3 & C_3\end{bmatrix}$

Eg : $B_2=(-1)^{2+2}\begin{vmatrix}a_1 & c_1\\ a_3 & c_3\end{vmatrix}=(a_1c_3-a_3c_1)$

✈ Determinant of Third Order Matrix

The determinant of a square matrix is equal to the sum of the products of the elements of a row (or column) with their corresponding cofactors

If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow \Delta = a_1 A_1 + a_2 A_2 + a_3 A_3$

$= b_1 B_1 + b_2 B_2 + b_3 B_3$
$= c_1 C_1 + c_2 C_2 + c_3 C_3$

Properties of determinants:

i) The determinant of a square matrix changes its sign when any two rows (or columns) are interchanged

ii) If two rows (columns) of a square matrix are identical or proportional then the value of the determinant is zero.

iii) If all the elements of a row (column) of a square matrix are multiplied by a number K then the determinant of the resulting matrix is equal to K times the determinant of the original matrix.

iv) If each element of a row (column) of a square matrix is the sum of two terms then its determinant can be expressed as the sum of two determinants of two square matrices of the same order.

$$\begin{vmatrix} a_1+x & a_2 & a_3 \\ b_1+y & b_2 & b_3 \\ c_1+z & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} x & a_2 & a_3 \\ y & b_2 & b_3 \\ z & c_2 & c_3 \end{vmatrix}$$

v) If the elements of a row (column) of a square matrix are added with K times the corresponding elements of any other row (column) then the value of the determinant of the resulting matrix is unaltered

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} x_1+ky_1 & y_1 & z_1 \\ x_2+ky_2 & y_2 & z_2 \\ x_3+ky_3 & y_3 & z_3 \end{vmatrix}$$

vi) The sum of the products of the elements of any row (column) of a squre matrix with the cofactors of the corresponding elements of any other row (column) is zero.

$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow a_1 A_2 + b_1 B_2 + c_1 C_2 = 0$

$a_1 A_3 + b_1 B_3 + c_1 C_3 = 0$
$a_2 A_1 + b_2 B_1 + c_2 C_1 = 0$

vii) If the elements of a square matrix are Polynomials in x and two rows (columns) become identical when $x = a$ then $x-a$ is a factor of its determinant and if three rows are identical then $(x-a)^2$ is a factor

Note: i) $|A| = |A^T|, |AB| = |A||B| = |B||A|$

ii) $|KA| = K^n|A|$, $n =$ order of A.

iii) If $\Delta = |a_{ij}|$ is a determinant of order n, then the value of the determinant $[A_{ij}]$, where A_{ij} is the cofactor of a_{ij} is, Δ^{n-1}.

iv) Determinant of nilpotent matrix is 0

v) Determinant of an orthogonal matrix = 1or -1

vi) Determinant of a Skew - symmetric matrix of odd order is 0.
vii) Determinant of Hermitian matrix is purely real.

viii) Determinant of triangular matrix is zero

✈ Product of Determinants of the same Order :

Let $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$

Then row by row multiplication of Δ_1 *and* Δ_2 is given by

$$\Delta_1\Delta_2 = \begin{vmatrix} a_1\alpha_1+b_1\beta_1+c_1\gamma_1 & a_1\alpha_2+b_1\beta_2+c_1\gamma_2 & a_1\alpha_3+b_1\beta_3+c_1\gamma_3 \\ a_2\alpha_1+b_2\beta_1+c_2\gamma_1 & a_2\alpha_2+b_2\beta_2+c_2\gamma_2 & a_2\alpha_3+b_2\beta_3+c_2\gamma_3 \\ a_3\alpha_1+b_3\beta_1+c_3\gamma_1 & a_3\alpha_2+b_3\beta_2+c_3\gamma_2 & a_3\alpha_3+b_3\beta_3+c_3\gamma_3 \end{vmatrix}$$

Note: Multiplication can also be performed row by column; column by row or column by column as required in the problem.

✈ Derivative of Determinants :

Let $\Delta(x) = \begin{vmatrix} f_1(x) & g_1(x) \\ f_2(x) & g_2(x) \end{vmatrix}$,

where $f_1(x), f_2(x), g_1(x)$ and $g_2(x)$ are the functions of x. then,

$$\Delta'(x) = \begin{vmatrix} f_1'(x) & g_1'(x) \\ f_2(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) \\ f_2'(x) & g_2'(x) \end{vmatrix}$$

Also

$$\Delta'(x) = \begin{vmatrix} f_1'(x) & g_1(x) \\ f_2'(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1'(x) \\ f_2(x) & g_2'(x) \end{vmatrix}$$

Thus, to differentiate a determinant, we differentiate one row (or column) at a time, keeping others unchanged.

✈ Integration of Determinants :

If $\Delta(x) = \begin{vmatrix} f(x) & g(x) \\ \lambda_1 & \lambda_2 \end{vmatrix}$,then

$$\int_a^b \Delta(x)dx = \begin{vmatrix} \int_a^b f(x)dx & \int_a^b g(x)dx \\ \lambda_1 & \lambda_2 \end{vmatrix}$$

Here, $f(x)$ and $g(x)$ are functions of x and λ_1, λ_2 are constants.

Note: This formula is only applicable, if there is a variable only in one row or column, otherwise expand the determinant and then integrate.

✈ Singular and non-singular matrix :

If the determinant of a square matrix is zero then it is called a singular matrix otherwise non-singular matrix.

Note: i) A is singular $\Leftrightarrow A^T$ is singular

A is non-singular $\Leftrightarrow A^T$ is non-singular

ii) If A and B are non-singular matrices of the same type, then AB is non-singular of the same type.

iii) If product of two non-zero square matrices is a zero matrix, then both of them must be singular matrices.

✈ Adjoint matrix of a square matrix :

If the elements of a square matrix are replaced by corresponding co-factors then the transpose of the resulting matrix is called the adjoint of the matrix. Adjoint matrix of A is denoted by Adj A

If $P = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ then $\text{Adj}P = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$

Where $A_1, B_1, C_1.....$ are the co-factors of $a_1, b_1, c_1.....$

✈ Properties of adjoint matrix :

If A, B are square matrices of order n and I_n is corresponding unit matrix, then

$(i)\ A(adjA) = |A| I_n = (adjA)A$

(Thus A $adj(A)$ is always a scalar matrix)

ii) $|adjA| = |A|^{n-1}$

iii) $adj(adjA) = |A|^{n-2} A; |A| \neq 0$

iv) $|adj(adjA)| = |A|^{(n-1)^2}$

v) $|adj(adj(adjA)......r\ times| = |A|^{(n-1)^r}$

vi) $adj(A^T) = (adjA)^T$

vii) $adj(AB) = (adjB)(adjA)$

viii) $adj(A^m) = (adjA)^m, m \in N$

ix) $adj(kA) = k^{n-1}(adjA), k \in R$

x) $adj(I_n) = I_n$ xi) $adj(O) = O$

xii) A is symmetric matrix $\Rightarrow$ adj A is also symmetrix matrix.

xiii) A is diagonal matrix $\Rightarrow$ adj A is also diagonal matrix.

xiv) A is triangular matrix $\Rightarrow adjA$ is also triangular matrix.

xv) A is singular $\Rightarrow |adjA| = 0$

✈ **Inverse of a matrix :** Let A be a non-singular square matrix of order n, if there exist a square matrix B of the same order such that $AB = BA = I_n$ then B is called the inverse of A and we write it as A^{-1}.

The inverse of A given by $A^{-1} = \dfrac{adj\ A}{\det A}$

A matrix is said to be invertible, if it possesses inverse.

✈ **Properties of inverse matrix :** If A and B are invertible matrices of the same order, then

i) $\left(A^{-1}\right)^{-1} = A$ ii) $\left(A^{T}\right)^{-1} = \left(A^{-1}\right)^{T}$

iii) $(AB)^{-1} = B^{-1}A^{-1}$

iv) $\left(A^{k}\right)^{-1} = \left(A^{-1}\right)^{k}, k \in N$

v) $adj\left(A^{-1}\right) = (adjA)^{-1}$ vi) $\left|A^{-1}\right| = \dfrac{1}{|A|} = |A|^{-1}$

vii) If A= diag($a_1a_2...a_n$)

then $A^{-1} = diag\left(a_1^{-1}a_2^{-1}....a_n^{-1}\right)$

viii) If A is symmetric matrix then A^{-1} is also symmetric matrix.

ix) The inverse of a skew symmetric matrix of odd order does not exist.

x) A is a non singular scalar matrix $\Rightarrow A^{-1}$ is also a scalar matrix.

xi) A is triangular matrix, $|A| \neq 0 \Rightarrow A^{-1}$ is also triangular matrix.

xii) If A,B are symmetric matrices and commute then $A^{-1}B$, AB^{-1}, $A^{-1}B^{-1}$ are also symmetric matrices

Cancellation law with respect to multiplication : If A is a non -singular matrix i.e., if $|A| \neq 0$, then A^{-1} exist and

$AB = AC \Rightarrow A^{-1}(AB) = A^{-1}(AC)$

$\Rightarrow \left(A^{-1}A\right)B = \left(A^{-1}A\right)C \Rightarrow IB = IC \Rightarrow B = C$

$\therefore AB = AC \Rightarrow B = C \Leftrightarrow |A| \neq 0$.

✈ **Elementary Transformations :** Any one of the following operation on a matrix is called an elementary transformation.

a) Interchanging any two rows (or columns) this transformation is indicated as

$R_i \leftrightarrow R_j \left(C_i \leftrightarrow C_j\right)$

b) Multiplication of the elements of any row (or columns) by a non-zero scalar quantity this could be indicated as

$R_i \to kR_i \left(C_i \to kC_i\right)$

c) Addition of a constant multiple of the elements of any row to the corresponding element of any other row, indicated as $R_i \to R_i + kR_j$.

✈ **Equivalent Matrices :** Two matrices A and B are said to be equivalent, if one is obtained from the other by elementary transformations. It is denoted by $A \sim B$

✈ **Solution of Linear Equations by Determinants :**

1. **Cramer's Rule:** (Solution of system of linear equations in **two** unknowns) The solution of the system of equations

$a_1x + b_1y = c_1,\ a_2x + b_2y = c_2$

is given by $x = \dfrac{\Delta_1}{\Delta}$ and $y = \dfrac{\Delta_2}{\Delta}$,where

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \Delta_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} and\ \Delta_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

provided $\Delta \neq 0$

2. **Cramer's Rule:** (Solution of system of linear equations in **three** unknowns) The solution of the system of equations

$a_1x + b_1y + c_1z = d_1,\ a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$ its matrix form is

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \Rightarrow AX = B$$

is given by $x = \dfrac{\Delta_1}{\Delta}, y = \dfrac{\Delta_2}{\Delta}$ and $z = \dfrac{\Delta_3}{\Delta}$, where

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}; \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix};$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Provided $\Delta \neq 0$

Sub matrix: A matrix obtained by deleting finite number of rows or columns or both of a given matrix A is called sub matrix of A.

Rank of a matrix : Let A be a non - zero matrix. The rank of A is defined as the maximum of the orders of the non - singular square submatrices of A and is denoted by rank (A).

Note: i) If A is a non-zero matrix of order 3, then the rank of A is

a) 3 if A is non-singular
b) 2 if A is singular and there exist atleast one of its 2×2 submatrices is non-singular
c) 1 if A is singular and every 2×2 submatrix is singular

ii) A positive integer r is said to be the rank of a non-zero matrix A, if
a) There exists at least one minor in A of order r which is not zero. and
b) Every minor in A of order greater than r is zero. It is written rank (A) = r.

iii) The rank of a null matrix is defined as zero
iv) The rank of identy matrix of order n is n
v) If A is a non-singular matrix of order n then $rank(A) = n$.
vi) Elementary operations do not change the rank of a matrix.
vii) If A^T is a transpose of A, then $\text{rank}(A^T) = \text{rank}(A)$
viii) If A^θ is the transposed conjugate of A, then $rank(A^\theta) = rank(A)$.
ix) $rank(A+B) \leq rank(A) + rank(B)$.
x) If A and B are two matrices such that the product AB is defined, then rank (AB) can not exceed the rank of the either matrix.
xi) If $A = \left[a_{ij}\right]_{m \times n}$ is a matrix of Rank r then $r \leq \min\{m, n\}$

✈ **Echelon form of a matrix :** In a matrix if the number of zeros before the first non zero element in any row doesnot exceed number of such type of zeros in the very next row, then the matrix is said to be in echelon form.

Eg : the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 3 & 5 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in the Echelon form.

Note: The number of non zero rows of a matrix given in echelon form is its rank.

✈ **Characteristic Equation of a Matrix:**

If A is a matrix of order $n \times n$ then $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its characteristic equation is $|A - \lambda I| = 0$ takes the form

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

$S_1 = a_{11} + a_{22} + a_{33}$ = sum of the leading diagonal elements of A.

$$S_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

= sum of the minors of the leading diagonals elements of A

$S_3 = |A|$=determinant of the matrix A.

- ✈ **Solution of system of non-homogenoeous linear equations in two unknowns :**

$a_1x+b_1y=c_1$, $a_2x+b_2y=c_2$

Its matrix form is $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$\Rightarrow AX=D$

i) $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}\Rightarrow$ system has unique solutions (consistent) $\Rightarrow X=A^{-1}D$

ii) $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}\Rightarrow$ System has no solution (inconsistent)

iii) $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}\Rightarrow$ System has infintely many solutions (consistent)

- ✈ **Solution of system of non-homogeneous linear equations in three unknowns :**

$a_1x+b_1y+c_1z=d_1$, $a_2x+b_2y+c_2z=d_2$

$a_3x+b_3y+c_3z=d_3$ Its matrix form is

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}\Rightarrow AX=D$$

(i) $|A|\neq 0$ then the system has unique solution

(ii) $|A|=0$ then the system also has infinitely many solutions or no solution

Consistent System: A system of equations is said to be **consistent** if its solution (one or more) exist.

Inconsistent system:A system of equations is said to be **inconsistent** if its solution does not exist.

Solution of system of homogeneous linear equations in two unknowns :

- ✈ $a_1x+b_1y=0$, $a_2x+b_2y=0$

Its matrix form is $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow AX=O$

(i) $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}\Rightarrow$ System has unique solution that is x = 0 = y (trivial solution)

(ii) $\frac{a_1}{a_2}=\frac{b_1}{b_2}\Rightarrow$ System has infinitely many solutions (non-trivial solutions)

(iii) The above system of equations is always consistent.

- ✈ **Solution of system of homogeneous linear equations in three unknowns :**

$a_1x+b_1y+c_1z=0$, $a_2x+b_2y+c_2z=0$

$a_3x+b_3y+c_3z=0$ Its matrix form is

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\Rightarrow AX=O$$

(i) $|A|\neq 0$ then the system has unique solution i.e $x=y=z=0$ (trivial solution)

(ii) $|A|=0$ then the system also has infinitely many solutions (non-trivial solutions)

(iii) The above system of equations is always consistent.

- ✈ **Solution of a system of Linear Equations by Matrix-Rank method :**

Let AX=B be a system of 'n' linear equations in 'n' variables.

1. Write the augmented matrix [A B]
2. Reduce the augmented matrix to echelon form using elementary row-opeations
3. Determine the rank of the coefficient matrix A and augmented matrix [A B] by counting the number of non-zero rows in A and [A B].

i) If $rank(A) \neq rank(AB)$, then the system of equations is inconsistent.

ii) If $rank(A) = rank(AB) =$ the number of unknowns, then the system of equations is consistent and has a unique solution.

iii) If $rank(A) = rank(AB) <$ the number of unknowns, then the system of equations is consistent and has infinitely many solutions.

✈ Let $AX = O$ be a homogeneous system of linear equations.

a) If $rank(A) =$ number of variables, then $AX = O$ have a trivial solution i.e. zero solution.

b) If $rank(A) <$ number of variables, then $AX = O$ have a non-trivial solution. It will have infinitely many solutions.

✈ **Conditions for consistency :**

The following cases may arise:

(i) If $\Delta \neq 0$, or Rank (A) = Rank [AB] =3 then the system is consistent and has a unique solution, which is given by Cramer's rule:

$$x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$$

(ii) If $\Delta = 0$ and at least one of the determinants $\Delta_1, \Delta_2, \Delta_3$ is non -zero,(or) Δ=0 and $(adjA)B \neq 0$ (or) rank (A) $\neq$ rank $[AB]$ the given system is inconsistent i.e., it has no solution.

(iii) If $\Delta = 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = 0$,

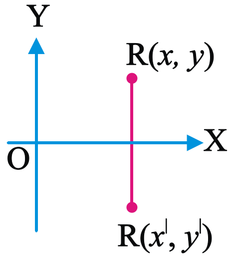
(or) $\Delta = 0$ and $(adjA)B = 0$

(or) rank (A) = rank $[AB] < 3$ then the system is consistent and dependent, and has infinitely many solutions.

✈ **Geometrical Transformations :**

Reflection of a point $R(x, y)$ in x-axis :

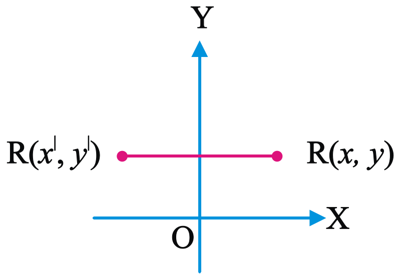
Let $R'(x', y')$ is the reflection of the point R (x, y) on x-axis. The matrix of reflection is given by



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Reflection of a point $R(x, y)$ in y-axis:

$$R(x', y') = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$



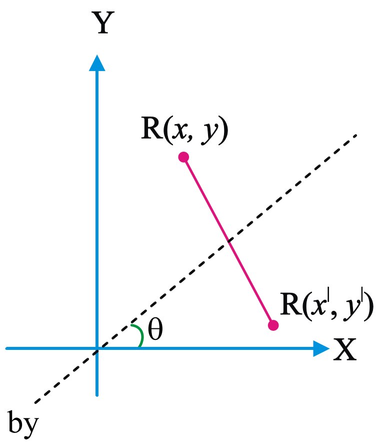
Note:(i) Reflection of a point $R(x, y)$ through origin is given by

$$\begin{bmatrix} x^1 \\ y^1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

(ii) Reflection of point $R(x, y)$ in the line y = x is given by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(iii) Reflection of point $R(x, y)$ in the line y = -x is given by $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

(iv) Reflection of point $R(x, y)$ in $y = (\tan\theta)x$ i.e $y = mx$ or $y = (\tan\theta)x$ *is*



given by

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} = \begin{pmatrix} \frac{1-\tan^2\theta}{1+\tan^2\theta} & \frac{2\tan\theta}{1+\tan^2\theta} \\ \frac{-2\tan\theta}{1+\tan^2\theta} & \frac{1-\tan^2\theta}{1+\tan^2\theta} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{-2m}{1+m^2} & \frac{1-m^2}{1+m^2} \end{pmatrix}$$

(v) Rotation of axes through an angle $'\theta'$ is given by $R'(x', y') = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Standard Results :

1. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$

2. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$

3. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = (a-b)(b-c)(c-a)$

4. $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix} =$

$(a-b)(b-c)(c-a)(a+b+c)$

5. $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$

$= (a-b)(b-c)(c-a)(ab+bc+ca)$

6. $\begin{vmatrix} 1 & a & a^4 \\ 1 & b & b^4 \\ 1 & c & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)$

$(a^2 + b^2 + c^2 + ab + bc + ca)$

7. $\begin{vmatrix} n^2 & (n+1)^2 & (n+2)^2 \\ (n+1)^2 & (n+2)^2 & (n+3)^2 \\ (n+2)^2 & (n+3)^2 & (n+4)^2 \end{vmatrix} = -8$

8. $\begin{vmatrix} n^2 & (n+1)^2 & (n+2)^2 \\ (n+3)^2 & (n+4)^2 & (n+5)^2 \\ (n+6)^2 & (n+7)^2 & (n+8)^2 \end{vmatrix} = -216$

EXAMPLES

1. A matrix X has (a+b) rows and (a+2) columns, while the matrix Y has (b+1) rows and (a+3) columns. Both matrices XY and YX exist. Find a and b.

Sol. Type of X is $(a+b) \times (a+2)$

Type of Y is $(b+1) \times (a+3)$

Since both XY, YXexist

$\therefore$ a+2=b+1 and a+b=a+3 $\therefore$ a=2, b=3

order of X is 5×4, order of Y is 4×5

$\therefore XY \neq YX$

($\because$ order of XY is 5×5 and YX is 4×4)

2.

If $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ ***then A is__***

Sol: $A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$$A.A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = I$$

Similarly $A^T A = I$

$\therefore$ A is Orthogonal matrix

3:

In a square matrix, the elements of a column are 2, 5k+1, 3 and the cofactors of another column are 1-5k, 2, 4k-2. Then find k

Sol. The sum of the products of the elements of any column of a square matrix with the cofactors of the corresponding elements of any other column is zero.

$\therefore\ 2(1-5k)+(5k+1)2+3(4k-2)=0$

$12k-2=0 \Rightarrow k=\dfrac{1}{6}$

4:

If α,β,γ **are roots of** $x^3+px^2+q=0$**,**

where $q\neq 0$ **then** $\begin{vmatrix} \frac{1}{\alpha} & \frac{1}{\beta} & \frac{1}{\gamma} \\ \frac{1}{\beta} & \frac{1}{\gamma} & \frac{1}{\alpha} \\ \frac{1}{\gamma} & \frac{1}{\alpha} & \frac{1}{\beta} \end{vmatrix} = -$

Sol. we have $\alpha\beta+\beta\gamma+\gamma\alpha=0 \Rightarrow \dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}=0$

Let $\Delta = \begin{vmatrix} \frac{1}{\alpha} & \frac{1}{\beta} & \frac{1}{\gamma} \\ \frac{1}{\beta} & \frac{1}{\gamma} & \frac{1}{\alpha} \\ \frac{1}{\gamma} & \frac{1}{\alpha} & \frac{1}{\beta} \end{vmatrix}$ $\quad C_1 \to C_1+C_2+C_3$

$$= \begin{vmatrix} \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} & \frac{1}{\beta} & \frac{1}{\gamma} \\ \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} & \frac{1}{\gamma} & \frac{1}{\alpha} \\ \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} & \frac{1}{\alpha} & \frac{1}{\beta} \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{\beta} & \frac{1}{\gamma} \\ 0 & \frac{1}{\gamma} & \frac{1}{\alpha} \\ 0 & \frac{1}{\alpha} & \frac{1}{\beta} \end{vmatrix} = 0$$

5:

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} =$$

Sol. $\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$

$c_2 \to c_2 - pc_1$ and $c_3 \to c_3 - qc_1$

$$\Delta = \begin{vmatrix} 1 & 1 & 1+q \\ 2 & 3 & 4+3p \\ 3 & 6 & 10+6p \end{vmatrix}_{C_3 \to C_3 - PC_2}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} = 1$$

6:

If $f(x)=\begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x(x-1) & x(x-1)(x-2) & x(x^2-1) \end{vmatrix}$

then $f(2015)=$ **[EAM-2020]**

Sol. Taking x common from C_2, $x+1$ from C_3 and (x -1) from R_3, we get

$$f(x) = x(x+1)(x-1)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x-1 & x \\ 3x & x-2 & x \end{vmatrix}$$

Applying $C_1 \to C_1 - C_3,\ C_2 \to C_2 - C_3$, we get

$$f(x) = x(x^2-1)\begin{vmatrix} 0 & 0 & 1 \\ x & -1 & x \\ 2x & -2 & x \end{vmatrix}$$

=0 [$\because C_1$ and C_2 are proportional]

Thus $f(2015)=0$

7:

If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$ **are the two given determinants then** $\dfrac{d}{dx}(\Delta_1) =$

Sol: Given $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix} \left(\begin{array}{l} \because \frac{d}{dx}(constant) = 0 \\ \frac{d}{dx}(x) = 1 \end{array} \right)$

$$\frac{d}{dx}(\Delta_1) = \begin{vmatrix} 1 & 0 & 0 \\ a & x & b \\ a & a & x \end{vmatrix} + \begin{vmatrix} x & b & b \\ 0 & 1 & 0 \\ a & a & x \end{vmatrix} + \begin{vmatrix} x & b & b \\ a & x & b \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x & b \\ a & x \end{vmatrix} + \begin{vmatrix} x & b \\ a & x \end{vmatrix} + \begin{vmatrix} x & b \\ a & x \end{vmatrix} = 3\Delta_2$$

8:

If A= $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ **then** $A^3 - 3A^2 - I =$

Sol. Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 2 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[-\lambda(2-\lambda)-1\right] - 1[-\lambda-2] = 0$$

$$\Rightarrow \lambda^3 - 3\lambda^2 - I = 0 \ (or)\ A^3 - 3A^2 - I = 0$$

9:

If the system of equations $x+y+z=1$**,** $x+2y+4z=n$ ***and*** $x+4y+10z=n^2$ **are consistent.then the values of 'n' are__**

Sol. $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{vmatrix} = \mathbf{0}$ $\therefore$ system has no solution or infinitely many solutions. Given system is consistent $\therefore$ It has infinitely many solutions

$$\therefore \Delta_3 = 0 \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & n \\ 1 & 4 & n^2 \end{vmatrix} = 0$$

$\Rightarrow$ (1 - 2) (2 - n) (n - 1) = 0 $\Rightarrow$ n = 1, 2.

10:

If α_1, α_2 **are roots of** $ax^2+bx+c=0$**,** β_1, β_2 **are roots of** $px^2+qx+r=0$ **and equations** $\alpha_1 y + \alpha_2 z = 0,\ \beta_1 y + \beta_2 z = 0$ **have a non trivial solution, then** $\frac{b^2}{q^2} =$ _____

Sol. $\alpha_1 y + \alpha_2 z = 0$ *and* $\beta_1 y + \beta_2 z = 0$ have a non trivial solution

$$\therefore \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = 0 \Rightarrow \alpha_1\beta_2 - \alpha_2\beta_1 = 0$$

$$\Rightarrow \frac{\alpha_1}{\alpha_2} = \frac{\beta_1}{\beta_2} \quad \Rightarrow \frac{\alpha_1+\alpha_2}{\alpha_1-\alpha_2} = \frac{\beta_1+\beta_2}{\beta_1-\beta_2}$$

$$\Rightarrow \frac{(\alpha_1+\alpha_2)^2}{(\alpha_1+\alpha_2)^2 - 4\alpha_1\alpha_2} = \frac{(\beta_1+\beta_2)^2}{(\beta_1+\beta_2)^2 - 4\beta_1\beta_2}$$

$$\Rightarrow \frac{\frac{b^2}{a^2}}{\frac{b^2}{a^2} - 4\frac{c}{a}} = \frac{\frac{q^2}{p^2}}{\frac{q^2}{p^2} - \frac{4r}{p}}$$

$$\Rightarrow \frac{b^2}{b^2-4ac} = \frac{q^2}{q^2-4pr} \Rightarrow \frac{b^2}{q^2} = \frac{b^2-4ac}{q^2-4pr}$$

11:

If $\begin{vmatrix} x+a & a^2 & a^3 \\ x+b & b^2 & b^3 \\ x+c & c^2 & c^3 \end{vmatrix}$ **= 0 and a ≠ b ≠ c then** $x =$

Sol: By given result

$$\begin{vmatrix} x & a^2 & a^3 \\ x & b^2 & b^3 \\ x & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$$

$$x\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} + abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$$x(a-b)(b-c)(c-a)(ab+bc+ca) + abc(a-b)(b-c)(c-a) = 0$$

$$x = \frac{-abc}{(a-b)(b-c)(c-a)} \quad (\because a \neq b \neq c)$$

Excerise - I

1. If $a_{ij} = \frac{1}{2}(3i-2j)$ **and** $A = \left[a_{ij}\right]_{2\times 2}$**, then A is equal to**

1) $\begin{bmatrix} 1/2 & 2 \\ -1/2 & 1 \end{bmatrix}$ 2) $\begin{bmatrix} 1/2 & -1/2 \\ 2 & 1 \end{bmatrix}$

3) $\begin{bmatrix} 2 & 2 \\ 1/2 & -1/2 \end{bmatrix}$ 4) $\begin{bmatrix} -2 & -2 \\ -1/2 & 1/2 \end{bmatrix}$

2. If $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-z \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$,

$(x+y+z+a) =$

1) -1 2) 0 3) 1 4) 8

3. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **+ 2X =** $\begin{bmatrix} 3 & 5 \\ 5 & 9 \end{bmatrix}$, $\Rightarrow$ **X =**

1) $\begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix}$ 2) $\begin{bmatrix} 1 & 3/2 \\ 1 & 5/2 \end{bmatrix}$

3) $\begin{bmatrix} -2 & -3 \\ 2 & 5 \end{bmatrix}$ 4) $\begin{bmatrix} 1 & 3/2 \\ -1 & -5/2 \end{bmatrix}$

4. If A-2B = $\begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}$ **and 2A-3B =** $\begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}$ **then** B =

1) $\begin{pmatrix} -5 & 7 \\ 5 & 1 \end{pmatrix}$ 2) $\begin{pmatrix} -5 & 7 \\ -5 & -1 \end{pmatrix}$

3) $\begin{pmatrix} -5 & 7 \\ 5 & -1 \end{pmatrix}$ 4) $\begin{pmatrix} -5 & -7 \\ -5 & -1 \end{pmatrix}$

5. If I = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$**, B =** $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ **and**

$C = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ then C=

1) $I\cos\theta + B\sin\theta$ 2) $I\sin\theta + B\cos\theta$

3) $I\cos\theta - B\sin\theta$ 4) $-I\cos\theta + B\sin\theta$

6. If a matrix has 13 elements, then the possible dimensions (orders) of the matrix are

1) 1 × 13 or 13 × 1 2) 1 × 26 or 26 × 1
3) 2 × 13 or 13 × 2 4) 13 × 13

7. If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ **and** $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$ **then**

1) AB, BA exist and equal
2) AB, BA exist and are not equal
3) AB exist and BA does not exist
4) AB does not exist and BA exist

8. If $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ **and** $A^n = 0$, **then the minimum value of 'n' is**

1) 2 2) 3 3) 4 4) 5

9. If A = diagonal (3,3,3) then $A^4 =$

1)12A 2) 81A 3) 684A 4) 27A

10. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$**,then** $A^2 - 4A$ **is equal to**

1) $2I_3$ 2) $3I_3$ 3) $4I_3$ 4) $5I_3$

11. If $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$**, then** $(aI+bE)^3 =$

1) $aI + bE$ 2) $a^3I + b^3E$
3) $a^3I + 3ab^2E$ 4) $a^3I + 3a^2bE$

12. If A = $\begin{bmatrix} o & c & -b \\ -c & o & a \\ b & -a & o \end{bmatrix}$ **and B =** $\begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$

then AB =

1) A 2) B 3) I 4) O

13. If $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$, then $A^n = ..., n \in N$

1) $\begin{bmatrix} 2^n x^n & 2^n x^n \\ 2^n x^n & 2^n x^n \end{bmatrix}$ 2) $\begin{bmatrix} 2^{n-1} x^n & 2^{n-1} x^n \\ 2^{n-1} x^n & 2^{n-1} x^n \end{bmatrix}$

3) $\begin{bmatrix} 2^{n-2}x^n & 2^{n-2}x^n \\ 2^{n-2}x^n & 2^{n-2}x^n \end{bmatrix}$ 4) $\begin{bmatrix} 2^{n-1}x^{n-1} & 2^{n-1}x^{n-1} \\ 2^{n-1}x^{n-1} & 2^{n-1}x^{n-1} \end{bmatrix}$

14. If $A=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ **and** $I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$**, then which one of the following holds for all** $n\geq 1$ **(by the principle of mathematical induction)**

1) $A^n = nA+(n-1)I$

2) $A^n = 2^{n-1}A+(n-1)I$

3) $A^n = nA-(n-1)I$

4) $A^n = 2^{n-1}A-(n-1)I$

15. If Tr(A) = 6 $\Rightarrow$ **Tr (4A)= [EAM-2019]**

1) 3/2 2) 2 3) 12 4) 24

16. If Tr (A)=3, Tr(B)=5 then Tr(AB) =

1)15 2)5 3)3/5 4) Cannot say

17. If $A=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 1 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix}$, $Tr(BA)=...$

1) 40 2) 45 3) 39 4) 5

18. If A is a 3 × 4 matrix and B is matrix such that A^TB **and** BA^T **are both define then order of B is**

1) 3 × 4 2) 4 × 3 3) 3 × 3 4) 4 × 4

19. If 3A + 4B^T=$\begin{pmatrix} 7 & -10 & 17 \\ 0 & 6 & 31 \end{pmatrix}$ **and**

2B-3A^T =$\begin{pmatrix} -1 & 18 \\ 4 & -6 \\ -5 & -7 \end{pmatrix}$ **then B = [EAM-2019]**

1) $\begin{pmatrix} 1 & 3 \\ -1 & 0 \\ -2 & -4 \end{pmatrix}$ 2) $\begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 2 & 4 \end{pmatrix}$

3) $\begin{pmatrix} 1 & 3 \\ -1 & 0 \\ 2 & 4 \end{pmatrix}$ 4) $\begin{pmatrix} 1 & -3 \\ 1 & 0 \\ 2 & 4 \end{pmatrix}$

20. If A=$\begin{bmatrix} 2 & x-3 & x-2 \\ 3 & -2 & -1 \\ 4 & -1 & -5 \end{bmatrix}$ **is a symmetric matrix then** $x=$

1) 0 2)3 3)6 4)8

21. $\begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} = P+Q$**, where P is a symmetric and Q is a skew-symmetric then Q=**

1) $\begin{pmatrix} 0 & \frac{-1}{2} & 2 \\ \frac{1}{2} & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}$ 2) $\begin{pmatrix} 0 & \frac{1}{2} & 1 \\ \frac{-1}{2} & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

3) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$ 4) $\begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{pmatrix}$

22. If A =$\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ **then A is**

1) an idempotent marix
2) nilpotent matrix
3) involutary
4) orthogonal matrix

23. $\begin{vmatrix} 24 & 25 & 26 \\ 25 & 26 & 27 \\ 26 & 27 & 27 \end{vmatrix}$ is equal to **(EAM-2018)**

1) 0 2) -1 3) 1 4) 2

24. $\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} =$ **[EAM -2020]**

1) $a^2b^2c^2$ 2) 4abc 3) $4a^2b^2c^2$ 4) $2a^2b^2c^2$

25. If $\begin{vmatrix} \lambda^2+3\lambda & \lambda-1 & \lambda+3 \\ \lambda+1 & 2-\lambda & \lambda-4 \\ \lambda-3 & \lambda+4 & 3\lambda \end{vmatrix}$

$= p\lambda^4+q\lambda^3+r\lambda^2+s\lambda+t$ **then** $t=$

1) 16 2) 17 3) 18 4) 19

26. If a, b, c are in A.P.then

$\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} =$ **[EAM -2018]**

1)1 2) 0 3) -1 4) 2

27. If $a \neq 6$, b,c **satisfy** $\begin{vmatrix} a & 2b & 2c \\ 3 & b & c \\ 4 & a & b \end{vmatrix}$ **= 0, then**

$abc =$

1) $a+b+c$ 2) 0

3) b^3 4) $ab+b-c$

28. $\begin{vmatrix} \log e & \log e^2 & \log e^3 \\ \log e^2 & \log e^3 & \log e^4 \\ \log e^3 & \log e^4 & \log e^5 \end{vmatrix} =$

1) 0 2) 1 3) 4 loge 4) 5 loge

29. The value of a third order determinant is 11, then the value of the square of the determinant formed by the cofactors will be

1) 11 2) 121 3) 1331 4) 14641

30. In a third order determinant, each element of the first column consists of sum of two terms, each element of the second column consists of sum of three terms and each element of the third column consits of sum of four terms,.Then it can be decomposed into n determinants, where n has value

1) 1 2) 9 3) 16 4) 24

31. $\begin{vmatrix} (b+c)^2 & a^2 & b^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$

$= \lambda abc(a+b+c)^3$ then $\lambda = ...$

1) 0 2) 1 3) 2 4) 3

32. $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} =$

1) $(x+y+z)^3$ 2) $2(x+y+z)^3$

3) $x+y+z$ 4) $(x+y+z)^2$

33. If A,B,C are the angles of triangle ABC, then $\begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix} =$ **[EAM-2019]**

1) 1 2) 0 3) -1 4) $\frac{3\sqrt{3}}{8}$

34. If $\begin{vmatrix} 1+x & 2 & 3 \\ 1 & 2+x & 3 \\ 1 & 2 & 3+x \end{vmatrix} = 0$ **then** $x =$

1) 1 2) -1 3) -6 4) 6

35. If A_{3x3} **and det A = 5 then det (adj A) =**

1) 5 2) 25 3) 125 4) 1/5

36. If A is a square matrix such that

A $adj(A) = diag(k,k,k)$ **then** $|adjA| =$

1) k 2) k^2 3) k^3 4) k^4

37. If A= $\begin{bmatrix} 2 & 2 \\ -3 & 2 \end{bmatrix}$ **B =** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ **then** $(B^{-1}A^{-1})^{-1} =$

1) $\begin{bmatrix} 2 & 2 \\ -3 & 2 \end{bmatrix}$ 2) $\begin{bmatrix} 2 & -2 \\ 2 & 3 \end{bmatrix}$ 3) $\begin{bmatrix} 2 & 2 \\ -2 & 3 \end{bmatrix}$ 4) $\begin{bmatrix} -2 & 2 \\ 2 & 3 \end{bmatrix}$

38. If A is a 3×3 non singular matrix and $|adjA| = |A|^x$, $|adj(adjA)| = |A|^y$, $|A^{-1}| = |A|^z$, **then the values of** x,y,z **in descending order.**

1) x,y,z 2) z,y,x 3) z,x,y 4) y,x,z

39. If A = $\begin{bmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{bmatrix}$ **then** $|Adj\ A| =$

1) 8 2) 16 3) 64 4) 128

40. If A is a square matrix such that A (AdjA) $= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ **then det (AdjA) = (EAM-2017)**

1) 4 2) 16 3) 64 4) 256

41. If det $(A_{3\times3}) = 6$**, then det (adj 2A)=**

1) 144 2) $2^2 \times 3^8$ 3) $3^3 \times 2^4$ 4) $3^2 \times 2^8$

42. The inverse of

$\begin{bmatrix}1 & a & b\\0 & x & 0\\0 & 0 & 1\end{bmatrix} is \begin{bmatrix}1 & -a & -b\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} then\ x =$

1) a 2) b 3) 0 4) 1

43. If $|A| \neq 0$ **and** $(A-2I)(A-3I)=0$ **then** $A^{-1} =$

1) $\frac{A-5I}{6}$ 2) $\frac{5I-A}{5}$ 3) $\frac{5A-I}{6}$ 4) $\frac{5I-A}{6}$

44. The rank of the matrix A = $\begin{bmatrix}-1 & 2 & 3\\-2 & 4 & 6\end{bmatrix}$ **is**

1) 3 2) 2 3) 1 4) 0

45. If I **is a (9 × 9) unit matrix, then rank (** I **) =**

1) 0 2) 3 3) 6 4) 9

46. The ranks of the matrices in descending order

A. $\begin{bmatrix}1 & 4 & -1\\2 & 3 & 0\\0 & 1 & 2\end{bmatrix}$ **B.** $\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$ **C.** $\begin{bmatrix}1 & 2 & 3\\2 & 3 & 4\\0 & 1 & 2\end{bmatrix}$

1) A,B,C 2) A,C,B 3) B,C,A 4) C,A,B

47. The system of equations

$2x+6y+11=0$, $6y-18z+1=0$

$6x+20y-6z+3=0$

1) is consistent 2) has unique solution
3) is inconsistent 4) cannot be determined

48. The value of 'a' for which the equations $3x-y+az=1$, $2x+y+z=2$, $x+2y-az=-1$ **fail to have unique solution is**

1) 7/2 2) -7/2 3) 2/7 4) -2/7

49. The system of equations $3x+2y+z=6$, $3x+4y+3z=14$ **and** $6x+10y+8z=a$**, has infinite number of solutions, if** a **is equal to (EAM-2020)**

1) 8 2) 12 3) 24 4) 36

50. If x, y, z **not all zeros and the equations** $x+y+z=0$, $(1+a)x+(2+a)y-8z=0$, $x-(1+a)y+(2+a)z=0$ **have non-trivial solution then a =**

1) $2+\sqrt{15}$ 2) $3\pm\sqrt{15}$
3) $\sqrt{15}$ 4) $-5\pm 2\sqrt{2}$

51. If $a \neq b \neq c \neq 1$ **and the system** $ax+y+z=0$, $x+by+z=0$, $x+y+cz=0$ **have non trivial solutions then** $a+b+c-abc=$ **...............**

1) 0 2) 1 3) 2 4) 4

KEY

01) 2	02) 3	03) 2	04) 2	05) 1	06) 1
07) 2	08) 1	09) 4	10) 4	11) 4	12) 4
13) 2	14) 3	15) 4	16) 4	17) 1	18) 1
19) 3	20) 3	21) 1	22) 1	23) 3	24) 3
25) 3	26) 2	27) 3	28) 1	29) 4	30) 4
31) 3	32) 2	33) 2	34) 3	35) 2	36) 2
37) 2	38) 4	39) 3	40) 2	41) 4	42) 4
43) 4	44) 3	45) 4	46) 2	47) 3	48) 2
49) 4	50) 4	51) 3			

SOLUTIONS

1. $a_{ij} = \frac{1}{2}(3i-2j) \Rightarrow a_{11} = \frac{1}{2}, a_{12} = \frac{-1}{2}$

and $a_{21} = 2, a_{22} = 1$

$\therefore A = [a_{ij}]_{2\times 2} = \begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix}$

$\therefore A = \begin{bmatrix}1/2 & -1/2\\2 & 1\end{bmatrix}$

2. Equating corresponding elements

3. $2X = \begin{bmatrix}2 & 3\\2 & 5\end{bmatrix}$

4. $2A-4B = \begin{bmatrix}2 & -4\\6 & 0\end{bmatrix}$

$2A-3B = \begin{bmatrix}-3 & 3\\1 & -1\end{bmatrix}$ subtract and find B.

5. $C = \begin{pmatrix}\cos\theta & 0\\0 & \cos\theta\end{pmatrix} + \begin{pmatrix}0 & \sin\theta\\-\sin\theta & 1\end{pmatrix}$

$= I\cos\theta + B\sin\theta$

6. possible order of the matrices are 1×13 or 13×1

7. AB, BA exists and not equal

8. $A^2 = \begin{bmatrix} a^2b^2 - a^2b^2 & ab^3 - ab^3 \\ -a^3b + a^3b & -a^2b^2 + a^2b^2 \end{bmatrix} = 0$

$\Rightarrow A^3 = A.A^2 = 0$ and $A^n = 0$, for all $n \geq 2$

9. $A^4 = 81I = 27A$

10. $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, A^2 = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$

$\therefore A^2 - 4A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = 5I_3$

11. find aI, bE
12. find AB
13. put n = 3
14. $A^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \therefore A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$

$nA = \begin{bmatrix} n & 0 \\ n & n \end{bmatrix}, (n-1)I = \begin{bmatrix} n-1 & 0 \\ 0 & n-1 \end{bmatrix}$

$nA - (n-1)I = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} = A^n$

15. $Tr(4A) = 4Tr(A)$
16. $Tr(AB) \neq Tr(A).Tr(B)$
17. Find BA
18. Verification
19. find $\left(3A + 4B^T\right)^T$ and adding
20. $A^T = A$
21. $Q = \dfrac{A - A^T}{2}$
22. $A^2 = A$
23. $\begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \end{matrix}$ we get $\begin{vmatrix} 24 & 25 & 26 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1$
24. put a = 1, b = 1, c = 2 verify option
25. put $\lambda = 0$
26. Put a=1, b=2, c=3
27. expand
28. $\begin{vmatrix} \log e & 2\log e & 3\log e \\ 3\log e & 3\log e & 4\log e \\ 3\log e & 4\log e & 5\log e \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} (\log e)^3$

$R_2 \to R_2 - R_1, R_3 \to R_3 - R_2$

29. It is square of

$\det(adj\ A) = \left((\det A)^{n-1}\right)^2 = \left(11^2\right)^2 = 11^4$

= 14641

30. 2 x 3 x 4 = 24
31. put $a = b = c = 1$ find λ
32. $x = y = z = 1$
33. put $A = B = C = 60^0$
34. expand
35. $|adjA| = |A|^{n-1} = 5^2 = 25$
36. $A(AdjA) = |A| I$
37. $\left(B^{-1}A^{-1}\right)^{-1} = AB$
38. $x = n-1, y = (n-1)^2, z = -1$
39. $|AdjA| = |A|^{n-1}$
40. $A(adjA) = |A|.I$
41. $|AdjkA| = |kA|^{n-1} = (k^n |A|)^{n-1}$
42. vertify with $AA^{-1} = I$
43. $A^2 - 5A + 6I = 0$

$5A = A^2 + 6I \because A^{-1} = \dfrac{5I - A}{6}$

44. All 2×2 sub matrices are singular
45. Rank of I is its order
46. Rank of the matrixof order 3×3
47. Rank $(A) \neq Rank(AD)$
48. Rank $(A) = Rank(AD) = 3$
49. Rank $(A) = Rank(AD) < 3$

50. $\begin{vmatrix} 1 & 1 & 1 \\ 1+a & 2+a & -8 \\ 1 & -(1+a) & (2+a) \end{vmatrix} = 0$

51. $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$

EXERCISE - II

1. $A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$ **and** $A^2 = \lambda I$ $then\, \lambda =$

1) 0 2) 1 3) 1/2 4) -2

2. $A = \begin{bmatrix} 0 & 0 & x \\ 0 & x & 0 \\ x & 0 & 0 \end{bmatrix}, A^{100} =$ **[EAM -2017]**

1) $\begin{bmatrix} 0 & 0 & x^{100} \\ 0 & x^{100} & 0 \\ x^{100} & 0 & 0 \end{bmatrix}$ 2) $\begin{bmatrix} x^{100} & 0 & 0 \\ 0 & x^{100} & 0 \\ 0 & 0 & x^{100} \end{bmatrix}$

3) $\begin{bmatrix} o & x^{100} & o \\ o & o & x^{100} \\ x^{100} & o & o \end{bmatrix}$ 4) $\begin{bmatrix} o & x^{100} & o \\ x^{100} & o & o \\ o & o & x^{100} \end{bmatrix}$

3. If A= $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ **then the value of**

$A + A^2 + A^3 + ... + A^n =$

1) A 2) nA 3) $(n+1)A$ 4) 0

4. The number of 2×2 **matrices that can be formed by using 1,2,3,4 when repetitions are allowed is**

1) 24 2) 12 3) 6 4) 256

5. A square matrix A is said to be nilpotent of index m. If $A^m = 0$ **now, if for this A,** $(I - A)^n = I + A + A^2 + A^3 + ... + A^{m-1}$**, then n is equal to**

1) 0 2) m 3) -m 4) -1

6. If $A_K = \begin{bmatrix} k & k-1 \\ k-1 & k \end{bmatrix}$ **then**

$|A_1| + |A_2| + ... + |A_{2015}| =$

1) 0 2) 2015 3) $(2015)^2$ 4) $(2015)^3$

7. Matrix A is given by $A = \begin{bmatrix} 6 & 11 \\ 2 & 4 \end{bmatrix}$ **then the determinant of** $A^{2015} - 6A^{2014}$ **is**

1) 2^{2016} 2) $(-11)2^{2015}$

3) $-2^{2015} \times 7$ 4) $(-9)2^{2014}$

8. If $\begin{vmatrix} x^n & x^{n+2} & x^{n+3} \\ y^n & y^{n+2} & y^{n+3} \\ z^n & z^{n+2} & z^{n+3} \end{vmatrix}$

$= (x-y)(y-z)(z-x)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$ **then value of n is**

1) -1 2) -2 3) 1 4) 2

9. If x, y, z **are integers in A.P lying between 1 and 9 and** $x15, y41$ **and** $z31$ **are three digit numbers, then the value of**

$\begin{vmatrix} 5 & 4 & 3 \\ x51 & y41 & z31 \\ x & y & z \end{vmatrix}$ **is:**

1) $x + y + z$ 2) $x - y + z$

3) 0 4) $x + 2y + z$

10. If the entries in a 3×3 **determinant are either 0 or 1, then the greatest value of their determinants is**

1) 1 2) 2 3) 3 4) 9

11. **If a,b,c are the p^th^, q^th^, r^th^ terms in H.P. then**

$$\begin{vmatrix} bc & p & 1 \\ ca & q & 1 \\ ab & r & 1 \end{vmatrix} =$$

1) 1 2) 0 3) abc 4) $a^2+b^2+c^2$

12. **If [.] denotes the greatest integer less than or equal to the real number under consideration, and $-1 \le x < 0$; $0 \le y < 1$; $1 \le z < 2$, then the value of the determinant** $\begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix}$ **is**

1) $[x]$ 2) $[y]$ 3) $[z]$ 4) $[x]+[y]+[z]$

13. **If** $D_k = \begin{vmatrix} 1 & n & n \\ 2k & n^2+n+2 & n^2+n \\ 2k-1 & n^2 & n^2+n+2 \end{vmatrix}$ **and** $\sum_{k=1}^{n} D_k = 48$, **then 'n' equals**

1) 4 2) 6 3) 8 4) 10

14. **If** $A = \begin{vmatrix} 1+\cos\alpha & 1+\sin\alpha & 1 \\ 1+\cos\beta & 1+\sin\beta & 1 \\ 1 & 1 & 1 \end{vmatrix} \ne 0,$ **then**

1) $\alpha = \beta$
2) $\alpha \ne \beta + n\pi$, n being any integer
3) $\alpha \ne \beta + \pi/2$ 4) $\alpha \ne \beta - \pi/2$

15. **The value of the determinant**

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$ **is**

1) 0 2) $\sin\theta$
3) $\cos\theta$ 4) $\sin\theta+\cos\theta$

16. **If** $f(\theta) = \begin{vmatrix} \cos\theta & 1 & 0 \\ 1 & 2\cos\theta & 1 \\ 0 & 1 & 2\cos\theta \end{vmatrix}$ **then range of $f(\theta)$ is**

1) $[0,1]$ 2) $[-1,0]$ 3) $[-1,1]$ 4) $\left[0,\frac{1}{2}\right]$

17. **If A is orthogonal matrix of order 3 then** $\det(adj 2A) =$

1) 4 2) 16 3) 27 4) 64

18. **A is an involutary matrix given by** $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ **then the inverse of $\frac{A}{2}$ will be**

1) 2A 2) $\frac{A^{-1}}{2}$ 3) $\frac{A}{2}$ 4) A^2

19. **If the product of the matrix** $B = \begin{bmatrix} 2 & 6 & 4 \\ 1 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix}$ **with a matrix A has inverse** $C = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 3 \\ 2 & 0 & 2 \end{bmatrix}$ **then** $A^{-1} =$

1) $\begin{bmatrix} -3 & -5 & 5 \\ 0 & 9 & 14 \\ 2 & 2 & 6 \end{bmatrix}$ 2) $\begin{bmatrix} -3 & 5 & 5 \\ 0 & 0 & 9 \\ 2 & 14 & 16 \end{bmatrix}$

3) $\begin{bmatrix} -3 & -5 & -5 \\ 0 & 9 & 2 \\ 2 & 14 & 6 \end{bmatrix}$ 4) $\begin{bmatrix} -3 & -3 & -5 \\ 0 & 9 & 2 \\ 2 & 14 & 16 \end{bmatrix}$

20. If $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix}=0$ **and** α **is not a root of** $ax^2+2bx+c=0$ **,then**

1) a,b,c are in A.P 2) a,b,c are in G.P

3) a,b,c are in H.P 4) a,b,c are in A.G..P

21. If $f(x)=\begin{vmatrix} 2\cos x & 1 & 0 \\ x-\dfrac{\pi}{2} & 2\cos x & 1 \\ 0 & 1 & 2\cos x \end{vmatrix}$ **, then** $f^1(\pi)=$

1) 0 2) 2 3) $\dfrac{\pi}{2}$ 4) $\pi-6$

22. Let $\Delta(x)=\begin{vmatrix} \cos^2 x & \cos x\sin x & -\sin x \\ \cos x\sin x & \sin^2 x & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$

then $\int\limits_0^{\pi/2}\left[\Delta(x)+\Delta^1(x)\right]dx$ **equals**

1) $\pi/3$ 2) $\pi/2$ 3) 2π 4) $3\pi/2$

23. If $A=\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$ **then** $\det\left(adj\left(adjA\right)\right)$ **is**

1) $(14)^4$ 2) $(14)^3$ 3) $(14)^2$ 4) $(14)^1$

24. If $A=\begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$**,then** $[A(adjA)A^{-1}]A=$

1) $2\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ 2) $\begin{bmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$

3) $\begin{bmatrix} 0 & \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$ 4) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

25. If the trivial solution is the only solution of the system of equations $x-ky+z=0$ **,** $kx+3y-kz=0$**,** $3x+y-z=0$**. Then the set of all values of k is**

1) $\{2,-3\}$ 2) $R-\{2\}$

3) $R-\{2,-3\}$ 4) $R-\{-3\}$

26. If the system of equations $ax+y+z=0$**,** $x+by+z=0$**,** $x+y+cz=0(a,b,c\neq 1)$ **has a non trivial solution then**

$\dfrac{1}{1-a}+\dfrac{1}{1-b}+\dfrac{1}{1-c}=$ **[EAM-2019]**

1)1 2)1- 3)2 4)-2

27. The system of equations $x+y+z=6$ **,** $x+2y+3z=10$**,** $x+2y+\lambda z=k$ **is inconsistent if** $\lambda=.....,\ k\neq..........$ **[EAM - 2018]**

1) 3,7 2) 3,10 3) 7,10 4) 10,3

28. Let λ **and** α **be real. The set of all values of** λ **for which the system of linear equations**

$\lambda x+(\sin\alpha)y+(\cos\alpha)z=0,$

$x+(\cos\alpha)y+(\sin\alpha)z=0$ **,**

$-x+(\sin\alpha)y-(\cos\alpha)z=0$

has a non trival solution is [EAM-2020]

1) $\left[0,\sqrt{2}\right]$ 2) $\left[-\sqrt{2},0\right]$ 3) $\left[-\sqrt{2},\sqrt{2}\right]$ 4) $\left[1,\sqrt{2}\right]$

KEY

01) 1	02) 2	03) 2	04) 4	05) 4	06) 3
07) 2	08) 1	09) 3	10) 2	11) 2	12) 3
13) 1	14) 2	15) 1	16) 3	17) 4	18) 1
19) 3	20) 2	21) 2	22) 2	23) 1	24) 1

25) 3 26) 1 27) 2 28) 3

SOLUTIONS

1. $A^2 = \lambda I$ equating 1st row x 1st column eleme nts on both sides

2. $A = \begin{bmatrix} 0 & 0 & x \\ 0 & x & 0 \\ x & 0 & 0 \end{bmatrix} \Rightarrow A^{100} = \begin{bmatrix} x^{100} & 0 & 0 \\ 0 & x^{100} & 0 \\ 0 & 0 & x^{100} \end{bmatrix}$

3. $A^2 = A^3 = A^4 = ... = A$

4. $\begin{bmatrix} \underline{4} & \underline{4} \\ \underline{4} & \underline{4} \end{bmatrix}$

5. Let $B = I + A + A^2 + ...A^{m-1}$

$\Rightarrow B(I-A) = \left(I + A + A^2 + ... + A^{m-1}\right)$

$(I-A) = 1 - A^m = I$

$\Rightarrow B = (I-A)^{-1} \Rightarrow n = -1$

6. $|A_k| = (2K-1)$

7. $\left|A^{2015} - 6A^{2014}\right| = |A|^{2014}|A - 6I| =$

$2^{2014}\begin{vmatrix} 0 & 11 \\ 2 & -2 \end{vmatrix} = (-22)2^{2014} = -11(2)^{2015}$

8. Order of determinant is

$n + n + 2 + n + 3 = 3n + 5$

Order of R.H.S = 1+1+1-1=2

$\Rightarrow 3n + 5 = 2 \Rightarrow 3n = -3 \Rightarrow n = -1$.

9. Put x = 5, y = 4, z = 3

10. Keep least of given values in principal diagonal, highest of given values in other places.

$\Rightarrow \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2$

11. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P, $\frac{1}{a} = A + (P-1)D$

$\frac{1}{b} = A + (q-1)D$, $\frac{1}{c} = A + (r-1)D$

$\begin{vmatrix} bc & p & 1 \\ ca & q & 1 \\ ab & r & 1 \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} \frac{1}{a} & p & 1 \\ \frac{1}{b} & q & 1 \\ \frac{1}{c} & r & 1 \end{vmatrix} = 0$

12. [x] = - 1, [y] = 0 , [z] = 1

Det = $\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{vmatrix} = 1 = [z]$

13. $\sum_{k=1}^{n} 1 = n$, $\sum_{k=1}^{n} 2k = n(n+1)$, $\sum_{k=1}^{n}(2k-1) = n^2$

Exp =

$\begin{vmatrix} n & n & n \\ n(n+1) & n^2+n+2 & n^2+n \\ n^2 & n^2 & n^2+n+2 \end{vmatrix} = 48$

By $C_2 \to C_2 - C_1$

$C_3 \to C_3 - C_1$, $n(2n+4) = 48 \Rightarrow n = 4$

14. Expanding the det, we get

$\sin(\alpha - \beta) \neq 0$

$\alpha \neq n\pi + \beta; n \in Z$

15. Applying $R_2 \to R_2 + R_3$

$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ -\sin\theta & -\cos\theta & -\sin 2\theta \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix} = 0$

16.. $f(\theta) = \cos 3\theta \Rightarrow$ range of f is $[-1,1]$

17 $|adj(2A)| = 4^3|A|^{n-1} = 64(1) = 64$

18. $A^2 = I \Rightarrow A = A^{-1}$ also $(KA)^{-1} = \frac{1}{K}A^{-1}$

Hence $\left(\frac{1}{2}A\right)^{-1} = 2A^{-1} = 2A$

19. $BA = C^{-1}$, $A = B^{-1}C^{-1}$

$$A^{-1} = CB = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 3 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 6 & 4 \\ 1 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

20. $C_3 \to C_3 - (C_1\alpha + C_2)$

21. Expand

22. Applying $C_1 \to C_1 - (Sinx)C_3$

$C_2 \to C_2 + (Cosx)C_3$

$$\Delta(x) = \begin{vmatrix} 1 & 0 & -Sinx \\ 0 & 1 & Cosx \\ Sinx & -Cosx & 0 \end{vmatrix}$$

applying $R_3 \to R_3 - SinxR_1 + CosxR_2$

$$\Delta(x) = \begin{vmatrix} 1 & 0 & -Sinx \\ 0 & 1 & Cosx \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$2A + 4B + 7C = 24 - 52 + 28 = 0$

23. We know that

$adj(adjA) = |A|^{n-2} A$, if $|A| \neq 0$; provided order of A is n

$\therefore adj(adjA) = |A| A (as\, n = 3)$

$\therefore \det(adj(adjA)) = |A|^3 \det A = |A|^4$

But $|A| = \begin{vmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{vmatrix} = 14$

$\therefore \det(adj(adjA)) = (14)^4$

24. A(adjA)=|A|I

25. $\Rightarrow \begin{vmatrix} 1 & -k & 1 \\ k & 3 & -3 \\ 3 & 1 & -1 \end{vmatrix} \neq 0 \Rightarrow k^2 + k - 6 \neq 0$

$\Rightarrow (k+3)(k-2) \neq 0 \Rightarrow k \neq \{2, -3\}$

26. det A=0

27.. $[A.B] = \begin{bmatrix} 1 & 1 & 1:6 \\ 1 & 2 & 3:10 \\ 1 & 2 & \lambda:k \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1:6 \\ 0 & 1 & 2:4 \\ 0 & 0 & \lambda-3:k-10 \end{bmatrix}$

$R_2 \to R_2 - R_1, R_3 \to R_3 - R_2$

Given System is in consistent

rank$(A) \neq rank(B)$. Hence $K \neq 10, \lambda = 3$

28. det A=0 $\Rightarrow \lambda = \sin 2\alpha + \cos 2\alpha$

Range of λ is $\left[-\sqrt{2}, \sqrt{2}\right]$

EXERCISE - III

1. **If** $A = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}; B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ **and** $C = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$ **then**

$tr(A) + tr\left(\frac{ABC}{2}\right) + tr\left(\frac{A(BC)^2}{4}\right) + tr\left(\frac{A(BC)^3}{8}\right) + ...\infty =$

1) 6 2) 9 3) 12 4) 15

2. **Matrix** $A = \begin{bmatrix} x & 3 & 2 \\ 1 & y & 4 \\ 2 & 2 & z \end{bmatrix}$, **if** $xyz = 60$ **and** $8x + 4y + 3z = 20$, **then** $A(adjA)$ **is equal to**

1) $\begin{bmatrix} 64 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 64 \end{bmatrix}$ 2) $\begin{bmatrix} 88 & 0 & 0 \\ 0 & 88 & 0 \\ 0 & 0 & 88 \end{bmatrix}$

3) $\begin{bmatrix} 68 & 0 & 0 \\ 0 & 68 & 0 \\ 0 & 0 & 68 \end{bmatrix}$ 4) $\begin{bmatrix} 34 & 0 & 0 \\ 0 & 34 & 0 \\ 0 & 0 & 34 \end{bmatrix}$

3. **If A, B and C are $n\times n$ matrices and $\det(A)=2, \det(B)=3\, and \det(C)=5$, then the value of the $\det\left(A^2BC^{-1}\right)$ is equal to**

1) $\frac{6}{5}$ 2) $\frac{12}{5}$ 3) $\frac{18}{5}$ 4) $\frac{24}{5}$

4. **If** $\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} \frac{1}{2} & 4 \\ \frac{1}{3} & 3 \end{vmatrix} + \begin{vmatrix} \frac{1}{4} & 4 \\ \frac{1}{9} & 3 \end{vmatrix} + ...\infty =$

1) 1 2) -1 3) 0 4) ∞

5. **The maximum and minimumvalues of (3×3) determinant whose elements belong to {0,1,2,3}is**

1) ± 9 2) ± 15 3) ± 54 4) ± 32

6. **If** $A=\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$ **and** $\det A = 6$,

If $B=\begin{bmatrix} p+x & q+y & r+z \\ a+x & b+y & c+z \\ a+p & b+q & c+r \end{bmatrix}$ **,then**

1) $\det B = 6$ 2) $\det B = -6$
3) $\det B = 12$ 4) $\det B = -12$

7. **If** $x^a y^b = e^m, x^c y^d = e^n$, $\Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}$,

and $\Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$ **and** $\Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ **,then the values of x and y are**

1) $\frac{\Delta_1}{\Delta_3}\, and\, \frac{\Delta_2}{\Delta_3}$ 2) $\frac{\Delta_2}{\Delta_1}\, and\, \frac{\Delta_3}{\Delta_1}$

3) $\log\left(\frac{\Delta_1}{\Delta_3}\right) and \log\left(\frac{\Delta_2}{\Delta_3}\right)$ 4) $e^{\Delta_1/\Delta_3}\, and\, e^{\Delta_2/\Delta_3}$

8.Let $\{D_1, D_2, D_3.....D_n\}$ be the set of all third order determinants that can be formed with the distinct non-zero real numbers $a_1, a_2, ..., a_9$ then

1) $\sum_{i=1}^{n} D_i = 1$ 2) $\sum_{i=1}^{n} D_i = 0$

3) $D_i = D_j$ for all i, j 4) None of the above

9. **If** $\Delta(x)=\begin{vmatrix} e^x & \sin 2x & \tan x^2 \\ \ln(1+x) & \cos x & \sin x \\ \cos x^2 & e^x-1 & \sin x^2 \end{vmatrix} = A+Bx+Cx^2+......$

then B is equal to

1) 0 2) 1 3) 2 4) 4

10. **If** $f(x)=\begin{bmatrix} \sin x & \cos ecx & \tan x \\ \sec x & x\sin x & x\tan x \\ x^2-1 & \cos x & x^2+1 \end{bmatrix}$ **then** $\int_{-a}^{a} |f(x)|\,dx$ **equals**

1) 1 2) -1 3) 2a 4) 0

11. **D$_r$ =** $\begin{vmatrix} r-1 & n & 6 \\ (r-1)^2 & 2n^2 & 4n-2 \\ (r-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix} \Rightarrow \sum_{r=1}^{n}$ **D$_r$ =**

1) nr 2) 0 3) $\frac{n(n-1)}{2}-r^2$ 4) 2n-n^2

12. **If** $P=\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ **and**

$Q=PAP^T$ **and** $X=P^TQ^{2015}P$, **then** X **is**

1) $\begin{bmatrix} 1 & 2015 \\ 0 & 1 \end{bmatrix}$ 2) $\begin{bmatrix} 1 & 0 \\ 2015 & 1 \end{bmatrix}$

3) $\begin{bmatrix} 2015 & 1 \\ 0 & 1 \end{bmatrix}$ 4) $\begin{bmatrix} 1 & 1 \\ 0 & 2015 \end{bmatrix}$

13. **If** $A=\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

then $A^{-1}+(A-5I)(A-I)^2 =$

1) $\frac{1}{5}\begin{bmatrix} 4 & 2 & -1 \\ -1 & 3 & 1 \\ -1 & 2 & 4 \end{bmatrix}$ 2) $\frac{1}{5}\begin{bmatrix} 4 & -2 & -1 \\ -1 & 3 & -1 \\ -1 & -2 & 4 \end{bmatrix}$

3) $\frac{1}{3}\begin{bmatrix} 4 & 2 & -1 \\ -1 & 3 & 1 \\ -1 & 2 & 4 \end{bmatrix}$ 4) $\frac{1}{3}\begin{bmatrix} 4 & -2 & -1 \\ -1 & 3 & -1 \\ -1 & -2 & 4 \end{bmatrix}$

14. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ **if** u_1 *and* u_2 **are column matrices such that** $Au_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ **and** $Au_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ **then** $u_1 + u_2$ **equal to** **[AIE-2012]**

1) $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ 2) $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ 3) $\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$ 4) $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

15. If a, b, c **are non zero real numbers and if the equations** $(a-1)x = y + z$, $(b-1)y = z + x, (c-1)z = x + y$ **has a non trivial solution then** $ab + bc + ca$ **equals**

1) a+b+c 2) abc 3)1 4) a+b-c

KEY

01) 1 02) 3 03) 2 04) 3 05) 3 06) 3
07) 4 08) 2 09) 1 10) 4 11) 2
12) 1 13) 2 14) 4 15) 2

SOLUTIONS

1. $BC = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} \Rightarrow BC = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$tr(A) + tr\left(\frac{A}{2}\right) + tr\left(\frac{A}{2^2}\right) + \ldots$

$= tr(A) + \frac{1}{2}tr(A) + \frac{1}{2^2}tr(A) + \ldots.$

$= \frac{tr(A)}{1-(1/2)} = 2tr(A) = 2(2+1) = 6$

2. $A.adjA = |A|I$

$|A| = xyz - 8x - 3(z-8) + 2(2-2y)$

$|A| = xyz - (8x + 3z + 4y) + 28$

$\Rightarrow 60 - 20 + 28 = 68$

3. $|A| = 2; |B| = 3; |C| = 5$

$\det\left(A^2BC^{-1}\right) = \left|A^2BC^{-1}\right| = \frac{|A|^2|B|}{|C|} = \frac{4.3}{5} = \frac{12}{5}$

4. $\begin{vmatrix} 1+\frac{1}{2}+\frac{1}{4}+\ldots.\infty & 4 \\ 1+\frac{1}{3}+\frac{1}{9}+\ldots.\infty & 3 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ \frac{3}{2} & 3 \end{vmatrix} = 0$

5. Keep least of given values in principal diagonal, highest of given values in other places.

$\Rightarrow \begin{vmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{vmatrix} = 54$

6. Consider the $\det B$, using $R_1 \to R_1 + R_2 + R_3$

$B = 2\begin{vmatrix} a+p+x & b+q+y & c+r+z \\ a+x & b+y & c+z \\ a+p & b+q & c+r \end{vmatrix}$

Using $R_2 \to R_2 - R_1$ *and* $R_3 \to R_3 - R_1$

$= \begin{vmatrix} a+p+x & b+q+y & c+r+z \\ -p & -q & -r \\ -x & -y & -z \end{vmatrix}$

using $R_2 \to R_1 + R_2 + R_3$

$B = 2\det A = 2.6 = 12$

7.. Given that, $x^a y^b = e^m$ and $x^c y^d = e^n$

$\Rightarrow a\log x + b\log y = m$, and

$c\log x + d\log y = n$

Using cramer's rule, we have

$$\log x = \frac{\begin{vmatrix} m & b \\ n & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \text{ and } \log y = \frac{\begin{vmatrix} a & m \\ c & n \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$\Rightarrow \log x = \frac{\Delta_1}{\Delta_3} \, and \, \log y = \frac{\Delta_2}{\Delta_3}$$

$$\Rightarrow x = e^{\Delta_1/\Delta_3} \, and \, y = e^{\Delta_2/\Delta_3}$$

8..Total no. of third order determinants with distinct non-zero real numbers $a_1, a_2 .... a_9$ as elements is 9!. These determinants can be grouped into two groups each containing $\frac{9!}{2}$ determinansts such that correpsonding to each determinant in a group there is another determinant in the other group which is obtained by interchanging two consecutive rows of the determinant in the first group.

$\therefore$ Sum of the values of the determinants is 0.

9. Differentiating w.r.t 'x' on both sides $\Rightarrow$

$B + 2Cx + ... = || + || + ||$

Put x = 0 $\Rightarrow$

$$B = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix};$$

B = 0.

10. $f(x) = \begin{bmatrix} \sin x & \operatorname{cosec} x & \tan x \\ \sec x & x\sin x & x\tan x \\ x^2-1 & \cos x & x^2+1 \end{bmatrix}$ then

$$|f(x)| = \begin{vmatrix} \sin x & \operatorname{cosec} x & \tan x \\ \sec x & x\sin x & x\tan x \\ x^2-1 & \cos x & x^2+1 \end{vmatrix} = g(x)$$

assume $g(-x) = \begin{vmatrix} -\sin x & -\operatorname{cosec} x & -\tan x \\ \sec x & x\sin x & x\tan x \\ x^2-1 & \cos x & x^2+1 \end{vmatrix}$

$= -g(x),\ g(-x) = -g(x)$

$g(x) = |f(x)|$ is an odd function

$\therefore \int_{-a}^{a} |f(x)|\,dx = 0$.

11. $$\sum_{r=1}^{n} Dr = \begin{vmatrix} \frac{n(n-1)}{2} & n & 6 \\ \frac{n(n-1)(2n-1)}{6} & 2n^2 & 4n-2 \\ \frac{n^2(n-1)^2}{4} & 3n^3 & 3n^2-3n \end{vmatrix}$$

taking common $\frac{n(n-1)}{2}$ from c_1

then c_1 and c_3 are proportional $\therefore \sum_{r=1}^{n} Dr = 0$

12. Here $P^T.P = I \Rightarrow P^T = P^{-1}$

$Q^2 = PA^2P^T \Rightarrow Q^{2015} = PA^{2015}P^T$

$\therefore X = A^{2015}$, $X = \begin{bmatrix} 1 & 2015 \\ 0 & 1 \end{bmatrix}$

13. $|A - \lambda I| = 0$, i.e $(\lambda-5)(\lambda-1)^2 = 0$

$(A-5I)(A-I)^2 = 0$, $A^{-1} = \frac{adj\ A}{\det A}$

14. $A(u_1+u_2) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; |A| = 1$, But $A^{-1} = \frac{1}{|A|}(adjA)$

$$u_1 + u_2 = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

15. Rewritten the given equations are

$(a-1)x - y - z = 0$

$x - (b-1)y + z = 0, \qquad x + y - (c-1)z = 0$

has a non trivial solution then

$$\Delta = \begin{vmatrix} a-1 & -1 & -1 \\ 1 & -(b-1) & 1 \\ 1 & 1 & -(c-1) \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & 0 & -1 \\ b & -b & 1 \\ 0 & c & 1-c \end{vmatrix} \begin{matrix} C_1 \to C_1 - C_2 \\ C_2 \to C_2 - C_3 \end{matrix} = 0$$

$$\Rightarrow a(-b+bc-c)-0-1(bc)=0$$

$$\Rightarrow -b+abc-ca-bc=0$$

$$\Rightarrow ab+bc+ca=abc$$

JEE MAINS QUESTIONS

1. Let A = $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ and B = A^{20}. Then the sum of the elements of the first colum of B is [2018]

1) 210 2) 211 3) 231 4) 251

2 . If A = $\begin{pmatrix} \cos\theta - \sin\theta \\ \sin\theta - \cos\theta \end{pmatrix}$ then the matrix a^{-50} when $\theta = \dfrac{\pi}{12}$ [2019]

1) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ 2) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

3) $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ 4) $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

3. Let α be a root of the equation $x^2+x+1=0$ and t he matrix $A = \dfrac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha^4 \end{bmatrix}$ then the matrix $A^{31} =$ [2020]

1) A 2) I_3 3) A^2 4) A^3

4. Let $A=\begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ $x \in R$ and $A^4=\left[a_{ij}\right]$ if $a_{11}=109$ then $a_{22}=$ [2020]

5. If $A=\begin{pmatrix} 2 & 2 \\ 9 & 4 \end{pmatrix}$ and $I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $10A^{-1}=$ [2020]

1) $A-AI$ 2) $6I-A$

3) $A-6I$ 4) $4I-A$

6. The number of all 3×3 matrices A, with enteries from the set $\{-1,0,1\}$ such the sum of the diagonal elements of $A\,A^T$ is 3 is

________________________ [2020]

7. Let A and B be two inventible matrices of order 3×3 If $\det\left(ABA^T\right)$=8. $\det\left(AB^{-1}\right)=8$ then $\det\left(BA^{-1}B^T\right)=$ [2020]

1) 1/4 2) 16 3) 1/16 4) 1

8. Let $P=\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 3 & 1 \end{pmatrix}$ and $Q=\left(q_{ij}\right)$ two 3×3 matrices such that $Q-P^5=I_3$ then $\dfrac{q_{21}+q_{31}}{q_{32}}=$ [2019]

1)15 2)9 3)135 4)10

9. If A $=\begin{pmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{pmatrix}$ then for all $\theta\in\left(\dfrac{3\pi}{4},\dfrac{5\pi}{4}\right)$ $|A|$ lies in the interval [2019]

1) $\left[\dfrac{3}{2},3\right]$ 2) $\left[0,\dfrac{3}{2}\right]$ 3) $\left[\dfrac{5}{2},4\right]$ 4) $\left[1,\dfrac{5}{2}\right]$

10. Let $d\in R$ and

$$A=\begin{pmatrix} -2 & 4+d & \sin\theta-2 \\ 1 & \sin\theta+2 & d \\ 5 & 2\sin\theta-d & -\sin\theta+2+2d \end{pmatrix} \theta\in[0,2\pi] \text{ If}$$

the minimum value of det(A) is 8 then a value of d is

1) -7 2) $2\left(2+\sqrt{2}\right)$ 3) -5 4) $2\left(2-\sqrt{2}\right)$

11. Suppose A is any 3×3 non-singular martrix and (A-3I)(A-5I)=0 where $I=I_3$ and $0=0_3$ .If $\alpha A+\beta A^{-1}=4I$ then $\alpha+\beta$ is equal to [2018]

1)8 2)7 3)13 4)12

12. If $\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix}=(A+Bx)(x-A)^2$ then the order pair (A,B) = [2020]

1) (4,5) 2) (-4,-5) 3) (-4,3) 4) (-4,5)

13. If the system of linear equations $x+ay+z=3, x+2y+2z=6, x+5y+3z=b$ has no solutions then [2018]

1) $a=-1,\ b=9$ 2) $a=-1, b\neq 9$

3) $a\neq -1, b=9$ 4) $a=1, b\neq 9$

14. If the system of linear equations x+ky+3z=0, 3x+ky-2z=0,x+4y-3z=0 has a non-zero solutions (x,y,z), then $\frac{xz}{y^2}$ is equal to [2020]

1) 30 2) -10 3) 10 4) -30

15. The number of values of K for which the system of linear equations, $(k+2)x+10y=k, kx+(k+3)y=k-1$ has no solution is [2018]

1) 1 2) 2 3)3 4) infinitely many

16. The system of linear equations x+y+z = 2, 2x+3y+2z = 5, $2x+3y+(a^2-1)z=a+1$ [2019]

1) Infinitely many solutions for a = 4
2) a unique solution for $|a|=\sqrt{3}$
3) Is inconsistant when $|a|=\sqrt{3}$
4) Is inconsistant when a = 4

17. If the system of equations $x-4y+7z=g, 3y-5z=h\ -2x+5y-9z=k$ is consistant then [2020]

1) $g+2h+k=0$ 2) $g+h+2k=0$

3) $g+h+k=0$ 4) $2g+h+k=0$

18. If the system of equations $x+y+z=5, x+2y+3z=9, x+3y+\alpha z=\beta$ has infinitety many solutions then $\beta-\alpha=$ [2020]

19. The number of values of $\theta\in(0,\pi)$ for which the system of linear equations x+3y+7z = 0, -x+4y+7z = 0 $(\sin 3\theta)x+(\cos 2\theta)y+2z=0$ has a non trivial solution is [2019]

1) 2 2)1 3)4 4)3

20. The set of all values of λ for which the system of linear equations $x-2y-2z=\lambda x, x+2y+z=\lambda y - x - y=\lambda z$ has non-trivial solutions [2019]

1) Is an empty set
2) contains more than two elements
3) Is a singleton
4) contains exactly two elements

KEY

1. 3	2.3	3.4	4.10	5. 3
6.672	7. 3	8.4	9.1	10.3
11.1	12. 4	13.2	14. 3	15.1
16.3	17.4	18.1	19. 1	20. 3

SOLUTIONS

1. A=$\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}$, now

$$A^2=A.A=\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\2&1&0\\3&2&1\end{bmatrix}$$

$$A^3=A^2.A=\begin{bmatrix}1&0&0\\2&1&0\\3&2&1\end{bmatrix}\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\3&1&0\\6&3&1\end{bmatrix}$$

$$A^4=A^3.A=\begin{bmatrix}1&0&0\\3&1&0\\6&3&1\end{bmatrix}\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\4&1&0\\10&4&1\end{bmatrix}$$

similary $A^{20}=\begin{bmatrix}1&0&0\\20&1&0\\210&20&1\end{bmatrix}$

$\therefore$ sum of elements of first column = 1+20+210=231

2. Given $A=\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}$

$$A^{-1}=\frac{1}{|A|}(adjA)=\frac{1}{1}\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}$$

$$\left(A^{-1}\right)^{50}=\begin{pmatrix}\cos 50\theta&\sin 50\theta\\-\sin 50\theta&\cos 50\theta\end{pmatrix}\ \theta=\frac{\pi}{12}$$

$$\left(A^{50}\right)^{-1}=\begin{pmatrix}\cos 750^0&\sin 750^0\\-\sin 750^0&\cos 750^0\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{3}}{2},&\frac{1}{2}\\\frac{-1}{2}&\frac{\sqrt{3}}{2}\end{pmatrix}$$

3. Given equation $x^2+x+1=0$. then roots w,w^2.

let $\alpha=w,\alpha^4=w^4=w^3.w=w$

$$A^2=A.A=\frac{1}{3}\begin{bmatrix}1&1&1\\1&w&w^2\\1&w^2&w\end{bmatrix}\begin{bmatrix}1&1&1\\1&w&w^2\\1&w^2&w\end{bmatrix}=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

$$A^4=A^2.A^2=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I_3$$

$$A^3=A^{28}.A^3=\left(A^4\right)^7A^3=I.A^3=A^3$$

4. Given $A=\begin{bmatrix}x&1\\1&0\end{bmatrix}$,

$$A^2=A.A=\begin{bmatrix}x&1\\1&0\end{bmatrix}\begin{bmatrix}x&1\\1&0\end{bmatrix}=\begin{bmatrix}x^2+1&x\\x&1\end{bmatrix}$$

$$A^4=A^2.A^2=\begin{bmatrix}x^2+1&x\\x&1\end{bmatrix}\begin{bmatrix}x^2+1&x\\x&1\end{bmatrix}=\begin{bmatrix}\left(x^2+1\right)^2+x^2&x\left(x^2+1\right)+x\\x\left(x^2+1\right)+x&x^2+1\end{bmatrix}$$

Given $\left(x^2+1\right)^2+x^2=109$

$$x^4+3x^2-108=0$$

$$\left(x^2+12\right)\left(x^2-9\right)=0\ \therefore x^2=9$$

$$a_{22}=x^2+1=9+1=10$$

5. Given $A=\begin{pmatrix}2 & 2\\ 9 & 4\end{pmatrix}$ characteristics equation of matrix A is $|A-\lambda I|=0$

$$\begin{vmatrix}2-\lambda & 2\\ 9 & 4-\lambda\end{vmatrix}=0\Rightarrow \lambda^2-6\lambda-10=0$$

$$\begin{aligned}&A^2-6A-10I=0\\ &A^{-1}A^2-6A\,A^{-}=10I\;\;A^{-1}=0\\ &A=6I-10A^{-1}=0\\ &A-6I=10A^{-1}\end{aligned}$$

6. Let $A=\left[a_{ij}\right]_{3\times 3}$ it is given that sum of diagonal elements of AA^T is 3 i.e $t\Omega\left(AA^T\right)=3$

$$a_{11}^2+a_{12}^2+a_{13}^2+a_{21}^2+............+a_{33}^2=3$$

possible cases are

$$\left.\begin{array}{l}0,0,0,0,0,0,1,1,1\quad\to 1\\ 0,0,0,0,0,0,-1,-1,-1\to 1\\ 0,0,0,0,0,0,-1,1,1\quad\to 3\\ 0,0,0,0,0,0,-1,1,-1\quad\to 3\end{array}\right\}{}^9C_6\times 8=84\times 8=672$$

7. Given $\left|AB\,A^T\right|=8\Rightarrow |A||B|\left|A^T\right|=8$

$$\left|A^2\right||B|=8\text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_}\boxed{1}$$

$$\left|AB^{-1}\right|=8$$

$$\Rightarrow |A|\frac{1}{|B|}=8,\Rightarrow |A|=8|B|\qquad |A|^2=64|B|^2$$

sub in $\boxed{1}$

$$64|B|^3=8$$

$$|B|^3=\frac{1}{8}\Rightarrow |B|=\frac{1}{2},|A|=4$$

now $\left|BA^{-1}B^T\right|=|B|\frac{1}{|A|}|B|^T=\frac{|B|^2}{|A|}=\frac{1}{4/4}=1/16$

8. Given

$$P=\begin{bmatrix}1&0&0\\3&1&0\\9&3&1\end{bmatrix},P^2=P.P=\begin{bmatrix}1&0&0\\3&1&0\\9&3&1\end{bmatrix}\begin{bmatrix}1&0&0\\3&1&0\\9&3&1\end{bmatrix}=\begin{bmatrix}1&0&0\\6&1&0\\27&3&1\end{bmatrix}$$

$$P^5=\begin{bmatrix}1&0&0\\15&1&0\\135&15&1\end{bmatrix}$$

$a_{21}=15,\ a_{31}=135\quad a_{32}=15$

Now $\dfrac{q_{21}+q_{31}}{q_{32}}=\dfrac{15+135}{15}=\dfrac{150}{15}=10$

9. Given $A=\begin{pmatrix}1 & \sin\theta & 1\\ -\sin\theta & 1 & \sin\theta\\ -1 & -\sin\theta & 1\end{pmatrix}$

$$|A|=2\left(1+\left[0,\frac{1}{2}\right]\right)=2\left[1,\frac{1}{2}\right]\quad |A|\in[2,3]$$

10. Apply $R_3\to R_3+R_1-2R_2$

$$|A|=\begin{vmatrix}-2 & 4+d & \sin\theta-2\\ 1 & 2+\sin\theta & d\\ 1 & 0 & 0\end{vmatrix}$$

$$|A|=d^2+4d+4-\sin^2\theta$$

$$|A|=(d+2)^2-\sin^2\theta$$

Given min of det(A) = 8 = $\left(d+2\right)^2-1$

$$\left(d+2\right)^2=9$$

$$\left(d+2\right)=\pm 3$$

d = 1(or) d=-5

11. Given (A-3I) (A-5I)=0

$$\Rightarrow A^2-8A+15I=0$$

$$\Rightarrow A-8I+15A^{-1}=0$$

$$\Rightarrow 4I=\frac{1}{2}A+\frac{15}{2}A^{-1}$$

Given $4I=\alpha A+\beta A^{-1}\quad \alpha=\frac{1}{2},\beta=\frac{15}{2}$

Now $\alpha+\beta=\frac{1}{2}+\frac{15}{2}=\frac{16}{2}=8$

12. Apply $C_1\to C_1+C_2+C_3=$

$$\begin{vmatrix}5x-4 & 2x & 2x\\ 5x-4 & x-4 & 2x\\ 5x-4 & 2x & x-4\end{vmatrix}=\left(5x-4\right)\begin{vmatrix}1 & 2x & 2x\\ 1 & x-4 & 2x\\ 1 & 2x & x-4\end{vmatrix}$$

apply $R_2\to R_2-R_1, R_3\to R_3-R_1$

$$=\left(5x-4\right)\begin{vmatrix}1 & 2x & 2x\\ 0 & -\left(x+4\right) & 0\\ 0 & 0 & -\left(x+4\right)\end{vmatrix}=\left(5x-4\right)\left(x+4\right)^2=\left(A+Bx\right)\left(x-A\right)^2$$

$$\Rightarrow A=-4,\ B=5$$

13.

$$\Delta=\begin{vmatrix}1 & 2 & 2\\ 1 & 5 & 3\\ 1 & a & 1\end{vmatrix}=1\left(5-3a\right)-2\left(-2\right)+2\left(a-5\right)=-a-1$$

$$\Delta_1=\begin{vmatrix}6 & 2 & 2\\ 6 & 5 & 3\\ 3 & a & 1\end{vmatrix}=6\left(5-3a\right)-2\left(b-9\right)+2\left(ab-15\right)=2\left(9-b\right)\left(1-a\right)$$

$$\Delta_2=\begin{vmatrix}1 & 6 & 2\\ 1 & 6 & 3\\ 1 & 3 & 1\end{vmatrix}=1\left(b-9\right)-6\left(-2\right)+2\left(3-b\right)=9-b$$

$$\Delta_3=\begin{vmatrix}1 & 2 & 6\\ 1 & 5 & b\\ 1 & a & 3\end{vmatrix}=1\left(15-ab\right)-2\left(3-b\right)+6\left(a-5\right)=15-ab-6+2b+6a-30=6a+2b-ab-21$$

For no.solution $\Delta=0$ and atleast one of $\Delta_1,\Delta_2,\Delta_3$ is non-zero $a=-1\ \ b\neq 9$

14. For non-zero solution $\Delta=0$

$$\Delta=\begin{vmatrix}1 & k & 3\\ 3 & k & -2\\ 2 & 4 & -3\end{vmatrix}=0\Rightarrow 1\left(-3k+8\right)-k\left(-9+4\right)+3\left(12-2k\right)=0$$

$$\Rightarrow -4k+44=0,\ \ k=11$$

$$\Delta=\begin{vmatrix}0 & 3 & -5\\ -2 & 5 & -9\end{vmatrix}=-2-40+42=0$$

hence the equations are $x+11y+3z=0, 3x+11y-2z=0$

and $2x+4y-3z=0$ let $z=t$ then $x=\frac{5}{2}t$

and $y=\frac{-t}{2}$ now $\frac{xz}{y^2}=\frac{\left(\frac{5}{2}t\right)t}{\frac{t^2}{4}}=10$

15. For no solutions $\frac{k+2}{k}=\frac{10}{k+3}\neq\frac{k}{k-1}$,

$(k+2)(k+3)=10k$

$k^2-5k+6=0$

k = 2,3 so $k\neq 2$ become for k = 2 both lines are identical so k = 3 only solution.

16.

$$\Delta\begin{vmatrix}1 & 1 & 1\\ 2 & 3 & 2\\ 2 & 3 & a^2-1\end{vmatrix}=1\left(\left(3a^2-3\right)-6\right)-1\left(2a^2-2-4\right)+1(0)$$

$=a^2-3$

for inconsistant $\Delta=0\Rightarrow|a|=\sqrt{3}$

17 .

$$\Delta=\begin{vmatrix}1 & -4 & 7\\ 0 & 3 & -5\\ -2 & 5 & -9\end{vmatrix}=1(-27+25)+4(0-10)+7(0+6)$$

$=-2-40+42=0$

If system is consistant then $\Delta_1=\Delta_2=\Delta_3=0$

$$\Rightarrow\begin{vmatrix}1 & -4 & g\\ 0 & 3 & h\\ -2 & 5 & k\end{vmatrix}=0\Rightarrow 1$$

$\Rightarrow 1(3k-5h)+4(0+2h)+g(0+6)=0$

$\Rightarrow 3k+6g+3h=0\Rightarrow 2g+h+k=0$

18.

$$\Delta=\begin{vmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 3 & \alpha\end{vmatrix}=1(2\alpha-9)-1(\alpha-3)+1(3-2)=0$$

$\alpha-5=0\Rightarrow\alpha=5$

$$\Delta=\begin{vmatrix}1 & 1 & 5\\ 1 & 2 & 9\\ 1 & 3 & \beta\end{vmatrix}=0\Rightarrow 1(2\beta-27)-1(\beta-9)+5(3-2)=0$$

$\beta-13=0$

$\beta=13$ now $\beta-\alpha=13-5=8$

19. $$\Rightarrow\begin{vmatrix}1 & 3 & 7\\ -1 & 4 & 7\\ \sin 3\theta & \cos 2\theta & 2\end{vmatrix}=0$$

$$1(8-7\cos\theta)-3(-2-7\sin 3\theta)+7(-\cos 2\theta-4\sin 3\theta)=0$$

$\Rightarrow 4\sin^3\theta+4\sin^2\theta-3\sin\theta=0$

$\sin\theta=0(or)4\sin\theta=\frac{1}{2}(or)\sin\theta=\frac{-3}{2}$

or $\theta\in(0,\pi)$ $\theta=\frac{\pi}{6}$ and $\theta=\frac{5\pi}{6}$

are satisfy the equation number of values of θ is 2

20. Given system of equations has non-trivial

$$\begin{vmatrix} \lambda-1 & 2 & 2 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-1)\left(2\lambda-\lambda^2-1\right)-2(\lambda-1)+2(1-2+\lambda)=0$$

$$\Rightarrow (\lambda-1)\left(2\lambda-\lambda^2-1\right)-2(\lambda-1)+2(\lambda-1)=0$$

$$(\lambda-1)\left(2\lambda-\lambda^2-1\right)=0 \ (\lambda-1)^3=0$$

$\lambda-1$ is a singleton

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ADVANCED LEVEL QUESTINS

SINGLE ANSWER TYPE QUESTIONS

1. If $A=\begin{bmatrix}1 & 0\\ -1 & 7\end{bmatrix}, I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ **and** $A^2=8A+\alpha I$ **, then, the value of** α **is**

A) 7 B) 8 C) -7 D) -8

2. Let $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ **and** $B=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}$ $a,b\in N$ **then** **[IIT 2017]**

A) there exists exactly one B such that AB=BA

B) There exists infinitely many B's such that AB =BA

C) there cannot exist any B such that AB=BA

D) there exists more than one but finite number

of B's such that AB=BA

3. Let A is a 3×3 **matrix and** $A=\left[a_{ij}\right]_{3\times 3}$**. If for every column matrix X, if** $X^T AX=O$ **and** $a_{23}=-2009$ **then** a_{32} = **[IIT - 2016]**

A) 2009 B) -2009 C) 0 D) 2008

4. If α,β,γ **are the roots of the equation** $x^3-12x^2+47x-60=0$ and $P(\alpha,\beta,\gamma)$ lines in the plane $8x+4y+3z=20$ and

A=$\begin{bmatrix}\alpha & 3 & 2\\ 1 & \beta & 4\\ 2 & 2 & \gamma\end{bmatrix}$ then $A(adjA)=$

A) $\begin{bmatrix}64 & 0 & 0\\ 0 & 64 & 0\\ 0 & 0 & 64\end{bmatrix}$ B) $\begin{bmatrix}34 & 0 & 0\\ 0 & 34 & 0\\ 0 & 0 & 34\end{bmatrix}$

C) $\begin{bmatrix}68 & 0 & 0\\ 0 & 68 & 0\\ 0 & 0 & 68\end{bmatrix}$ D) $\begin{bmatrix}20 & 0 & 0\\ 0 & 20 & 0\\ 0 & 0 & 20\end{bmatrix}$

5. If $A=\left[a_{ij}\right]_{4\times 4}$ **such that**

$$a_{ij}=\begin{Bmatrix}2 \ when\ i=j\\ 0\ when\ i\neq j\end{Bmatrix}$$ **then**

$\left\{\frac{\det\left(adj\left(adjA\right)\right)}{7}\right\}$ **is (where** $\{\ \}$ **represents fractional part of x) is**

A) $\frac{1}{7}$ B) $2/7$ C) $3/7$ D) $9/7$

6. If $A\neq A^2=I$ **then det (I+A) =**

A) 0 B) 1 C) -1 D) 2

7. Let P be a non-singular matrix and $I+P+P^2+.......+P^n=O$ **then** P^{-1} **is**

a) P^n b) P c) P^{n-1} d) I

8. If 2ax - 2y + 3z = 0, x + ay + 2z = 0 and 2x + az = 0 have a non-trivial solution then

A) a = 2 B) a = 1 C) a = 0 D) a = -1

9. If the equation 2x + 3y + 1 = 0, 2x + y - 1 = 0 and ax + 2y - b = 0 are consistent then

A) a - b = 2 B) a + b + 1 = 0

KEY

1. C 2 .b 3. a 4. c 5.a 6 .a 7..a 8 . a 9 .a

SOLUTIONS

1. $A^2 = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix}$

$8A + \alpha I = \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix} + \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$

$= \begin{bmatrix} 8+\alpha & 0 \\ -8 & 56+\alpha \end{bmatrix}$

$8 + \alpha = 1$

$+\alpha = -7$

2. $AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 2b \\ 3a & 4b \end{pmatrix}$

$BA = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$= \begin{pmatrix} a & 2b \\ 3a & 4b \end{pmatrix}$ $AB = BA \Rightarrow a = b$

3.. Let $X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$

$\therefore X^1.A.X = 0$

$\Rightarrow a_{11}X_1^2 + a_{22}X_2^2 + a_{33}X_3^2 + (a_{12} + a_{21})X_1X_2$

$+(a_{13} + a_{31})X_1X_3 + (a_{23} + a_{32})X_2X_3 = 0$

This is true X_i

$a_{11} = a_{22} = a_{33} = 0$

$a_{12} + a_{21} = 0$

$a_{13} + a_{31} = 0$

$a_{23} + a_{32} = 0$

$\Rightarrow a_{32} = -a_{23} = -(-2009) = 2009$

4. $|A| = \alpha\beta\gamma - (8\alpha + 4\beta + 2\gamma) + 28$

$= 60 - 20 + 28$

$(\therefore \alpha\beta\gamma = product\ of\ roots = 60)$

=68 , $8\alpha + 4\beta + 2\gamma = 20$ as p lines on the plane

5. $|A| = 2^4 = 16 \Rightarrow |adJ(adJA)| = 16^9$

$\therefore \left\{\frac{16^9}{7}\right\} = \left\{\frac{2^9(1+7)^9}{7}\right\} = \left\{\frac{2^9}{7}\right\} = \left\{\frac{8^3}{7}\right\} = \frac{1}{7}$

6. $A^2 - I = 0 \Rightarrow (A+I)(A-I) = 0$

det $|A+I| \neq 0 \Rightarrow A - I = 0$

$\Rightarrow A = I$

$\therefore \det(A+I) = 0$

7. $I + P + P^2 + - + P^n = 0$

$\Rightarrow P^{-1} + I + P + - - + P^{n-1} = 0$

$\Rightarrow p^{-1} = -(I + P + - + P^{n-1}) = -(-P^n) = P^n$

8. $\begin{vmatrix} 2a & -2 & 3 \\ 1 & a & 2 \\ 2 & 0 & a \end{vmatrix} = 0$

$2a^3 - 4a - 8 = 0$

$a^3 - 2a - 4 = 0$

$(a - 2)(a^2 + 2a + 2) = 0$

$a = 2$

9. $2x + 3y + 1 = 0$

$2x + y - 1 = 0$ on solving $x = 1, y = -1$

If equations are consistent, then

$a(1) + 2(-1) - b = 0$

$a - b - 2 = 0$

MULTIPULE ANSWER TYPE QUESTIONS

1. **Let** $A=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$, $B=\begin{bmatrix}\cos 2\beta & \sin 2\beta\\ \sin 2\beta & -\cos 2\beta\end{bmatrix}$

where $0<\beta<\dfrac{\pi}{2}$ **then which of the following is(are) true** [IIT 2016]

A) $(AB)^2=I$ B) $(AB)^{-1}=AB$ C) $BAB=A^{-1}$

D) The least positive value of α for which $BA^4B=A^{-1}$ is $\dfrac{2\pi}{3}$

2. **If the matrix** $\begin{bmatrix}l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\\ l_3 & m_3 & n_3\end{bmatrix}$ **is orthogonal then**

A) $(l_1,m_1,n_1),(l_2,m_2,n_2),(l_3,m_3,n_3)$ can be the Dc's of a line

B) The rays with Dc's (l_1,m_1,n_1), $(l_2,m_2,n_2),(l_3,m_3,n_3)$ are orthogonal

C) The rays are parallel

D) The rays lie on the same plane

3. **Which of the following statement (s) is/are true about square matrix A of order n?**

A) $(-A)^{-1}$ is equal to $-A^{-1}$ when n is odd only

B) If $A^n=0$, then $I+A+A^2.....+A^{n-1}=(I-A)^{-1}$

C) If A is a skew symmetric matrix of odd order then its inverse does not exist

D) $\left(A^T\right)^{-1}=\left(A^{-1}\right)^T$ holds always

4. **If A is symmetic and B is skew symmetric and A+B is non singular and also** $C=(A+B)^{-1}(A-B)$ **then** **[IIT 2020]**

A) $C^T(A+B)C=A+B$

B) $C^T(A-B)C=A-B$

C) $C^TAC=A$ D) $C^TAC=O$

5. **Let** $[A_k]_{n\times n}$ **be a square matrix of order n x n, such that** $a_{ij}=\begin{cases}0, i\neq j\\ \dfrac{1}{k+j}, i=j\end{cases}$ **and** $[B_k]_{n\times n}$ **is its inverse matrix, then which is/are true?**

A) $\underset{m\to\infty}{Lt}\left(\dfrac{\sum_{n=1}^{m}\text{trace}(B_k)_{n\times n}}{\text{m}^3}\right)=\dfrac{1}{6}$

B) $\sum_{n=1}^{10}\text{trace}(B_2)_{n\times n}=320$

C) $\underset{m\to\infty}{Lt}\left(\dfrac{\sum_{n=1}^{m}\text{trace}(B_k)_{n\times n}}{\text{m}^3}\right)=\dfrac{1}{3}$

D) $\sum_{n=1}^{10}\text{trace}(B_2)_{n\times n}=330$

6. Let A be the 2 x 2 matrix given by $A=(a_{ij})$ where $a_{ij}\in\{0,1,2,3,4\}$ such that $a_{11}+a_{12}+a_{21}+a_{22}=4$ then which of the following statement(s) is /are true?

A) Number of matrices A such that the trace of A is equal to 4, is 5

B) Number of matrices A, such that A is invertible is 18

C) Absolute difference between maximum value and minimum value of det(A) is 8

D) Number of matrices A such that A is either symmetric (or) skew symmetric and det (A) is divisible by 2, is 5

KEY

1.ABCD	2.AB	3. BC	4.ABC
5. AD	6. ABCD		

SOLUTIONS

1. $AB=\begin{bmatrix}\cos(\alpha+2\beta) & \sin(\alpha+2\beta)\\ \sin(\alpha+2\beta) & -\cos(\alpha+2\beta)\end{bmatrix}$

$\Rightarrow (AB)^2=I$

$\Rightarrow (AB)(AB)=I\Rightarrow A^{-1}=BAB$

$(AB)^{-1}=AB$

$\Rightarrow B^{-1}A^{-1}=AB$

$\Rightarrow BB^{-1}A^{-1}=BAB\Rightarrow A^{-1}=BAB$

$BA^4B=A^{-1}\Rightarrow B^{-1}BA^4B=B^{-1}A^{-1}$

$\Rightarrow A^4B=(AB)^{-1}=AB$

$\Rightarrow A^4=A\Rightarrow \cos 4\alpha=\cos\alpha, \sin 4\alpha=\sin\alpha$

$4\alpha=2\pi+\alpha\Rightarrow \alpha=\dfrac{2\pi}{3}$

2. Every row vector is unit vector and the vectors of any two rows are orthogonal.

$SAS^{-1}=\begin{bmatrix}2a & 0 & 0\\ 0 & 2b & 0\\ 0 & 0 & 2c\end{bmatrix}\therefore$ diagonal also

Invertible

3. $(-A^T)^{-1}=\dfrac{adj(-A)}{|-A|}=$

$\dfrac{(-1)^{n-1}adj(A)}{(-1)^n|A|}=\dfrac{adj(A)}{-|A|}$

$=A^{-1}(\forall n)$

given $A^n=0$ now

$(I-A)(I+A+A^2......+A^{n-1})=I-A^n=I$

$\therefore (I-A)^{-1}=I+A+.....+A^{n-1}$

4. $(A+B)C=(A+B)(A+B)^{-1}(A-B)$

$= A-B$

$C^T = (A-B)^T\left((A+B)^{-1}\right)^T$

$= (A-B)^T\left((A+B)^T\right)^{-1}$

$= (A-B)^T(A-B)^{-1} = (A+B)(A-B)^{-1}$

$C^T(A+B)C$

$= (A+B)(A-B)^{-1}(A-B) = A+B$ -------1

Taking transpose

$\Rightarrow C^T(A+B)^T\left(C^T\right)^T = (A+B)^T$

$\Rightarrow C^T(A-B)C = A-B$ ----------2

$1+2 \Rightarrow C^T.2A.C = 2A \Rightarrow C^TAC = A$

5. $[B_k]_{n\times n}$ is a matrix, such that

$b_{ij} = 0, i \neq j$ and $b_{ii} = k+i$

$$\text{trace}\,(B_k)_{n\times n} = \sum_{i=1}^{n}(k+i) = k_n + \frac{n(n+1)}{2}$$

$$\sum_{n=1}^{m}\text{trace}(B_k)_{n\times n} = \frac{km(m+1)}{2} + \frac{m(m+1)(m+2)}{6}$$

$$\sum_{n=1}^{10}\text{trace}(B_2)_{n\times n} = \frac{2\times10\times11}{2} + \frac{10\times11\times12}{6} = 330$$

$$\underset{m\to\infty}{lt}\left(\frac{\sum_{n=1}^{m}\text{trace}(B_k)_{n\times n}}{m^3}\right) = \frac{1}{6}$$

6. a) $\begin{pmatrix}1&0\\0&3\end{pmatrix},\begin{pmatrix}3&0\\0&1\end{pmatrix},\begin{pmatrix}2&0\\0&2\end{pmatrix}\begin{pmatrix}4&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&4\end{pmatrix}$

b)using $0,0,2,2 \rightarrow$ there are two matrices which are invertible $\begin{pmatrix}2&0\\0&2\end{pmatrix},\begin{pmatrix}0&2\\2&0\end{pmatrix}$

using $0,0,1,3 \rightarrow$ there are four matrices which are invertible

using $0,1,1,2 \rightarrow$ there are twelve matrices which are invertible

using $0,0,0,4$ and using $1,1,1,1$ no matrix is formed, which is invertible

$\therefore$ total 18

c) $|4-(-4)| = 8$

d) there are five matrices, which are either symmetric or skew symmetric and whose determinant is divisible by 2

$\begin{pmatrix}2&0\\0&2\end{pmatrix},\begin{pmatrix}0&2\\2&0\end{pmatrix},\begin{pmatrix}0&0\\0&4\end{pmatrix}\begin{pmatrix}4&0\\0&0\end{pmatrix},\begin{pmatrix}1&1\\1&1\end{pmatrix}$

COMPREHENSION TYPE QUESTIONS

Passage - 1

Let $a_{11}x_1 + a_{12}x_2 + ...a_{1n}x_n = b_1$
$a_{21}x_1 + a_{22}x_2 + ...a_{2n}x_n = b_2$

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$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n$ **be a system of n linear equations in n unknowns. Then this can be written in the matrix form as AX=B Where**

A= $[a_{ij}]_{n\times n}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ \cdot \\ . \\ x_n \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ . \\ \cdot \\ . \\ b_n \end{pmatrix}$ **Then**

(I) If |A| $\neq$ 0, the system is consistent, and has a unique solution given by $X = A^{-1}B$

(II) If |A| =0 and (adj A) B =0, then the system is consistent and has infinitely many solutions.

(III) If $|A|$ = 0 and (adj A) B$\neq$ 0, then the system is inconsistent.

1. **The system of equations**
$2x - y + 3z = 1, x + y - 2z = 5, x + y + z = -1$
has
A) a unique solution
B) infinitely many solutions
C) no solution
D) finite number of solutions.

2 .**Let** $2x - y + z = 4, \quad x + 3y + 2z = 12,$
$3x + 2y + kz = 10$. **The value of k in the above system of equations so that system does not have a unique solution is**
A) 2 B) 3 C) -1 D) -2

3. **If** $x + y + z = 6, \quad x + 2y + 3z = 10, \quad x + 2y + \lambda z = \mu$ **the values of** λ **and** μ **for which the system has infinitely many solutions is**
A) $\lambda = 3, \mu = 9$ B) $\lambda = 3, \mu = 10$
C) $\lambda = 2, \mu = 10$ D) $\lambda = 10, \mu = 3$

Passage - 2

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ **be a square matrix and**

C_1, C_2, C_3 **be 3 column matrices satisfying**

$AC_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, AC_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ **and** $AC_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ **of matrix B.**

If the matrix $C = \frac{1}{3}AB$ **then**

4. **The value of sum of elements of** B^{-1} **is**
A) -1 B) 0 C) 4 D) 2

5. **The ratio of trace of matrix A to the det of matrix B is**
A) 1 : 3 B) 2 :3 C) 1 : 1 D) 3 : 1

6. **The value of** $\sin^{-1}|A| + \cos^{-1}|C|$ **is (where** $|A|$ **is determinant of A)**
A) $\frac{\pi}{2}$ B) $\frac{\pi}{3}$ C) $\frac{\pi}{4}$ D) 1

KEY

1.A	2. B	3.B
4. B	5. C	6.A

SOLUTIONS

1,2,3

$$A=\begin{bmatrix}2 & -1 & 3\\ 1 & 1 & -2\\ 1 & 1 & 1\end{bmatrix}$$

$$\Rightarrow |A| = 2(1+2)+1(1+2)+3(1-1) \neq 0$$

The solution is unique

. If the system does not have a unique solution the value of the determinant of coefficients =0

$$\therefore \begin{vmatrix}2 & -1 & 1\\ 1 & 3 & 2\\ 3 & 2 & k\end{vmatrix} = 0 \Rightarrow k = 3$$

. The required conditions are $|A| = 0$ and (Adj A) B = 0

$$\therefore \begin{vmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 1 & 2 & \lambda\end{vmatrix} = 0, \quad \text{and}$$

$$\begin{bmatrix}2\lambda-6 & 2-\lambda & 1\\ -\lambda+3 & \lambda-1 & -2\\ 0 & -1 & 1\end{bmatrix}\begin{pmatrix}6\\ 10\\ \mu\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

i.e., $(2\lambda-6)+(3-\lambda)+0=0$ and 0.6 - 10 + $\mu = 0$

$\Rightarrow \lambda = 3, \mu = 10$

4,5,6

$$\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 2 & 1\end{bmatrix}\begin{bmatrix}a\\ b\\ c\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

$\Rightarrow a = 1, 2a+b=0 \Rightarrow b=-2$

$3a+2b+c=0 \Rightarrow c=1$

$$\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 2 & 1\end{bmatrix}\begin{bmatrix}d\\ e\\ f\end{bmatrix}=\begin{bmatrix}2\\ 3\\ 0\end{bmatrix}$$

$\Rightarrow d=2, 2d+e=3 \Rightarrow e=-1$

$3d+2e+f=0 \Rightarrow 6-2+f=0 \Rightarrow f=-4$

$$\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 2 & 1\end{bmatrix}\begin{bmatrix}p\\ q\\ r\end{bmatrix}=\begin{bmatrix}2\\ 3\\ 1\end{bmatrix}$$

$\Rightarrow p=2, 2p+q=3 \Rightarrow q=-1$

$3p+2q+r=1 \Rightarrow r=-3$

$$B=\begin{bmatrix}1 & 2 & 2\\ -2 & -1 & -1\\ 1 & -4 & -3\end{bmatrix}$$

$$|B| = 1(-1)-2(7)+2(9)=3$$

$$C=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ -2 & -1 & -1\\ 1 & -4 & -3\end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix}1 & 2 & 2\\ 0 & 3 & 3\\ 0 & 0 & 1\end{bmatrix} \qquad |A|=1, |C|=1$$

$$\begin{matrix}+(-1) & -7 & +9\\ -2 & +(-5) & -(-6)\\ +0 & -3 & +3\end{matrix}$$

$$B^{-1}=\frac{1}{3}\begin{bmatrix}-1 & -2 & 0\\ -7 & -5 & -3\\ 9 & 6 & 3\end{bmatrix}$$

Sum of elements = 0

MATRIX MATCHING TYPE

1.If $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix}\begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2+3a \\ 3b^2+3b \\ 3c^2+3c \end{bmatrix}$ **where f(x) is a quadratic function and** $f(x)=ax^2+bx+c$ **whose maximum value occurs at a point V say** (α,β)**. Let A be the point of intersection of** $y=f(x)$ **with negative x-axis, say (p,o) and point B is such that the chord AB subtends a right angle at V. Let B be (r,s). Let** Δ **be the area enclosed by** $y=f(x)$ **and the chord AB. Then**

COLUMN-I	COLUMN-II
A) $\alpha+\beta=$	P) 125/3
B) $p=$	Q) -7
C) $r+s=$	R) -2

2. Let $A=\begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$

COLUMN I

A) A^{-1}

B) $(adjA)^{-1}$

C) $adj(adjA)$

D) $adj(2A)$

COLUMN II

P) $\begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$

Q) $2\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

R) $\frac{1}{2}\begin{bmatrix} 1+\cos 2x & -\sin x \\ \sin 2x & 1+\cos 2x \end{bmatrix}$

S) $\begin{bmatrix} 1+\cos x & \sin 2x \\ -\sin 2x & 1+\cos 2x \end{bmatrix}$

KEY

01) A-S; B-R; C-Q; D-P

02) A-R; B-S; C-P, D-Q

SOLUTIONS

1. $4a^2f(-1)+4af(1)+f(2)=3a^2+3a$

$4b^2f(-1)+4bf(1)+f(2)=3b^2+3b$

$4f(-1)(a^2-b^2)+4f(1)(a-b)$

$=3(a^2-b^2)+3(a-b)$

$4f(-1)(a+b)+4f(1)=3(a+b)+3$

$4f(-1)=3 \qquad 4f(1)=3 \qquad f(2)=0$

$f(-1)=\frac{3}{4}, \qquad f(1)=\frac{3}{4}, \qquad f(2)=0$

$4a+2b+c=0$

$a-b+c=\frac{3}{4}$

$a+b+c=\frac{3}{4}$

$2b=0\Rightarrow b=0$

$4a+c=0$

$a+c=\frac{3}{4}$

$3a=-\frac{3}{4}\Rightarrow a=-\frac{1}{4}$

$c=\frac{3}{4}+\frac{1}{4}=1$

$f(x)=-\frac{1}{4}x^2+1$

$(\alpha,\beta)=(0,1)$

$A=(-2,0)\Rightarrow P=-2$

$\frac{-\frac{x^2}{4}}{x}\times\frac{1}{2}=-1\Rightarrow x=8 \qquad B(8,-15)$

2. $A=\begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$

$\therefore adj(A)=\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

$\therefore A^{-1}=\dfrac{adj(A)}{|A|}$

$=\dfrac{1}{1+\tan^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

$=\begin{bmatrix} \cos^2 x & -\sin x\cos x \\ \sin x\cos x & \cos^2 x \end{bmatrix}$

$=\dfrac{1}{2}\begin{bmatrix} 1+\cos 2x & -\sin 2x \\ \sin 2x & 1+\cos 2x \end{bmatrix}$

$\therefore (A)\to(R)$

$Adj(AdjA)=\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}=A$

$\therefore (AdjA)^{-1}=\dfrac{Adj(AdjA)}{|AdjA|}$

$=\dfrac{1}{2}\begin{bmatrix} 1+\cos 2x & \sin 2x \\ -\sin 2x & 1+\cos 2x \end{bmatrix}$

$\therefore (B)\to(S); and (C)\to(P)$

$2A=2\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$

$\therefore (adj\,2A)=2^{2-1}\,adj\,(A)$

$=2\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} \therefore D-Q$

INTEGER TYPE QUESTIONS

1. If $A=\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$ **and if** $I+2A+3A^2+.......\infty=\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ **then numerical value of** $|\alpha+\beta+\gamma+\delta|$ **is**

2. The number of all possisble value of θ**, where** $0<\theta<\pi$ **, for which the system of equations** $(y+z)\cos 3\theta = xyz\sin 3\theta$ **,**

$x\sin 3\theta=\dfrac{2\cos 3\theta}{y}+\dfrac{2\sin 3\theta}{z}$,

$(xyz)\sin 3\theta=(y+2z)\cos 3\theta + y\sin 3\theta$ **have a solution** (x_0, y_0, z_0) **with** $y_0 z_0 \neq 0$, **is**

3. The values of k $\left(say\ k_{1,}\ k_2\ and\ k_1<k_2\right)$ **for which the planes** $kx+4y+z=0$**,** $2x+2y+z=0$ **and** $4x+ky+2z=0$ **intersect in a line and** $P=\begin{pmatrix} k_2 & k_1 \\ k_1 & k_2 \end{pmatrix}$ **then** $\dfrac{1}{24}$ **det(P**$^{-1}$**) = ...**

4. Let $\alpha, \beta\ (\alpha<\beta)$ **be the solutions of** $\sum_{n=1}^{6}\dfrac{1}{\sin\left(\theta+(n-1)\dfrac{\pi}{4}\right)\sin\left(\theta+\dfrac{n\pi}{4}\right)}=4\sqrt{2}$ **and determinant value of A where** $A=\begin{pmatrix} \sin\beta & \sin\alpha \\ \sin\alpha & \sin\beta \end{pmatrix}$ **is k then** $8k^2=$

KEY

1. 2 2. 1 3. 2 4. 6

SOLUTIONS

1. Let $S = I + 2A + 3A^2 + ...\infty$

$AS = \quad A + 2A^2 + ...\infty$

$(I-A)S = I + A + A^2 + ...\infty$

$= \dfrac{I}{I-A} = I(I-A)^{-1}$

$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$

$= \begin{bmatrix} -1 & -1 \\ 5 & 3 \end{bmatrix}$

$S = \dfrac{(I-A)^{-1}}{I-A}$

$(I-A)^{-1} = \dfrac{1}{2}\begin{bmatrix} 3 & 1 \\ -5 & -1 \end{bmatrix}$

$= \left((I-A)^{-1}\right)^2$

$= \dfrac{1}{4}\begin{bmatrix} 3 & 1 \\ -5 & -1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -5 & -1 \end{bmatrix}$

$= \dfrac{1}{4}\begin{bmatrix} 4 & 2 \\ -10 & -4 \end{bmatrix} = \begin{bmatrix} 1 & \dfrac{1}{2} \\ -\dfrac{5}{2} & -1 \end{bmatrix}$

$|\alpha + \beta + \gamma + \delta| = 2$

2. from the equations we get $y\cos 3\theta - y\sin 3\theta = 0$

$\Rightarrow \tan 3\theta = 1 (y \neq 0)$ $\Rightarrow \pi/4 + n\pi = 3\theta$

$\Rightarrow \theta = \dfrac{\pi}{12}, \dfrac{5\pi}{12}, \dfrac{9\pi}{12}$ are the solutions.

3. $\Delta = \begin{vmatrix} k & 4 & 1 \\ 2 & 2 & 1 \\ 4 & k & 2 \end{vmatrix} = -(k-4)(k-2) = 0$

$\Rightarrow k = 4, 2 \quad \therefore |p| = 12$

$24\left(\dfrac{1}{|p|}\right) = 24 \times \dfrac{1}{12} = 2$

4. $\sum_{n=1}^{6} \operatorname{cosec}\left(\theta + (n-1)\dfrac{\pi}{4}\right)\operatorname{cosec}\left(\theta + \dfrac{n\pi}{4}\right)$

$= \sqrt{2}\left[\cot\theta - \cot\left(\theta + \dfrac{3\pi}{2}\right)\right]$

$\Rightarrow \cot\theta + \tan\theta = 4 \Rightarrow \theta = 15^0$ or 75^0

$\therefore \sin^2\beta - \sin^2\alpha = \dfrac{\sqrt{3}}{2}$

MATHEMATICAL INDUCTION

SYNOPSIS

✈ **Principle of Finite Mathematical Induction:**

For $n \in N$, let P(n) be a statement in terms of n. If P(1) is true and P(k) is true $\Rightarrow$ P(k + 1) is true, then P(n) is true, for all $n \in N$.

✈ **Principle of Complete Mathematical Induction:**

For $n \in N$, let P(n) be a statement in terms of n. If P(1), P(2), P(3),.... P(k-1) are true $\Rightarrow$ P(k) is true, then P(n) is true, for all $n \in N$.

✈ $a, (a+d), (a+2d), \ldots$ form an A.P. then

(i) n^{th} term $t_n = a+(n-1)d$, Where a is the first term and d is the common difference.

(ii) Sum of n terms $S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

$= \frac{n}{2}[a+l]$

Where a = first term, l = last term

\✈ $a, ar, ar^2, \ldots$ form a G.P then

(i) n^{th} term $t_n = a.r^{n-1}$, Where a = first term r = common ratio

(ii) Sum of n terms $S_n = a\frac{(r^n-1)}{r-1}$;

(iii) In an infinite G.P, Sum of Infinite terms is

$S_\infty = \frac{a}{1-r}$ where $|r|<1$

✈ $a+(a+d)r+(a+2d)r^2+\ldots$

$\ldots\ldots+\left[a+(n-1)d\right]r^{n-1}$ form A.G.P. then

(i) n^{th} term $t_n = \left[a+(n-1)d\right]r^{n-1}$

(ii) Sum of n terms

$$S_n = \frac{a}{1-r} + \frac{dr\left(1-r^{n-1}\right)}{(1-r)^2} - \frac{\left[a+(n-1)d\right]r^n}{(1-r)}$$

(iii) Sum of Infinite terms

$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2} \quad where\ |r|<1$$

SOME IMPORTANT POINTS

i) Sum of first n natural numbers i.e.

$$\sum n = 1+2+3+\ldots\ldots n = \frac{n(n+1)}{2}, \forall n \in N$$

ii) Sum of the squares of first n natural numbers is

$$\sum n^2 = 1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}, \forall n \in N$$

iii) Sum of the cubes of first n natural numbers is

$$\sum n^3 = 1^3+2^3+3^3+\ldots+n^3 = \frac{n^2(n+1)^2}{4}$$

$$= \left(\sum n\right)^2, \forall n \in N$$

iv) $1^4+2^4+3^4+\ldots\ldots+n^4 = \sum n^4$

$$= \frac{n\left(n^4-1\right)}{5} + \frac{n(n+1)\left(3n^2-n+1\right)}{6}$$

v) Sum of the first 'n' odd +ve integers = 1 + 3 + 5 + ... + (2n-1) $= n^2$

vi) Sum of the first 'n' even +ve integers = 2 + 4 + 6 + ... + 2n $= n(n+1)$

vii) $\sum n^2 \neq \left(\sum n\right)^2$ ... etc.,

viii) $\sum n(n+1)=\dfrac{n(n+1)(n+2)}{3}$

ix) $\sum n(n+1)(n+2)=\dfrac{n(n+1)(n+2)(n+3)}{4}$

x) $1^2-2^2+3^2-4^2+5^2-6^2+.....$ 'n' terms

$=-\dfrac{n(n+1)}{2}$; if n is even $=\dfrac{n(n+1)}{2}$; if n is odd

xi) $(n+3)^2\le 2^{n+3}$

xii) The sum in the n^{th} bracket of

$1+(2+3)+(4+5+6)+...$ is $\dfrac{n(n^2+1)}{2}$

xiii) $\sum_{k=1}^{n}k\left(1+\dfrac{1}{n}\right)^{k-1}=n^2$

✈ The inequality

i) $2^n<n!$ is true for all $n\ge 4$

ii) $2^n>2n+1$ is true for all $n\ge 3$

iii) $2^n\le(n+1)!$ is true for all $n\in N$

iv) $2n-3\le 2^{n-2}$ is true for all $n\ge 5$

✈ $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+...+$

$+\cos(\alpha+(n-1)\beta)$

$$=\frac{\sin\left(n\beta/2\right)}{\sin\left(\beta/2\right)}\times\cos\left[\alpha+(n-1)\beta/2\right]$$

✈ $\cos\alpha.\cos 2\alpha.\cos 4\alpha...\cos\left(2^{n-1}\alpha\right)=$

$$\frac{\sin\left(2^n\alpha\right)}{2^n\sin\alpha}$$

✈ $\left(1+\dfrac{3}{1}\right)\left(1+\dfrac{5}{4}\right)\left(1+\dfrac{7}{9}\right)...\left(1+\dfrac{2n+1}{n^2}\right)$

$=(n+1)^2$

✈ For a sequence $T_1,T_2,T_3,T_4,....$, the difference of two consecutive terms $T_2-T_1,T_3-T_2,T_4-T_3...$ are in A.P or G.P then n^{th} term of given series are in the form of an^2+bn+c or $a.r^n+b$ where a,b,c to be determined

SOME IMPORTANT POINTS

i) The sum of cubes of three consecutive natural numbers is always divisible by 9

ii) For all positive integral values of n , x^n-y^n is divisible by $x-y$.

iii) For all positive integral values of n, $x^{2n+1}+y^{2n+1}$ is divisible by $x+y$.

iv) For all positive integral values of n, $x^{2n-1}+y^{2n-1}$ is divisible by $x+y$.

v) $P^{n+1}+(P+1)^{2n-1}$ is divisible by $P^2+P+1,\forall n\ge 2$

vi) n^p-n is divisible by $P\ \forall n\ge 2$ where P is prime.

vii) n is any odd integer then $n(n^2-1)$ is divisible by 24.

viii) The product of "n" consecutive natural numbers is always divisible by n!.

✈ If x,y,m are positive integers then x is said to be congruent of y modulo m if $x-y$ is divisible by m and is denoted by $x\equiv y(\bmod m)$

EXERCISE - I

1. $1^3-2^3+3^3-4^3+.....+9^3=$

1) 425 2) -425 3) 475 4) -475

2. $\dfrac{1}{2.5}+\dfrac{1}{5.8}+\dfrac{1}{8.11}+.....n\ \ terms=$

1) $\dfrac{n}{6n+4}$ 2) $\dfrac{n}{3n+2}$

3) $\frac{n}{4n+6}$ 4) $\frac{1}{2(2n+3)}$

3. 2+7+14+......+ (n^2 + 2n - 1) =

1) $\frac{n(2n^2+9n+1)}{6}$ 2) $\frac{2n^2+9n+1}{6}$

3) $\frac{2n^2+9n+1}{12}$ 4) $\frac{2n^2+9n+1}{24}$

4. 1+3+6+10+ ....+ $\frac{(n-1)n}{2}+\frac{n(n+1)}{2}=$

1) $\frac{n(n+1)(n+2)}{3}$ 2) $\frac{(n+1)(n+2)}{6}$

3) $\frac{n(n+1)(n+2)}{6}$ 4) $\frac{(n+2)(n+1)^2}{3}$

5. 3.6+6.9+9.12+.......+ 3n (3n+3) =

1) $\frac{n(n+1)(n+2)}{3}$ 2) 3n (n+1) (n+2)

3) $\frac{(n+1)(n+2)(n+3)}{3}$ 4) $\frac{(n+1)(n+2)(n+4)}{4}$

6. 1.6+2.9+3.12+.......+ n(3n+3) =

1) $n(n+1)(n+2)$ 2) $(n+1)(n+2)(n+3)$

3) $(n+2)(n+3)(n+4)$ 4) $(n-1)n(n+1)$

7. 1^3+1^2 + $1+2^3$ + 2^2 + $2+3^3$ + 3^2 + 3 + ... 3n terms =

1) $\frac{n(n+1)(n^2+12n+5)}{12}$

2) $\frac{n(n+1)(3n^2+7n+8)}{12}$

3) $\frac{n(n+1)(n+2)(n^2+5n+6)}{12}$

4) $\frac{(n+1)(n+2)(n+3)}{4}$

8. $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+$**......(n-3) terms**

1) $\frac{n}{n+2}$ 2) $\frac{n+1}{n(n+5)}$ 3) $\frac{n-3}{2n-5}$ 4) $\frac{n-1}{n(2n-3)}$

9. 2.4+4.7+6.10+.......(n-1) terms =

1) $2n^3 - 2n^2$ 2) $\frac{n^3+3n^2+1}{6}$

3) $2n^3+2n$ 4) $2n^3-n^2$

10. $\frac{1}{4.7}+\frac{1}{7.10}+\frac{1}{10.13}+.....$ **upto n terms =**

1) $\frac{n}{4(4n+3)}$ 2) $\frac{n}{4(6n+1)}$

3) $\frac{n}{4(3n+4)}$ 4) $\frac{n}{4(3n-4)}$

11. $\forall n \in N, 49^n+16n-1$ **is divisible by**

1) 64 2) 49 3) 132 4) 32

12. $\forall n \in N, 7^{2n}+3^{n-1}.2^{3n-3}$ **is divisible by**

1) 50 2) 25 3) 2425 4) 2550

13. $\forall n \varepsilon N, 3^{2n}+7$ **is divisible by**

1) 8 2) 16 3) 24 4) 64

14. For every natural number n, $3^{2n+2}-8n-9$ **is divisible by**

1) 16 2) 128 3) 256 4) 512

KEY

01) 1 02) 1 03) 1 04) 3 05) 2 06) 1
07) 2 08) 3 09) 1 10) 3 11) 1 12) 2
13) 1 14) 1

SOLUTIONS

1. $\sum_{n=1}^{9} n^3-2\left(2^3+4^3+6^3+8^3\right)$

$=\frac{9^2(9+1)^2}{4}-2^4\left(1^3+2^3+3^3+4^3\right)$

$=\frac{81\times100}{11}-2^4\times\frac{4^2(4+1)^2}{4}$

$=\frac{8100-6400}{4}=\frac{1700}{4}=425$

2. Given $\frac{1}{2.5}+\frac{1}{5.8}+----$ n terms

put $n=2$ take two terms

$$L.H.S=\frac{1}{2.5}+\frac{1}{5.8}=\frac{5}{40}=\frac{1}{8}$$

option $\frac{n}{6n+4}=\frac{2}{16}=\frac{1}{8}$

3. Put n = 2 and verify the options.
4. Put n = 2 and verify the options
5. Put n = 2 and verify the options.
6. Put n = 2 and verify the options.
7. Put n = 2 and verify the options.

8. Given $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}-----(n-3)$ terms

put $n-3=1 \Rightarrow n=4$ in option

$$L.H.S=\frac{1}{1.3}=\frac{1}{3}$$

option (3) $\frac{n-3}{2n-5}=\frac{4-3}{8-5}=\frac{1}{3}$

9. Put n = 2 and verify the options.

10. Given $\frac{1}{4.7}+\frac{1}{7.10}+\frac{1}{10.13}+----$ upto n terms

multiply and divide by 3

$$=\frac{1}{3}\left[\frac{3}{4.7}+\frac{3}{7.10}+\frac{3}{10.13}---\frac{3}{(3n+1)(3n+4)}\right]$$

$$=\frac{1}{3}\left[\begin{array}{l}\frac{1}{4}-\frac{1}{7}+\frac{1}{7}-\frac{1}{10}+\frac{1}{10}-\frac{1}{13}\\ ----\frac{1}{3n+1}-\frac{1}{3n+4}\end{array}\right]$$

$$=\frac{1}{3}\left[\frac{1}{4}-\frac{1}{3n+4}\right]=\frac{1}{3}\left[\frac{3n}{4(3n+4)}\right]=\frac{n}{4(3n+4)}$$

11. Put n = 1 and verify the options.
12. Put n =1, 2 and verify the options.
13. Put n = 1, $3^2+7=16$

Put n = 2, $3^4+7=88$, G.C.D. of 16 & 88 = 8

14. $3^{2n+2}-8n-9,\ \forall\ n\in N$

put $n=2$, $f(2)=704$, divisible by 16

EXERCISE - II

1. $\left(\sum n^3\right)\left(\sum n\right)=\left(\sum n^2\right)^2$ **if**

1) n = 3 2) n = 1 3) $n^2=3$ 4) n= -1

2. Sum of n^{th} bracket of
(1) + (2+3+4)+(5+6+7+8+9) +........ is

1) $(n-1)^3+n^3$ 2) $(n-1)^3+8n^2$

3) $\frac{(n+1)(n+2)}{6}$ 4) $\frac{(n+3)(n+2)}{12}$

3. $S_n=1^3+2^3+3^3+........+n^3$ **and**
$T_n=1+2+3+4.......n$ **[Eam-2007]**

1) $S_n=T_n^3$ 2) $S_n=T_{n^2}$ 3) $S_n=T_n^2$ 4) $S_n=T_n^3$

4. $1+(1+3)+(1+3+5)+...$ **n brackets =**

1) $\frac{n(n+1)(n+2)}{6}$

2) $\frac{n(n+1)(3n^2+23n+46)}{12}$

3) $\frac{n(27n^3+90n^2+45n-50)}{4}$

4) $\frac{n(n+1)(2n+1)}{6}$

5. If $2^3+4^3+6^3+.....+(2n)^3=Kn^2(n+1)^2$ **then k =**

1) 1/2 2) 1 3) 3/2 4) 2

6. If $a_k=\frac{1}{k(k+1)}$ **for k = 1,2,3,... n, then**

$$\left(\sum_{k=1}^{n} a_k\right)^2 =$$

1) $\frac{n}{n+1}$ 2) $\frac{n^2}{(n+1)^2}$ 3) $\frac{n^4}{(n+1)^4}$ 4) $\frac{n^6}{(n+1)^6}$

7. $1^2+(1^2+2^2)+(1^2+2^2+3^2)$ **+ ....+ n brackets =**

1) $\frac{n(n+1)^2(n+2)^2}{12}$ 2) $\frac{n(n+1)^2(n+2)}{12}$

3) $\frac{n^2(n+1)(n+2)}{12}$ 4) $\frac{(n+1)}{2}$

8. If $t_n=\sum_1^n n,$ **then** $t^{|}{}_n=\sum_1^n t_n=$

1) $\frac{n(n+1)}{2}$ 2) $\frac{n(n+3)}{2}$

3) $\frac{n(n+1)(n+2)}{6}$ 4) $\frac{n(n+4)}{3}$

9. Let the statement $m^2>100$**, the statement** $P(k+1)$ **will be true if**

1) $P(1)$ is true 2) $P(2)$ is true

3) $P(k)$ is true 4) none of these

10. $1+\frac{1}{2}(1+2)+\frac{1}{3}(1+2+3)+\frac{1}{4}(1+2+3)+...$ **upto 20 terms is**

1) 110 2) 111 3) 115 4) 116

11. If $\frac{1}{2\times4}+\frac{1}{4\times6}+\frac{1}{6\times8}+......n$ \

' c// terms $=\frac{kn}{n+1}$

then $k=$ **[Eam-2012]**

1) $\frac{1}{4}$ 2) $\frac{1}{2}$ 3) 1 4) $\frac{1}{8}$

12. If $10^n+3.4^n+x$ **is divisible by 9 for all** $n\in N$ **,then least positive value of 'x' is**

1) 1 2) 5 3) 14 4) 23

13. $\forall n\varepsilon N, 5^{2n+2}-24n-25$ **is divisible by**

1) 576 2) 25 3) 24 4) 50

14. The remainder left out when $8^{2n}-(62)^{2n+1}$ **is divided by 9 is**

1) 2 2) 7 3) 8 4) 0

15. $\forall n\varepsilon\ N$**,**

1) |sin (nx)| < |sin x | 2) | sin (nx) |<n|sinx|
3) | sin (nx)|≤n|sinx| 4) sin (nx) ≤ sin n

KEY

01) 2 02) 1 03) 3 04) 4 05) 4 06) 2
07) 2 08) 3 09) 3 10) 3 11) 1 12) 2
13) 1 14) 1 15) 3

SOLUTIONS

1. Put n = 1 and verify the options.
2. Put n = 2 and verify the options.
3. Put n = 2 and verify the options.
4. put $n=2$ take two terms

L.H.S $=+(1+3)=1+4=5$

Option (4) $\frac{n(n+1)(2n+1)}{6}=\frac{2(3)(5)}{6}=5$

5. $2^3\left(1^3+2^3+3^3+...+n^3\right)=kn^2(n+1)^2$

6. Given $a_k=\frac{1}{k(k+1)}$ $for\ k=1,2,3,---n$

$$a_k=\frac{1}{k}-\frac{1}{k+1}$$

$$\sum_{k=1}^{n}a_k=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+---\frac{1}{n}-\frac{1}{n+1}$$

$$=1-\frac{1}{n+1}=\frac{n}{n+1}$$

$$\left(\sum_{k=1}^{n}a_k\right)^2=\frac{n^2}{(n+1)^2}$$

7. Put n = 2 and verify the options.

8. $\sum_1^n t_n=\frac{1}{2}\left[\sum n^2+\sum n\right]$

9. $P(r)$ is true

$\Rightarrow r^2 > 100 \Rightarrow r^2 + 2r + 1 > 100 + 2r + 1$

$\Rightarrow (r+1)^2 > 100 \Rightarrow P(k+1)$ is true.

$P(k+1)$ is true when every $P(k)$ is so.

10. $t_n = \frac{1}{n}(1+2+3+...+n) = \frac{1}{n}\frac{n(n+1)}{2} = \frac{n+1}{2}$

Now find $\sum_1^n \left(\frac{n+1}{2}\right) = \frac{1}{2}\frac{n(n+1)}{2} + \frac{n}{2} = S_n$

now put $n = 20$

11. $\frac{kn}{n+1} = \left[\frac{1}{2.4} + \frac{1}{4.6} + \frac{1}{6.8} + ...n \text{ terms}\right]$

$= \frac{1}{2}\left[\frac{4-2}{2.4} + \frac{6-4}{4.6} + \frac{8-6}{6.8} + ... + \frac{1}{2n(2n+2)}\right]$

$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{4} + \frac{1}{4} - \frac{1}{6} + \frac{1}{6} - \frac{1}{8} + ... + \frac{1}{2n} - \frac{1}{2n+2}\right]$

$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{2(n+1)}\right] = \frac{n}{4(n+1)} \Rightarrow k = \frac{1}{4}$

12. n = 1 $\Rightarrow 10 + 3.4 + x = 9m \Rightarrow x = 5$

13. Put n = 1 and verify the options.

14. Given $8^{2n}_n - (62)^{2n+1}$ $2n+1$ is divided by 9

$= (64)^4 - (62)^{2n+1}$

$= (63+1)^n - (63-1)^{2n+1}$

$= n_{c_0} + n_{c_1}(63) + n_{c_2}(63)^2 --- +$
$(2n+1_{c_0} - 2n+1_{c_1}(63) ---)$

$= 1 + n(63) + n_{c_2}(63)^2 --- + 1 -$
$(2n+1)63 ----$

$= 2 + 63(\text{some integer})$

$= 2 + \text{divided by}$
Remainder 2

15. $|\sin nx| \le 1 \le n|\sin x|$

EXERCISE - III

1. $7 + 77 + 777 + ... + (777......7 \text{ n times}) =$

1) $\frac{7}{81}(10^{n+1} - 9n - 10)$ 2) $\frac{7}{81}(10^n - 9n - 10)$

3) $\frac{7}{81}(10^{n+1} + 9n + 10)$ 4) $\frac{7}{81}(10^{n+1} + 9n - 10)$

2. $\forall n \in N$, $1 + 2x + 3x^2 + ... + n.x^{n-1} =$
$(x \in R, x \ne 1)$

1) $\frac{1-(n+1)x^n + n.x^{n+1}}{(1-x)^2}$ 2) $\frac{(n+1)x^n}{(1-x)^2}$

3) $\frac{1-(n+1)x^n + n.x^{n+1}}{(1+x)^2}$ 4) $\frac{(n-1)x^n}{(1+x)^2}$

3. $\frac{1.2^2 + 2.3^2 + 3.4^2 + ... + n(n+1)^2}{1^2.2 + 2^2.3 + 3^2.4 + ... + n^2(n+1)} =$

1) $\frac{3n+1}{3n+5}$ 2) $\frac{3n+5}{3n+1}$

3) $(3n+1)(3n+5)$ 4) $\frac{3n+5}{3n+7}$

4. For any $n \in N$, the value of the expression $\sqrt{2+\sqrt{2+\sqrt{2+...ntimes}}}$ is

1) $2\cos\left(\frac{\pi}{2^{n+1}}\right)$ 2) $2\sin\left(\frac{\pi}{2^{n+1}}\right)$

3) $\sqrt{2}\cos(2^{n+1}\pi)$ 4) $2Cos(2^n\pi)$

5. If the sum to 'n' terms of an A.P. is $\frac{4n^2 - 3n}{4}$, then the n^{th} term of the A.P. is

1) $\frac{5n-1}{4}$ 2) $\frac{8n-7}{4}$ 3) $\frac{3n^2-2}{4}$ 4) $\frac{7n-8}{4}$

6. $(1+x)^n - nx - 1$ **is divisible by (where** $n \in N$**)**

1) $2x$ 2) x^2 3) $2x^3$ 4) all of these

7. **If** $t_n = \frac{1}{4}(n+2)(n+3)$ **for n = 1, 2, 3...**

then $\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+........+\frac{1}{t_{2003}}=$

1) $\frac{4006}{3006}$ 2) $\frac{4003}{3007}$ 3) $\frac{4006}{3008}$ 4) $\frac{4006}{3009}$

8. The value of the sum in the 50th **bracket of** $(1)+(2+3)+(4+5+6)+(7+8+9+10)+....$ **is**

1) 62525 2) 65225 3) 56255 4) 55625

9. $\forall$ **n** $\in$ **N, x** $\in$ **R,**

$\tan^{-1}\left[\frac{x}{1.2+x^2}\right]+\tan^{-1}\left[\frac{x}{2.3+x^2}\right]+..........+\tan^{-1}\left[\frac{x}{n(n+1)+x^2}\right]=$

1) $\tan^{-1}\left[\frac{x}{n}\right]-\tan^{-1}\left[\frac{x}{n+1}\right]$

2) $\tan^{-1}[x]-\tan^{-1}\left[\frac{x}{n+1}\right]$

3) $\tan^{-1}[n+1]-\tan^{-1}[x]$ 4) $\tan^{-1}[x]$

10. $Tan^{-1}\left(\frac{1}{1+1+1^2}\right)+Tan^{-1}\left(\frac{1}{1+2+2^2}\right)+....+Tan^{-1}\left(\frac{1}{1+n+n^2}\right)=$

1) $Tan^{-1}(n+1)+\pi$ 2) $Tan^{-1}(n+1)+\frac{\pi}{4}$

3) $Tan^{-1}(n+1)$ 4) $Tan^{-1}(n+1)-\frac{\pi}{4}$

11. $\frac{1}{2}\tan\left(\frac{x}{2}\right)+\frac{1}{4}\left(\tan\frac{x}{4}\right)+...+\frac{1}{2^n}\tan\left(\frac{x}{2^n}\right)=$

1) $\frac{1}{2^n}\cot\left(\frac{x}{2^n}\right)$ 2) $\frac{1}{2^n}\cot\left(\frac{x}{2^n}\right)+\cot x$

3) $\frac{1}{2^n}\cot\left(\frac{x}{2^n}\right)-\cot x$ 4) $\cot\left(\frac{x}{2^n}\right)-\cot x$

3) $n+1-2^{-n}$ 4) $n-1-2^{-n}$

12. **If n is even, then the sum of first 'n' terms of the series**

$1^2+2.2^2+3^2+2.4^2+5^2+2.6^2+......$ is

1) $\frac{n(n+1)^2}{3}$ 2) $\frac{n(n+1)^2}{4}$

3) $\frac{n(n+1)^2}{2}$ 4) $\frac{n^2(n+1)}{2}$

13. **If** $S_1=\{2\}$, $S_2=\{3,6\}$, $S_3=\{4,8,16\}$, $S_4=\{5,10,20,40\}$,....... **then the sum of numbers in the set** S_{15} **is**

1) $5(2^{15})$ 2) $16(2^{15}-1)$

3) $16(2^{16}-1)$ 4) $15(2^{15}-1)$

14. **The sets** $S_1, S_2, S_3,....$ **are given by**

$S_1=\left\{\frac{2}{1}\right\}$, $S_2=\left\{\frac{3}{2},\frac{5}{2}\right\}$,

$S_3=\left\{\frac{4}{3},\frac{7}{3},\frac{10}{3}\right\}, S_4=\left\{\frac{5}{4},\frac{9}{4},\frac{13}{4},\frac{17}{4}\right\}$, **...then the sum of the numbers in the set** S_{25} **is**

1) 322 2) 324 3) 325 4) 326

15. **Sum of the series**

$S=1^2-2^2+3^2-4^2+....-2002^2+2003^2$ **is**

1) 2007006 2) 1005004

3) 2000506 4) none

16. $\sum_{k=1}^{2n+1}(-1)^{k-1}k^2=$

1) $(n+1)(2n+1)$ 2) $(n+1)(2n-1)$

3) $(n-1)(2n-1)$ 4) $(n-1)(2n+1)$

17. **When** 2^{301} **is divided by 5, the least +ve remainder is**

1) 4 2) 8 3) 2 4) 6

KEY

01) 1	02) 1	03) 2	04) 1	05) 2	06) 2
07) 4	08) 1	09) 2	10) 4	11) 3	12) 3
13) 2	14) 4	15) 1	16) 1	17) 3	

1. Put n = 1 and verify the options.
2. Put n = 2 and verify the options.
3. Put n = 2 and verify the options.
4. Put n=2

$$1+\cos\frac{\pi}{4}=2\cos^2\frac{\pi}{8}$$

$$2\left(1+\frac{1}{\sqrt{2}}\right)=\left(2\cos\frac{\pi}{8}\right)^2$$

$$\sqrt{2+\sqrt{2}}=2\cos\frac{\pi}{8}$$

5. Given $S_n=\dfrac{4n^2-3n}{4}$

$$s_{n-1}=\frac{4(n-1)^2-3(n-1)}{4}$$

$$=\frac{4(n^2+1-2n)-3n+3}{4}$$

$$=\frac{4n^2-11n+7}{4}$$

We know $t_n=s_n-s_{n-1}$

$$=\frac{4n^2-3n-4n^2+11n-7}{4}=\frac{8n-7}{4}$$

6. Put $n=2,$ and $x=3$
not divisible by 6, 54 but divisible by 9.

7. Given $t_n=\frac{1}{4}(n+2)(n+3)$ for $n=1,2,3---$

$$\frac{1}{t_n}=\frac{4}{(n+2)(n+3)}=4\left(\frac{1}{n+2}-\frac{1}{n+3}\right)$$

Now $\dfrac{1}{t_1}+\dfrac{1}{t_2}+\dfrac{1}{t_3}+----\dfrac{1}{t_{2003}}$

$$=4\left[\frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\frac{1}{5}+---\frac{1}{2005}-\frac{1}{2006}\right]$$

$$=4\left[\frac{1}{3}-\frac{1}{2006}\right]\quad=4\left[\frac{2003}{3\times2006}\right]=\frac{4006}{3009}$$

8. First term of 50 th bracket

$$(1+2+3+4---49)+1=\frac{49(50)}{2}+1=1226$$

sum of 50 th brack is

$$s_{50}=\frac{50}{2}\left[2(1226)+(50-1)1\right]$$

$$25[24\ 52+49]=25\times25\ 01=6\ 25\ 25$$

9. Put n = 1 and verify the options.

10. $\tan^{-1}\left(\dfrac{1}{1+n+n^2}\right)=\tan^{-1}(n+1)-\tan^{-1}(n)$

$$\tan^{-1}\left(\frac{1}{1+1+1^2}\right)+\tan^{-1}\left(\frac{1}{1+2+2^2}\right)$$

$$+\cdots\cdots+\tan^{-1}\left(\frac{1}{1+n+n^2}\right)$$

$$=\tan^{-1}(n+1)-\tan^{-1}(1)\ =\tan^{-1}(n+1)-\frac{\pi}{4}$$

11. Put n = 1 and verify the options.
12. Put n=2
Take two terms
L.H.S $=1^2+2.2^2=1+8=9$

Option (3) $\dfrac{n(n+1)^2}{2}=\dfrac{2\times(2+1)^2}{2}=9$

13. Given $s_1=[2],s_2=[3,6],s_3=[4,8,16]$

$s_4[5,10,15,20]----$

$s_{15}=[16,32,64----]$ are in G.P

$\gamma=2$

Sum of $s_{15}=16+32+64+-----$

$$=(1+2+2^2+---)$$

$$=16\frac{(2^{15}-1)}{2^{-1}}=16(2^{15}-1)$$

14. $s_{25} = \left[\frac{26}{25}, \frac{51}{25}, \frac{76}{25} ------\right]$

Sum $s_{25} = \frac{1}{25}\left(26 + 51 + 76 ----- 25 \text{ terms}\right)$

$= \frac{1}{25}\left[\frac{25}{2}\left(2\left(26\right) + 24 \times 25\right)\right]$

$= 26 + 300 = 326$

15. We can write S as

$S = (1-2)(1+2) + (3-4)(3+4) + ... +$

$(2001-2002)(2001+2002) + 2003^2$

$= -[1+2+3+4+...+2002] +$

$2003^2 = -\frac{1}{2}(2002)(2003) + 2003^2 = 2007006$

16. $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + ... + (2n+1)^2$

$= 1^2 + 2^2 + 3^2 + ... + (2n+1)^2$

$-2\left[2^2 + 4^2 + ... + (2n)^2\right]$

$= \frac{(2n+1)(2n+2)(4n+3)}{6}$

$-\frac{8.n(n+1)(2n+1)}{6} = (n+1)(2n+1)$

17. $\overset{301}{2}$ is divided by 5

$= 2^{301} = \left(2^2\right)^{150} 2^1 = 2\left[(1-5)^{150}\right]$

$= 2\left[150_{c_0} - 150_{c_1}(5) + 150_{c_2}(5)^2 ---\right]$

$= 2\left[1 - 150(5) + 150_{c_2}(5)^2 ----\right]$

$= 2 - 5(\text{some integer})$

$= 2 - \text{divisible by } 5$

Remainder=2

FUNCTIONS

SYNOPSIS

- **Function or mapping :**

  A relation 'f' from a set A to a set B is said to be function or mapping if every element of set A has associated with unique element in set B. It is denoted by $f : A \to B$.

- **Image and Pre-Image:** If 'f' is a function from A to B and (a, b) $\in$ f, then $f(a) = b$, where 'b' is called the image of 'a' under f and 'a' is called the pre-image of 'b' under 'f'.

  Note: The number of functions from A to B is $\{n(B)\}^{n(A)}$

- **Domain, codomain, Range :** Let $f : A \to B$ be a function then A is called the domain and B is called the co-domain of the function f.

- If $f : A \to B$ is a function, then $f(A)$, the set of all *f*-images of elements in *A*, is called the range of *f*

- The range of a function f denoted by $f(A)$,and $f(A) \subseteq B$.

- **One-one function (Injection) :**

  A function $f : A \to B$ is one-one (injection) if distinct elements of A have distinct images in B.

  Let $a_1, a_2 \in A$ and $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$

  (or) $a_1, a_2 \in A$ and $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$ only.

- Let $n(A) = r$ and $n(B) = n$ then the condition to define an injection from *A* to B is $r \leq n$ and the number of such injections is ${}^nP_r$.

- **Note:** If $r > n$ then the number of injections is 0.

- **Many-one function:** (i) Let $x_1, x_2 \in$ domain of f. If $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ for every x_1, x_2 in the domain,then f is one-one ,other wise many-one.

(ii) Consider any two points $x_1, x_2 \in X$. Take $f(x_1) = f(x_2)$ and solve the equation if we get $x_1 = x_2$ only then f is one-one other wise it is many -one .

(iii) **Horizontal line test :** If any straight line parallel to x-axis intersects the graph of the function atmost at one point then the function is one-one, other wise it is many-one. (i.e., intersects the graph of the function in atleast two points).

(iv) All even functions are many-one

(v) If the function is continuous and entirely strictly increasing or strictly decreasing in the domain, then f is one - one, other wise many-one .

(vi) All periodic functions are many one.

- **Onto function (Surjection) :**

  $f : A \to B$ is called an onto or a surjection if every element of B has atleast one pre-image in A.

- If $f : A \to B$ is onto (a surjection) then the range = codomain of 'f' i.e., $f(A) = B$.

- The condition for a function $f : A \to B$ to be a surjection from A to B is $n(A) \geq n(B)$.

- If n(A) = n(≥ 2) , n(B) = 2 then the number of onto functions from A to $B = 2^n - 2$

- If $n(A) = r$ and $n(B) = n(\geq 2)$. Then the number of surjections from A to B is

  $n^r - {}^nC_1(n-1)^r + {}^nC_2(n-2)^r - {}^nC_3(n-3)^r + \ldots + (-1)^{n-1}.{}^nC_{n-1}$.

- **Note:** If $n(A) < n(B)$ then the no. of onto functions is zero.
- **Into function :** If $f: A \to B$ is not onto then it is called an into function. i.e., there is no pre-image for atleast one element of B in A then *f* is into function.
- **Working Rule for checking whether the function $f: A \to B$ is onto or into:**

(i) Let $y \in B$ and $y = f(x)$ and from this find x in terms of y. $\forall\ y \in B$ if there exists atleast one $x \in A$, then f is onto. Otherwise into. i.e., if range of $f = B$, then f is onto, other wise it is into.

(ii) **Horizontal line test :** If every straight line parallel to x-axis from points in the codomain intersects the graph of the function at atleast one point then f is onto. If the line does not cut the graph of $y = f(x)$ then f is into.

- **Bijection (or) one-one & onto function:**

  If $f: A \to B$ is both an injection and a surjection then f is said to be bijection or one to one and onto from A to B.

i) If A,B are finite sets and $f: A \to B$ is a bijection then $n(A) = n(B)$.

ii) Identity function on any non empty set A is bijection.

iii) If A,B are finite sets and $n(A) = n(B)$ then number of bijective functions defined from A to B is $(n(A))!$

iv) A constant function is bijection if n(A)= n(B)=1

v) **Horizontal line test:** If every straight line parallel to x-axis from the points in codomain intersects the graph of the function at only one point then f is bijection.

- **Equality of Functions:** The functions f and g are said to be equal if,

i) the domain of f = the domain of g

ii) $f(x) = g(x)$ for every x in domain.

- **Constant function:** A function $f: A \to B$ is a constant function if the range of f contains only one element.
- The number of constant functions from A to B is n(B).
- The graph of constant function is a line parallel to x-axis.
- Range of any constant function is a singleton set.
- **Identity function:** Let A be a non - empty set then $f: A \to A$ defined by $f(x) = x, \forall\ x \in A$ is called the identity function on A and it is denoted by I_A.
- The graph of identity function is a straight line passing through origin and inclined at an angle of 45^0 with x-axis.

  E.g: Let $A = \{1,2,3,4\}$, then the identity function on A is $I_A = \{(1,1)(2,2)(3,3)(4,4)\}$
- **Composite function:** If $f: A \to B$ and $g: B \to C$ are functions then $gof: A \to C$ is defined by $(gof)(x) = g\{f(x)\}\ \forall x \in A$ is called the composite function of $f\ \&\ g$.
- **Inverse function:** A function $f: X \to Y$ is defined to be invertible if there exists a function $g: Y \to X$ such that $gof = I_X$ and $fog = I_Y$. The function g is called the inverse of f,and is denoted by f^{-1}.
- If f is invertible then f must be bijective i.e one-one and onto.
- If the inverse of a function exists then it is said to be invertible. The inverse of a function if exists then it is unique.
- Graph of $y = f(x)$ and $y = f^{-1}(x)$ are symmetrical about $y = x$ and intersects on the line $y = x$, and also $f(x) = f^{-1}(x) = x$ when ever graph intersects.
- **Working Rule to find the inverse of a function**

  Let $f: X \to Y$ be a bijection function.

put $f(x)=y$. Solve the equation $y=f(x)$ to obtain x in terms of y. Interchange x and y to obtain the inverse of f.

- **Properties of composite function :**

i) If $f: A\to B,\ g: B\to C$ are one-one then $gof: A\to C$ is also one-one.

ii) If $f: A\to B,\ g: B\to C$ are onto then $gof: A\to C$ is also onto.

iii) If $gof: A\to C$ is one-one then f is one-one.

iv) If $gof: A\to C$ is onto then g is onto.

v) If $f: A\to B$ is a function then $foI_A = I_B of = f$.

vi) If $f: A\to B$ and $g: B\to C$ are bijections then $gof: A\to C$ is a bijection & $(gof)^{-1} = f^{-1}og^{-1}$.

vii) If $f: A\to B$ is a bijection then $fof^{-1} = I_B, f^{-1}of = I_A$.

viii) If $f: A\to B$ and $g: B\to A$ are functions such that $g\,of = I_A$ and $f\,og = I_B$ then $g = f^{-1}$.

ix) If $f: A\to A$ is a bijection then $f\,of^{-1} = f^{-1}of = I_A$.

x) In general $f\,og \neq g\,of$.

xi) If $f: A\to B,\ g: B\to C$ and $h: C\to D$ are functions then $ho(gof)=(hog)\,of$.

- **Even & odd functions :**

f:A $\to$ R,A $\subseteq$ R is said to be an even function if $f(-x)=f(x), \forall x\in A$.

E.g: $y=|x|,\ y=x^2,\ y=\cos x$ are some even functions.

- **Note:** (i)The graph of an even function is symmetric about Y-axis.

(ii) If (x,y) is point on the even function then $(-x,y)$ is also point on even function.

- f:A $\to$ R, A $\subseteq$ R is said to be an odd function if $f(-x)=-f(x)\ \forall x\in A$.

E.g: $y=x, y=x^3, y=\sin x$ are some odd functions.

- **Note:** (i) The graph of an odd function is symmetric about origin (symmetrical in opposite quadrants)

(ii) If (x,y) is a point on an odd function graph then $(-x,-y)$ is also a point on the same odd function graph.

- **Important points of odd and even functions :**

(i) A function which is even or odd, when even power is always even function.

(ii) The derivative of an odd function is an even function and derivative of an even function is an odd function.

(iii) Every function can be uniquely expressed as the sum of an even and an odd function. i.e.,

$$f(x)=\frac{1}{2}\{f(x)+f(-x)\}+\frac{1}{2}\{f(x)-f(-x)\}$$

= {even function} + {odd function}

- **Note:** (i) A function may be neither even nor odd. for example $f(x)=\sin x+x^2$ is neither even nor odd.

(ii) $f(x)=0$ is the function which is both even and odd.

(iii) $f(x)=c$ is an even function

(iv) Every even function $y=f(x)$ is not one-one $\forall x\in D_f$.

- **Polynomial function :** If f: $f: R\to R$ is defined by

$f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+..+a_{n-1}x+a_n$,

$a_0\neq 0, a_1, a_2\ .....\ a_n\in R$, n is a non-negative integer is a polynomial function of degree n in x

E.g: (i) x^4-x+2 is a polynomial function.

(ii) $x^4-\sqrt{x}+2$ is not a polynomial function

- **Rational Function :** A function of the form $\frac{f(x)}{g(x)}$ where f(x) and g(x) are polynomial functions and $g(x)\neq 0$ is called a rational function.

- **Algebraic functions:** A function f is said to be an algebraic function if it arises due to a finite number of fundamental operations like addition, subtraction, multiplication, division and root extraction etc, on polynomial functions.
- **Transcendental function:** The functions which are not algebraic are called transcendental functions.

E.g: i) Exponential functions

ii) Logarithmic functions

iii) Trigonometric functions

iv) inverse trigonometric functions

E.g: $f(x)=x^3+3x^2-\sqrt{x^2-x}+\sqrt[3]{x}$ is an algebraic function where as $g(x)=x+\sin^{-1}x$.

$h(x)=e^x.\cosh^{-1}x$ are transcendental functions.

- **Algebra of real valued functions :**
- **Definition :** Let $f:A\to B$ is a function and

i) if $A\subseteq R$ then f is called a real variable function.

ii) if $B\subseteq R$ then f is called a real valued function.

iii) if $A\subseteq R, B\subseteq R$ then f is called a real function.

- **Properties:** If f and g are real valued functions with domain A and B respectively, then both f and g are defined on $A\cap B$ when $A\cap B\neq\phi$.
- Let $f:A\to R$ & $g:B\to R$ then

(i) $(f\pm g)(x)=f(x)\pm g(x), \forall x\in A\cap B$

(ii) $(fg)(x)=f(x).g(x), \forall x\in A\cap B$

(iii) $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}, \forall x\in A\cap B$ & $g(x)\neq 0$

(iv) $(f\pm k)(x)=f(x)\pm k, k\in R$

(v) $(kf)(x)=kf(x), k\in R$

(vi) $f^n(x)=\{f(x)\}^n, n>0$

(vii) $|f|(x)=|f(x)|, x\in A$

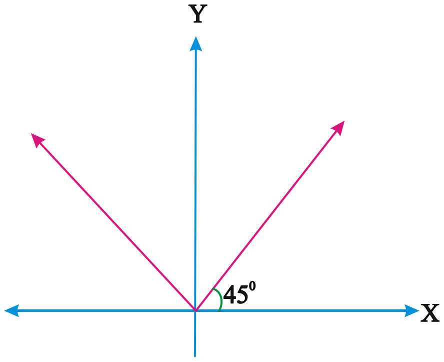
- **Modulus function(Absolute value function) :**

The absolute value or numerical value or the modulus of real number x denoted by | x | is defined as

$$|x|=\begin{cases}-x & if \quad x<0\\ 0 & if \quad x=0\\ x & if \quad x>0\end{cases}$$

Thus we have $|x|\geq 0$ and $|-x|=|x|$

- The graph of $f(x)=|x|$ is



- The domain of $|x|$ is R and range is $[0,\infty)$
- **Properties of modulus function :**

(i) $x^2=|x|^2=|x^2|$

(ii) $|xy|=|x||y|$

(iii) $\left|\frac{x}{y}\right|=\frac{|x|}{|y|}$, provided $y\neq 0$

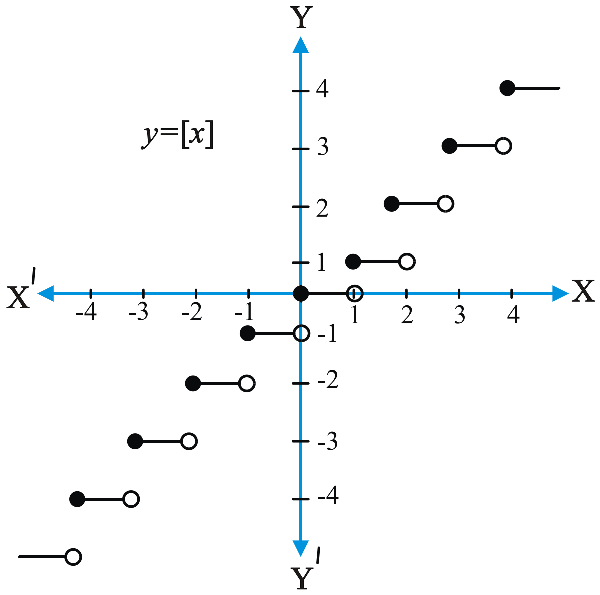
(iv) $|x|\leq a\Rightarrow -a\leq x\leq a;(a\geq 0)$

(v) $|x|\geq a\Rightarrow x\leq -a \ and \ x\geq a;(a\geq 0)$

(vi) $|x\pm y|\leq|x|+|y|$ and if $xy\geq 0$ then $|x+y|=|x|+|y|$

- **Step function(Greatest integer function) (or) floor function :**

The function $f:R\to R$ defined by f(x) $=[x]$ is called the greatest integer function, where $[x]$ equal to integral part of x or greatest integer less than or equal to x. (or) If x is any real number then there exist integers n and n+1 such that $n\leq x<n+1$. Then the integral part of x is defined as n. It is denoted by $[x]$.



→ From the definition of $[x]$, we have

(i) $[x]=-1$, for $-1\le x<0$

(ii) $[x]=0$, for $0\le x<1$

(iii) $[x]=1$, for $1\le x<2$

(iv) $[x]=2$, for $2\le x<3$

→ The domain of $[x]$ is R and range is Z

→ **Properties of greatest integer function :**

(i) If $f(x)=[x+n]=[x]+n$, where $n\in I$ and [.] denotes the greatest integer function

(ii) $x-1<[x]\le x$

(iii) $[x]+[-x]=\begin{cases}0 & if\ \ x\in I\\ -1 & if\ \ x\notin I\end{cases}$

(iv) $[x+y]\ge[x]+[y]$.

(v) $[x]\ge k\Rightarrow x\ge k$, where $k\in Z$

(vi) $[x]\le k\Rightarrow x<k+1$, where $k\in Z$

(vii) $[x]>k\Rightarrow x\ge k+1$, where $k\in Z$

(viii) $[x]<k\Rightarrow x<k$, where $k\in Z$

(ix) $[x]\le x<[x]+1$

(x) $[x+y]=[x]+[y]$ if either x (or) y (or) both are integers

(xi) $[[x]]=[x]$

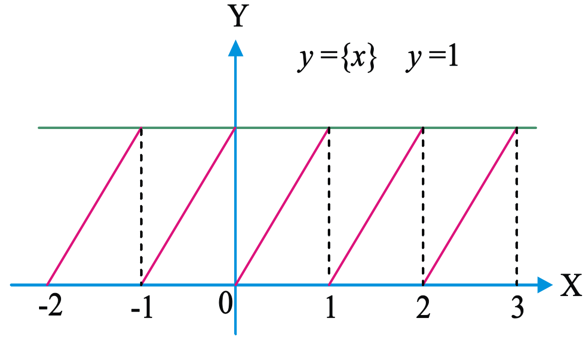
(xii) $\left[\frac{x}{n}\right]+\left[\frac{x+1}{n}\right]+\left[\frac{x+2}{n}\right]+....+\left[\frac{x+n-1}{n}\right]=[x], n\in N$

(xiii) $[x]+\left[x+\frac{1}{n}\right]+\left[x+\frac{2}{n}\right]+...+\left[x+\frac{n-1}{n}\right]=[nx]$

→ **Fractional Part of x:**

For $x\in R$, the function **f(x) = x - [x]** is called fractional part of x, it is denoted by **]x[** or **{x}**. and is defined by $\{x\}=f$ if $x=n+f$ where $n\in I$ and $0\le f<1$

Eg : {2.7} = 0.7, {-3.6} = 0.4, {3} = 0



→ Domain of $\{x\}$ is R and Range of $\{x\}$ is [0,1)

→ The function $\{x\}$ is neither even nor odd

$$\{x\}=\begin{cases}x+2, -2\le x<-1\\ x+1, -1\le x<0\\ x, \quad 0\le x<1\\ x-1, 1\le x<2\\ x-2, 2\le x<3\end{cases}$$

Properties of fractional part function :

i) $\{x\pm n\}=\{x\};\ n\in I$

ii) $\{x\}=0;\ x\in I$

iii) $\{x\}+\{-x\}=\begin{cases}0;\ x\in I\\ 1;\ x\notin I\end{cases}$

iv) $\{x\}-\{-x\}=\begin{cases}0;\ x\in I\\ 2\{x\}-1;\ x\notin I\end{cases}$

v) $\{[x]\}=0$

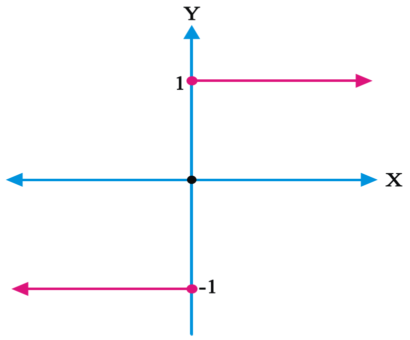
vi) If $f(x)=\frac{1}{\{x\}}$ then $D_f = R - I$

$R_f = (1,\infty)$

✈ **Signum Function:** The signum function or signature function is defined as

$$Sgn(x)=\begin{cases}\frac{|x|}{x} & if \quad x\neq 0\\ 0 & if \quad x=0\end{cases}$$

$$i.e \quad Sgn(x)=\begin{cases}1 & if \quad x>0\\ -1 & if \quad x<0\\ 0 & if \quad x=0\end{cases}$$



✈ The domain of $Sgn(x)$ is R and its Range is $\{-1,0,1\}$

Properties of signum function :

i) $\text{sgn}(Kx)=\text{sgn}(x);\ K\in N$

ii) $x.\text{sgn}(x)=|x|$

iii) $|x|.\text{sgn}(x)=x$

iv) $x.\text{sgn}(x).\text{sgn}(x)=x$

✈ **Periodic function :** A function f(x) is said to be periodic function if there exists a positive real number T, such that $f(x+T)=f(x)$, $\forall x\in D_f$, where the least positive real number T is called its fundamental period.

Range:

✈ The set of values of 'x' for which $f(x)$ is defined is called the domain of $f(x)$ and is denoted by D_f. The set of images of the elements of the domain (out comes) is called the range of $f(x)$ and is denoted by R_f.

Study of range is very important. There are no fixed methods to find range. Some of the methods to find the range are given as follows.

Type (1) Usage of $A.M \geq G.M \geq H.M$ on the given positive quantities

Type (2) The range of $y=ax^2+bx+c$ is $\left[\frac{4ac-b^2}{4a},\infty\right)$ (or) $\left(-\infty,\frac{4ac-b^2}{4a}\right]$ according as $a>0$ or $a<0$

Type (3) To find the range of a function whose domain is $(-\infty,\infty)$, we proceed as follows. Let $y=f(x)$

Now express 'x' interms of 'y' i.e, $x=g(y)$ (if possible)

Now find 'y' for which 'x' is defined.

Type (4) To find the range f. $f(x)$ in $[a,b]$, find $f'(x)$ and put $f'(x)=0\Rightarrow x=\alpha,\beta,\gamma,......$ accept these values if they line in $[a,b]$ find these images and also find $f(a),f(b)$. The least, greatest values thus obtained are called Max, Min. values of $f(x)$ in $[a,b]$

To find range of $f(x)$ in (a,b) follow the above procedure is

$f'(x)=0\Rightarrow x=\alpha,\beta,\gamma,.....$

If m, M are the least, greatest values.

If $\underset{x\to a^+}{Lt}(f(x))$ and $\underset{x\to b^-}{Lt}(f(x))$ both are greater than 'm' then 'm' is least value. Similarly, if $\underset{x\to a^+}{Lt}(f(x))$ and $\underset{x\to b^-}{Lt}(f(x))$ both are less than 'M', then M is the greatest value. Otherwise we cannot decide the least, greatest values.

Type (5) To find the range of $f(x)=\frac{P(x)}{Q(x)}$, where degree is '2' then $yQ(x)=P(x) \Rightarrow$ we have a quad. equation in 'x' $\Delta \geq 0$ ($\because$ 'x' is real) $\Rightarrow$ range will come

Type (6) If various functions of different natures are present in the given problem, then find the common domain and have the range.

Type (7) If $f(x)$ is strictly increasing function in $[a,b]$ then its range $=[f(a),f(b)]$

If $f(x)$ is strictly decreasing in $[a,b]$ then its range $=[f(a),f(b)]$

Note : If $a^2+b^2+c^2=k$, then range of $ab+bc+ca$ is $\left[-\frac{k}{2}, k\right]$

✈ Domain and range of some standard functions :

S.No	*Function*	*Domain*	*Range*
1.	$\sqrt{a^2-x^2}$	[-a,a]	[0,a]
2.	$\frac{1}{\sqrt{a^2-x^2}}$	(-a,a)	$\left[\frac{1}{a},\infty\right)$
3.	$\sqrt{x^2-a^2}$	$R-(-a,a)$	$[0,\infty)$
4.	$\frac{1}{\sqrt{x^2-a^2}}$	$R-[-a,a]$	$(0,\infty)$
5.	$a^x,(a>0, a\neq 1)$	R	$(0,\infty)$
6.	e^x	R	$(0,\infty)$
7.	$\log_a x$ $(a>0, a\neq 1)$	$(0,\infty)$	R
8.	$\lvert x\rvert$	R	$[0,\infty)$
9.	$[x]$	R	Z
10.	$\{x\}=x-[x]$	R	[0,1)
11.	$\sqrt{x}$	$[0,\infty)$	$[0,\infty)$
12.	$\sin x$	R	[-1,1]
13.	$\cos x$	R	[-1,1]
14.	$\tan x$	$R-\left\{(2n+1)\frac{\pi}{2}: n\in Z\right\}$	R
15.	*cosec x*	$R-\{n\pi: n\in Z\}$	$R-(-1,1)$
16.	*sec x*	$R-\left\{(2n+1)\frac{\pi}{2}: n\in Z\right\}$	$R-(-1,1)$
17.	*cot x*	$R-\{n\pi: n\in Z\}$	R
18.	$Sin^{-1}x$	[-1,1]	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
19.	$Cos^{-1}x$	[-1,1]	$[0,\pi]$
20.	$Tan^{-1}x$	R	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
21.	$Cosec^{-1}x$	$R-(-1,1)$	$\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$
22.	$Sec^{-1}x$	$R-(-1,1)$	$\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$
23.	$Cot^{-1}x$	R	$(0,\pi)$
24.	*acosx+bsinx+c*	R	$\left[c-\sqrt{a^2+b^2},c+\sqrt{a^2+b^2}\right]$
25.	*sinh x*	R	R
26.	*cosh x*	R	$[1,\infty)$
27.	*tanh x*	R	(-1,1)
28.	*cosech x*	$R-\{0\}$	$R-\{0\}$
29.	*sech x*	R	(0,1]
30.	*coth x*	$R-\{0\}$	$R-[-1,1]$
31.	$sinh^{-1}x$	R	R
32.	$cosh^{-1}x$	$[1,\infty)$	$[0,\infty)$
33.	$tanh^{-1}x$	(-1,1)	R
34.	$cosech^{-1}x$	$R-\{0\}$	$R-\{0\}$
35.	$sech^{-1}x$	(0,1]	$[0,\infty)$
36.	$coth^{-1}x$	$R-[-1,1]$	$R-\{0\}$

✈ **Functional equations:** If f(x) is a function such that

i) $f(x+y)=f(x)f(y)$ then $f(x)=k^x$ $(k\in R^+)$

ii) $f(x+y)=f(x)+f(y)$ then $f(x)=kx$

iii) $f(xy)=f(x)+f(y)$ then $f(x)=k\log_a x$ $(a\neq 1, a>0)$

iv) If f(x) is a polynomial function such that

$f(x)+f\left(\frac{1}{x}\right)=f(x)f\left(\frac{1}{x}\right)$ then $f(x)=1\pm x^{n}$.

v) If f(x) is a function such that

$f(x+y)+f(x-y)=2f(x)f(y)$ then

$f(x)=\frac{k^{x}+k^{-x}}{2}$ or $f(x)=\cos x$

✈ Some more points to observe :

1. Any function , which is entirely increasing or decreasing in its whole domain , is said to be one-one function
2. If any line parallel to the X-axis cuts the graph of the function at one point , then the function is one-one
3. Any function which is neither increasing nor decreasing in whole domain , then f(x) is many-one. (OR) any continuous function f(x) which has atleast one local maxima or local minima , is many-one
4. Every odd continuous function passes through the origin and it is symmetrical in opp.quadrants
5. $\sin^{n} x, \cos^{n} x, \sec^{n} x, \cos ec^{n} x$ periodic functions with period 2π (or) π according as 'n' is odd or even
6. $\tan^{n} x, \cot^{n} x$ are periodic functins with period π , if 'n' is even or odd
7. $|\sin x|$, $|\cos x|$, $|\tan x|$, $|\cot x|$, $|\sec x|$, $|\cos ecx|$ are periodic functions with the period π
8. Algebraic functions i.e , $x^{2}, 2x^{2}+5x+4, \sqrt{x}, ---etc$ are not periodic
9. Every constant function is always periodic with no fundamental period
10. A function can have infinite periods , but among them the least positive value is called the fundamental period
11. If f(x) is the periodic function with the period T, then the function f(ax+b) is periodic with the period $\frac{T}{|a|}$
12. If f(x) is periodic with T , then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ are also periodic with the same period T
13. If $f(x)$ is periodic with period T then $cf(x), f(x\pm c), f(x)\pm c$ are also periodic with the period T where 'c' is any constant
14. Inverse of a periodic function does not exist. But here by restricting the domain , we can have inverse
15. Strictly increasing and strictly decreasing functions are non-periodic
16. If the given problem is a combination of two or more functions, then find their periods separately and take their L.C.M (if L.C.M is possible). Here L.C.M is the period. IF L.C.M is not possible , then period does not exist
17. L.C.M of

$$\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, ---- = \frac{L.C.M \ of \ a, c, e ----}{H.C.F \ of \ b, d, f ---}$$

18. L.C.M of a rational number with irrational number is not posssible
19. If $h(x)=f_{1}(x)+f_{2}(x)$ $where$ T_{1} and T_{2} are the periods of $f_{1}(x)$ and $f_{2}(x)$

Now period of $h(x)=L.C.M \ of \ T_{1}, T_{2}$

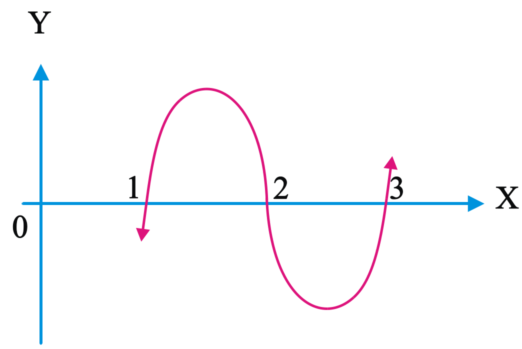
$=\frac{1}{2}(L.C.M \ of \ T_{1}, T_{2})$ if $f_{1}(x)$ and $f_{2}(x)$ are even and pair wise complementary functions.

20. The graphs of f(x) and its inverse $f^{-1}(x)$ are symmetrical about the line y=x
21. Every even function is many-one

EXAMPLES

1. $f: R \rightarrow R$ **defined by** $f(x)=(x-1)(x-2)(x-3)$ **then** $f(x)$ **is**

Sol:

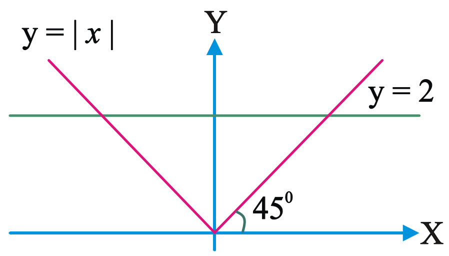


Graphically, $y = f(x) = (x-1)(x-2)(x-3)$, which is clearly many - one and onto.

2:

$f: R \to [0,\infty), f(x) = |x|$ **is an onto function**

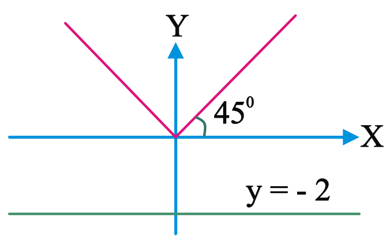
Sol:



3:

$f: R \to R$, $f(x) = |x|$ **is an into function**

Sol:



4:

1) State whether $f(x) = x;\ g(x) = \sqrt{x^2}$ **are identical or not**

Sol. $D_f = D_g = R$ but $R_f = R$ where

$R_g = [0,\infty) \Rightarrow f(x) \neq g(x)$

2) State where $f(x) = \text{sgn}(x^2+1)$ **and** $g(x) = \sin^2 x + \cos^2 x$ **are equal (or) not**

Sol. $D_f = R = D_g$ and $f(x) = 1\ \left(\therefore x^2+1>0\right)$

$g(x) = 1 \Rightarrow f(x) = g(x)$

5:

$f: R \to (-\infty, 1)$ **given by** $f(x) = 1 - 2^{-x}$. **Then** $f^{-1}(x)$ **is**

Sol: Here $f(x)$ is both one-one and onto

Let $y = 1 - 2^{-x}$ or $2^{-x} = 1 - y$ or $-x = \log_2(1-y)$

$f^{-1}(x) = g(x) = -\log_2(1-x)$

6:

Identify whether the function $f(x) = xg(x)g(-x) + \tan(\sin x)$ **is odd or even.**

Sol: $f(-x) = (-x)g(-x).g(x) + \tan(\sin(-x))$

$= -\left(xg(x)g(-x) + \tan(\sin x)\right) = -f(x)$

Hence $f(x)$ is an odd function.

7:

Solve the equation $4\{x\} = x + [x]$

Sol. $4\{x\} = [x] + \{x\} + [x]$

$\Rightarrow \{x\} = \frac{2}{3}[x] \Rightarrow 0 \le \frac{2}{3}[x] < 1$

$0 \le [x] < \frac{3}{2}$

$\Rightarrow \{x\} = 0, \frac{2}{3} \Rightarrow x = 0, \frac{5}{3}$

8:

Find the domain and range of $f(x) = \text{sgn}(x^2+1)$

Sol:

$$f(x) = \text{sgn}(x^2+1) = \begin{cases} 1; & x^2+1>0 \\ -1; & x^2+1<0\ (Not\ possible) \\ 0; & x^2+1=0\ (Not\ possible) \end{cases}$$

$\therefore f(x) = 1,\ x \in R \Rightarrow D_f = R$; $R_f = 1$

9:

The period (if periodic) of the function $f(x) = x - [x-b], b \in R$ **([.] denotes the greatest integer function) is**

Sol: $f(x) = x - [x-b]$

$x - b - [x-b] + b = b + \{x-b\}$

So, $f(x)$ has period 1. $\left(\because Period\ \{x\}\ is\ 1\right)$

10:

The domain of $f(x)=\sqrt{\sin^{-1}(\log_2 x)}$ **is**

Sol: $\sin^{-1}(\log_2 x)\geq 0$ and $-1\leq \log_2 x\leq 1$ and $x>0$

$\log_2 x\geq 0\ \&\ \log_2 x\leq 1\ \&\ x>0$.

$0\leq \log_2 x\leq 1$, $x>0$

$2^0\leq x\leq 2^1 \qquad \therefore x\in[1,2]$

11:

Find the range of $x+\frac{1}{x}$

Sol. Let $x>0$, $A.M\geq G.M$

$$\frac{x+\frac{1}{x}}{2}\geq\sqrt{x.\frac{1}{x}}\Rightarrow x+\frac{1}{x}\geq 2$$

Let $x<0$, Let $x=-y$ where $y>0$

$$y+\frac{1}{y}\geq 2\Rightarrow -x-\frac{1}{x}\geq 2$$

$$\Rightarrow x+\frac{1}{x}\leq -2$$

$\therefore$ Range $=(-\infty,-2]\cup[2,\infty)$

12:

Find the range of $y=\dfrac{x^2+2x+3}{x}$

Sol. $x^2+(2-y)x+3=0$

$\Rightarrow \Delta\geq 0\Rightarrow (2-y)^2-4(3)\geq 0$

$\Rightarrow 4+y^2-4y-12\geq 0\Rightarrow y^2-4y-8\geq 0$

$\Rightarrow y\in\left(-\infty,2-2\sqrt{3}\right]\cup\left[2+2\sqrt{3},\infty\right)$

13:

The range of $f(x)=\sin^{-1}x+\tan^{-1}x$ **is**

Sol: Domain of $f(x)$ is $[-1,1]$

$$\sin^{-1}x\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right],\ \tan^{-1}x\in\left[\frac{-\pi}{4},\frac{\pi}{4}\right]$$

$$-\frac{\pi}{2}-\frac{\pi}{4}\leq \sin^{-1}x+\tan^{-1}x\leq\frac{\pi}{2}+\frac{\pi}{4}$$

$$-\frac{3\pi}{4}\leq f(x)\leq\frac{3\pi}{4}$$

14:

The range of $f:A\to B$ **where A={1,2,3,4} and** $f(x)$ **is defined as** $f(x)=x^2+x+1$ **is**

Sol. f(1) = 1+1+1 = 3

Similarly $f(2)=7,\ f(3)=13, f(4)=21$

Range = {3,7,13,21}

ii) If the domain of $y=f(x)$ is R(i.e. the set of real numbers) or R-{some finite points}or an infinite interval, express x in terms of y. From this,

(a) find y for x to be defined or real, or

(b) If a quadratic equation is formed in terms of x then apply the condition of real roots. $(\Delta\geq 0)$

15:

The range of the function $f(x)=6^x+3^x+6^{-x}+3^{-x}+2$ **is**

Sol: $6^x+6^{-x}\geq 2\sqrt{6^x.6^{-x}}=2$

$3^x+3^{-x}\geq 2\sqrt{3^x.3^{-x}}=2$

$\therefore\ f(x)=6^x+3^x+6^{-x}+3^{-x}+2\geq 2+2+2$,

$f(x)\geq 6$

$\therefore$ range of $f(x)$ is $[6,\infty)$

EXERCISE - I

1. **A function whose graph is symmetrical about the y axis is given by**

   1) $f(x)=\sin\left[\log\left(x+\sqrt{x^2+1}\right)\right]$

   2) $f(x)=\dfrac{\sec^4 x+\operatorname{cosec}^4 x}{x^3+x^4\cot x}$

   3) $f(x+y)=f(x)+f(y)\forall x,y\in R$

   4) $f(x)=x^2$

2. **Let f :{(1,1),(2,3),(0,-1),(-1,-3)} be a function from z to z defined by f(x) = ax+b,for some integers a,b then (a,b) =**

1) (-1,2) 2) (2,-1) 3) (3,-2) 4) (0,3)

3. If $e^{f(x)} = \frac{10+x}{10-x}, x \in (-10,10)$

and $f(x) = k.f\left(\frac{200x}{100+x^2}\right)$ then $k =$

1) 0.5 2) 0.6 3) 0.7 4) 0.8

4. If $f(x) = \frac{\cos^2 x + \sin^4 x}{\sin^2 x + \cos^4 x}$ **for** $x \in R$ **then** $f(2016) =$

1) 1 2) 2 3) 3 4) 4

5. If $f = \{(-2,4),(0,6),(2,8)\}$ **and** $g = \{(-2,-1),(0,3),(2,5)\}$**, then**

$\left(\frac{2f}{3g} + \frac{3g}{2f}\right)(0) =$

1) 1/12 2) 25/12 3) 5/12 4) 13/12

6. If f = {(-1,3), (0,2), (1,1)} then the range of $f^2 - 1$ is

1) {0, 8} 2) {0,3,8} 3) {0,1,3} 4) {0,2,8}

7. If f(x) = sin $\left(\frac{\pi}{3}[x] - x^2\right)$ **then the value of** $f\left(\sqrt{\frac{\pi}{3}}\right)$ **is**

1) 1 2) -1 3) 0 4) $\frac{-3}{4}$

8. If $f = \{(a,1),(b,-2),(c,3)\}$, $g = \{(a,-2),(b,0),(c,1)\}$

then $f^2 + g^2 =$

1) $\{(a,-1),(b,-2),(c,4)\}$

2) $\{(a,3),(b,-2),(c,2)\}$

3) $\{(a,-4),(b,-4),(c,9)\}$

4) $\{(a,5),(b,4),(c,10)\}$

9.The function y = f(x) such that f $\left(x+\frac{1}{x}\right) = x^2 + \frac{1}{x^2}$

1) $2 - x^2$ 2) $x^2 - 2$ 3) $x^2 + 4$ 4) $4x^2 - 2$

10. If $f(x+y, x-y) = xy$ **then the arithmetic mean of** $f(x,y)$ **and** $f(y,x)$ **is**

1) x 2) y 3) 0 4) xy

11. If $f(x) = ax^5 + bx^3 + cx + d$ **is odd then**

1) a = 0 2) b = 0 3) c = 0 4) d = 0

12. Let $f(x) = \frac{x}{e^x - 1} + \frac{x}{2} + 1$ **, then** f **is**

1) an odd funciton 2) an even function
3) both odd and even 4) neitheer odd nor even

13. Which of the following is an even function

1) $f(x) = \frac{a^x + a^{-x}}{a^x - a^{-x}}$ 2) $f(x) = \frac{a^x + 1}{a^x - 1}$

3) $f(x) = x\frac{a^x - 1}{a^x + 1}$ 4) $f(x) = \log_2\left(x + \sqrt{x^2+1}\right)$

14. If A = {1, 2, 3}, B = {1, 2}then the number of functions from A to B are

1) 6 2) 8 3) 9 4) 32

15. Let $A = \{1,2,3\}$ **and** $B = \{a,b,c\}$ **. If** l **is number of funcitons from A to B and** m **is number of one-one functions from A to B, then**

1) l is 9 2) m is 9 3) l is 27 4) m is16

16. The number of one-one functions that can be defined from A = $\{4,8,12,16\}$ **to B is 5040, then n(B)=**

1) 7 2) 8 3) 9 4) 10

17. The number of non-bijective mappings that can be defined from A = $\{1,2,7\}$ **to itself is**

1) 21 2)27 3) 6 4) 9

18. The number of constant functions possible from R to B where B = $\{2,4,6,8,....24\}$ **are**

1) 24 2) 12 3) 8 4) 6

19. If B = $\{1,2,3\}$ **and A =** $\{4,5,6,7,8\}$ **then the number of surjections from A to B is**

1) 81 2) 64 3) 48 4) 150

20. If n (A) = 4 and n(B) = 6, then the number of surjections from A to B is

1) 4^6 2) 6^4 3) 0 4) 24

21. The number of possible many to one functions from A = {6, 36} to B = {1, 2, 3, 4, 5} is

1) 32 2) 25 3) 5 4) 20

22. The number of non-surjective mappings that can be defined from A = $\{1,4,9,16\}$ **to**

$B = \{2,8,16,32,64\}$ **is**

1) 1024 2) 20 3) 505 4) 625

23. If f(x) = 2x + 1 **and** g(x) = x² + 1 **then** go(fof)(2) =

1) 112 2) 122 3) 12 4) 124

24. If $f(x)=[x], g(x)=x-[x]$ **then which of the following functions is a zero function**

1) $(f+g)(x)$ 2) $(fg)(x)$

3) $(f-g)(x)$ 4) $(fog)(x)$

25. Let $f(x)=\frac{Kx}{x+1}(x\neq -1)$ **then the value of K for which** $(\text{fof})(\text{x})=\text{x}$ **is**

1) 1 2) -1 3) 2 4) $\sqrt{2}$

26. If $f(x)=(a-x^n)^{\frac{1}{n}}$ **then** $fof(x)$ **is**

1) x 2) $a-x$ 3) x^n 4) $x^{\frac{-1}{n}}$

27. If $f(x)=\frac{x}{\sqrt{1+x^2}}$ **then** fofof(x) =

1) $\frac{x}{\sqrt{1+3x^2}}$ 2) $\frac{x}{\sqrt{1-x^2}}$

3) $\frac{2x}{\sqrt{1+2x^2}}$ 4) $\frac{x}{\sqrt{1+x^2}}$

28. If f : $[1,\infty)\to B$ **defined by** $f(x) = x^2 - 2x + 6$ **is a surjection then B =**

1) $[1,\infty)$ 2) $[5,\infty)$ 3) $[6,\infty)$ 4) $[2,\infty)$

29. $f:(-\infty,\infty)\to(-\infty,\infty)$ **is defined by** $f(x)=ax+b, a,b\in R \quad (a\neq 0)$ **then f is**

1) injective but not surjective
2) surjective but not injective
3) bijective
4) neither injective nor surjective

30. $f:Q\to Q$ **is defined by f(x) = 15x + 7 is**

1) injective only 2) surjective only
3) bijective
4) neither injective nor surjective

31. $f:(0,\infty)\to[0,\infty)$ **defined by** $f(x)=x^2$ **is**

1) one-one but not onto
2) onto but not one-one
3) bijective 4) neither one-one nor onto

32. f : Z→Z defined as f(x) =[x] then f is

1) not a function
2) many-to-one function
3) into function 4) identity function

33. $f:R\to R$ **defined by** $f(x)=\frac{x}{x^2+1}, \forall x\in R$ **is**

1) one - one 2) onto
3) bijective
4) neither one one nor onto

34. $f:(-\infty,\infty)\to(0,1]$ **defined by** $f(x)=\frac{1}{x^2+1}$ **is**

1) one-one but not onto
2) onto but not one-one
3) bijective
4) neither one-one nor onto

35. The function $f:R\to R$ **defined by** $f(x)=4^x+4^{|x|}$ **is**

1) One - one and into 2) Many - one and into
3) One - one and onto 4) Many-one and onto

36. Period of $f(x)=e^{\cos\{x\}}+\sin\pi[x]$ **is (**$[\bullet]$ **and** $\{\bullet\}$ **denote the greatest integer function and fractional part function respectively)**

1) 1 2) 2 3) π 4) 2π

37. The period of $f(x)=\sqrt{x-[x]}$ **is**

1) no fundamental period

2) $\frac{1}{2}$ 3) 1 4) 2

38. Let $f(x)$ **be periodic and** k **be a positive real number such that** $f(x+k)+f(x)=0$ **for all** $x\in R$**. Then the period of** $f(x)$ **is**

1) k 2) 2k 3) 4k 4) 8k

39. The period of $x\cos x$ **is**

1) 2π 2) π 3) $\frac{\pi}{2}$ 4) non periodic

40. The domain of $f(x)=\frac{x^2+2x+1}{x^2-x-6}$

1) R – {3, –2} 2) R – {–3, 2}
3) R – {3, –2] 4) R – (3, –2)

41. The domain of f(x) = $\frac{1}{\log|x|}$ **is**

1) $R-\{0\}$ 2) $R-\{0,1\}$
3) R - {-1, 0, 1} 4) $(-\infty,\infty)$

42. The domain of the function f(x) = $\sqrt{\log_{16} x^2}$ **is**

1) $x=0$ 2) $|x|\geq 4$ 3) $|x|\geq 1$ 4) $|x|\geq 2$

43. The domain of $f(x)=\frac{1}{[x]-x}$ **is**

1) R 2) Z 3) R - Z 4) Q-{0}

44. The domain of f(x) = $\sqrt{x-2}+\frac{1}{\log(4-x)}$ **is**

1) $[2,\infty)$ 2) $(-\infty,4)$
3) $[2,3)\cup(3,4)$ 4) $[3,\infty)$

45. The domain of $\log_a \sin^{-1} x$ **is (a>0,** $a\neq 1$**)**

1) $0<x\leq 1$ 2) $0\leq x\leq 1$
3) $0\leq x<1$ 4) $0<x<1$

46. The domain of $f(x)=\log\{(x-3)(6-x)\}$ **is**

1) $(3,\infty)$ 2) $(3,6)$ 3) $(0,\infty)$ 4) $(-\infty,\infty)$

47. The domain of $f(x)=\sqrt{-x^2}$ **is**

1) $(0,\infty)$ 2) $(-\infty,0)$ 3) $\{0\}$ 4) $(1,\infty)$

48. Range of $\frac{|x-4|}{x-4}$ **is**

1) R – {4} 2) R 3) {–1, 1} 4) R – {–1, 1}

49. If x is positive, the values of $f(x)=-3\cos\sqrt{3+x+x^2}$ **lie in the intervel**

1) [-1, 3] 2) [-3, 3] 3) [0 , 3] 4) [-3,0]

50. The range of $f(x)=\frac{\sin\pi[x^2-1]}{x^4+1}$ **is, where [.] is greatest integer function**

1) R 2) [-1,1] 3) {0,1} 4) {0}

51. Range of $\sqrt{9-x^2}$ is

1) [0, 3] 2) [–3, 3] 3) [–3, 0] 4) R

52. Range of $f(x)=\frac{1}{1-2\cos x}$ **is**

1) $\left[\frac{1}{3},1\right]$ 2) $\left[-1,\frac{1}{3}\right]$

3) $(-\infty,-1]\cup\left[\frac{1}{3},\infty\right)$ 4) $\left[-\frac{1}{3},1\right]$

53. $f=\{(x,\frac{x^2}{x^2+1}):x\in R\}$ **,be a function R into R,range of 'f'**

1) [0,1) 2) $(-\infty,\infty)$ 3) $(0,\infty)$ 4) R^+

54. Range of the function $f(x)=\sqrt{[x]-x}$ **is**

1) R 2) $\{1\}$ 3) $\{0\}$ 4) $(0,\infty)$

55. If $f:\{1,2,3,.....\}\to\{0,\pm1,\pm2,....\}$ **is defined by** $f(n)=\begin{cases} n/2 & if\ n\ is\ even \\ -\left(\frac{n-1}{2}\right) & if\ n\ is\ odd\end{cases}$

then $f^{-1}(-100)$ **is**

1) 100 2) 199 3) 201 4) 200

56. If $f(x)=Sin^{-1}\{3-(x-6)^4\}^{1/3}$ **then** $f^{-1}(x)=$

1) $6+\sqrt[4]{3+\sin^3 x}$ 2) $6+\sqrt[4]{3-\sin^3 x}$

3) $6+\sqrt[4]{3+\sin x}$ 4) $6+\sqrt[4]{3-\sin x}$

57. $f:R\to R$ **is a function defined by** $f(x)=10x-7.$ *If* $g=f^{-1}$ *then* $g(x)=$

1) $\frac{1}{10x-7}$ 2) $\frac{1}{10x+7}$ 3) $\frac{x+7}{10}$ 4) $\frac{x-7}{10}$

58. If the function $f:R\to R$ **be such that** $f(x)=x-[x]$**, where** $[x]$ **denotes the greatest integer less than or equal to x then** $f^{-1}(x)$ **is**

1) $\frac{1}{x-[x]}$ 2) $[x]-x$

3) not defined 4) $x-[x]$

59. If $f(x+y)=f(x)f(y)$ **and** $f(5)=32$ **then** $f(7)=$

1) 35 2) 36 3) $\frac{7}{5}$ 4) 128

60. If $f(x)$ **is a function such that**

$f(xy)=f(x)+f(y)$ **and** $f(2)$**=1 then** $f(x)=$

1) x^2 2) 2^x 3) $\log_2 x$ 4) $\log_x 2$

61. If f satisfies the relation

$f(x+y)+f(x-y)=2\ f(x).f(y)\forall\ x,y\in R$

and $f(0)\neq 0$; then $f(10)-f(-10)=$

1) 0 2) 1 3) 2 4) 3

62. If $f(x)$ **is a polynomial in** $x(>0)$ **satisfying the equation**

$f(x)+f(1/x)=f(x).f(1/x)$ **and**

$f(2)=-7$**, then** $f(3)=$

1) -26 2) -27 3) -28 4) -29

KEY

1) 4	2) 2	3) 1	4) 1	5) 2	6) 2
7) 3	8) 4	9) 2	10) 3	11) 4	12) 2
13) 3	14) 2	15) 3	16) 4	17) 1	18) 2
19) 4	20) 3	21) 3	22) 4	23) 2	24) 4
25) 2	26) 1	27) 1	28) 2	29) 3	30) 3
31) 1	32) 4	33) 4	34) 2	35) 1	36) 1
37) 3	38) 2	39) 4	40) 1	41. 3	42) 3
43) 3	44) 3	45) 1	46) 2	47) 3	48) 3
49) 2	50) 4	51) 1	52) 3	53) 1	54) 3
55) 3	56) 2	57) 3	58) 3	59) 4	60) 3
61) 1	62) 1				

SOLUTIONS

1. A funciton whose graph is symmbrical about the y-axis must be even since sinx and $\log\left(x+\sqrt{x^2+1}\right)$ are odd function.

   Therefore $\sin\left(\log\left(x+\sqrt{x^2+1}\right)\right)$ must be odd

   Also, $\frac{\sec^4+\cos ec^4 x}{x^3+x^4\cot x}$ is an odd function Now,

   let $f(x)+y=f(x)+f(y)\forall x,y\in R$

   $\therefore f(0+0)=f(0)+f(0)\therefore f(0)=0$

   $f(x-x)=f(x)+f(-x)\text{or}\,0=f(x)+f(-1)$

   i.e., $f(-x)=-f(x)\therefore f(x)$ is odd.

2. Given $f:\{(1,1)(2,3),(0,-1),(-1,-3)\ f(x)=ax+b$

   $f(x)=ax+b$

   $f(1)=a(1)+b=1$

   $a+b=1$__________(1)

   $f(2)=a(2)+b=3$

   $2a+b=3$__________(2)

   $(1)-(2)\Rightarrow -a=-2$

   $a=2,\ b=-1$s

3. Given $e^{f(x)=\frac{10+x}{10-x}}$

   $$f(x)=\log_e\left(\frac{10+x}{10-x}\right)$$

   $$f\left(\frac{200x}{100+x^2}\right)=\log\left(\frac{10+\frac{200x}{100+x^2}}{10-\frac{200x}{100+x^2}}\right)$$

   $$\log_e\left(\frac{100+x^2+20x}{100+x^2-20x}\right)=\log_e\left(\frac{10+x}{10-x}\right)^2=2\log_e\left(\frac{10+x}{10-x}\right)$$

   $$f(x)=\frac{1}{2}f\left(\frac{200x}{100+x^2}\right)$$

   $$k=\frac{1}{2}$$

4. $f(x)=\frac{\cos^2 x+\sin^2 x\left(1-\cos^2 x\right)}{\sin^2 x+\cos^2 x\left(1-\sin^2 x\right)}=1$

5. $\left(\frac{2f}{3g}+\frac{3g}{2f}\right)(0)=\frac{2f(0)}{3g(0)}+\frac{3g(0)}{2f(0)}=\frac{25}{12}$

6. Range of $f^2-1=\{3^2-1,2^2-1,1^2-1\}$

7. $\left[\sqrt{\pi/3}\right]=1\Rightarrow f(x)=\sin\left(\frac{\pi}{3}-x^2\right)$

8. $f^2+g^2=\{(a,1+4),(b,4+0),(c,9+1)\}$

9. $f\left(x+\frac{1}{x}\right)=\left(x+\frac{1}{x}\right)^2-2$

10. Let $x+y=p, x-y=q$. Then

$$f(p,q)=\frac{p+q}{2}.\frac{p-q}{2}=\frac{p^2-q^2}{4}$$

$\therefore\ f(x,y)=\frac{x^2-y^2}{4}$ and $f(y,x)=\frac{y^2-x^2}{4}$

11. $f(-x)=-f(x)$

12. Given $f(x)=\frac{x}{e^x-1}+\frac{x}{2}+1$

now $f(-x)=\frac{-x}{e^{-x}-1}-\frac{x}{2}+1=\frac{-xe^x}{1-e^x}-\frac{x}{2}+1$

$=\frac{-xe^x+x-x}{1-e^x}-\frac{x}{2}+1=x+\frac{x}{e^x-1}-\frac{x}{2}+1$

$f(-x)=\frac{x}{e^x-1}+\frac{x}{2}+=f(x)$

$\therefore f(x)$ is even function

13. $f(x)=a\frac{a^x-1}{a^x+1}$ now

$f(-x)=-x\frac{a^{-x}-1}{a^{-x}+1}=-x\frac{(1-a^x)}{1+a^x}=\frac{(a^x-1)}{a^x+1}=f(x)$

f (x) is even function

14. $n(B)^{n(A)}$

15. The no. of functions from A to B = 3^3 = 27

16. Given $A=\{4,8,12,16\}$ given one-one functions from A to B is 5040

$n(B)_{P_4}=5040\,n(B)_{P_4}=10\times9\times8\times7=10_{P_4}$

$n(B)=10$

17. $3^3-3!$

18. The no. of constant functions from A to B is n(B).

19. The number of surjections from A to B is

$n^r-{}^nC_1(n-1)^r+{}^nC_2(n-2)^r-{}^nC_3(n-3)^r+$

$....+(-1)^{n-1}.{}^nC_{n-1}$ where $n(A)=r$ & $n(B)=n$

20. If n(A) < n(B) then the no. of surjections from A to B = 0

21. No. of many-one functions from A to B

$=n(B)^{n(A)}-{}^{n(B)}P_{n(A)}$

22. $5^4-0=625$

23. $g(f(f(2)))=g(f(5))=g(11)=121+1=122$

24. $(fog)(x)=f[g(x)]=f[x-[x]]$

$=f(\{x\})=[\{x\}]=0, \sin 0\le\{x\}<1$

25. Given $f(x)=\frac{kx}{x+1}$ and $fof(x)=x$

$$f(f(x))=x$$

$f\left(\frac{kx}{x+}\right)=x$

$$\Rightarrow\frac{k\left(\frac{kx}{x+1}\right)}{\frac{kx}{x+1}+1}=x\Rightarrow k^2=kx+x+1=k=-1$$

26. $fof(x)=f[f(x)]=f\left[\left(a-x^n\right)^{\frac{1}{n}}\right]=x$

27. $f(f(.........n\ \ times))=\frac{x}{\sqrt{1+nx^2}}$

28. $x^2-2x+1+5=(x-1)^2+5\ge5$

29. $f(x_1)=f(x_2)\Rightarrow x_1=x_2\Rightarrow f$ is one-one

let $f(x)=y\Rightarrow ax+b=y\Rightarrow x=\frac{y-b}{a}\in R$

$\forall y\in R\Rightarrow f$ is onto

30. $f(x_1)=f(x_2)\Rightarrow x_1=x_2\Rightarrow f$ is one-one

Let $f(x)=y\Rightarrow 15x+7=y\Rightarrow x=\frac{y-7}{15}\in Q$

$\forall y\in Q\Rightarrow f$ is onto

31. $f(x_1)=f(x_2)\Rightarrow x_1^2=x_2^2\Rightarrow x_1=x_2\Rightarrow f$ is one-one

Range of $f = (0, \infty)$ not equal to codomain

32. $f(x) = [x] = x \ \forall x \in R$, f is identity function

33. $f(2) = f\left(\frac{1}{2}\right)$, f(1) has no pre image.

34. $f(-1) = f(1) \Rightarrow$ f is not one one
range = codomain $\Rightarrow$ f is onto

35. Since for different $x, 4^x$ *and* $4^{|x|}$ are different postive numbers
$\therefore$ f is one one also, f is not onto as its range $(0, \infty)$ is a proper subset of its co domain.

36. Given $f(x) = e^{\cos\{x\}} + \sin \pi[x]$ period of $\{x\}$
i.e. of $x - [x]$ is
$\therefore$ period of f (x) is 1

37. $f(x) = \sqrt{\{x\}}$ period is one

38. We have, $f(x+k) + f(x) = 0$ ...(1) for all $x \in R$. Putting $x + k$ for x, weget

$f(x+2k) + f(x+k) = 0$ ...(2)

$[(2) - (1)] \Rightarrow f(x+2k) - f(x) = 0 \Rightarrow$

$f(x+2k) = f(x)$

The shows that $f(x)$ is periodic with period 2k.

39. no value of T exist such that $f(x+T) = f(x)$

40. $f(x) = \dfrac{x^2+2x+1}{x^2-x-6}$ is define if $x^2 - x - 6 \neq 0$

$x^2 - 3x + 2x - 6 \neq 0$

$x(x-3) + 2(x-3) \neq 0$

$(x+2)(x-3) \neq 0$

$x \neq -2 (or) x \neq 3$

domain R - $\{-2, 3\}$

41. $\log|x| \neq 0$ and $x \neq 0 \Rightarrow x \neq \{0, 1, -1\}$

42. $x \neq 0$ and $x^2 \geq 1 \Rightarrow |x| \geq 1$

43.

$f(x) = \dfrac{1}{x - [x]}$ is define if $x - [x] \neq 0$ $x \neq [x]$

domain R - Z

44. $f(x) = \sqrt{x-2} + \dfrac{1}{\log(4-x)}$ is define if

$x - 2 \geq 0$ *and* $4 - x \neq 1, 4 - x > 0$

$x \geq 2$ *and* $x \neq 3 \quad x < 4$

domain of f(x) is $[2,3] \cup [3,4]$

45. $\sin^{-1} x > 0 \Rightarrow 0 < x \leq 1$

46. $(x - 3)(6 - x) > 0$

47. $\sqrt{f(x)}$ defined $f(x) \geq 0$.

48. $x \geq 4$ then $f(x) = 1$; $x < 4$ then $f(x) = -1$

49. Range of cos x = [–1, 1]

50. $[x^2 + 1]$ is an integer $\Rightarrow$

$\sin n\pi = 0 \forall x \in R \Rightarrow f(x) = 0 \forall x \in R$

51. substitute domain values

52. $-2 \leq -2\cos x \leq 2$, $-1 \leq 1 - 2\cos x \leq 3$

53. $x^2 < x^2 + 1 \ \therefore \dfrac{x^2}{x^2+1} \in [0, 1)$

54. Domain of x = z, $\therefore [x] - x = 0 \ \forall x \in z$

55. Take $f^{-1}(x) = y$

$\Rightarrow x = f(y) = 5x - 6 \Rightarrow y = f^{-1}(x) = \dfrac{x+6}{5}$

Let $f^{-1}(-100) = x \Rightarrow f(x) = -100$

$\Rightarrow \dfrac{-(x-1)}{2} = -100 \Rightarrow x = 201$

56. $y = f(x) \Rightarrow x = f^{-1}(y)$

$y = \sin^{-1}\left\{3 - (x-6)^4\right\}^{1/3}$

$\sin y = \left\{3 - (x-6)^4\right\}^{1/3}$

$\sin^3 y = 3-(x-6)^4 \Rightarrow (x-6)^4 = 3-\sin^3 y$

$x-6 = (3-\sin^3 y)^{1/4}$

$f^{-1}(x)=y \Rightarrow x=f(y)=\sin^{-1}\left(3-(y-6)^4\right)^{1/3}$

$f^{-1}(x) = 6+\sqrt[4]{3-\sin^3 x}$

$\Rightarrow f^{-1}(x) = 6+\sqrt[4]{3-\sin^3 x}$.

57. Let $f^{-1}(x) = y \Rightarrow x = f(y) = 10y-7$

$\Rightarrow y = f^{-1}(x) = \frac{x+7}{10} = g(x)$

58. $x-[x]=0$ for all integral values of x. Therefore, the function is many one-and, therefore, not defined.

$\Rightarrow y^2 - xy + 1 = 0$

59. Take $f(x) = k^x$

60. Take $f(x) = \log_a x$

61. If y = x then $f(2x)+f(0)=2[f(x)]^2$
if y = -x then

$f(2x)+f(0) = 2f(x).f(-x)$

$\Rightarrow 2.[f(x)]^2 = 2.f(x).f(-x) \Rightarrow f(x) = f(-x)$

$\Rightarrow f(x)-f(-x)=0 \Rightarrow f(10)-(-10)=0$.

62. $f(x)+f(1/x)=f(x)f(1/x)$ and $f(2)=-7$ take $f(x)=1-x^n$

$f(2)=1-2^n=-7$

$2^n = 8 = 2^3$

$n=3 \therefore f(x)=1-x^3$

$f(3)=1-3^3=1-27=-26$

EXERCISE - II

1. If $f\left(2x+\frac{y}{8}, 2x-\frac{y}{8}\right)=xy,$ then

$f(m,n)+f(n,m)=0$

1) Only when m = n 2) Only when $m \neq n$
3) Only when m = - n 4) For all m and n

2. Let $g(x)$ be a function defined on $[-1,1]$. If the area of the equilateral triangle with two of its vertices at $(0,0)$ and $(x,g(x))$ is $\sqrt{3}/4$, then the function $g(x)$ is

1) $g(x)=\pm\sqrt{1-x^2}$ 2) $g(x)=\sqrt{1-x^2}$
3) $g(x)=-\sqrt{1+x^2}$ 4) $g(x)=\sqrt{1+x^2}$

3. If $f:R\to R$ is defined by $f(x)=x-[x]-\frac{1}{2}$ for $x\in R$, where $[x]$ is the greatest integer not exceeding x, then $\left\{x\in R: f(x)=\frac{1}{2}\right\}=$

1) Z 2) N 3) ϕ 4) R

4. Suppose $f:[2,2]\to R$ is defined by

$$f(x)=\begin{cases} -1 & for\ -2\le x\le 0 \\ x-1 & for\ \ 0\le x\le 2 \end{cases}$$

then the $\{x\in(-2,2): x\le 0$ and $f(|x|)=x\}=$

1) $\{-1\}$ 2) $\{0\}$ 3) $\{-1/2\}$ 4) ϕ

5. If $f(x)=\begin{cases} x^2\sin\frac{\pi x}{2} & |x|<1 \\ x\,|x| & |x|\ge 1\end{cases}$ then $f(x)$ is

1) an even funciton 2) an odd function
3) a periodic function
4) neither odd nor even

6. $f(x)=\dfrac{\cos x}{\left[\dfrac{2x}{\pi}\right]+\dfrac{1}{2}}$, **where x is not an integral multiple of π and $[\bullet]$ denotes the greatest integer function is**

1) an odd function
2) even function
3) neither odd nor even
4) both even and odd

7. **Let $f(x)=|x-2|+|x-3|+|x-4|$ and $g(x)=f(x+1)$. Then $g(x)$ is**

1) an even function
2) an odd function
3) neither even nor odd
4) periodic

8. **If $f(x)=sgn\{x\}$ (where $\{.\}$ denotes the fractional part of x), is**

1) even function
2) odd function
3) neither even nor odd
4) constant function

9. **If $f(x)=\log_a x$ and $F(x)=a^x$, then $F[f(x)]$ is**

1) $f[F(x)]$
2) $f[F(2x)]$
3) $f|F(2x)|$
4) $F(x)$

10. **Let $g(x)=1+x-[x]$ and**

$$f(x)=\begin{cases}-1, & x<0\\ 0, & x=0\\ 1, & x>0\end{cases}$$, **then for all $x, f(g(x))=$**

1) x
2) 1
3) $f(x)$
4) $g(x)$

11. **If $f:R\to R$ and $g:R\to R$ are defined by $f(x)=2x+3$ and $g(x)=x^2+7$, then the values of x such that $g(f(x))=8$ are.**

1) 1,2
2) -1,2
3) -1,-2
4) 1,-2

12. **If $f(x)$ and $g(x)$ are two functions with $g(x)=x-\dfrac{1}{x}$ and $fog(x)=x^3-\dfrac{1}{x^3}$, then $f(x)=$**

1) x^3+3x
2) $x^2-\dfrac{1}{x^2}$
3) $1+\dfrac{1}{x^2}$
4) $3x^2+\dfrac{3}{x^4}$

13. **If $f(x)=\sin^2 x$ and the composite functions $g\{f(x)\}=|\sin x|$, then the function $g(x)=$**

1) $\sqrt{x-1}$
2) $\sqrt{x}$
3) $\sqrt{x+1}$
4) $-\sqrt{x}$

14. **If $f:R\to R$ and $g:R\to R$ are given by $f(x)=|x|$ and $g(x)=[x]$ for each $x\in R$, then $\{x\in R: g(f(x))\le f(g(x))\}=$**

1) $Z\cup(-\infty,0)$
2) $(-\infty,0)$
3) Z
4) R

15. **Let $g:R\to R$ be given by $g(x)=3+4x$. If $g^n(x)=gogo....og(x)$, and $g^n(x)=A+Bx$ then A and B are**

1) $2^{n+1}-1, 2^{n+1}$
2) $4^n-1, 4^n$
3) $3^n, 3^n+1$
4) $5^n-1, 5^n$

16. **If f(x) = $|x-1|+|x-2|+|x-3|$ when 2<x<3 is**

1) one one function only
2) an onto function only
3) into function
4) identify function

17. **Let A = [-1,1]= B then which of the following function from A to B is bijective function**

1) $f(x)=\dfrac{x}{2}$
2) $g(x)=|x|$
3) $h(x)=x^2$
4) $k(x)=\sin\dfrac{\pi x}{2}$

18. $f(x)=\begin{cases}0, if\ x\ is\ rational\\ x, if\ x\ is\ irrational\end{cases}$

$g(x)=\begin{cases}0, if\ x\ is\ irrational\\ x, if\ x\ is\ rational\end{cases}$ **Then $f-g$ is**

1) one-one and into
2) neither one-one nor onto
3) many one and onto
4) one-one and onto

19. **Which of the following function is not periodic**

1) $\dfrac{2^x}{2^{[x]}}$
2) $\sin^{-1}(\{x\})$
3) $\sin^{-1}(\sqrt{\cos x})$
4) $\sin^{-1}(\cos(x^2))$

20. **Let $f(x)=nx+n-[nx+n]+\tan\dfrac{\pi x}{2}$, where $[x]$ is the greatest integer $\le x$ and $n\in N$. It is**

1) a periodic function of period 1

2) a periodic function of period 4
3) not periodic
4) a periodic function of period 2

21. If f is periodic, g is polynomial function and $f\left(g(x)\right)$ is periodic and $g(2)=3, g(4)=7$ then $g(6)$ is

1) 13 2) 15 3) 11 4) 21

22. period of

$$f(x)=[x]+[2x]+[3x]+...+[nx]-\frac{n(n+1)}{2}x,$$

where $n \in N$ is

1) n 2) 1 3) $\frac{1}{n}$ 4) 2

23. Let $f(x)=\cos 3x+\sin\sqrt{3}x$. Then $f(x)$ is

1) a periodic function of period 2π
2) a periodic function of period $\sqrt{3}\pi$
3) not a periodic function
4) a periodic function of period π

24. If $f(x)=e^{x-[x]+|\cos\pi x|+|\cos 2\pi x|+...+|\cos n\pi x|}$ then the period of $f(x)$ is

1)1 2)$\frac{1}{n}$ 3)$\frac{2}{n}$ 4) no fundamental period

25. If $f(x)$ and $g(x)$ are periodic functions with period 7 and 11, respectively. Then the period of $F(x)=f(x)g\left(\frac{x}{5}\right)-g(x)f\left(\frac{x}{3}\right)$ is

1) 177 2) 222 3) 433 4) 1155

26. The domain of f(x) = $\log_x$ **(9-x^2) is**

1) (-3,3) 2) $(0,\infty)$
3) $(0,1)\cup(1,\infty)$ 4) $(0,1)\cup(1,3)$

27. The domain of $f(x)=\sqrt{1-\sqrt{1-\sqrt{1-x^2}}}$ is

1) $[0,1]$ 2) $[-1,1]$ 3) $(-\infty,\infty)$ 4) $(-1,1)$

28. The domain of the function

$$f(x)=\frac{1}{\sqrt{\{\sin x\}+\{\sin(\pi+x)\}}}$$ **where $\{\bullet\}$ denotes the fractional part, is**

1. $[0,\pi]$ 2. $(2n+1)\frac{\pi}{2}, n\in Z$
3. $(0,\pi)$ 4. $R-\left\{\frac{n\pi}{2}, n\in Z\right\}$

29. Domain of $\frac{1}{\sqrt{[x]^2-[x]-2}}$

1) $R/[-1,3)$ 2) $(-\infty,-3)\cap[3,\infty)$
3) $[2,\infty)$ 4) $(-\infty, 3]$

30. If $\alpha\in(0,\frac{\pi}{2})$, then $\sqrt{x^2+x}+\frac{\tan^2\alpha}{\sqrt{x^2+x}}$ is always greater than or equal to ($x\neq 0,-1$)

1) 2 2) 1 3) $2\tan\alpha$ 4) $2Sec^2\alpha$

31. The range of $f(x)=\mathrm{Sin}^{-1}\left[\frac{1}{2}+x^2\right]$ is $[\cdot]$ denotes greatest integer function)

1) $\left\{-\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$ 2) $\left\{0, \frac{\pi}{2}\right\}$
3) $\left\{\frac{\pi}{2}\right\}$ 4) $\{0, \pi\}$

32. The range of $x^2+4y^2+9z^2-6yz-3xz-2xy$ is

1) ϕ 2) R 3) $[0,\infty)$ 4) $(-\infty,0)$

33. The range of
$f(x)=\sin^{-1}x+\cos^{-1}x+\tan^{-1}x$ is

1) $(0,\pi)$ 2) $\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$
3) $\left[\frac{-\pi}{4},\frac{\pi}{4}\right]$ 4) $\left[0,\frac{3\pi}{4}\right]$

34. The range of f(x) = $8\sqrt{2}$ sin $\sqrt{\frac{\pi^2}{16}-x^2}$ is

1) [-1,1] 2) [0, 1] 3) [0,8] 4) [0,4]

35. The range of f(x) = $\frac{x^4}{1+x^8}$ is

1) $[0,\infty)$ 2) $\left[0,\frac{1}{2}\right]$ 3) [0, 1] 4) $(-\infty,\infty)$

36. If $a^2+b^2+c^2=1$ **then the range of** ab+bc+ca **is**

1) $[1,\infty)$ 2) $\left[\frac{-1}{2},\infty\right]$

3) $\left(-\frac{1}{2},1\right)$ 4) $\left[\frac{-1}{2},1\right]$

37. If f(x)=1+x+x^2+x^3+....∞ for $|x|<1$ then $f^{-1}(x)=$

1) $\frac{x-1}{x+1}$ 2) $\frac{x+1}{x}$ 3) $\frac{x}{x-1}$ 4) $\frac{x-1}{x}$

38. Let $f(x)=\sin x+\cos x, g(x)=x^2-1$. Then $g(f(x))$ is invertible for $x\in$

1) $\left[-\frac{\pi}{2},0\right]$ 2) $\left[-\frac{\pi}{2},\pi\right]$

3) $\left[-\frac{\pi}{2},\frac{\pi}{4}\right]$ 4) $\left[0,\frac{\pi}{2}\right]$

39. Let 'f' be an injective function with domain $\{x,y,z\}$ and range {1,2,3} such that exactly one of the follwowing statements is correct and the remaining are false $f(x)=1$, $f(y)\neq 1$, $f(z)\neq 2$ the value of $f^{-1}(1)$ is

1) x 2) y 3) z 4) x or z

40. If the function $f:[1,\infty)\to[1,\infty)$ is defined by $f(x)=2^{x(x-1)}$ then $f^{-1}(x)$ is

1) $\left(\frac{1}{2}\right)^{x(x-1)}$ 2) $\frac{1}{2}\left(1+\sqrt{1+4\log_2 x}\right)$

3) $\frac{1}{2}\left(1-\sqrt{1+4\log_2 x}\right)$ 4) $\frac{1}{2}\left(1\pm\sqrt{1+4\log_2 x}\right)$

41. f is a function defined as

$\sum_{k=1}^{n} f(a+k)=16\left(2^n-1\right)$ **and** $f(x+y)=f(x).f(y)$

and f(1) = 2 then integral value of a

1) 3 2) 0 3) 2 4) 1

42. A function $f:R\to R$ satisy the equation $f(x)f(y)-f(xy)=x+y$ for all $x,y\in R$ and f(y)>0, then

1) $f(x)=x+\frac{1}{2}$ 2) $f(x)=\frac{x}{2}+1$

3) $f(x)=\frac{x}{2}-1$ 4) $f(x)=x+1$

KEY

1) 4	2) 2	3) 3	4) 3	5) 2	6) 1
7) 3	8) 1	9) 1	10) 2	11) 3	12) 1
13) 2	14) 4	15) 2	16) 4	17) 4	18) 4
19) 4	20) 4	21) 3	22) 2	23) 3	24) 1
25) 4	26) 4	27) 2	28) 4	29) 1	30) 3
31) 2	32) 3	33) 2	34) 3	35) 2	36) 4
37) 4	38) 3	39) 2	40) 2	41) 1	42) 4

SOLUTIONS

1. Let $2x+\frac{y}{8}=\alpha$ and $2x-\frac{y}{8}=\beta$, then $x=\frac{\alpha+\beta}{4}$ and $y=4(\alpha-\beta)$ Given

$f\left(2x+\frac{y}{8},2x-\frac{y}{b}\right)=xy\Rightarrow f(\alpha,\beta)=\alpha^2-\beta^2$

$\Rightarrow f(m,n)+f(n,m)=m^2-n^2+n^2-m^2=0$ for all m,n

2. Side of the triangle with vertices $(0,0)$ and $(x,g(x))$ is a $\sqrt{x^2+(g(x))^2}$. Area of equilateral triangle, whose side a is $\frac{\sqrt{3}}{4}a^2$.

$\therefore \frac{\sqrt{3}}{4}\left[x^2+g(x)^2\right]=\frac{\sqrt{3}}{4}\Rightarrow x^2+g(x)^2=1$

$\Rightarrow g(x)=\pm\sqrt{1-x^2}$. Thus $g(x)=\sqrt{1-x^2}$

3. $0\le x-[x]<1\Rightarrow -\frac{1}{2}\le x-[x]-\frac{1}{2}<\frac{1}{2}$

$\Rightarrow -\frac{1}{2}\le f(x)<\frac{1}{2}\Rightarrow f(x)\neq\frac{1}{2}\forall x$

$\Rightarrow\left\{x\in R: f(x)=\frac{1}{2}\right\}=\phi$

4. By verification,

$f\left(\left|-\frac{1}{2}\right|\right)=f\left(\frac{1}{2}\right)=\frac{1}{2}-1=-\frac{1}{2}$

hence $f(|x|)=x$.

5. Here, $f(x)=x|x|, x\le -1$

$x^2\sin\left(\frac{\pi x}{2}\right), -1<x<1$, $x|x|, x\ge 1$.

Let $k\ge 0$. Then

and $f(1+k)=(1+k)|1+k|=(1+k)^2$.

Therefore, $f(1+k)=-f(-1+k)$

$f(-1+k)=(-1+k)^2\sin\frac{\pi}{2}(-1+k)=$

$-(1-k)^2\sin\frac{\pi}{2}(1-k)=-f(1-k)=-f|-(-1+k)|$

$\therefore f(x)=-f(-x)$ for all x. Also, none of the pieces of definition are periodic.

6 $f(-x)=\dfrac{\cos(-x)}{\left[-\frac{2x}{\pi}\right]+\frac{1}{2}}=\dfrac{\cos x}{-1-\left[\frac{2x}{\pi}\right]+\frac{1}{2}}$

(as x is not an integral multiple of π)

$\Rightarrow f(-x)=-\dfrac{\cos x}{\left[\frac{2x}{\pi}\right]+\frac{1}{2}}=-f(x)$

$\Rightarrow f(x)$ is an odd function

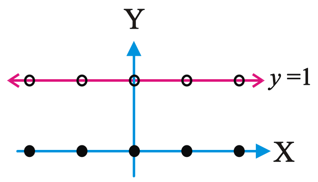
7. $g(x)=f(x+1)=|x-1|+|x-2|+|x-3|$.

If $x<1, g(x)=-x+1-x+2-x+3=-3x+6$

If $1\le x<2, g(x)=x-1-x+2-x+3=-x+4$

If $2\le x<3, g(x)=x-1+x-2-x+3=x$

If $x\ge 3, g(x)=x-1+x-2+x-3=3x-6$

8. 

9. $F[f(x)]=F(\log_a x)=a^{\log_a x}=x$

$f(F(x))=f(a^x)=\log_a a^x=x\log_a a=x.$

10. Here $g(x)=1+n-n=1, x=n\in Z$

$1+n+k-n=1+k, x=n+k$

(where $n\in Z, 0<k<1$)

Now $f(g(x))=\begin{cases}-1, & g(x)<0\\ 0, & g(x)=0\\ 1, & g(x)>0\end{cases}$

Clearly $g(x)>0\ \forall x$.so, $f(g(x))=1\ \forall x$

11. $g[f(x)]=8$ *or* $g(2x+3)=8$

$\Rightarrow (2x+3)^2+7=8 \Rightarrow 2x+3=\pm 1\Rightarrow -1,-2.$

13. $(gof)(x)=|\sin x|$ *and* $f(x)=\sin^2 x$

$\Rightarrow g(\sin^2 x)=|\sin x| \quad \therefore g(x)=\sqrt{x}$

14. $g(f(x))\le f(g(x))\Rightarrow g(|x|)\le f[x]$

$[|x|]\le |[x]|$ this is true for $x\in R$..

15. Since $g(x)=3+4x$

$\therefore g^2(x)=(gog)(x)=g\{g(x)\}=g(3+4x)$

$=3+4(3+4x)=15+4^2x=(4^2-1)+4^2x$

$g^3(x)=(gogog)x=g\{g^2(x)\}=g(15+4^2x)$

$=3+4(15+4^2x)=63+4^3x=(4^3-1)+4^3x$

similarly, we get $g^n(x)=(4^n-1)+4^n x$

16. f(x) = x when $2<x<3\Rightarrow$ f is a Identify function (Bijection)

17. $\sin\frac{\pi x_1}{2}=\sin\frac{\pi x_2}{2}\Rightarrow x_1=x_2\Rightarrow$ f is one-one

18. $(f-g)(x)=f(x)-g(x)$

$=\begin{cases}0-x=-x, & \textit{if } x \textit{ is rational}\\ x-0=x & \textit{if } x \textit{ is irrational}\end{cases}$

Clearly $(f-g)(x)$ is one-one and onto.36.

$fog(x)=x^3-\frac{1}{x^3}$,

$f\left(x-\frac{1}{x}\right)=\left(x-\frac{1}{x}\right)^3+3\left(x-\frac{1}{x}\right)$

let $x-\frac{1}{x}=t,\ f(t)=t^3+3t$

Thus $f(x)=x^3+3x, f'(x)=3x^2+3.$

19. Period of $\frac{2^x}{2^{[x]}}$ is 1. period of $\sin^{-1}\{x\}$ is 1. period of $\sin^{-1}\left(\sqrt{\cos x}\right)$ is 2π where as $\sin^{-1}(\cos x^2)$ is non -periodic.

20. $nx+n-[nx+n]$ has the period 1/n and $\tan\frac{\pi x}{2}$ has the period $\frac{\pi}{\pi/2}$ i.e, LCM of 1/n, 2 is 2.

21. From the given data $g(x)$ must be linear function.

Hence, $g(x)=ax+b$

Also $g(2)=2a+b=3$ and $g(4)=4a+b=7$

solving, $a=2$ and $b=-1$

Hence, $g(x)=2x-1$ Then, $g(6)=11$

22. $f(x)=[x]+[2x]+[3x]+...+[nx]-(x+2x+3x+..nx)\arcsin\theta$

$=-(\{x\}+\{2x\}+\{3x\}+...+\{nx\})$

period of $f(x)=L.C.M\left(1,\frac{1}{2},\frac{1}{3},...\frac{1}{n}\right)=1$

23. $\cos 3x$ has the period $\frac{2\pi}{3}$ and $\sin\sqrt{3}x$ has the period $\frac{2\pi}{\sqrt{3}}$.

As $\frac{2\pi}{3}$ and $\frac{2\pi}{\sqrt{3}}$ do not have a common multiple, $f(x)$ is not periodic

24. Period of $|\cos n\pi x|=\frac{1}{2}\times\frac{2\pi}{n\pi}=\frac{1}{n}$.

$\because$ period of $|\cos\pi x|$ is half of that of $\cos\pi x$]

L.C.M of $1,1,\frac{1}{2},\frac{1}{3},...\frac{1}{n}$ is 1].

25. The period of $f(x)$ is 7 $\Rightarrow$ The period of $f\left(\frac{x}{3}\right)$ is $\frac{7}{1/3}=21$

The period of $g(x)$ is 11 $\Rightarrow$ The period of $g\left(\frac{x}{3}\right)$ is $\frac{11}{1/5}=55$, Hence, $T_1=$ period of $f(x)g\left(\frac{x}{5}\right)=7\times55=385$ and

$T_2=$ period of $g(x)f\left(\frac{x}{3}\right)=11\times21=231$

$\therefore$ period of $F(x)=LCM(T_1,T_2)=LCM\{385,231\}$

$=7\times11\times3\times5=1155$.

26. $f(x)=\log_x(9-x^2)$ is define

$9-x^2>0,\ x>0\ and\ x\neq1\quad x^2-9<0$

$(x+3)(x-3)<0, x>0\ and\ x\neq1$ domain

$(0,1)\cup(1,3)$ $\quad 9-x^2>0$, $x>0$ and $x\neq1$

27. $f(x)=\sqrt{1-\sqrt{1-\sqrt{1-x^2}}}$ is define

$1-x^2\geq0$, $x^2-1\leq0$, $(x+1)(x-1)\leq0$

Domain of $f(x)$ is $[-1,1]$

28. $f(x)=\frac{1}{\sqrt{\{\sin x\}+\{\sin(\pi+x)\}}}$

$=\frac{1}{\sqrt{\{\sin x\}+\{-\sin x\}}}$

Now $\{\sin x\}+\{-\sin x\}=\begin{cases}0 & \sin x \text{ is an integer}\\ 1 & \sin x \text{ is not an integer}\end{cases}$

For $f(x)$ to get defined

$\{\sin x\}+\{-\sin x\}\neq0$

$\Rightarrow\sin x\neq$ integer

$\Rightarrow\sin x\neq\pm1,0\Rightarrow x\neq\frac{n\pi}{2}, n\in I$

Hence, the domain is R-$\left\{\frac{n\pi}{2}/n\in I\right\}$

29. $[x]^2-[x]-2>0\Rightarrow(([x]+1)([x]-2))>0$

$[x]<-1$ or $[x]>2$

$4-x^2<0$ & $[x]+2<0$

30. $A.M.\geq G.M.$

31.

$f(x)=\sin^{-1}\left[\frac{1}{2}+x^2\right]$

$0\leq\left[\frac{1}{2}+x^2\right]\leq1\Rightarrow\left[\frac{1}{2}+x^2\right]=0,1$

Range of $f(x)=\{\sin^{-1}0,\sin^{-1}1\}=\left\{0,\frac{\pi}{2}\right\}$

32. $x^2+(2y)^2+(3z)^2-(2y)(3z)-(x)(3z)-x(2y)\geq 0$

33. Domain $[-1,1]$ $\quad -1\leq x\leq 1\Rightarrow \frac{-\pi}{4}\leq \tan^{-1}x$

$\leq\frac{\pi}{4}$ $\qquad \frac{\pi}{2}-\frac{\pi}{4}\leq f(x)\leq\frac{\pi}{2}+\frac{\pi}{4}$

$\Rightarrow\frac{\pi}{4}\leq f(x)\leq\frac{3\pi}{4}$

34. $-\frac{\pi}{4}\leq x\leq\frac{\pi}{4}\Rightarrow 0\leq\sqrt{\frac{\pi^2}{16}-x^2}\leq\frac{\pi}{4}$

$\Rightarrow 8\sqrt{2}\sin\sqrt{\frac{\pi^2}{16}-x^2}\in[0,8]$

35. $\frac{x^4}{1+x^8}=y\Rightarrow yx^8-x^4+y=0$ and x is real.

36. If $a^2+b^2+c^2=k$ then $ab+bc+ca\in\left[-\frac{k}{2},k\right]$

37. Let $f(x)=\frac{1}{1-x}=y \Rightarrow f^{-1}(y)=x$

$\Rightarrow x=f^{-1}(y)=\frac{y-1}{y}$

38. By defination of composition of functions

$g(f(x))=(\sin x+\cos x)^2-1$

$g(f(x))=\sin 2x$

We know $\sin x$ is bijection only, when

$x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Thus $g(x)$ is bijection if

$-\frac{\pi}{2}\leq 2x\leq\frac{\pi}{2}\Rightarrow\frac{-\pi}{4}\leq x\leq\frac{\pi}{4}.$

39. Suppose f(x)=1, then f(y)=1, f(z)=2 $\Rightarrow$ f is not an injection.

Suppose $f(y)\neq 1$, then

$f(z)=2, f(y)=3, f(x)\neq 1$ A contradiction

Suppose $f(z)\neq 2$, then

f(y)=1, f(z)=3, f(x)=2 this is true $\therefore f^{-1}(1)=y$

40. $y=2^{x(y-1)}\Rightarrow x(x-1)=\log_2 y$

$\Rightarrow x^2-x-\log_2 y=0\ \therefore x=\frac{1\pm\sqrt{1+4\log_2 y}}{2}$

$\therefore f^{-1}(x)=\frac{1}{2}\left\{1+\sqrt{1+4\log_2 y}\right\}\quad [\because x\geq 1]$

41.G iven

$\sum_{k=1}^{n}2^{a+k}=16(2^n-1)\qquad f(x)=a^x$

$f(1)=a^1=2$

$2^9(2^1+2^2+.....2^n)=16(2^n-1)\quad f(x)=2^x$

$2^{a+1}(1+2+2^2+....2^{n-1})=16(2^n-1)$ above series is a G.P $r=2, a=1$

$2^{a+1}.1\frac{(2^n-1)}{2-1}=16(2^n-1)$,

$2^{a+1}=2^4\quad a+1=4\quad a=3$

42. Take $f(x)=x+1$ and verify

JEE MAINS QUESTIONS

1. The domain of the function $f(x)=\sin^{-1}\left(\frac{1\times 1+5}{x^2+1}\right)$ is $[-\infty,-a]\upsilon[a,\infty]$ then a is equal to [2020]

1) $\frac{1+\sqrt{17}}{2}$ 2) $\frac{\sqrt{17}}{2}+1$

3) $\frac{\sqrt{17}-1}{2}$ 4) $\frac{\sqrt{17}}{2}$

2. Suppose that a function $f:R\to R$ satisfies $f(x+y)=f(x)f(y)$ for all $x,y\in R$ and $f(1)=3$ .If $\sum_{i=1}^{n} f(i)=363$ then n =

[2020]

3. If $g(x)=x^2+x-1$ and $(gof)(x)=4x^2-10x+5$ then $f\left(\frac{5}{4}\right)$ is equal to

[2020]

1) 1/2 2) -1/2

3) -1/3 4) 1/3

4. Let $f(x)=\frac{8^{2x}-8^{-2x}}{8^{2x}+8^{-2x}}$ then inverse of $f(x)$ is -------------- [2020]

1) $\frac{1}{4}\log_8\left(\frac{1+x}{1-x}\right)$ 2) $\frac{1}{2}\log_8\left(\frac{1-x}{1+x}\right)$

3) $\frac{1}{4}\log_8\left(\frac{1-x}{1+x}\right)$ 4) $\frac{1}{2}\log_8\left(\frac{1+x}{1-x}\right)$

5. Let $f:(1,3)\to R$ beafunction defined by $f(x)=\frac{x[x]}{1+x^2}$ where $[x]$ denotes the greatest integer $\leq x$ then the range of ‘f’ is [2020]

1) $\left(0,\frac{1}{2}\right)\cup\left(\frac{3}{5},\frac{7}{5}\right)$ 2) $\left(\frac{2}{5},\frac{1}{2}\right)\cup\left(\frac{3}{5},\frac{4}{5}\right)$

3) $\left(\frac{2}{5}1\right)\cup\left(1,\frac{4}{5}\right)$ 4) $\left(0,\frac{1}{3}\right)\cup\left(\frac{2}{5},\frac{4}{5}\right)$

6. Let $f:R\to R$ be defined by $f(x)=\frac{x}{1+x^2}x\in R$. Then the range of ‘f’ is

[2019]

1) $\left[\frac{-1}{2},\frac{1}{2}\right]$ 2) $R-\left[\frac{-1}{2},\frac{1}{2}\right]$

3) $(-1,1)-\{0\}$ 4) $R-[-1,1]$

7. The number of functions ‘f’ from $\{1,2,3------20\}$ on to $\{1,2,---20\}$ such that $f(k)$ is a multiple of 3 ,when ever k is a multiple of 4 is [2019]

1) $6^5\times 15!$ 2) $5^6\times 15$

3) $15!\times 6!$ 4) $5!\times 6!$

8. Let F be a differentiable function such that $f(x)=2$ and $f^1(x)=f(x)\forall x\in R$ .If $h(x)=f(f(x))$,then $h^1(1)=$ [2019]

1) 4e 2) $2e^2$

3) $4e^2$ 4) 2e

9. Let $A=\{x\in R : x\, is\ not\, a\ positive\ \text{int}\, eger\}$ define a function $f:A\to R$ as $f(x)=\dfrac{2x}{x-1}$ then f is [2019]

1) Not injective
2) Susjective but injective
3) injective but not susjective
4) neither injective nor susjective

10. Let N be the set of natural numbers and two functions ‘f’ and ‘g’ be defined as $f,g:N\to N$ such that

$$f(n)=\begin{cases}\dfrac{n+1}{n} & if\ n\ is\, odd\\ \dfrac{n}{2} & if\ \ n\ is\ even\end{cases}$$ and

$g(n)=n-(1)^n$ then fog is [2019]

1) Neither one-one nor onto
2) onto but not one-one
3) Both one-one and onto
4) one-one but not on to

11 .Let $f(x)=2^{10}x+1$ and $g(x)=3^{10}x-1$ .If $fog(x)=x$ then x is equal to [2018]

1) $\dfrac{3^{10}-1}{3^{10}-2^{-10}}$ 2) $\dfrac{2^{10}-1}{2^{10}-3^{-10}}$

3) $\dfrac{1-3^{10}}{2^{10}-3^{-10}}$ 4) $\dfrac{1-2^{-10}}{3^{10}-2^{-10}}$

12. Let $f:A\to B$ be a function defined as $f(x)=\dfrac{x-1}{x-2}$ where $A=R-\{2\}$ and $B=R-\{1\}$, then f is [2018]

1) Invertible and $f^{-1}(y)=\dfrac{3y-1}{y-1}$

2) Invertible and $f^{-1}(y)=\dfrac{2y-1}{y-1}$

3) Invertible and $f^{-1}(y)=\dfrac{2y+1}{y-1}$

4) not invertible

KEY

1) 1	2) 5	3) 2	4) 1	5) 2
6) 1	7) 3	8) 1	9) 3	10) 4
11) 4	12) 2			

SOLUTIONS

1. $f(x)=\sin^{-1}\left(\dfrac{1\times1+5}{x^2+1}\right)\ \therefore -1\le\dfrac{1\times1+5}{x^2+1}\le1$

$\therefore x^2-1\times1-4\ge0$

$$\left(1\times1+\frac{1+\sqrt{17}}{2}\right)\left(1\times1-\frac{1+\sqrt{17}}{2}\right)\ge0$$

$$x\in\left(-\infty,-\frac{\left(1+\sqrt{17}\right)}{2}\right)\cup\left(\frac{1+\sqrt{17}}{2},\infty\right)$$

3. $g(x)=x^2+x-1 \Rightarrow g(f(x))=4x^2-10x+5$

$$(f(x))^2+f(x)-1=4x^2-10x+5$$

$$(f(5/4))^2+f(5/4)-1=4(5/4)^2-10(5/4)+5$$

$$\left(f\left(\frac{5}{4}\right)\right)^2+f\left(\frac{5}{4}\right)-1=\frac{25}{4}-\frac{50}{4}+5$$

$$\Rightarrow\left(f\left(\frac{5}{4}\right)\right)^2+f\left(\frac{5}{4}\right)-1=\frac{-5}{4}$$

$$\left(f\left(\frac{5}{4}\right)\right)^2+f\left(\frac{5}{4}\right)+\frac{1}{4}=0 \qquad \left(f\left(\frac{5}{4}\right)+\frac{1}{2}\right)^2=0$$

$$f\left(\frac{5}{4}\right)+\frac{1}{2}=0 \qquad f\left(\frac{5}{4}\right)=\frac{-1}{2}$$

2. $f(x+y)=f(x)f(y)\forall x\in R, f(1)=3$

$$f(x)=3^x \Rightarrow f(i)=3^i \Rightarrow \sum_{i=1}^{n} f(i)=363 \Rightarrow 3+3^2+3^3+------3^n=363$$

$$3\left(\frac{3^n-1}{2}\right)=363 \qquad \begin{aligned}3^n-1&=2\times|2|\\ 3^n-1&=242\\ 3^n&=3^5\end{aligned}$$

n = 5

4. Given $y=f(x)=\frac{8^{2x}-8^{-2x}}{8^{2x}+8^{-2x}}$ $\quad \frac{1}{y}=\frac{8^{2x}+8^{-2x}}{8^{2x}-8^{-2x}}$ apply componendo and divideno

$$\frac{1+y}{1-y}=\frac{8^{2x}+8^{-2x}+8^{2x}-8^{-2x}}{8^{2x}+8^{-2x}-8^{2x}+8^{-2x}} \Rightarrow \frac{1+y}{1-y}=8^{4x}$$

$$4x=\log_8\left(\frac{1+y}{1-y}\right) \quad , \quad f^{-1}(x)=\frac{1}{4}\log_8\left(\frac{1+x}{1-x}\right)$$

5. $f(x)=\begin{cases}\frac{x}{1+x^2} & x\in(1,2)\\ \frac{2x}{1+x^2} & x\in[2,3]\end{cases}$

$\therefore f(x)$ is a decreasing function

$$\therefore y\in\left(\frac{2}{5},\frac{1}{2}\right)\upsilon\left(\frac{6}{10},\frac{4}{5}\right)$$

the range of $f(x)$ is $\left(\frac{2}{5},\frac{1}{2}\right)\cup\left(\frac{3}{5},\frac{4}{5}\right)$

6 .Given $f(x)=\frac{x}{1+x^2} x\in R$ let

$$y=\frac{x}{1+x^2} \Rightarrow y+yx^2=x$$

$$yx^2-x+y=0 \Rightarrow 1-4y^2\geq 0 \quad \forall x\in R$$

$$y^2-\frac{1}{4}\leq 0$$

$$y\in\left[\frac{-1}{2},\frac{1}{2}\right]$$

7. $k\in\{4,8,12,16,20\}$ $f(k)$ can take value from the set $\{3,6,9,12,15,18\}$ this can be done in $6_{c_5}\times\underline{|5}=\underline{|6}$ ways for remaining elements $\underline{|15}$ so total number of functions $=\underline{|6}\ \underline{|15}$

8. $f^1(x)=f(x)\Rightarrow \log f(x)=x+c$

$$f(x)=e^{x+c}\Rightarrow f(x)=k\ e^x\Rightarrow f(1)=ke\Rightarrow k=\frac{2}{e}$$

$$\therefore f(x)=\frac{2}{e}e^x,\ f^1(x)=\frac{2}{e}e^x$$

$$h(x)=f(f(x))\Rightarrow h^1(x)=f^1(f(x))f^1(x)$$

$$h^1(1)=f^1(f(1))+f^1(1)=f^1(2).\frac{2}{e}e$$

$$h^1(1)=\frac{2}{e}e^2.2=4e$$

9. $f(x)=\dfrac{2x}{x-1}\Rightarrow f(x)=2+\dfrac{2}{x-1}$

$$f^1(x)=\frac{-2}{(x-1)^2}<0\ \forall x\in R$$

hence $f(x)$ is strictly decreasing function . $f(x)$ is one-one

$$y=\frac{2x}{x-1}\Rightarrow xy-y=2x\Rightarrow x(y-2)=y,\ x=\frac{y}{y-2}$$

given that $x\in R:x$ is not a+ve integers if y = 3 then x = 3 $\notin A$ $\therefore$ f is not susjection

10. If n = 1 then $fog(1)=f(g(1))=f(2)=\dfrac{2}{2}=1$

if $n=2$ then $fog(2)=f(g(2))=f(1)=\dfrac{1+1}{1}=2$

If n = 3 then $fog(3)=f(g(3))=f(4)=\dfrac{4}{2}=2$

$\therefore$ fog is not one-one but it is onto

11. $fog(x)=x\Rightarrow f(g(x))=x$

$$f(3^{10}x-1)=x\Rightarrow x=2^{10}(3^{10}x-1)+1\Rightarrow x(2^{10}3^{10}-1)=2^{10}-1$$

$$\Rightarrow x=\frac{2^{10}-1}{2^{10}3^{10}-1}=\frac{1-2}{3^{10}-2^{-10}}$$

12. $f(x)$ is bisectible function of invertible function

$$y=\frac{x-1}{x-2}\Rightarrow xy-2y=x-1$$

$$x(y-1)=2y-1\quad y=f(x)$$

$$x=\frac{2y-1}{y-1}\ x=f^{-1}(y)$$

$$f^{-1}(y)=\frac{2y-1}{y-1}$$

EXERCISE - III

1. **If** $f(x+y)=f(x).f(y)$ **for all real** x,y **and** $f(0)\neq 0$**, then the function** $g(x)=\dfrac{f(x)}{1+\{f(x)\}^2}$ **is**
1) even function
2) odd function
3) odd if $f(x)>0$
4) neither even nor odd

2. **Let** $f(x)=\dfrac{2x(\sin x+\tan x)}{2\left[\dfrac{x+21\pi}{\pi}\right]-41}$, $x\neq n\pi$, **then** f **is (where [.] represents greatest integer function)**
1) an odd function
2) an even function
3) both odd and even
4) neither odd nor even

3. **If the real valued function** $f(x)=\dfrac{a^x-1}{x^n\left(a^x+1\right)}$ **is even, then n =**
1) 2 2) $\dfrac{2}{3}$ 3) $\dfrac{1}{4}$ 4) 3

4. **Let** $f:[-3,3]\to$ **R where** $f(x)=x^3+\sin x+\left[\dfrac{x^2+2}{a}\right]$ **be an odd function then the value of a is (where [.] represents greatest integer function)**
1) less than 11 2) 11
3) greater than 11 4) 12

5. **If f is an even function defined on the interval** $(-5,5)$ **then find the total number of real values of x satisfying the equations** $f(x)=f\left(\dfrac{x+1}{x+2}\right)$ **are**
1) 1 2) 2 3) 4 4) 8

6. **If for nonzero x,** $af(x)+bf\left(\dfrac{1}{x}\right)=\dfrac{1}{x}-5,$ **where** $a\neq b$**, then f (2) =**
1) $\dfrac{3(2b+3a)}{2\left(a^2-b^2\right)}$ 2) $\dfrac{3(2b-3a)}{2\left(a^2-b^2\right)}$
3) $\dfrac{3(3a-2b)}{2\left(a^2-b^2\right)}$ 4) $\dfrac{6}{a+b}$

7. **If** $f(x)=64x^3+\dfrac{1}{x^3}$ **and a,b are the roots of** $4x+\dfrac{1}{x}=3$**, then**
1) $f(a)=12$ 2) $f(b)=11$
3) $f(a)=f(b)$ 4) $f(a)\neq f(b)$

8. **If** $f(x)=\dfrac{9^x}{9^x+3}$ **then** $f\left(\dfrac{1}{1996}\right)+f\left(\dfrac{2}{1996}\right)+..........+\left(\dfrac{1995}{1996}\right)=$
1) 997 2) 997.5 3) 998 4) 998.5

9. **If [x] stands for the greatest integer function, then**
$\left[\dfrac{1}{2}+\dfrac{1}{1000}\right]+\left[\dfrac{1}{2}+\dfrac{2}{1000}\right]+....+\left[\dfrac{1}{2}+\dfrac{999}{1000}\right]=$
1) 498 2) 499 3) 500 4) 501

10. **If** $f:R\to R$ **is a function satisfying the property** $f(2x+3)+f(2x+7)=2,\forall x\in R$**, then the period of** $f(x)$ **is**
1) 2 2) 4 3) 8 4) 12

11. **The period of the function** $f(x)=x[x]$ **is**
1) 1 2) 2
3) Non periodic 4) 4

12. **If the period of the function** $f(x)=\sin\left(\sqrt{[n]}x\right)$ **where [n] denotes the greatest integer less than or equal to n is** 2π**, then**
1) $1\leq n<2$ 2) $1<n<2$
3) $1\leq n\leq 2$ 4) $0\leq n\leq 1$

13. **If f (x) is an odd periodic function with period 2, then f (4) =**

1) -4 2) 4 3) 2 4) 0

14. **The domain of the function** $f(x)=\sqrt{\log^{x}_{\left(x^2-1\right)}}$ **is**

1) $\left(\sqrt{2},\infty\right)$ 2) $(0,\infty)$ 3) $(1,\infty)$ 4) R

15. **If** $f(x)=\sqrt{3|x|-x-2}$ **and** $g(x)=\sin x$**, then domain of** $(fog)(x)$ **is**

1) $\{2n\pi+\pi/2\},n\in z$

2) $\left(2n\pi+\frac{7\pi}{6},2n\pi+\frac{11\pi}{6}\right),n\in Z$

3) $\left\{2n\pi+\frac{7\pi}{6}\right\},n\in Z$

4) $\left(2n\pi+\frac{7\pi}{6},2n\pi+\frac{11\pi}{6}\right)\cup\left(2m\pi+\frac{\pi}{2}\right);n,m\in Z$

16. **The function** $f(x)=\cot^{-1}\left(\sqrt{(x+3)x}\right)+\cos^{-1}\left(\sqrt{x^2+3x+1}\right)$ **is defined on the set S, then S is equal to**

1) $\{-3,0\}$ 2) $[-3,0]$

3) $[0,3]$ 4) $(-3,0)$

17. **Domain of** $\sqrt{x\left(1-e^x\right)(x+2)(x-3)^2}$

1) $[-2,3]$ 2) $(-2,0]$

3) $(-\infty,-2]\cup\{0,3\}$ 4) $(-\infty,-2)\cup[0,3]$

18. **If** $f(x)$ **is defined on** $(0,1]$**, then the domain of** $f(\sin x)$ **is**

1) $\left(2n\pi,(2n+1)\pi\right),n\in Z$

2) $\left((2n+1)\frac{\pi}{2},(2n+3)\frac{\pi}{2}\right);n\in Z$

3) $\left((n-1)\pi,(n+1)\pi\right),n\in Z$

4) $\left(n\pi,(2n+1)\pi\right)n\in Z$

19. **If** $b^2-4ac=0,\ a>0,$ **then the domain of** $y=\log[ax^3+(a+b)x^2+(b+c)x+c]$ **is**

1) $R-\left\{-\frac{b}{2a}\right\}$

2) $R-\left\{\left\{-\frac{b}{2a}\right\}\cup\{x:x\geq-1\}\right\}$

3) $R-\left\{-\frac{b}{2a}\right\}\cup\{x>-1\}$

4) $R-\left\{\frac{b}{2a}\right\}$

20. **The range of** $f(x)=\log_e\left(3x^2-4x+5\right)$ **is**

1) $\left(-\infty,\log_e^{\frac{11}{3}}\right]$ 2) $\left[\log_e^{\frac{11}{3}},\infty\right)$

3) $\left(-\log_e^{\frac{11}{3}},\log_e^{\frac{11}{3}}\right)$ 4) $[1,\infty)$

21. **The range of** $f(x)=x^2+\frac{1}{x^2+1}$ **is**

1) $[1,\infty)$ 2) $[2,\infty)$ 3) $[\frac{3}{2},\infty)$ 4) R

22. **If** $x\in R$ **and** $P=\frac{x^2}{x^4-2x^2+4}$**, then P lies interval**

1) $\left[0,\frac{1}{2}\right]$ 2) $\left[\frac{3}{4},\frac{4}{5}\right]$ 3) $\left[0,\frac{1}{3}\right]$ 4) $\left[0,\frac{1}{4}\right]$

23. **If** $f:R\to R$ **is defined by** $f(x)=[2x]-2[x]$ **for** $x\in R$**, where** $[x]$ **is the greatest integer not exceeding** x**, then the range of** f **is**

1) $\{x\in R:0\leq x\leq1\}$ 2) $\{0,1\}$

3) $\{x\in R:x>0\}$ 4) $\{x\in R:x<0\}$

24. **The range of** $\sin^{-1}\left[x^2+\frac{1}{2}\right]+\cos^{-1}\left[x^2-\frac{1}{2}\right]$**, where** $[.]$ **denotes the greatest integer function, is**

1) $\left\{\frac{\pi}{2},\pi\right\}$ 2) $\{\pi\}$ 3) $\left\{\frac{\pi}{2}\right\}$ 4) $\left[\frac{\pi}{2},\pi\right]$

25. The range of

$f(x)=\left[\sin x+\left[\cos x+\left[\tan x+\left[\sec x\right]\right]\right]\right]$,

$x\in(0,\pi/4)$, **where** $[.]$ **denotes the greatest integer function** $\le x$, **is**

1) $\{0,1\}$ 2) $\{-1,0,1\}$ 3) $\{1\}$ 4) $\{0\}$

26. The function $f:R\to B$ **is defined by** $f(x)=[x]+[-x]$ **where [.] is G.I.F is surjective then B =**

1) R 2) $[0,1]$ 3) $[-1,0]$ 4) $\{-1,0\}$

27. If $f(x)=ax^7+bx^3+cx-5$ **(**a,b,c **are real constants) and** $f(-7)=7$, **then the range of** $f(7)+17\cos x$ **is**

1) $[-34,0]$ 2) $[0,34]$ 3) $[-34,34]$ 4) $\{-34,34\}$

28. If $f(x)=\sin^2 x+\sin^2(x+\pi/3)+\cos x\cos(x+\pi/3)$ **and** $g(5/4)=1$ **then** $(gof)(x)=$

1) 1 2) 0

3) sinx 4) $-\cos x$

29. If $f:(4,8)\to(5,9)$ **is a function defined by** $f(x)=x+[\frac{x}{4}]$ **where [.] is G.I.F then** $f^{-1}(x)=$

1) $1-x$ 2) $x-1$ 3) $x-3$ 4) $3-x$

30. If the function $f:[2,\infty)\to[-1,\infty)$ **is defined by** $f(x)=x^2-4x+3$ **then** $f^{-1}(x)=$

1) $2-\sqrt{x+1}$ 2) $2+\sqrt{x+1}$

3) $\frac{2-\sqrt{x+1}}{5}$ 4) $\frac{2+\sqrt{x+1}}{5}$

31. Let $f(x)=x^2-x+1, x\ge 1/2$, **then the solution of the equation** $f^{-1}(x)=f(x)$ **is**

1) $x=1$ 2) $x=2$ 3) $x=\frac{1}{2}$ 4) x= 0

32. If $f(x)=\frac{(x-a)(x-b)}{x}$ **and**

$$\frac{f(x)}{(x-y)(x-z)}+\frac{f(y)}{(y-z)(y-x)}+\frac{f(z)}{(z-x)(z-y)}=\frac{k}{xyz}$$ **then** $k=$

1) a 2) b 3) ab 4) 3ab

KEY

1) 1	2) 1	3) 4	4) 3	5) 3	6) 2
7) 3	8) 2	9) 3	10) 2	11) 3	12) 1
13) 4	14) 1	15) 4	16) 1	17) 3	18) 1
19) 3	20) 2	21) 1	22) 1	23) 2	24) 2
25) 3	26) 4	27) 1	28) 1	29) 2	30) 2
31) 1	32) 3				

SOLUTIONS

1. Given $f(x+y)=f(x)f(y)$. Put $x=y=0$, then $f(0)=1$. Put $y=-x$, then

$$f(0)=f(x)f(-x)\Rightarrow f(-x)=\frac{1}{f(x)}$$

Now, $g(x)=\frac{f(x)}{1+\{f(x)\}^2}$

$$\Rightarrow g(-x)=\frac{f(-x)}{1+\{f(-x)\}^2}=\frac{\frac{1}{f(x)}}{1+\frac{1}{\{f(x)\}^2}}$$

$$=\frac{f(x)}{1+\{f(x)\}^2}=g(x)$$

2. The denominator is

$$=2\left[\frac{x+21\pi}{\pi}\right]-41=2\left[\frac{x}{\pi}+21\right]-41$$

$$=2\left\{21+\left[\frac{x}{\pi}\right]\right\}-41=2\left[\frac{x}{\pi}\right]+1$$

$$\therefore f(x)=\frac{x(\sin x+\tan x)}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}$$

$$\Rightarrow f(-x)=\frac{-x\{\sin(-x)+\tan(-x)\}}{\left[-\frac{x}{\pi}\right]+\frac{1}{2}}$$

$$=\frac{x(\sin x+\tan x)}{-1-\left[\frac{x}{\pi}\right]+\frac{1}{2}}=-\frac{x(\sin x+\tan x)}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}=-f(x)$$

3. $f(-x)=f(x)\Rightarrow\frac{a^{-x}-1}{(-x)^n(a^{-x}+1)}=\frac{a^x-1}{x^n(a^x+1)}$

$$\Rightarrow\frac{-1}{(-1)^n}=1\Rightarrow n=3$$

4. $f(x)=x^3+\sin x+\frac{x^2+2}{a}$

$\therefore f(-x)=-x^3-\sin x+\left[\frac{x^2+2}{a}\right]$

Now $f(-x)=-f(x)$ given,

$\therefore f(x)+f(-x)=0$

$\Rightarrow 0\le\frac{x^2+2}{a}<1,\forall -3\le x\le 3$ $\therefore$ a > 11

($\because$ maximum of x^2+2 in $-3\le x\le 3$ is 11)

5. $f(-x)=f(x)$

6. Replace x by $\frac{1}{x}$ then solve.

7. We have, $f(a)=64a^3+\frac{1}{a^3}=(4a)^3+\frac{1}{a^3}$

$=\left[4a+\frac{1}{a}\right]^3-3.4a.\frac{1}{a}\left[4a+\frac{1}{a}\right]$

$=(3)^3-12.3=27-36=-9$

[since a, b all roots of $4x+\frac{1}{x}=3$,

$\therefore 4a+\frac{1}{a}=3$]

Similarly $f(b)=-9$, $\therefore f(a)=f(b)=-9$

8. $f(x)+f(1-x)=\frac{9^x}{9^x+3}+\frac{9^{1-x}}{9^{1-x}+3}=1$

9. if $1\le x<500$ $=1$ if $500\le x\le 999$

$GE=0+0+...+1+1+1+...+1\,(500\text{ times})=500$

10. Wehave

$f(2x+3)+f(2x+7)=2$ ______ (1)

Replance x by x + 1,

$f(2x+5)+f(2x+9)=2$ ______ (2)

Replace x by x + 2

$f(2x+7)+2(2x+11)=2$ ______ (3)

From (1) and (3)

We get $f(2x+3)-f(2x+11)=0$

i.e., $f(2x+3)=f(2x+11)\Rightarrow T=4$

$\therefore f(x)$ is periodic with period 2k.

11. Let $n\le x<n+1$

Then, $f(x)=x$, n where n changes with x clearly no constant k > 0 is possible for which $f(x)=f(x+k)$ corresponding to all x.

$\therefore f(x)$ is a non periodic function.

12. Sin x is a periodic function with period 2π, therefore $\sin\left(\sqrt{[n]}x\right)$ is a periodic function with period $\frac{2\pi}{\sqrt{[n]}}$

But the period of f(x) is 2π (given)

$\therefore\frac{2\pi}{\sqrt{[n]}}=2\pi\Rightarrow\sqrt{[n]}=1$ $\Rightarrow[n]=1\Rightarrow 1\le n<2$

13. $f(-x)=-f(x)$; $f(x+2)=f(x)$

14. $\log_b^a$ is defind for a , b > 0 and $b\ne 1$

15. We have,

$f(x)=\sqrt{3|x|-x-2}$ *and* $g(x)=\sin x$

$\therefore fog(x)=\sqrt{3|\sin x|-\sin x-2}$

$\Rightarrow -1\le\sin x\le-\frac{1}{2}$

$\therefore x\in\left(2n\pi+\frac{7\pi}{6},2n\pi+\frac{11\pi}{6}\right)\cup\left\{2m\pi+\frac{\pi}{2}\right\},n,m\in Z$

16. For the two components to be meaningful, we must have $x(x+3)\ge 0$ *and* $0\le x^2+3x+1\le 1$

Hence, $(x+3)x=0$ *i.e.,* $x=0,-3$

$\therefore S=\{-3,0\}$

17. $f(x)\ge 0$

18. Since the domain of f is (0,1]
$\therefore 0<\sin x\le 1\Rightarrow 2n\pi<x<(2n+1)\pi, n\in Z$

19. $y=\log[ax^3+(a+b)x^2+(b+c)x+c]$
$=\log[(ax^2+bx+c)(x+1)]$

since $a>0$ $\therefore$ y is defined if $x\ne -\dfrac{b}{2a}$ and

$x>-1\Rightarrow x\in R-\left\{-\dfrac{b}{2a}\right\}\cup x>-1$

20. $f(x)$ is defined if $3x^2-4x+5>0, \forall x\in R$

$16-12\left(5-e^y\right)\ge 0\Rightarrow 12e^y\ge 44\Rightarrow e^y\ge\dfrac{11}{3}$

$\Rightarrow y\ge \log_e^{\frac{11}{3}}$ Range of $f=\left[\log_e^{\frac{11}{3}}.\infty\right)$

21. $x+\dfrac{1}{x}\ge 2$

22. We have, $p=\dfrac{x^2}{x^4-2x^2+4}=\dfrac{x^2}{\left(x^2-1\right)^2+3}\ge 0$

Also, $p=\dfrac{1}{x^2-2+\dfrac{4}{x^2}}=\dfrac{1}{\left[x-\dfrac{2}{x}\right]^2+2}\le\dfrac{1}{2}$

$\therefore p\in\left[0,\dfrac{1}{2}\right]$

23. $x\in R\Rightarrow\exists n\in Z \ni n\le x<n+1$
$\ni [x]=n\Rightarrow 2n\le 2x<2n+2\Rightarrow[2x]=2n$ or

$2n+1\Rightarrow[2x]=2[x]$ or

$2[x]+1\Rightarrow[2x]-2[x]=0$ or

$1\Rightarrow f(x)=0$ *or* $1\Rightarrow$ *range* $=\{0,1\}$

24. Thus, from domain point of view,

$\left[x^2-\dfrac{1}{2}\right]=0,-1\Rightarrow\left[x^2+\dfrac{1}{2}\right]=1,0$

$\Rightarrow f(x)=\sin^{-1}(1)+\cos^{-1}(0)$

or $\sin^{-1}(0)+\cos^{-1}(-1)\Rightarrow f(x)=\{\pi\}$.

25. $f(x)=\left[\sin x+\left[\cos x+\left[\tan x+[\sec x]\right]\right]\right]$

$=[\sin+p]$, where

$p=\left[\cos x+\left[\tan x+[\sec x]\right]\right]$

$=[\sin x]+p$, (as p is an integer)

$=[\sin x]+\left[\cos x+\left[\tan x+[\sec x]\right]\right]$

$=[\sin x]+[\cos x]+[\tan x]+[\sec x]$

Now, for

$x\in(0,\pi/4), \sin x\in\left(0,\dfrac{1}{\sqrt{2}}\right), \cos x\in\left(\dfrac{1}{\sqrt{2}},1\right),$

$\tan x\in(0,1), \sec x\in\left(1,\sqrt{2}\right)$

$\Rightarrow[\sin x]=0, [\cos x]=0$

$[\tan x]=0$ *and* $[\sec x]=1$

$\Rightarrow$ The range of $f(x)$ is 1.

$x^2+x+a\ge 0\Rightarrow 1-4a\le 0\Rightarrow a\ge 1/4$

26. verify by taking integer& decimal number.

27. $f(7)+f(-7)=-10 \qquad \Rightarrow f(7)=-17$

$\Rightarrow f(7)+17\cos x=-17+17\cos x$ which has the range $[-34,0]$

28. $f(x)=\sin^2 x+(\sin x\cos\frac{\pi}{3}+\cos x\sin\frac{\pi}{3})^2+\cos x$

$(\cos x\cos\frac{\pi}{3}-\sin x\sin\frac{\pi}{3})$

$=\sin^2 x+\left[\dfrac{\sin x}{2}+\dfrac{\sqrt{3}\cos x}{2}\right]+\dfrac{\cos^2 x}{2}-\dfrac{\sqrt{3}}{2}\cos x\sin x$

$$= \sin^2 x + \frac{\sin^2 x}{4} + \frac{3}{4}\cos^2 x + \frac{\sqrt{3}}{2}$$

$$\sin x \cos x + \frac{\cos^2 x}{2} - \frac{-\sqrt{3}}{2}$$

$$\cos x \sin x = \frac{5}{4}(\sin^2 x + \cos^2 x) = \frac{5}{4}$$

$$\therefore [gof](x) = g[f(x)] = g(5/4) = 1$$

29. $f(x) = x + 1$

30. Let f(x)=y

31. $f^{-1}(x) = \sqrt{x - \frac{3}{4}} - \frac{1}{2}$

32. Put $y = a, z = b$. Then $f(y) = 0, f(z) = 0$

$$\frac{f(x)}{(x-a)(x-b)} = \frac{k}{xab} \Rightarrow \frac{1}{x} = \frac{k}{xab} \Rightarrow k = ab$$

ADVANCED QUESTIONS

SINGLE ANSWER

1. Let $f: R \to R$ be a function defined by

$f(x) = \dfrac{x^2 + 2x + 5}{x^2 + x + 1}$ **is**

a) one-one and into b)one-one and onto

c) many-one and onto d)many-one and into

2. $f: R \to R, f(x) = x^3 - 3x^2 + 6x - 5$ **is**

a)one-one and onto

b)one-one and into

c)onto but not one-one

d)neither one-one nor onto

3. $f: R \to R$ **is a function defined by**

$f(x) = \dfrac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$ **then f is**

a) a bijection b) an injection only

c) surjection only

d) neither injection nor surjection

4. $f(x) = \ln\left(x + \sqrt{1 + x^2}\right)$ **is**

a) even function b) odd function

c) Neither even nor odd d) Constant function

5. *If* $f(x)$ **is a function that is odd and even simultaneously then** $f(3) - f(2)$ *is*

a)1 b) -1 c) 0 d)2

6. The entire graph of $y = x^2 + kx - x + 9$ **is strictly above the X-axis if and only if**

[ADV- 2019]

a) K<7 b) $-5 < K < 7$

c) $K > -5$ d) K > 7

7. $f(x) = \dfrac{\cos x}{\left[\dfrac{x}{\pi}\right] + \dfrac{1}{2}}$ **(where x is not an integral multiple of π and [.] is G.I.F) is**

a) even function b) odd function

c) Neither even nor odd d) cannot decide

8. *Let* $f: [-10, 10] \to R$ **where**

$f(x)=\sin x+\left[\frac{x^2}{a}\right]$ **where [.] is G.I.F be an odd function then** $a \in$

a) (-10,10)-$\{0\}$ b)$(0,10)$

c)$[100,\infty)$ d)$(100,\infty)$

9. **The domain of** $f(x)=\frac{1}{x}+2^{\sin^{-1}x}+\frac{1}{\sqrt{x-2}}$ **is**

a)$(0,\infty)$ b)$(-\infty,0)$ c)$(1,3)$ d)ϕ

10. **The domain of** $f(x)=\frac{\log_2(x+3)}{x^2+3x+2}$ **is**

a) $(3,\infty)$ b)$(-3,\infty)$

c)$(1,\infty)$ d)$(-3,\infty)-\{-1,-2\}$

11. **The domain of** $f(x)=\sqrt{\ln_{(|x|-1)}(x^2+4x+4)}$ **is**

a) $[-3,-1]\cup[1,2]$

b) $(-2,-1)\cup[2,\infty)$

c) $(-\infty,-3]\cup(-2,-1)\cup(2,\infty)$

d) $(-\infty\ \infty)$

12. **The domain of** $f(x)=\sqrt{\sin^{-1}(\log_2 x)}$ **is**

a)[1,2] b)[-1,1] c)[0,1] d)(1,2)

13. **The domain of** $f(x)=\cos^{-1}([x])$

(where [.] is G.I.F) is

a)[-1,1] b)[0,1] c)[-1,2] d)[-1,2)

14. **The domain of** $f(x)=\sqrt{\frac{\log_{0.3}|x-2|}{|x|}}$ **is**

a)[1,2] b)[2,3] c)$[1,2)\cup(2,3]$ d)$[0,3]$

15. **The domain of**

$f(x)=\log_{10}\log_{10}\log_{10}-----\log_{10}x$

('log' n times) is

a)$\left(10^{10^{10^{10^{-^{(n-2)times}}}}},\infty\right)$ b)$\left(10^{n-2},\infty\right)$

c)$\left(10^{10^{10^{-^{(n-1)times}}}},\infty\right)$ d)$\left(10^{10^{10^{-^{-^{n-3}}}}},\infty\right)$

16. **The function**

$f(x)=\cot^{-1}\left(\sqrt{x(x+3)}\right)+\cos^{-1}\left(\sqrt{x^2+3x+1}\right)$

is defined on the set S , where S is

a)$[-3,0]$ b)$\{-3,0\}$ c)$[0,3]$ d)ϕ

17. **The domain of** $f(x)=\sin^{-1}[2-4x^2]$

(where [.] is G.I.F) is

a)$\left[\frac{-\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right]$ b)$\left[\frac{-3}{2},\frac{3}{2}\right]-\{0\}$

c)$\left[\frac{-\sqrt{3}}{2},0\right)\cup\left(0,\frac{\sqrt{3}}{2}\right]$ d) ϕ

18. **The domain of**

$f(x)=\sqrt{\cos(\sin x)}+\sqrt{\log_x\{x\}}$ **(where {.} is fractional part of x) is**

a)$[1,\pi)$ b)$(0,2\pi)-[1,\pi)$

c)$\left(0,\frac{\pi}{2}\right)-\{1\}$ d)(0,1)

19. **The range of** $f(x)=\sin^2 x-5\sin x-6$ **is**

a)$[-10,0]$ b)[-1,1]

c)$[0,\pi]$ d)$\left[\frac{-49}{4},0\right]$

20. **The range of** $f(x)=\frac{x-[x]}{1-[x]+x}$

(where [.] is G.I.F) is

a)$\left[0,\frac{1}{2}\right]$ b)[0,1] c)$(0,\frac{1}{2}]$ d)$[0,\frac{1}{2})$

21. **If** $f:R\to S$ **defined by**

$f(x)=\sin x-\sqrt{3}\cos x+1$ **is an onto function, then S=**

a)[1,3] b) [-1,3] c)[0,1] d)[-1,1]

22. The range of $f(x)=(7-x)_{P_{(x-3)}}$ is

a) $\{1,2,3,4,5\}$ b) $\{3,4,5\}$

c) $\{1,2,3\}$ d) $\{1,2,3,4\}$

23. The range of $f(x)=\dfrac{x^2-x+1}{x^2+x+1}$ is

a) $\left[\dfrac{1}{3},3\right]$ b) $\left[\dfrac{1}{2},2\right]$

c $[0,1]$ d)[-1,1]

24. The range of $f(x)=\left[\dfrac{1}{\sin\{x\}}\right]$ is (where{.} is fractional part and [.] is G.I.F)

a) $\{1,-1\}$ b) $\{0\}$ c)N d)Z

25. The range of $f(x)=\dfrac{\tan\left(\pi\left[x^2-x\right]\right)}{1+\sin(\cos x)}$ (where [.] is G.I.F) is

a) $\left(0,\dfrac{\pi}{2}\right)$ b) $\{0,1\}$

c) $\{0\}$ d) $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$

26. If $f(x)=x^2+2bx+2c^2$ and $g(x)=-x^2-2cx+b^2$ such that minimum of f(x)> maximum of g(x) then [IIT 2003]

a) $|c|<|b|\sqrt{2}$ b) $|c|>|b|\sqrt{2}$

c) $|c|<|b|$ d)None

27. The sum of the maximum and minimum values of

$f(x)=\sin^{-1}(2x)+\cos^{-1}(2x)+\sec^{-1}(2x)$ *is*

a) π b) $\dfrac{\pi}{2}$ c) 2π d) $\dfrac{3\pi}{2}$

28. If a,b,c,d ,e are +ve real numbers such that $a+b+c+d+e=8$ and $a^2+b^2+c^2+d^2+e^2=16$ then the range of 'e' is

a) $\left[0,\dfrac{16}{5}\right]$ b) $\left[1,\dfrac{5}{3}\right]$ c) $\left(0,\dfrac{5}{16}\right]$ d) [-1 1]

29. If $f(x)=\log_{[x-1]}\left(\dfrac{|x|}{x}\right)$, where[.] is G.I.F then domain and range are

a) $(2,\infty),(0,1)$ b) $[3,\infty),\{0\}$

c) $[3,\infty),\{0,1\}$ d) $(-\infty\ \infty);\{0\}$

30. If $f(x)=x^3+3x^2+4x+a\sin x+b\cos x$ $\forall x\in R$ is an injection then the greatest value of a^2+b^2 is

a)1 b)2 c) $\sqrt{2}$ d) $2\sqrt{2}$

31. If $f(x)=\sin x+\cos x$, $g(x)=x^2-1$ then $g(f(x))$ is invertible in the domain [ADV 2018]

a) $\left[0,\dfrac{\pi}{2}\right]$ b) $\left[\dfrac{-\pi}{4},\dfrac{\pi}{4}\right]$ c) $\left[\dfrac{-\pi}{2},\dfrac{\pi}{2}\right]$ d) $[0,\pi]$

32. If the function $f(x)=\sin x+\cos(ax)$ is periodic , then 'a' is

a) any real unmber b) any integer

c)any rational number d) no such 'a'

33. The period of

$$\frac{|\sin(4x)|+|\cos(4x)|}{|\sin(4x)-\cos(4x)|+\sin(4x)+\cos(4x)}\ is$$

a) $\dfrac{\pi}{4}$ b) $\dfrac{\pi}{2}$ c) $\dfrac{\pi}{8}$ d) π

34. If $f(x)=\cos x+\{x\}$ *where* $\{.\}$ is fractional part function then the period of $f(x)$ is

a) 2π b) 1

c) $\dfrac{\pi}{2}$ d) Does not exist

35. Period of $f(x)=\sin((\cos x)+x)$ *is*

a) Does not exist b) π

c) $\frac{\pi}{2}$ d) 2π

36. The period of

$f(x)=e^{x-[x]+|\cos(\pi x)|+|\cos(2\pi x)|+----|\cos(n\pi x)|}$

(where [.] is G.I.F) is

a) 1 b) $\frac{1}{2}$ c) $\frac{1}{4}$ d) 1 / 5

37. If 'f' is periodic, 'g' is non-periodic, $f(g(x))$ **is periodic and** $f(2)=3, g(4)=7$ **then** $g(6)$ **is**

a) 13 b) 15 c) 11 d) 0

38. The period of $f(x)=[x]+[2x]+[3x]+[4x]$ $+............[nx]-\frac{n(n+1)}{2}x$ **(where** $n\in N$**) is**

a) n b) 1 c) 1 / n d) 5

39. Consider the real valued function satisfying $2f(\sin x)+f(\cos x)=x$**, then** $f\left(\frac{1}{2}\right)=$

a) 1 b)2 c)0 d) 4

40. Let $f(x)$ **be defined for all x>0 and be continuous . Let f(x) satisfy the relation** $f\left(\frac{x}{y}\right)=f(x)-f(y)$ **for all 'x' and 'y' and f(e)=1 then [IIT - 1995]**

a) $f(x)$ is bounded b) $f\left(\frac{1}{x}\right)\to 0$ $as\ x\to 0$

c) $x f(x)\to 0$ $as\ x\to 0$ d) $f(x)=\ln x$

41. A function 'f' well defined $\forall x, y\in R$ **is such that f(1)=2 ,f(2)=8 and** $f(x+y)-kxy=f(x)+2y^2$**, where 'k' is some constant then f(x) is**

a) x^2 b) $3x^2$ c) $2x^2$ d) $4x^2$

42. If $f(x+y)=f(x)f(y)\ \forall x, y\in R$ **and** $f(0)\neq 0$ **then** $F(x)=\frac{f(x)}{1+(f(x))^2}$ **is**

a)an even function b)an odd function
c)both
d)Neither even nor odd function

43. For $x\in R$**, the function f(x) satisfies** $2f(x)+f(1-x)=x^2$ **then the value of f(4) is**

a) $\frac{13}{3}$ b) $\frac{43}{3}$ c) $\frac{23}{3}$ d) $\frac{23}{4}$

44. If [x] and {x} represent the integral and fractional parts of x respectively, then the value of $\sum_{r=1}^{2000}\frac{\{x+r\}}{2000}$ **is**

a) x b) $[x]$ c) $\{x\}$
d) $x+2001$

45. Total number of solutions of $2^x+3^x+4^x-5^x=0$ **is**

a) 0 b) 1
c) 2 d) infinitely many

46. Let R be the set of real numbers and $f:R\to R$ **be such that for all** $x, y\in R, |f(x)-f(y)|\leq|x-y|^3$ **then** $f(x)$ **is [ADV 2020]**

a) a quadratic function b)constant function
c)a cubic function d) linear function

47. Let $A=\{1,2,3,4,5\}$**. If 'f' is a bijective function from A to A , then the number of such functions for which** $f(k)\neq k$ **, k=1,2,3,4,5 is**

a) 5^5 b)120 c) 44 d) 5^5-120

48. If $f:R\to R$**,** $f(x)=x^3+3x^2+10x+2\sin x$

then the range of the function is given by

a) $(-\infty,\infty)$ b) $[0,\infty)$ c) $(-\infty,0]$ d) ϕ

KEY

1) d	2) a	3) d	4) b	5) c	6) b
7) b	8) d	9) d	10) d	11) c	12) a
13) d	14) c	15) a	16) d	17) c	18) d
19) a	20) d	21) b	22) c	23) a	24) c
25) c	26) b	27) c	28) a	29) b	30) a
31) b	32) c	33) c	34) d	35) d	36) a
37) c	38) b	39) c	40) d	41) c	42) a
43) c	44) c	45) b	46) b	47) c	48) a

SOLUTIONS

1. $f(x)=\dfrac{(x+1)^2+4}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}>0$

$range \neq codomain$

$f(x) is \text{ into}$

clearly $f^1(x)=0$ is possible for some 'x'

$\Rightarrow f(x)$ is many-one

2. $f^1(x)=3x^2-6x+6=3(x^2-2x+2)>0$

Disc <0 and leading coeff>0

f is one-one

f(x) is odd degree polynomial , f(x) is onto

3. Clearly $f(x)=0$ for all $x\le 0$ and $f(x)>0$ for all $x>0$

Neither one-one nor onto

4. $f(-x)=\ln\left(-x+\sqrt{1+x^2}\right)$

$=\ln\left(\left(\sqrt{1+x^2}-x\right)\dfrac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}+x}\right)$

$=\ln\left(\dfrac{1}{x+\sqrt{1+x^2}}\right)$

$=-f(x)$

f(x) is an odd function

5. Clearly $f(x)=0$

6. $f(x)>0\rightarrow x^2+(k-1)x+9>0\rightarrow Disc<0$

$\Rightarrow (k-1)^2-4(9)<0$

$\Rightarrow k^2-2k+1-36<0$

$k^2-2k-35<0$

$k\in(-5,7)$

7. $f(-x)=\dfrac{\cos(-x)}{\left[\frac{-x}{\pi}\right]+\frac{1}{2}}=\dfrac{\cos x}{-\left[\frac{x}{\pi}\right]-1+\frac{1}{2}}=-f(x)$

odd function

8. f(x) is an odd function

$\Rightarrow\left[\dfrac{x^2}{a}\right]=0\,\forall x\in[-10,10]$

$0\le\dfrac{x^2}{a}<1\Rightarrow a>100$

9. Domain of $\dfrac{1}{x}$ is $R-\{0\}$

Domain of $\dfrac{1}{\sqrt{x-2}}$ is $x-2>0$

$\Rightarrow x>2$

Domain of $2^{\sin^{-1}x}$ $is[-1,1]$

No common region

10. $x+3>0$ and $(x+1)(x+2)\neq 0$

$\Rightarrow x\in(-3,\infty)-\{-1,-2\}$

11. Case(i) $0<|x|-1<1\Rightarrow 1<|x|<2\Rightarrow x\in(-2,2)$ and

$x\in(-\infty,-1)\cup(1,\infty)$ ......(1)

$\Rightarrow x^2+4x+4\le 1\Rightarrow x^2+4x+3\le 0$

$\Rightarrow -3\le x\le -1$...........(2)

from (1)and(2)

$\therefore x\in(-2,-1)$

Case(ii) $|x|-1>1\Rightarrow |x|>2$

then $x^2+4x+4\ge 1\Rightarrow x^2+4x+3\ge 0$

$\Rightarrow x\ge -1$ or $x\le -3$

$\therefore x\in(-\infty,-3]\cup(2,\infty)$

$\therefore x \in (-\infty,-3] \cup (-2,-1) \cup (2,\infty)$

12. $\sin^{-1}\left(\log_2^x\right) \geq 0$ and

$-1 \leq \log_2 x \leq 1$

$\Rightarrow \log_2^x \geq 0$

$\Rightarrow x \geq 1$

$\Rightarrow -\log_2^2 \leq \log_2^x \leq \log_2^2 \Rightarrow \frac{1}{2} \leq x \leq 2$

$x > 0$

$\therefore Df = [1,2]$

13. $-1 \leq [x] \leq 1 \quad -1 \leq x < 2$

14. $\frac{\log_{0.3}|x-2|}{|x|} \geq 0$ and

$|x-2| \leq 1 \Rightarrow x \in [1,2) \cup (2,3]$

15. Use the definition of lagarithum several times
16. There is no intersectin of the values of ‘x’
17. $-1 \leq \left[2-4x^2\right] \leq 1$

$\Rightarrow -1 \leq 2-4x^2 < 2$

$\Rightarrow x^2 \leq \frac{3}{4}$

$\Rightarrow x \in \left[\frac{-\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$

but at x=0 , f(x) is not defined

18. cos(sinx)≥ 0 is true for all $x \in R; \log\{x\}_x \geq 0$

case(i) 0<x<1$\Rightarrow \{x\} \leq 1 \quad \Rightarrow x \in (0,1), \quad x > 1$ and $\{x\} \leq 1$ not possible

19. $f(x) = \left(\sin x - \frac{5}{2}\right)^2 - \frac{49}{4}$ where

$\frac{9}{4} \leq \left(\sin x - \frac{5}{2}\right)^2 \leq \frac{49}{4}$

20. $f(x) = \frac{\{x\}}{1+\{x\}} \quad \Rightarrow \{x\} = \frac{y}{1-y}$

$\Rightarrow 0 \leq \frac{y}{1-y} < 1 \quad \Rightarrow y \in \left[0, \frac{1}{2}\right)$

21. Here S = co-domain = onto

$\because$ f is onto

use

$a\sin x + b\cos x + c \in \left[c - \sqrt{a^2+b^2}, c + \sqrt{a^2+b^2}\right]$

22. $7-x > 0$ and $x-3 \geq 0$ and $7-x \geq x-3 \Rightarrow x \in \{3,4,5\}$ is the domain

Now $x = 3 \Rightarrow 4_{P_0} = 1$

$x = 4 \Rightarrow 3_{P_1} = 3$, $x = 5 \Rightarrow 2_{P_2} = 2$

23. Let

$y = \frac{x^2-x+1}{x^2+x+1} \Rightarrow yx^2 + yx + y - x^2 + x - 1 = 0$

$\Rightarrow (y-1)x^2 + (y+1)x + (y-1) = 0$

Now $\Delta \geq 0 \Rightarrow y \in \left[\frac{1}{3}, 3\right]$

24. $0 \leq \{x\} < 1$

$\Rightarrow \{x\} = 0$ is ruled out Now $0 < \{x\} < 1$

$\Rightarrow \sin 0 < \sin\{x\} < \sin 1$

$\Rightarrow \frac{1}{\sin\{x\}} > 1.18$, $\therefore \left[\frac{1}{\sin\{x\}}\right] \geq 1$

25. Since Nr is of the form $\tan(n\pi)$ which is zero

26. $\min.value\ of\ f(x) = 2c^2 - b^2$

$\max.value\ of\ g(x) = b^2 + c^2$

$Now\ 2c^2 - b^2 > b^2 + c^2$

$\Rightarrow c^2 > 2b^2$

$\Rightarrow |c| > |b|\sqrt{2}$

27. Celarly D_f *is* $\left\{-\frac{1}{2}, \frac{1}{2}\right\}$

$f(x)$ is minimum when $x = \frac{1}{2}$

and it is $f\left(\frac{1}{2}\right) = \frac{\pi}{2}$

$f(x)$ is maximum when $x=\frac{1}{2}$ and it is

$\left(\frac{-1}{2}\right)=\frac{3\pi}{2}$

$\therefore \frac{\pi}{2}+\frac{3\pi}{2}=2\pi$

28. As we know

$\left(\frac{a+b+c+d}{4}\right)^2 \le \frac{a^2+b^2+c^2+d^2}{4}$

(using tchebycheff's inequality)

$\left(\frac{8-e}{4}\right)^2 \le \frac{16-e^2}{4} \to e(5e-16)\le 0$

$\to e\in\left[0,\frac{16}{5}\right]$

29. $\frac{|x|}{x}>0$ and [x-1]>0 and [x-1] $\neq 1$

[x]>1 [x]-1 $\neq 1$

[x] $\neq 2$ $\Rightarrow x\in[3,\infty)$

clearly $\frac{|x|}{x}$=1 only $\Rightarrow$ range=log1=0

30. $f^1(x)=3x^2+6x+4+a\cos x-b\sin x\ge 0$

leading coeff>0

$\Rightarrow 3x^2+6x+4\ge b\sin x-a\cos x$

$\Rightarrow 3x^2+6x+4\ge \sqrt{a^2+b^2}$

$\Rightarrow 3(x+1)^2+1\ge\sqrt{a^2+b^2}$

$\max .value\, of\ a^2+b^2\ is\ 1$

31. $g(f(x))=(\sin x+\cos x)^2-1=\sin(2x)$

Clearly $g(f(x))$ is invertible in $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]\to x\in\left[\frac{-\pi}{4},\frac{\pi}{4}\right]$

32. Let T be the period of f(x) , then

$f(x+T)=\sin(x+T)+\cos(a(x+T))$

$=f(x)=\sin x+\cos(ax)$

$\sin(x+T)+\cos(a(x+T))$

$=\sin x+\cos(ax)$

put x=0 and x=-T respetively

$\sin T+\cos(aT)=1$

$-\sin T+\cos(aT)=1$ sove these euqations

$\sin(T)=0\ and\ \cos(aT)=1$

$T=n\pi; n\in z\ aT=2m\pi; m\in z$

$\frac{aT}{t}=\frac{2m\pi}{n\pi}=\frac{2m}{n}, a\, rational\ number$

33. Period of $|\sin(4x)|+|\cos(4x)|\ is\ \frac{\pi}{8}$

Period of

$|\sin(4x)-\cos(4x)|+|\sin(4x)+\cos(4x)|$

is $\frac{\pi}{8}$

$\therefore period\ =\pi/8$

34. $f(x)=\cos x+(x-[x])$

Here period of cosx is 2π and the period of $x-[x]$ is 1

But L.C.M of 2π and 1 does not exist

35. $f(x+2\pi)=\sin(\cos(2\pi+x)+(2\pi+x))$

$=\sin(2\pi+x+\cos x)=\sin(x+\cos x)=f(x)$

$\Rightarrow period\ of\ f(x)\ is\ 2\pi$

36. $period\ of\ x-[x] is 1$

$period\ of\,|\cos(\pi x)|\ is\ \frac{\pi}{\pi}=1$

$period\ of\,|\cos(2\pi)x|\ is\ \frac{\pi}{2\pi}=\frac{1}{2}.......$

$L.C.M\ of\ 1,\frac{1}{2},\frac{1}{3},.....\frac{1}{n}\ is\ 1$

37. From given data , g(x) must be linear function

$g(x)=ax+b$

Also $g(2)=3\Rightarrow 2a+b=3; g(4)=7$

$\Rightarrow 4a+b=7$ solving $a=2, b=-1$

$\therefore g(x)=2x-1 \Rightarrow g(6)=11$

38. Conceptual

39. **Given** $2f(\sin x)+f(\cos x)=x$ **. ......(i)**

Replacing x **by** $\left(\frac{\pi}{2}-x\right)$ **in equation (i), we get**

$$2f\left(\sin\left(\frac{\pi}{2}-x\right)\right)+f\left(\cos\left(\frac{\pi}{2}-x\right)\right)=\frac{\pi}{2}-x$$

or $2f(\cos x)+f(\sin x)=\frac{\pi}{2}-x$ **......(ii)**

Multiplying in equation (i) by 2, then

$4f(\sin x)+2f(\cos x)=2x$ **.....(iii)**

Now subtraction Eq. (ii) from Eq. (ii), we get

$$3f(\sin x)=2x-\left(\frac{\pi}{2}-x\right)=3x-\frac{\pi}{2}$$

or $f(\sin x)=x-\frac{\pi}{6}$

or $f(x)=\sin^{-1}x-\frac{\pi}{6}$

Hence,

$$f(x)=\sin^{-1}x-\frac{\pi}{6}\ \forall x\in[-1,1].$$

40. By observation , it is easy to decide that $f(x)=\ln x$

41. $f(x+y)-kxy=f(x)+2y^2$; replace 'x' by '-x' , then we get

$f(0)+kx^2=f(x)+2x^2$

$\Rightarrow f(x)=f(0)+kx^2-2x^2$

put $x=1$

$f(1)=f(0)+k-2=2$

$\Rightarrow f(2)=f(0)+4k-8=8$

solving $k=4$ *and* $f(0)=0$

$\therefore f(x)=2x^2$

42. $f(x+y)=f(x).f(y)-----(1)$

put $x=y=0\Rightarrow f(0)=1[\therefore f(0)\neq 0]$

put $y=-x\ in(1)$ then

$f(0)=f(x).f(y)\rightarrow 1=f(x).f(-x)$

$\Rightarrow f(-x)=\frac{1}{f(x)}----(2)$

Now $F(x)=\frac{f(x)}{1+(f(x))^2}$

$\Rightarrow F(-x)=\frac{f(-x)}{1+(f(-x))^2}$

$$=\frac{\frac{1}{f(x)}}{1+\frac{1}{(f(x))^2}}\ from\ (2)$$

$=F(x)\Rightarrow F$ is an even function

43. $2f(x)+f(1-x)=x^2----(1)$

replace 'x' by '1-x'

$2(1-x)+f(x)=(1-x^2)----(2)$

Now $(1)\times(2)$

$4f(x)+2f(1-x)=2x^2$

$(2)\Rightarrow \quad f(x)+2f(1-x)=(1-x)^2$

$3f(x)=2x^2-(1-x)^2$

$\Rightarrow f(x)=\frac{x^2+2x-1}{3}$

$\Rightarrow f(4)=23/3$

44. $\sum_{r=1}^{2000}\frac{\{x+r\}}{2000}=\sum_{r=1}^{2000}\frac{\{x\}}{2000}=2000\frac{\{x\}}{2000}=\{x\}$

45. Let $\quad Let\ f(x)=\left(\frac{2}{5}\right)^x+\left(\frac{3}{5}\right)^x+\left(\frac{4}{5}\right)^x-1$,

which is a decreasing function $\Rightarrow$ graph cuts the X-axis at onlny one point

46. Since

$|f(x)-f(y)|\le|x-y|^3$ *is true* $\forall x,y\in R$

we have for $x\ne y, \dfrac{|f(x)-f(y)|}{|x-y|}\le|x-y|^2$

$\underset{y\to x}{Lt}\left|\dfrac{f(x)-f(y)}{x-y}\right|\le\underset{y\to x}{Lt}|x-y|^2$

$\to\underset{y\to x}{Lt}\left|\dfrac{f(x)-f(y)}{x-y}\right|\le 0$

$|f^1(x)|\le 0 \quad \Rightarrow f^1(x)=0$

$f(x)$ is a constant function

47. Use derangement formula

$5_{p_5}\left[\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right]=44$

48. as x^3+3x^2+10x is a cubic polynomial

$\therefore -\infty<x^3+3x^2+10x<\infty$

$\Rightarrow -\infty<x^3+3x^2+10x+2\sin x$

$<\infty \because -1\le\sin x\le 1$

MULTIPULE ANSWER TYPE QUESTIONS

1. **f(x) =** $\begin{cases} x & \text{if x is rational} \\ 0 & \text{if x is irrational}\end{cases}$ **and**

g(x) = $\begin{cases} 0 & \text{if x is rational} \\ x & \text{if x is irrational}\end{cases}$ **then f – g is**

[IIT- 2005]

a) one–one and into
b) neither one–one nor onto
c) many one and onto
d) one–one and onto

2. **If f(x) =** $\cos[\pi^2]x+\cos[-\pi^2]x$ **(where [x] stands for the greatest integer function)then** **[IIT-1991]**

a) $f(\pi/2)=-1$ b) $f(\pi)=1$

c) $f(-\pi)=0$ d) $f(\pi/4)=2$

3. **The** $f:R\to R$ **be any function. Define** $g:R\to R$ **by g(x) = | f(x) | for all x then g is** **[IIT - 2000]**

a) onto if f is onto
b) one–one if f is one-one
c) continuous if f is continuous
d) differentiable if f is differentiable

4. **Let** $f(x)=[x]^2+[x+1]-3$ **where**

$[x]\le x$ *then and* $f:R\to R$

a) many-one b) one-one
c) onto d) into

5. **The domain of** $f(x)=\dfrac{1}{\sqrt{\log_{1/2}(x^2-7x+13)}}$ **is**

a) (3,4) b) [3,4]

c) $(-\infty,3]\cup[1,\infty)$ d)(1,3)

6. **The domain of definition of f(x) =** $\dfrac{\log_2(x+3)}{x^2+3x+2}$ **is** **[IIT - 2001]**

a) R-{-1, -2} b) $(-2, \infty)$

c) R-{-1, -2, -3} c) $(-3, \infty)-\{-1, -2\}$

7. For the function $f(x)=\log_{10}(3x^2-4x+5)$

a) Domain is $(0,\infty)$ b) range is R

c) Domain is R d) range is $\left[\log_{10}\left(\frac{11}{3}\right),\infty\right)$

8. For the function $f(x)=\cos^{-1}\sqrt{\log_{[x]}\left(\frac{|x|}{x}\right)}$ **(where [.] G.I.F)is**

a) Domain is $[1,\infty)$ b)Domain is $[2,\infty)$

c)range is $[0,\infty)$ d)range is $\left\{\frac{\pi}{2}\right\}$

9. Range of the function $f(x)=\dfrac{x^2+x+2}{x^2+x+1}; x\in R$ **is [ADV - 2017]**

s) $(1, \infty)$ b) (1, 11/7]

c) (1, 7/3] d) [1, 7/5]

10. $Let\ f: X\to Y, f(x)=\sin x+\cos x+2\sqrt{2}$ **is invertible then**

a) $X=\left[\frac{\pi}{4},\frac{5\pi}{4}\right]$ b) $X=\left[\frac{\pi}{4},\frac{\pi}{2}\right]$

c) $Y=[-1,1]$ d) $Y=\left[\sqrt{2},3\sqrt{2}\right]$

11. If f(x) = 3x - 5 then $f^{-1}(x)$ [IIT - 1998]

a) is given by $\frac{1}{3x-5}$ b) is given by $\frac{x+5}{3}$

c) does not exist because f is not one-one

d) does not exist because f is not onto

12. Let f(x) = $\frac{\alpha x}{x+1}, x\neq -1$ **then the value of α so that f(f(x)) = x is [ADV-2016]**

a) $\sqrt{2}$ b) $-\sqrt{2}$ c) 1 d) –1

13. Let f(x) = sin x + cos x and g(x) = x^2 – 1 then domain for which gof is invertible is [IIT- 2004]

a) $\left(0,\frac{\pi}{2}\right)$ b) $\left(\frac{\pi}{2},\frac{2\pi}{3}\right)$

c) $\left(-\frac{\pi}{2},\frac{\pi}{3}\right)$ d) $\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$

14. All periodic functions are

a) one-one b) many-one

c)invertible

d)invertible by restriciting the domain

15. If $\{x\}$ *and* $[x]$ denote the fractional and integral parts of 'x' and $\{x+1\}+2x=4[x+1]-6$ **then x is**

a)1 b)$\frac{8}{3}$ c)0 d)-1

16. The domain of definition of the function f(x) given by the equation $2^x + 2^y = 2$ **is [IIT - 2000]**

a) $0<x\le 1$ b) $0\le x\le 1$

c) $-\infty<x\le 0$ d) $-\infty<x<1$

KEY

01) d	02) a,c	03) c	04) a,d	05) a,c
06) d	07) c,d	08) b,d	09) c	10) a,d
11) B	12) d	13) d	14) b,d	15) a,b
16) d				

SOLUTIONS

1. We are given that

$$f: R\to R \text{ s.t. } f(x)=\begin{cases}0, & x\in \text{rational}\\ x, & x\in\text{irrational}\end{cases}$$

$$g: R\to R \text{ s.t. } g(x)=\begin{cases}0, & x\in \text{irrational}\\ x, & x\in\text{rational}\end{cases}$$

$$(f-g)(x)=\begin{cases}-x & \text{if } x\in\text{rational}\\ x & \text{if } x\in\text{irrational}\end{cases}$$

Since f . g : R $\to$ R for any x there is only one value of (f(x) - g(x)) whether x is rational or irrational. Moreover as x $\to$R, f(x) - g(x) also belong to R. Therefore (f - g) is one-one onto. (d) is the correct option.

2. (A, C)

f(x) = cos [π^2] x + cos [-π^2]x

We know $9 < \pi^2 < 10$ and $-10 < -\pi^2 < -9$

$[\pi^2] = 9$ and $[-\pi^2] = -10$

$\Rightarrow$ f(x) = cos9x + cos (-10x)

f(x) = cos9x + cos(-10x) $\Rightarrow$ f(x) = cos 9x + cos 10x

(a) $f\left(\frac{\pi}{2}\right) = \cos\frac{9\pi}{2} + \cos 5\pi = -1$ (True)

(b) $f(\pi) = \cos 9\pi + \cos 10\pi = -1 + 1 = 0$ (False)

(c) $f(-\pi) = \cos(-9\pi) + \cos(-10\pi) = \cos 9\pi + \cos 10\pi = -1 + 1 = 0$ (True)

(d) $f\left(\frac{\pi}{4}\right) = \cos\frac{9\pi}{4} + \cos\frac{5\pi}{2} =$

$\cos\left(2\pi + \frac{\pi}{4}\right) + 0$ (False)

Thus (A) and (C) are the correct options.

3. Let h(x) = $|x|$ then g(x) = $|f(x)| = h(f(x))$

Since composition of two continuous function is continuous, g is continuous if f is continuous.

4. Let $f(x) = [x]^2 + [x] + 1 - 3$

$= [x]^2 + [x] - 2 = ([x]+2)([x]-1)$

Clearly $f(x) = 0$

$[x] = -2\ (or)\ [x] = 1$

f is many-one and range contains only integers $\Rightarrow range \neq co-domain \Rightarrow f\ is$ into

5. $\log_{\frac{1}{2}}\left(x^2 - 7x + 13\right) > 0$ and $x^2 - 7x + 13 > 0$

$x^2 - 7x + 13 < 1$

$x^2 - 7x + 12 < 0$

$x \in (3, 4)$

$x^2 - 7x + 13 > 0$

always true i.e $\forall x \in R$

6. For domain of f(x) = $\frac{\log_2(x+3)}{x^2 + 3x + 2}$

$x^2 + 3x + 2 \neq 0$ and $x + 3 > 0$

$\Rightarrow x \neq -1, -2$ and $x > -3$

$\therefore D_f = (-3, \infty) - \{-1, -2\}$

7. $3x^2 - 4x + 5 > 0$ is tue $\forall x \in R \because Disc < 0$ and leading coeff > 0

Let $f(x) = y \Rightarrow \log_{10}\left(3x^2 - 4x + 5\right) = y$

$\Rightarrow 3x^2 - 4x + \left(5 - 10^y\right) = 0$

As x is real ,

$\Delta \geq 0 \Rightarrow 10^y \geq \frac{11}{3} \Rightarrow y \in [\log_{10}\left(\frac{11}{3}\right), \infty)$

8. $\frac{|x|}{x} > 0; [x] > 0\ and\ [x] \neq 1$

$\Rightarrow x > 0 \quad x \in [1, \infty) \Rightarrow x \in [2, \infty) = domain$

For $x \in [2, \infty) \Rightarrow \log_{[x]}\left(\frac{|x|}{x}\right) = \log 1 = 0$

$\Rightarrow \cos^{-1}(0) = \frac{\pi}{2}$

$\therefore range = \left\{\frac{\pi}{2}\right\}$

9. We have f(x) =

$\frac{x^2 + x + 2}{x^2 + x + 1} = \frac{(x^2 + x + 1) + 1}{x^2 + x + 1} = 1 + \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}$

We can see here that as $x \to \infty$, $f(x) \to 1$ which is the min value of f(x). Also f(x) is max when $\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$ is min which is so when x = -1/2 and then 3/4.

$f_{max} = 1 + \frac{1}{3/4} = \frac{7}{3}$, $R_f = (1, 7/3]$

10. $f(x) = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) + 2\sqrt{2}$ (or)

$\sqrt{2}\cos\left(x - \frac{\pi}{4}\right) + 2\sqrt{2} \Rightarrow -\frac{\pi}{2} \leq x + \frac{\pi}{4} \leq \frac{\pi}{2}$

$\frac{-\pi}{2} \leq x + \frac{\pi}{4} \Rightarrow \frac{-\pi}{2} - \frac{\pi}{4}$

$X = \left[\frac{-3\pi}{4}, \frac{\pi}{4}\right] (or) \Rightarrow X = \left[\frac{-\pi}{4}, \frac{5\pi}{4}\right]$

$Clearly\ Y = \left[\sqrt{2}, 3\sqrt{2}\right]$

11. f(x) = 3x – 5 (given)
Let y = f(x) = 3x – 5

$\Rightarrow y + 5 = 3x \Rightarrow x = \frac{y+5}{3}$
. . . (1)
and $y = f(x) \Rightarrow x = f^{-1}(y)$
. . . (2)
From (1) and (2)

$f^{-1}(y) = \frac{y+5}{3} \Rightarrow f^{-1}(x) = \frac{x+5}{3}$.

12. $f(x) = \frac{\alpha x}{x+1}, x \neq -1$

$f(f(x)) = x \Rightarrow \frac{\alpha\left(\frac{\alpha x}{x+1}\right)}{\frac{\alpha x}{x+1}+1} = x$

$\Rightarrow \frac{\alpha^2 x}{(\alpha+1)x+1} = x \Rightarrow (\alpha+1)x^2 + (1-\alpha^2)x = 0$
. . . (1)
$\Rightarrow \alpha + 1 = 0$ and $1 - \alpha^2 = 0$

$\left[\begin{array}{l}\text{As true } \forall x \neq -1 \\ \therefore \text{Eq.(1) is an identity}\end{array}\right]$

$\Rightarrow \alpha = -1$.

13. f(x) = sin x + cos x g(x) = $x^2 - 1$

$\Rightarrow g(f(x)) = (\sin x + \cos x)^2 - 1 = \sin 2x$

Clearly $g(f(x))$ is invertible in $-\frac{\pi}{2} \leq 2x \leq \frac{\pi}{2}$

$\left[\because \sin\theta \text{ is invertible when } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right]$

$\Rightarrow -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$.

14. Conceptual

15. $(x+1) - [x+1] + 2x = 4[x+1] - 6$

$3x = 5[x] - 2$...........(1)

$3([x] + \{x\}) = 5[x] - 2$

$\Rightarrow 3\{x\} = 2[x] - 2$........(2)

but

$O \leq \{x\} < 1 \Rightarrow 0 \leq \{3x\} < 3 \Rightarrow 0 \leq 2[x] - 2 < 3$

$\Rightarrow [x] = 1, 2\ Now\ [x] = 1 \Rightarrow x = 1$

$[x] = 2 \Rightarrow x = 8/3\left[\ from\ (1)\ and\ (2)\right]$

16. It is given that $2^x + 2^y = 2 \forall x, y \in R$

Therefore, $2^x = 2 - 2^y < 2 \Rightarrow 0 < 2^x < 2$

Taking log for both side with base 2

$\Rightarrow \log_2 0 < \log_2 2^x < \log_2 2$

Hence domain is $-\infty < x < 1$.

COMPREHENSION TYPE QUESTIONS

Passage - 1

$f: A \to B$ is said to be injective if distinct elements in A have distinct images in B and surjective if f(A)=B.Now answer the following.

1. **If the function $f: A \to B$ defined by $f(x) = \sqrt{\frac{1+\cos(2x)}{2}}$ is injective then A can be**

a)$[0, \pi]$ b)$[-\pi, \pi]$

c)$\left[-\frac{\pi}{2}, 0\right]$ d)$[-\pi, 0]$

2. **If the function $f: R \to B$ defined by $f(x) = \sqrt{x^2}$ is surjective then B is**

a) $(-\infty, 0]$ b) $[0, \infty)$ c) $(0, \infty)$ d)R

3. **The functions $f: R \to B$ defined by $f(x) = [x] + [-x]$(where [.] is G.I.F) is surjective then B=**

a)R b)[0,1] c)[-1,0] d)$\{-1, 0\}$

Passage - 2

Let $f(x) = \frac{1}{2}\left[f(xy) + f\left(\frac{x}{y}\right)\right] \quad \forall x, y \in R^+$

such that $f(1) = 0, f^1(1) = 2$**. Now answer**

the following.

4. $f(x)-f(y)=$

a) $f\left(\frac{y}{x}\right)$ b) $f\left(\frac{x}{y}\right)$ c) $f(2x)$ d) $f(2y)$

5. $f^1(3)=$

a) $\frac{1}{3}$ b) $\frac{2}{3}$ c) $\frac{1}{2}$ d) $\frac{1}{4}$

6. $f(e)=$

a) 2 b) 1 c) 3 d) 4

Passage - 3

Let $f:R\to R$ is a function satisfying $f(2-x)=f(2+x)$ and $f(20-x)=f(x)$ $\forall x\in R$. Now answer the following.

7. *If* $f(0)=5$ **then the minimum possible number of values of 'x' satisfying** $f(x)=5\ for\ x\in[0,170]\ is$

a) 21 b) 12 c) 11 d) 22

8. The graph of $y=f(x)$ is symmetrical about

a) $x=16$ b) $x=5$ c) $x=8$ d) $x=20$

9. If $f(2)\neq f(6)$ then the period of f(x) is

a) 1 b) may or may not be 1
c) can not be 1 d) Non periodic

KEY

01) c 02) b 03) b 04) b 05) b 06) a
07) a 08) a 09) c

SOLUTIONS

1. f(x)=$|\cos x|$ which will be in $\left[\frac{-\pi}{2},0\right]$

2. f(x)=$|x|\Rightarrow B=[0,\infty)$

3. [x]+[-x]=1; $x\notin Z$
=0; $x\in Z\Rightarrow B=\{-1,0\}$

4. $f(x)=\frac{1}{2}\left[f(xy)+f\left(\frac{x}{y}\right)\right]----(1)$

interchanging 'x' and 'y'

$f(y)=\frac{1}{2}\left[f(xy)+f\left(\frac{y}{x}\right)\right]-----(2)$

$(1)-(2)\Rightarrow f(x)-f(y)=\frac{1}{2}\left[f\left(\frac{x}{y}\right)-f\left(\frac{y}{x}\right)\right]$ $-----(3)$

$in(1)\ put\ x=1$

$f(1)=\frac{1}{2}\left[f(y)+f\left(\frac{1}{y}\right)\right]---(4)$

$f(1)=0\Rightarrow f(y)=-f\left(\frac{1}{y}\right)$

$\therefore f\left(\frac{x}{y}\right)=-f\left(\frac{y}{x}\right)$

$\Rightarrow f(x)-f(y)=f\left(\frac{x}{y}\right)$

5. $f^1x=\underset{h\to 0}{Lt}\frac{f(x+h)-f(x)}{h}$

$\Rightarrow f^1(3)=\frac{2}{3}$

6. $f^1(x)=\frac{2}{x}$

$\Rightarrow f(x)=2\log x+c$

$f(1)=0\Rightarrow c=0\Rightarrow f(x)=2\log x$

$\Rightarrow f(e)=2$

07,08,09.

$f(2-x)=f(2+x)---(1)$

replace 'x' by '2-x'

$\to f(x)=f(4-x)---(2)$ Also given

$f(2-x)=f(x)\quad ------(3)$

$from(2)and(3)\ f(4-x)=f(20-x)$

$Now replace\ 'x'\ by\ '4-x'\to f(x)=f(x+16)$

Hence period of $f(x)$ *is* 16

If 1 is a period then $f(x)=f(x+1)\ \forall x\in R$

$\Rightarrow\ f(2)=f(3)=f(4)=f(5)=f(6)$

which cantradicts the given hypothes is $f(2)\neq f(6)$

$\therefore 1$ can not be the period

MATRIX MATCHING TYPE QUESTIONS

1. Match the following functions with their ranges

COLUMN-I

a) $f(x)=\log_3\left(5+4x-x^2\right)$

b) $f(x)=\log_3\left(x^2-4x-5\right)$

c) $f(x)=\log_3\left(x^2-4x+5\right)$

d) $f(x)=\log_3\left(4x-5-x^2\right)$

COLUMN-II

p) function is not defined

q) $[0,\infty)$

r) $(-\infty,2]$

s) R

2. Let $f(x)=\dfrac{x^2-6x+5}{x^2-5x+6}$

Now match the following [IIT 2007]

COLUMN-I

a) *if* $-1<x<1$ *then* $f(x)$ *satisfies*

b) *if* $1<x<2$ *then* $f(x)$ *satisfies*

c) *if* $3<x<5$ *then* $f(x)$ *satisfies*

d) *if* $x>5$ *then* $f(x)$ *satisfies*

COLUMN-II

p) $0<f(x)<1$

q) $f(x)<0$

r) $f(x)>0$

s) $f(x)<1$

KEY

01) a-r, b-s, c-q, d-p

02) a-p,r,s; b-q; c-q; d-p,r,s

SOLUTIONS

1. $a\Rightarrow f(x)=\log_3\left(5+4x-x^2\right)$

$=\log_3\left(9-(x-2)^2\right)$

Now $-\infty<9-\left(x-2^2\right)\leq 9$ but for f(x) to get defined

$0<9-\left(x-2^2\right)\leq 9$

$\Rightarrow -\infty<\log_3\left(9-(x-2)^2\right)\leq\log_3 9$

$\Rightarrow range(-\infty,2^3]$

$b\Rightarrow f(x)=\log_3\left(x^2-4x+5\right)$

$=\log_3\left((x-2)^2-9\right)$

but here $0<(x-2)^2-9<\infty,-\infty<f(x)<\infty$

$c\Rightarrow\log_3\left(x^2-4x+5\right)$

$=\log_3\left((x-2)^2+1\right)$ but $(x-2)^2+1\in[1,\infty)$

$range = [0, \infty)$

$d \Rightarrow \log_3\left(4x-5-x^2\right)$ *but* $-x^2+4x-5$

is always Negative

disc < 0 and leading coeff < 0

2. Let $f(x)=\dfrac{(x-5)(x-1)}{(x-2)(x-3)}$ if $-1<x<1$ then

$$f(x)=\frac{(-ve)(-ve)}{(-ve)(-ve)}=+ve \to p,r,s$$

If $1<x<2 \to f(x)<0 \to q$

If $3<x<5 \to f(x)<0 \to q$

If $x>5 \to f(x)0 \to p,r,s$

INTEGER TYPE QUESTIONS

1. If 'f' is a polynomial such that $f\left(\dfrac{1-x}{1+x}\right)f\left(\dfrac{1+x}{1-x}\right)=f\left(\dfrac{1-x}{1+x}\right)+f\left(\dfrac{1+x}{1-x}\right)$ $\left(\textit{where } x\neq 0,\pm 1\right)$ **and** $f(3)=28$ **then the value of** $\dfrac{1}{605}\left(\sum_{n=1}^{10}\left(f(n)-1\right)\right)$ **is**

2. If $f^2(x)\,f\left(\dfrac{1-x}{1+x}\right)=x^3$ **[** $x\neq -1$ **and** $f(x)\neq 0$ **] then the value of** $|\left[f(-2)\right]|$ **is (where [.] is G.I.F)**

3. If $f^3(x)-3f^2(x)+3f(x)-1=x^6$ **then the value of** $f(0)$ **is**

4. If $f\left(x+\dfrac{1}{x}\right)=x^3+\dfrac{1}{x^3}-4\left(x^2+\dfrac{1}{x^2}\right)+13$ **then the value of** $f\left(2+\sqrt{3}\right)=$

5. The period of the function satisfying the relation $f(x)+f(x+3)=0,\ \forall x\in R$ **is**

6. If the period of $\dfrac{\cos\left(\sin(nx)\right)}{\tan\left(\dfrac{x}{n}\right)}; n\in N$ *is* 6π **then ' n' is**

7. If $f(x)=\operatorname{sgn}\left(x^2-2x+3\right)$ **then the value of** $f(x)$ **is**

8. If the range of the function $f(x)=\dfrac{\left(1+x+x^2\right)\left(1+x^4\right)}{x^3}$ **when** $x>0$ **is** $[K,\infty)$ **then** K **is**

KEY

01) 5 02) 2 03) 1 04) 9 05) 6 06) 6
07) 1 08) 6

SOLUTIONS

1. replace 'x' by $\dfrac{1-x}{1+x}$, *we have*

$$f(x).f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)$$

$$\Rightarrow f(x)=\pm x^n+1$$

$$\to (x)=x^3+1\left(\because f(3)=28\right)$$

Now $\sum_{n=1}^{10}\left(f(n)-1\right)=\sum_{n=1}^{10}n^3=325$

$$\Rightarrow \frac{1}{605}(3025)=5$$

2. $f^2(x).f\left(\dfrac{1-x}{1+x}\right)=x^3 ----(1)$

replace 'x' by $\dfrac{1-x}{1+x}$

$$f^2\left(\frac{1-x}{1+x}\right)f(x)=\left(\frac{1-x}{1+x}\right)^3 ----(2)$$

$from\ (1)\, and\, (2)$

$$f^3(x)=x^6\left(\frac{1+x}{1-x}\right)^3$$

$$\Rightarrow f(x)=x^3\left(\frac{1+x}{1-x}\right)$$

$$f(-2)=\frac{8}{3}\to |[f(-2)]|=2$$

3. $(f(x)-1)^3=x^6$

$\Rightarrow f(x)=x^2+1\to f(0)=1$

4. If $f(x)=x^3+\frac{1}{x^3}-4\left(x^2+\frac{1}{x^2}\right)+13$

$$=\left(x+\frac{1}{x}\right)^3-3\left(x+\frac{1}{x}\right)^2\ -4\left[\left(x+\frac{1}{x}\right)-2\right]+13$$

$$\Rightarrow f\left(2+\sqrt{3}\right)=9$$

5. $f(x)+f(x+3)=0$
--(1)
replace 'x' by 'x+3'

$f(x+3)+f(x+6)=0------(2)$

$(1)-(2)\Rightarrow f(x)=f(x+6)$

$\Rightarrow period\ of\ f(x)\ is\ 6$

6. Period of $\cos(\sin(nx))\ is\ \frac{\pi}{n}$ and the period of $\tan\left(\frac{x}{n}\right)\ is\ \pi n$

Now L.C.M of $\frac{\pi}{n},\pi n$ is $\pi n\ \Rightarrow n=6$

7. $b^2-4ac<0$.

8. $f(x)=\frac{1+x^4+x+x^5+x^2+x^6}{x^3}$

$$=\left(\frac{1}{x^3}+x^3\right)+\left(x+\frac{1}{x}\right)+\left(x^2+\frac{1}{x^2}\right)$$

$$\geq 2\sqrt{\frac{1}{x^3}.x^3}+2\sqrt{x.\frac{1}{x}}+2\sqrt{x^2.\frac{1}{x^2}}=6$$

$\left(use\ a+\frac{1}{a}\geq 2, a>0\right)$

$\Rightarrow f(x)\geq 6$

$\therefore R_f=[6,\infty)$.

5. $f(x)+f(x+3)=0$
--(1)
replace 'x' by 'x+3'

$f(x+3)+f(x+6)=0-----(2)$

$(1)-(2) \Rightarrow f(x)=f(x+6)$

$\Rightarrow \textit{period of } f(x) \textit{ is } 6$

6. Period of $\cos(\sin(nx))$ *is* $\frac{\pi}{n}$ and the period of

$\tan\left(\frac{x}{n}\right)$ *is* πn

Now L.C.M of $\frac{\pi}{n}, \pi n$ is $\pi n \Rightarrow n=6$

7. $b^2-4ac<0$.

8. $f(x)=\frac{1+x^4+x+x^5+x^2+x^6}{x^3}$

$=\left(\frac{1}{x^3}+x^3\right)+\left(x+\frac{1}{x}\right)+\left(x^2+\frac{1}{x^2}\right)$

$\geq 2\sqrt{\frac{1}{x^3}.x^3}+2\sqrt{x.\frac{1}{x}}+2\sqrt{x^2.\frac{1}{x^2}}=6$

$\left(use\ a+\frac{1}{a}\geq 2, a>0\right)$

$\Rightarrow f(x)\geq 6$

$\therefore R_f=[6,\infty)$.

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PROPERTIES OF TRIAGNLES

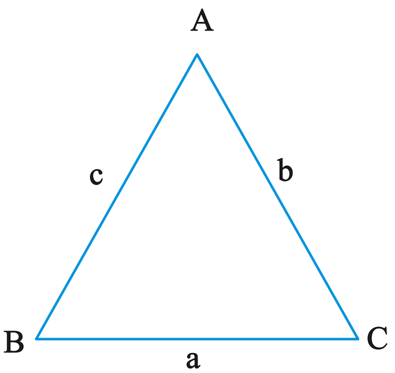
SYNOPSIS

Relation between the sides and angles of a triangle :

Notation:

(i) In triangle ABC the sides BC, CA, AB are denoted by a, b, c respectively. The angles BAC,CBA, ACB are denoted by A,B,C respectively. The semi perimeter of the triangle is denoted by 's'. The area of a triangle is denoted by Δ .

(ii) $s = \dfrac{a+b+c}{2}$



Sine Rule :

$$\frac{a}{Sin A} = \frac{b}{Sin B} = \frac{c}{Sin C} = 2R$$

a = 2R Sin A, b = 2R Sin B, c = 2R Sin C

(R is circumradius of triangle ABC) .

Cosine Rule :

(i) $a^2 = b^2 + c^2 - 2bc \text{ Cos } A$

(ii) $b^2 = c^2 + a^2 - 2ca \text{ Cos } B$

(iii) $c^2 = a^2 + b^2 - 2ab \text{ Cos } C$

(i) $\text{Cos } A = \dfrac{b^2 + c^2 - a^2}{2bc}$,

(ii) $\text{Cos } B = \dfrac{c^2 + a^2 - b^2}{2ca}$

(iii) $\text{Cos } C = \dfrac{a^2 + b^2 - c^2}{2ab}$

Mollweide's Rule :

(i) $\dfrac{b-c}{a} = \dfrac{\sin\left(\dfrac{B-C}{2}\right)}{\cos\dfrac{A}{2}}$; $\dfrac{b+c}{a} = \dfrac{\cos\left(\dfrac{B-C}{2}\right)}{\sin\dfrac{A}{2}}$

(ii) $\dfrac{c-a}{b} = \dfrac{Sin\left(\dfrac{C-A}{2}\right)}{Cos\dfrac{B}{2}}$; $\dfrac{c+a}{b} = \dfrac{Cos\left(\dfrac{C-A}{2}\right)}{Sin\dfrac{B}{2}}$

(iii) $\dfrac{a-b}{c} = \dfrac{\sin\left(\dfrac{A-B}{2}\right)}{\cos\left(\dfrac{C}{2}\right)}$; $\dfrac{a+b}{c} = \dfrac{\cos\left(\dfrac{A-B}{2}\right)}{\sin\left(\dfrac{C}{2}\right)}$

Napier's Formulae (Tangent rule) :

i) $\tan\left(\dfrac{B-C}{2}\right) = \left(\dfrac{b-c}{b+c}\right)\cot\dfrac{A}{2} = \left(\dfrac{b-c}{b+c}\right)\tan\left(\dfrac{B+C}{2}\right)$

ii) $\tan\left(\dfrac{C-A}{2}\right) = \left(\dfrac{c-a}{c+a}\right)\cot\dfrac{B}{2} = \left(\dfrac{c-a}{c+a}\right)\tan\left(\dfrac{C+A}{2}\right)$

iii) $\tan\left(\dfrac{A-B}{2}\right) = \left(\dfrac{a-b}{a+b}\right)\cot\dfrac{C}{2} = \left(\dfrac{a-b}{a+b}\right)\tan\left(\dfrac{A+B}{2}\right)$

Projection formulae :

(i) a = b Cos C + c Cos B

(ii) b = c Cos A + a Cos C

(iii) c = a Cos B + b Cos A

Half-angle results :

(i) $\sin\dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$

(ii) $\sin\dfrac{B}{2} = \sqrt{\dfrac{(s-c)(s-a)}{ca}}$

(iii) $\sin\dfrac{C}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$

✈ (i) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

(ii) $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$

(iii) $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

✈ (i) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{(s-b)(s-c)}{\Delta}$

$= \frac{\Delta}{s(s-a)} = \sqrt{\frac{rr_1}{r_2 r_3}} = \sqrt{\frac{r_1 - r}{r_2 + r_3}}$

(ii) $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \frac{(s-c)(s-a)}{\Delta}$

$= \frac{\Delta}{s(s-b)} = \sqrt{\frac{rr_2}{r_1 r_3}} = \sqrt{\frac{r_2 - r}{r_1 + r_3}}$

(iii) $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{(s-a)(s-b)}{\Delta}$

$= \frac{\Delta}{s(s-c)} = \sqrt{\frac{rr_3}{r_1 r_2}} = \sqrt{\frac{r_3 - r}{r_1 + r_2}}$

✈ (i) $\cot\frac{A}{2} = \frac{s(s-a)}{\Delta}$,

(ii) $Cot\frac{B}{2} = \frac{s(s-b)}{\Delta}$

(iii) $Cot\frac{C}{2} = \frac{s(s-c)}{\Delta}$

✈ **Area of Triangle :**

$\Delta = \frac{1}{2}$ bc Sin A $= \frac{1}{2}$ ca Sin B

$= \frac{1}{2}$ ab Sin C $= \sqrt{s(s-a)(s-b)(s-c)}$

$= 2R^2$ Sin A Sin B Sin C $= \frac{abc}{4R}$

✈ In a triangle ABC

(i) $CotA = \frac{b^2+c^2-a^2}{4\Delta}$, (ii) $CotB = \frac{c^2+a^2-b^2}{4\Delta}$

(iii) $CotC = \frac{a^2+b^2-c^2}{4\Delta}$

(iv) Cot A + Cot B + Cot C $= \frac{a^2+b^2+c^2}{4\Delta}$

✈ (i) Sin A $= \frac{2\Delta}{bc} = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$

(ii) Sin B $= \frac{2\Delta}{ca} = \frac{2}{ca}\sqrt{s(s-a)(s-b)(s-c)}$

(iii) Sin C $= \frac{2\Delta}{ab} = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$

✈ **Circles related with the triangle :**

(i) Incircle : The point of concurrence of the internal bisectors of the angles of a triangle is called the incentre and is denoted by I. It is equidistant from the sides. The circle with centre I and the length of the perpendicular from I to any side of the triangle as radius touches the sides of the triangle internally and this circle is known as Incircle. The radius of the incircle is called inradius and is denoted by ‘r’.

$$r = \frac{\Delta}{s} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2}$$

$$= (s-c)\tan\frac{C}{2}$$

$$= \frac{a}{\cot\frac{B}{2}+\cot\frac{C}{2}} = \frac{b}{\cot\frac{C}{2}+\cot\frac{A}{2}}$$

$$= \frac{c}{\cot\frac{A}{2}+\cot\frac{B}{2}}$$

(ii) Ex-circle: The point of concurrence of internal bisector of the angle A and externalbisectors of the angles B and C is called the excentre opposite to A and is denoted by ‘I_1’.

The circle with centre I_1 and the perpendicular distance r_1 from I_1 to any one of the three sides as radius is called the excircle opposite to A. Its radius r_1 is called the ex-radius. The Centres of the remaining two excircles opposite to B and C are denoted by I_2, I_3 and their radii are denoted by r_2, r_3.

✈ (i) $r_1 = \frac{\Delta}{s-a} = s\tan\frac{A}{2} = (s-b)Cot\frac{C}{2}$

$=(s-c)Cot\frac{B}{2}$

$= \frac{a}{\tan\frac{B}{2}+\tan\frac{C}{2}} = 4R\,Sin\frac{A}{2}Cos\frac{B}{2}Cos\frac{C}{2}$

(ii) $r_2 = \frac{\Delta}{s-b} = s\tan\frac{B}{2} = (s-c)Cot\frac{A}{2}$

$=(s-a)Cot\frac{C}{2}$

$= \frac{b}{\tan\frac{C}{2}+\tan\frac{A}{2}} = 4R\,Cos\frac{A}{2}Sin\frac{B}{2}Cos\frac{C}{2}$

(iii) $r_3 = \frac{\Delta}{s-c} = s\tan\frac{C}{2} = (s-a)Cot\frac{B}{2}$

$=(s-b)Cot\frac{A}{2}$

$= \frac{c}{\tan\frac{A}{2}+\tan\frac{B}{2}} = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

✈ DEF is the triangle formed by joining the point of contact of the in circle with the sides of the triangle ABC then

(i) Sides of Δ^{le} DEF are

$2r\cos\frac{A}{2}, 2r\cos\frac{B}{2}, 2r\cos\frac{C}{2}$

(ii) Angles of $\Delta^{le}DEF$ are $\frac{\pi}{2}-\frac{A}{2}, \frac{\pi}{2}-\frac{B}{2}, \frac{\pi}{2}-\frac{C}{2}$

(iii) Area of $\Delta^{le}DEF = \frac{r\Delta}{2R}$

✈ In a Δ^{le}ABC If excircle opposite to $\underline{|A}$ touches the sides BC, CA, AB at G, K, H respectively.

Then Area of $\Delta^{le}GHK$ is $\frac{r_1\Delta}{2R}$

✈ If Δ_0 is the Area of Δ^{1e} formed by contact points of in circle with sides of a Δ^{1e} ABC Δ_1, Δ_2, Δ_3, be the Area as of Δ^{1es} formed by contant points of Ex circles opposite $\angle A$, $\angle B$, $\angle C$ repectively with the sides then $\Delta_1+\Delta_2+\Delta_3-\Delta_0 = 2\Delta$

where Δ is area of Δ^{1e} ABC

✈ In an equilateral triangle ABC, P is any point inside the triangle ABC such that $PA = x$, PB=y PC=z then the side of the equilateral triangle is $\sqrt{x^2+z^2-2zx+\cos(60+\theta)}$

where $\cos\theta = \frac{x^2+z^2-y^2}{2zx}$

✈ Lengths of medians of a triangle :

(i) $AD = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$

$= \frac{1}{2}\sqrt{b^2+c^2+2bc\cos A}$

(ii) $BE = \frac{1}{2}\sqrt{2c^2+2a^2-b^2}$

(iii) $CF = \frac{1}{2}\sqrt{2a^2+2b^2-c^2}$.

(iv) Area of triangle in terms of lengths of medians m_1, m_2, m_3 is

$\frac{4}{3}\sqrt{m(m-m_1)(m-m_2)(m-m_3)}$

where $m = \frac{m_1+m_2+m_3}{2}$

✈ Lengths of altitudes :

Length of the altitude from

(i) C to AB = $\frac{2r_1r_2}{r_1+r_2}$ or $\frac{2r_3r}{r_3-r} = \frac{2\Delta}{c}$

(ii) B to AC $= \frac{2r_1r_3}{r_1+r_3}$ or $\frac{2r_2r}{r_2-r} = \frac{2\Delta}{b}$

(iii) A to BC $= \frac{2r_2r_3}{r_2+r_3}$ or $\frac{2r_1r}{r_1-r} = \frac{2\Delta}{c}$

✈ Lengths of angular bisectors :

(i) In triangle ABC if 'AD' is internal angular bisetor of angle 'A' meet BC at D, then $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$

(ii) Length of external angular bisector through the angle A meets BC produced at D_1 is $AD_1 = \left|\frac{2bc}{b-c}\right|\sin A/2$

✈ Concept of Pedal triangle :

Let ABC be any triangle and AD,BE,CF be perpendiculars from A,B,C to the opposite sides of the triangle. Let T be the orthocentre, the triangle DEF is called pedal triangle of triangle ABC.

(i) TB=2RcosB, TA=2RcosA, TC=2RcosC.

(ii) TD=2RcosBcosC, TE=2RcosCcosA, TF=2RcosAcosB

(iii) Sides of pedal triangle DE=ccosC, DF=bcosB, EF=acosA.

(iv) $\angle EDF = 180^0 - 2A, \angle DEF = 180^0 - 2B,$ $\angle DFE = 180^0 - 2C$.

(v) Area of pedal triangle = $2\Delta \cos A \cos B \cos C$

(vi) Radius of circumcircle of pedal triangle=R/2.

(vii) Radius of incircle of pedal triangle DEF $= 2R\cos A\cos B\cos C$

✈ EXCENTRAL TRIANGLE

Definition: If I_1, I_2, I_3 are centres of excircles opposite to vertices of $\angle A, \angle B, \angle C$ of $\Delta le\ ABC$ Then $\Delta le\ I_1I_2I_3$ is called Excentral triangle.

i) The Pedal triangle of $\Delta le\ I_1I_2I_3$ is Δle ABC and Incentre of Δle ABC is Ortho centre of $\Delta le\ I_1I_2I_3$.

ii) Vertical angles of $\Delta le\ I_1I_2I_3$ are respectively $\underline{|I_1} = 90 - A/2$, $\underline{|I_2} = 90 - B/2$, $\underline{|I_3} = 90 - C/2$.

iii) The circum radius of $\Delta le\ I_1I_2I_3$ is 2R.

iv) The sides of excentral triangle are $I_1I_3 = 4R\cos A/2$, $I_1I_3 = 4R\cos B/2$, $I_1I_2 = 4R\cos C/2$

v) Area of $\Delta^{le}\ I_1I_2I_3$ is

$\Delta_1 = 8R^2\cos A/2\cos B/2\cos C/2 = 2Rs.$

✈ If S is the circumcentre O is orthocentre, I is incentre, I_1, I_2, I_3 are the ex-centres of triangle ABC then

(i) AI = rcosec(A/2); BI = rcosec(B/2); CI = rcosec(C/2)

(ii) $AI_1 = r_1$cosecA/2; $BI_2 = r_2$cosec(B/2); $CI_3 = r_3$ cosec(C/2)

(iii) II_1 = a sec(A/2); II_2 = bsec(B/2); II_3 = c sec(C/2)

(iv) I_2I_3 = a cosec(A/2); I_3I_1 = bcosec(B/2); I_1I_2 = c cosec(C/2)

✈ (i) In a triangle circumcentre, centroid, orthocentre are always collinear.

(ii) Centroid divides circumcentre and orthocentre internally in the ratio 1:2.

(iii) Orthocentre divides the circumcentre and centroid externally in the ratio 3:2.

(iv) Distance between circumcentre(O) and orthocentre O' is given by

$$OO' = R\sqrt{1-8\cos A\cos B\cos C}$$

(v) Distance between circumcentre (O) and incentre(I) of the triangle ABC is

$$OI = \sqrt{2r^2 - 4R^2\cos A\cos B\cos C}$$

(vi) Distance between orthocentre(O') and incentre (I) is

$$O'I = R\sqrt{1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \sqrt{R^2 - 2Rr}$$

(vii) Distance between excentres and orthocentre is

$$O'I_1 = R\sqrt{1+8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \sqrt{R^2 + 2Rr_1}$$

$$O'I_2 = R\sqrt{1+8\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}} = \sqrt{R^2+2Rr_2}$$

$$O'I_3 = R\sqrt{1+8\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}} = \sqrt{R^2+2Rr_3}$$

✈ **Nine point circle :**

Acircle passing through the feet of the perpendiculars drawn from the vertices to the opposite sides of a triangle ABC, midpoints of sides AB,BC,CA and the midpoints of the segments joining the vertices to the orthocentre of triangle ABC is called the nine point circle. The centre of the nine point circle is called the nine point centre.The nine point centre N of triangle ABC is collinear with the circumcentre O and the orthocentre O' and bisects the joining them.

Radius of the nine point circle of a triangle is half the radius of the circumcircle.

(i) $NA = \frac{R}{2}\sqrt{1+8\cos A\sin B\sin C}$

(ii) $NB = \frac{R}{2}\sqrt{1+8\sin A\cos B\sin C}$

(iii) $NC = \frac{R}{2}\sqrt{1+8\sin A\sin B\cos C}$

✈ **Solutions of triangles (Ambiguous Cases) :**

(i) The sides a, b, c and angle A, B, C are called 6-elements of a triangle.

(ii) Solution of a triangle: When any 3 - elements (Except - 3 - angles) are given, the process of finding other 3 - elements is called solving a triangle. The values of other 3 - elements called solution of triangle.

(iii) Different cases of solution of triangles:

Case (i) : When 3-sides a,b,c are given. The solution is A, B, C angles which can be found by cosine rule.

$$CosB = \frac{a^2+c^2-b^2}{2ac}, CosA = \frac{b^2+c^2-a^2}{2bc}$$

$$CosC = \frac{a^2+b^2-c^2}{2ab}$$

Case(ii): When $\angle A, \angle B$ and side C are given.

$\rightarrow \angle C = 180 - A + B$

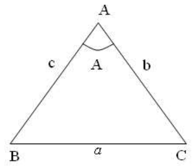
$c = 2R\sin C \Rightarrow R = \left.\frac{C}{2SinC}\right\} a = 2R\sin A, b = 2R\sin B$.

Case(iii):When two sides and their included angle are given.

Suppose b, c and $\angle A$ are given.

$\Rightarrow a$ can be found by cosine - rule.

$a^2 = b^2 + c^2 - 2bc\cos A \cdot$



Once a is known. Then B, C can be found by

$$\cos B = \frac{a^2+c^2-b^2}{2ac}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

Case-iv : When two sides and angle other than included angle is given.

Suppose $a, b, \angle A$ are given.

Then $a^2 = b^2 + c^2 - 2bc\cos A \cdot$

$c^2 - 2bc\cos A + b^2 - a^2 = 0$

$$c = \frac{2b\cos A \pm \sqrt{4b^2\cos^2 A - 4\left(b^2 - a^2\right)}}{2}$$

$c = b\cos A \pm \sqrt{a^2 - b^2\sin^2 A}$

$c-$ does exist only if $a^2 > b^2\sin^2 A$

$\Rightarrow a > b\sin A \cdot$

c - has two values

If $a > b\sin A \Rightarrow 2$ solutions exist

c - has only one value

If $a = b\sin A \Rightarrow 1$ solution exist

c- has no value

If $a < b\sin A \Rightarrow 0$ solution.

This is called ambiguous case.

✈ **m-n theorm:** If D is a point on the side BC of a triangle ABC such that BD:DC::m:n, let $\angle ADC = \theta$,

$\angle BAD = \alpha$, $\angle CAD = \beta$ then

(i) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

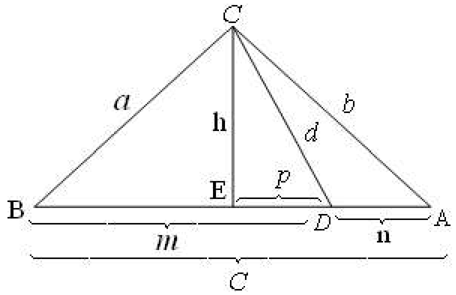
(ii) $(m+n)\cot\theta = n\cot B - m\cot C$

✈ **Stewart Theorem:**

In a triangle ABC, BC = a, AC = b, AB = c, CD = d, where D is a point on AB in to two segments BD=m and DA=n then $a^2n + b^2m = c\left(d^2 + mn\right)$.

Proof: Let $CE = h$

$ED = p \Rightarrow BD = m - p$



$\therefore$ In Δle BEC

$a^2 = h^2 + (m-p)^2$ ....(1)

In Δle CED $\Rightarrow d^2 = h^2 + p^2$ ....(2)

$a^2 = d^2 - p^2 + (m-p)^2$

$a^2 = d^2 + m^2 - 2pm$ ....(3)

Similarly In Δle CEA.

$h^2 + (p+n)^2 = b^2$ ....(4)

and In $\Delta le\, CED$ $h^2 + p^2 = d^2$ ....(2)

$(4)-(2) \Rightarrow b^2 = d^2 - p^2 + (h+p)^2$

$b^2 = d^2 + h^2 + 2np$ ....(5)

$(3)\times n + 5\times(m) \Rightarrow a^2n + b^2m =$

$d^2(m+n) + mn(m+n)$

$a^2n + b^2m = cd^2 + mnc$

✈ **Quadrilaterals :**

If ABCD is a quadrilateral with AB = a, BC = b, CD=c, AD = d and the quadrilateral inscribed in a circle of radius R. Then

(i) Area of ABCD =

$\sqrt{(s-a)(s-b)(s-c)(s-d)}$

where $s = \dfrac{a+b+c+d}{2}$

(ii) Length of diagonal

$$AC = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}$$

$$BD = \sqrt{\frac{(ab+cd)(ac+bc)}{ad+bc}}$$

(iii) Circum radius

$$R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}$$

(iv) Ptolemy's theorem

$(AC)(BD) = (AB)(CD) + (BC)(AD)$

✈ When a quadrilateral not necessarily inscribed in a circle then

(i) Area of quadrilateral =

$\sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\alpha}$

$\angle 2\alpha = \angle B + \angle D$ or $\angle A + \angle C$

(ii) Area of a quadrilateral which can be circumscribed about a circle is $\sqrt{abcd}\,\sin\alpha$.

(iii) The area of a quadrilateral which can be inscribed in a circle and circumscribed about another circle is $\sqrt{abcd}$.

(iv) Radius of circle inscribed in a quadrilateral

$$r = \frac{2\sqrt{abcd}}{a+b+c+d}.$$

✈ **Concept of regular polygon :**

A polygon is said to be regular polygon if all its sides are equal in measure (length) and the measure of all the angle are same

(i) sum of all angles of a regular polygon

= (2n-4) $\frac{\pi}{2}$ radian.

(ii) measure of each angle = $\frac{(2n-4)}{n}\frac{\pi}{2}$ radian

(iii) r = radius of incircle of polygon

$r = \frac{a}{2}\cot\frac{\pi}{n}$

(iv) R = radius of circum circle of polygon

$= \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$

(v) area of regular polygon of n sides

$$= n\times\frac{1}{2}\times a\times\frac{a}{2}\cot\frac{\pi}{4}$$

$$= \frac{na^2}{4}\cot\frac{\pi}{n} = nr^2\tan\frac{\pi}{n} = \frac{\pi}{2}R^2\sin 2\frac{\pi}{n}$$

(vi) In a regular polygon, centroid, circumcentre and incentre are coincide

✈ **Some standard results :**

✈ In a triangle ABC

(i) $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$,

(ii) $rr_1r_2r_3 = \Delta^2$

(iii) $r_1r_2+r_2r_3+r_3r_1 = s^2$

(iv) $r(r_1 + r_2 + r_3) = ab + bc + ca - s^2$

✈ In a triangle ABC

(i) $r_1 - r = 4R\sin^2\frac{A}{2}$, $r_2 + r_3 = 4R\cos^2\frac{A}{2}$;

(ii) $r_2 - r = 4R\,\mathrm{Sin}^2 B/2$, $r_1 + r_3 = 4R\,\mathrm{Cos}^2B/2$

(iii) $r_3 - r = 4R\,\mathrm{Sin}^2 C/2$, $r_1 + r_2 = 4R\,\mathrm{Cos}^2C/2$

✈ In a triangle ABC

(i) $r_1 + r_2 + r_3 - r = 4R$

(ii) $r - r_1 + r_2 + r_3 = 4R\cos A$

(iii) $r + r_1 - r_2 + r_3 = 4R\cos B$

(iv) $r + r_1 + r_2 - r_3 = 4R\cos C$

✈ In a triangle ABC

(i) $\frac{ab - r_1r_2}{r_3} = r$

(ii) $r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - (a^2 + b^2 + c^2)$

(iii) $a^2 = (r_1 - r)(r_2 + r_3)$.

✈ In a triangle ABC

(i) $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$

(ii) $\sin A + sinB + \sin C = \frac{s}{R}$

(iii) $\sum a\cot A = 2(r + R)$

✈ In a triangle ABC

(i) If $\cot\frac{A}{2}:\cot\frac{B}{2}:\cot\frac{C}{2} = k:l:m$ then

$a:b:c = l+m:m+k:k+l$

(ii) If $kr_1 = lr_2 = mr_3$ then

$a:b:c = l+m:m+k:k+l$

✈ (i) If $\mathrm{Cot}\frac{A}{2}, Cot\frac{B}{2}, Cot\frac{C}{2}$ are in A.P. then a,b,c are also in A.P.

(ii) If cot A, cot B, cot C are in A.P. then a^2,b^2,c^2 are also in A.P.

(iii) If $\sin^2\frac{A}{2}, \sin^2\frac{B}{2}, \sin^2\frac{C}{2}$ are in H.P. then a,b,c are also in H.P.

✈ (i) If $a\cos B = b\cos A$ then the triangle is isosceles.

(ii) If $a\cos A = b\cos B$ then the triangle is isosceles or right angled.

(iii) If $a^2 + b^2 + c^2 = 8R^2$ then the triangle is right angled.

(iv) If $\cos^2 A + \cos^2 B + \cos^2 C = 1$ then the triangle is right angled.

(v) If $\cos A = \frac{\sin B}{2\sin C}$ then the triangle is isosceles.

(vi) If $\frac{a}{\cos A} = \frac{b}{\cos B} = \frac{c}{\cos C}$ then the triangle is equilateral .

(vii) If $\cos A + \cos B + \cos C = 3/2$ then the triangle is equilateral.

(viii) If $\sin A + \sin B + \sin C = \frac{3\sqrt{3}}{2}$ then the triangle is equilateral.

(ix) If $\tan A + \tan B + \tan C = 3\sqrt{3}$ and the triangle is acute then the triangle is equilateral .

(x) If $\cot A + \cot B + \cot C = \sqrt{3}$ then the triangle is equilateral .

Eg 1: In ΔABC if $SinASinB = \frac{ab}{c^2}$ then C =

Sol: $\Rightarrow c^2 = \frac{a}{SinA}.\frac{b}{SinB}$

$\Rightarrow (2RSinC)^2 = 2R.2R \Rightarrow C = 90^0$

Eg 2:

If b=20, c=21, and $\sin A = \frac{3}{5}$ **then a=**

Sol: We have $\cos A = \frac{4}{5}$ then using cosine rule

$a^2 = b^2 + c^2 - 2bc\cos A. \Rightarrow a = 13.$

(substitute b,c cosA values & simplify)

Eg 3:

The sides of triangle are 4,5,6 then area is

Sol: $s = \frac{a+b+c}{2} = \frac{15}{2},$

$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \frac{15}{4}\sqrt{7}sq.units$

Eg 5:

$In\ \Delta ABC, \frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3} =$

Sol: $\sum \frac{b-c}{r_1} = \sum \frac{(b-c)(s-a)}{\Delta}$

$= \frac{1}{\Delta}\left[\sum s(b-c) - \sum a(b-c)\right] = 0$

Eg 5:

$If\ r_1^2 = r_1r_2 + r_2r_3 + r_3r_1$ **then the triangle is**

Sol: $r_1^2 = s^2 \Rightarrow r_1 = s \Rightarrow s\tan\frac{A}{2} = s \Rightarrow A = 90^0$

triangle is rightangled triangle

Eg 6:

If $R + r = r_3\ then\ C =$

Sol: $r_3 - r = R,\ 4R\sin^2\frac{C}{2} = R \Rightarrow C = 60^0$

Eg 7:

$If\ r_1 = 2, r_2 = 3, r_3 = 6\ then\ a =$

Sol: Using $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$ We get r =1.

then $a = \sqrt{(r_1 - r)(r_2 + r_3)} = 3$.

EXERCISE - 1

1. **In** $\Delta ABC, \sum \frac{\sin(B-C)}{bc} =$

   1) Δ 2) s 3) 0 4) s^2

2. **In** ΔABC **,** $\sum a^3\sin(B-C) =$

   1) $a^3b^3c^3$ 2) 3abc 3) $3a^2b^2c^2$ 4) 0

3. **In** ΔABC **,** $\sum$ **a^3cos (B-C) =**

   1) abc 2) 2abc 3) 3abc 4) 0

4. **In** ΔABC **,b^2 sin 2 C + c^2 sin 2 B =**

   1) Δ 2) 2Δ 3) 4Δ 4) $\Delta/2$

5. **In** ΔABC **,** $\frac{1}{2}a^2\frac{\sin B\ \sin C}{\sin\ A} =$

   1) Δ 2) 2Δ 3) 3Δ 4) 4Δ

6. **In** ΔABC, **if a cos A + b cos B + c cos C =** $\frac{2\Delta}{k}$ **then k =**

   1) r 2) R 3) s 4) R^2

7. **If a = 5 , b = 6, sin A = 5/6, then B =**

   1) π 2) $\pi/2$ 3) $\pi/3$ 4) $\pi/4$

8. **If the angles of a triangle** ABC **are in A.P., a = 2,c = 4, then b =**

   1) $2\sqrt{3}$ 2) $\sqrt{21}$ 3) 8 4) 14

9. **The sides of a right angled triangle are inA.P., then they are in the ratio**

   1) 2:3:1 2) 2:3:5 3)3:4:5 4)3:1:2

10. **In a** ΔABC **, If a is the arithmetic mean and** $b, c(b \neq c)$ **are the geometric means between any two positive real numbers then**

    $\frac{\sin^3 B + \sin^3 C}{\sin A \sin B \sin C} =$

    1) 0 2) 1 3) 2 4) 4

11. **In a triangle ABC if tan A + tanB+ tanC = 3** $\sqrt{3}$ **, then the triangle is**

    1) isosceles 2) right angled
    3) equilateral 4) scalene

12. **If the angles of a ΔABC are $30^0, 45^0$ and the included side is $\sqrt{3}+1$, then the remaining sides are**
1) $2,\sqrt{2}$ 2) $2,2\sqrt{3}$ 3) $\sqrt{2},4$ 4) $2,4\sqrt{3}$

13. **In ΔABC, If** $A = 60^o$ then $\frac{b}{c+a}+\frac{c}{a+b}=$
1) 1 2) 2 3) 3 4) 4

14. **If the angles of the ΔABC are in A.P., then a^2+c^2-ac =**
1) bc 2) b^2c 3) abc 4) b^2

15. **If a = 26, b = 30,** $\cos C = \frac{63}{65}$ **then c =**
1) 8 2) 25 3) 24 4) 6

16. **In triangle ABC, a = 4, b = 3 and** $\angle A = 60^0$. **Then c is a root of the equation** **(AIEEE-2002)**
1) $c^2-3c+7=0$ 2) $c^2+3c-7=0$
3) $c^2+3c+7=0$ 4) $c^2-3c-7=0$

17. **If the sides of a triangle are in the ratio x:y:$\sqrt{x^2+xy+y^2}$, then the greatest angle is**
1) 90^o 2) 120^o
3) $Cos^{-1}\left(\frac{x+y}{x-y}\right)$ 4) 30^o

18. **If in a** $\Delta ABC, \frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$ **then** $\underline{|C} =$ **(EAM-2013)**
1) $\frac{\pi}{3}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{2}$ 4) $\frac{\pi}{6}$

19. **Two sides of a triangle are given by the roots of the equation** $x^2-5x+6=0$ **and the angle between the sides is** $\pi/3$ **. Then the perimeter of the triangle is**
1) $5+\sqrt{2}$ 2) $5+\sqrt{3}$ 3) $5+\sqrt{5}$ 4) $5+\sqrt{7}$

20. **In ΔABC, If** $\sin A$ **and** $\sin B$ **are the roots of the equation** $c^2x^2-c(a+b)x+ab=0$ **then** $\sin C =$
1) 0 2) 1/2 3) $1/\sqrt{2}$ 4) 1

21. **In ΔABC, If** $\tan A, \tan B, \tan C$ **are in H.P., then** a^2, b^2, c^2 **are in**
1) H.P 2) G..P 3) A.P 4) A.G. P

22. **In a ΔABC, a = 2b and |A - B| =** $\frac{\pi}{3}$**, then** $\angle C$ **is**
1) $\frac{\pi}{6}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{2}$ 4) $\frac{\pi}{3}$

23. **In ΔABC if the angle A, B, C are in A.P. then**
$\frac{a+c}{\sqrt{a^2-ac+c^2}}=$
1) $Cos\left(\frac{A-C}{2}\right)$ 2) $2Cos\left(\frac{A-C}{2}\right)$
3) $2Sin\left(\frac{A-C}{2}\right)$ 4) $2Cos\left(\frac{A+C}{2}\right)$

24. **In a ΔABC**
$\sum(b+c)\tan\frac{A}{2}\tan\left(\frac{B-C}{2}\right)=$
1) a 2) b 3) c 4) 0

25. **If in a triangle ABC, in the usual notation,** $2a\cos\left(\frac{B-C}{2}\right)=b+c$ **and** $B \neq C$**, then the measure of the angle A is**
1) $\frac{\pi}{3}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{6}$ 4) $\frac{\pi}{2}$

26. **In ΔABC, If cot** $\frac{A}{2}=\frac{b+c}{a}$**, then the triangle is** **(EAM-94)**
1) isosceles 2) equilateral
3) right angled 4) scalene

27. $\frac{b-c}{b+c}\cot\frac{A}{2}+\frac{b+c}{b-c}\tan\frac{A}{2}=$
1) $2\sin(B-C)$ 2) $2\cos ec(B-C)$
3) $2\tan(B-C)$ 4) $2\cot(B-C)$

28. **In ΔABC, if a (b cos C + c cos B) = $2ka^2$, then k =**
1) 0 2) 1 3) $\frac{1}{2}$ 4) 2

29. **In ΔABC, Σ (b+c) cos A =**

1) s 2) 2s 3) 2(s-a) 4) 4 s

30. In a ΔABC,

$a(\cos^2 B+\cos^2 C)+\cos A(c\cos C+b\cos B)$

(EAM - 2005)

1) a 2) b 3) c 4) 0

31. In ΔABC, $(a-b)^2\cos^2\frac{C}{2}+(a+b)^2\sin^2\frac{C}{2}=$

1) c 2) c/2 3) 2c 4) c^2

32. In ΔABC, $(a+b+c)\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)=$

1) $2c\cot\frac{A}{2}$ 2) $2c\cot\frac{B}{2}$

3) $2c\cot\frac{C}{2}$ 4) $2c\tan\frac{C}{2}$

33. In ΔABC, If a, b, c are in A.P., then $\text{Cot}\frac{A}{2}$, $\text{Cot}\frac{B}{2}$, $\text{Cot}\frac{C}{2}$ are in

1) A.P. 2) G.P. 3) H.P. 4) A.G.P

34. In ΔABC, If a, b, c are in A.P. then $\text{Cot}\frac{A}{2}\text{Cot}\frac{C}{2}=$

1) 1 2) 2 3) 3 4) 4

35. In ΔPQR, $\underline{|R} = \frac{\pi}{4}$, $Tan\frac{P}{3}, Tan\frac{Q}{3}$ are the roots of the equation $ax^2+bx+c=0$ then

1) $a+b=c$ 2) $b+c=0$

3) $a+c=0$ 4) $b=c$

36. In ΔABC, if b + c = 3a then $\cot\frac{B}{2}.\cot\frac{C}{2}=$

(EAM-2003)

1) 1 2) 2 3) 3 4) 4

37. In a ΔABC, $\frac{s-a}{11}=\frac{s-b}{12}=\frac{s-c}{13}$, then $\tan^2\left(\frac{A}{2}\right)=$

1) 143/432 2) 13/33 3) 11/39 4) 12/37

38. If the length of each side of an equilateral triangle is 10 cm, then its area is

1) 75 2) $\frac{\sqrt{3}}{25}$ 3) $\frac{25}{\sqrt{3}}$ 4) $25\sqrt{3}$

39. In ΔABC if a=30, b=24, c=18 then $\Delta =$

1) 16 2) 216 3) $\sqrt{216}$ 4) 17

40. In ΔABC, If a = 2, B = 120°, C = 30°, then the area of the triangle is

1) $2\sqrt{3}$ 2) $\sqrt{3}$ 3) $\frac{\sqrt{3}}{2}$ 4) $4\sqrt{3}$

41. If $\Delta = a^2-(b-c)^2$, is the area of the triangle ABC, then $\tan A=$

1) 1/16 2) 8/15 3) 3/4 4) 4/3

42. In a ΔABC, If $\cot\frac{A}{2}:\cot\frac{B}{2}:\cot\frac{C}{2}=1:4:15$, then the greatest angle is

1) $\pi/3$ 2) $\pi/4$ 3) $\pi/6$ 4) $2\pi/3$

43. In a ΔABC, If $\tan\frac{A}{2},\tan\frac{B}{2},\tan\frac{C}{2}$ are in H.P., then a, b, c are in

1) H.P. 2) G.P 3) A.P 4) A.G.P.

44. In a ΔABC, If $A=60^0$, then the value of $\left(1+\frac{a}{c}+\frac{b}{c}\right)\left(1+\frac{c}{b}-\frac{a}{b}\right)=$

1) 3 2) 2 3) 1 4) 0

45. If α,β,γ are the lengths of the altitudes of ΔABC, then

$$\frac{1}{\alpha}+\frac{1}{\beta}-\frac{1}{\gamma}-\frac{2ab}{(a+b+c)\Delta}\cos^2\frac{C}{2}=$$

1) 0 2) 1 3) 2s 4) Δ

46. If α,β,γ are the lengths of the altitudes of ΔABC, then $\frac{\cos A}{\alpha}+\frac{\cos B}{\beta}+\frac{\cos C}{\gamma}=$

1) Δ 2) $1/\Delta$ 3) R 4) 1 / R

47. If p_1, p_2, p_3 are respectively the perpendiculars from the vertices of ΔABC to the opposite sides, then

(i) $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=$

1) $\frac{1}{r}$ 2) $\frac{1}{r_1}$ 3) $\frac{1}{r_2}$ 4) $\frac{1}{r_3}$

(ii). $\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}=$

1) $\frac{1}{r}$ 2) $\frac{1}{r_1}$ 3) $\frac{1}{r_2}$ 4) $\frac{1}{r_3}$

(iii) $\frac{\cos A}{p_1}+\frac{\cos B}{p_2}+\frac{\cos C}{p_3}=$

1) $\frac{1}{r}$ 2) $\frac{1}{R}$ 3) $\frac{1}{r_1}$ 4) $\frac{1}{r_3}$

(iv) $p_1p_2p_3 =$

1) $\frac{abc}{8R^2}$ 2) $\frac{a^2b^2c^2}{8R^2}$

3) $\frac{a^2b^2c^2}{8R^3}$ 4) $\frac{a^3b^3c^3}{8R^2}$

48. In triangle ABC, if $rr_1 = r_2r_3$, then the triangle is

1) equilateral 2) isosceles
3) right angled 4) scalene

49. In a triangle ABC, if $(r_1 + r_2)(r_2 + r_3)(r_3 + r_1 = 4RK$ then K=

1) s 2) Δ 3) Δ^2 4) s^2

50. If ΔABC is right angled at A then $r_2 + r_3 =$

1) $r_1 - r$ 2) $r_1 + r$ 3) $r - r_1$ 4) R

51. In an equilateral triangle $r : R : r_1$ is

1) 1 : 1 : 1 2) $1:\sqrt{2}:3$
3) 1 : 2 : 3 4) $2:\sqrt{3}:\sqrt{3}$

52. If $r : R : r_1 = 2 : 5 : 12$, then A =

1) 45° 2) 60° 3) 30° 4) 90°

53. In a triangle ABC, if $r_1 = 2r_2 = 3r_3$ then

$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=$ **(EAM - 2008)**

1) $\frac{75}{60}$ 2) $\frac{155}{60}$ 3) $\frac{176}{60}$ 4) $\frac{191}{60}$

54. $(r_1 - r)(r_2 - r)(r_3 - r) =$

1) 4Rr 2) 4Rs 3) $4R\Delta$ 4) $4Rr^2$

55. In ΔABC if $r_1 - r = r_2 + r_3$, then

1) A = 90° 2) B = 90° 3) C = 90° 4) A = 45°

56. If a,b,c are in A.P., then r_1, r_2, r_3 are in

1) A.P. 2) G.P. 3) H.P. 4) A.G.P

57. If the area of the triangle ABC is $a^2-(b-c)^2$, then its circumradius R =

1) $(a/6)\sin^2(A/2)$ 2) $(a/6)\operatorname{cosec}^2(A/2)$

3) $(b/16)\sin^2(B/2)$ 4) $(c/16)\sin^2(C/2)$

58. In triangle ABC, if a = 13, b = 14, c = 15, then r_1 =

1) 21/2 2) 14 3) 65/8 4) 4

59. In triangle ABC, if $r_1 = 3, r_2 = 10, r_3 = 15$, then c =

1) 5 2) 12 3) 13 4) 13/2

60. In triangle ABC, $\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right)=$

1) $\frac{abc}{\Delta^3}$ 2) 0 3) $4Rr^2$ 4) 1/r

61. In ΔABC, if $\sin^2 A+\sin^2 B=\sin^2 C$, then the triangle is

1) equilateral 2) right angled
3) isosceles 4) scalene.

62. In ΔABC, If
$\tan B\tan C+\tan C\tan A+\tan A\tan B=\sqrt{3}\tan A\tan B\tan C$ **then the triangle is**

1) isosceles 2) equilateral
3) right angled 4) right angled isosceles

63. In triangle ABC, if $\sin^2 A+\sin^2 B+\sin^2 C=9/4$, then the triangle is

1) isosceles 2) right angled
3) equilateral 4) scalene triangle.

KEY

01) 3	02) 4	03) 3	04) 3	05) 1	06) 2
07) 2	08) 1	09) 3	10) 3	11) 3	12) 1
13) 1	14) 4	15) 1	16) 4	17) 2	18) 1
19) 4	20) 4	21) 3	22) 4	23) 2	24) 4
25) 1	26) 3	27) 2	28) 3	29) 2	30) 1

31) 4 32) 3 33) 1 34) 3 35) 1 36) 2
37) 2 38) 4 39) 2 40) 2 41) 2
42) 4 43) 3 44) 1 45) 1 46) 4
47) i-1,ii-4, iii-2. iv-3 48) 3 49) 4 50) 1
51) 3 52) 4 53) 4 54) 4 55) 1 56) 3
57) 2 58) 1 59) 3 60) 1 61) 2 62) 2
63) 3

SOLUTIONS

1. A = B = C = 60^0, a = b = c = 1
2. A = B = C = 60^0, a = b = c = 1
3. A = B = C = 60^0, a = b = c = 1
4. A = B = C = 60^0, a = b = c = 1, $\Delta = \frac{\sqrt{3}}{4}$
5. A = B = C = 60^0, a = b = c = 1, $\Delta = \frac{\sqrt{3}}{4}$
6. $R = \frac{1}{\sqrt{3}}$, A = B = C = 60^0, $\Delta = \frac{\sqrt{3}}{4}$
7. Given $a = 5,\ b = 6\ \sin A = \frac{5}{6}$

   using sine rule $\frac{a}{\sin A} = \frac{b}{\sin B}$

   $\Rightarrow a\sin B = b\sin A$

   $5\sin B = 6\frac{5}{6}$

   $\sin B = 1 = \sin\frac{\pi}{2}$

   $\angle B = \frac{\pi}{2}$
8. Angles of triangles are in AP

   $\Rightarrow B = 60^0 \Rightarrow b^2 = a^2 + c^2 - 2\text{ac}\cos\text{B}$

   $\Rightarrow b = 2\sqrt{3}$
9. Let c be the hypotenuse.
   a,b,c are in A.P.,

   $\Rightarrow b = (a+c)/2\ ;\ a^2 + b^2 = c^2$

   $\Rightarrow a^2 + [(a+c)/2]^2 = c^2$

   $\Rightarrow 5a^2 + 2ac - 3c^2 = 0 \Rightarrow c = 5a/3, b = 4a/3$
10. Let x and y be two given positive real numbers,

    so $a = \frac{x+y}{2}$ *and* $b = xr, c = xr^2$ *then* $y = xr^3$

    Now $\frac{\sin^3 B + \sin^3 C}{\sin A \sin B \sin C} = \frac{b^3 + c^3}{abc} = \frac{2r^3\left(1+r^3\right)}{r^3\left(1+r^3\right)} = 2$
11. $A = B = C = 60^0$.
12. Let $\angle B = 30^0\ \angle C = 45^0 \Rightarrow \angle A = 105^0 \Rightarrow$

    $\frac{\sqrt{3}+1}{\sin 105^0} = \frac{b}{\sin 30^0} \Rightarrow b = \frac{(\sqrt{3}+1)\sin 30^0}{\sin 105^0}$

    $= \sqrt{2}$ *and* $\frac{c}{\sin 40^0} = \frac{\sqrt{2}}{\sin 30^0} \Rightarrow c = 2.$
13. Given $A = 60^0$

    We know $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

    $\Rightarrow \frac{1}{2} = \frac{b^2 + c^2 - a^2}{2bc}$

    $bc = b^2 + c^2 - a^2$

    $b^2 + c^2 = a^2 + bc$

    Now $\frac{b}{c+a} + \frac{c}{a+b} = \frac{b(a+b) + c(c+a)}{(c+a)(a+b)}$

    $= \frac{ab + b^2 + c^2 + ac}{ac + bc + a^2 + ab}$

    $= \frac{ab + a^2 + bc + ac}{ac + bc + a^2 + ba} = 1$
14. $2B = A + C \Rightarrow B = 60^0$

    $\cos B = \frac{c^2 + a^2 - b^2}{2ac}$
15. $c^2 = a^2 + b^2 - 2ab\cos C$
16. Given $a = 4, b = 3$ and $\angle A = 60^o$

    We know $a^2 = b^2 + c^2 - 2bc\cos A$

    $16 = 9 + c^2 - 6c.\cos 60^o$

    $16 = 9 + c^2 - 6c.\frac{1}{2}$

    $c^2 - 3c - 7 = 0$

17. x = y =1, $\sqrt{x^2+y^2+xy}=\sqrt{3}$;

$$CosC=\frac{a^2+b^2-c^2}{2ab}$$

18. $\frac{a+b+c}{a+c}+\frac{a+b+c}{b+c}=3\Rightarrow 1+\frac{b}{a+c}+1+\frac{a}{b+c}=3$

$$\Rightarrow\frac{a}{b+c}+\frac{b}{a+c}=1\Rightarrow a^2+b^2-ab=c^2\Rightarrow \underline{|C}=\frac{\pi}{3}$$

19. $x^2-5x-6=0\Rightarrow(x-2)(x-3)=0\Rightarrow$

$x=2,3\Rightarrow a=2,b=3.$

$c^2=a^2+b^2-2ab\cos C$

$=4+9-12\cos(\pi/3)=7\Rightarrow c=\sqrt{7}$

20. $\sin A+\sin B=\frac{c(a+b)}{c^2}=\frac{a+b}{c}\Rightarrow$

$$\frac{a+b}{2R}=\frac{a+b}{c}\Rightarrow c=2R.$$

21. Given, $\tan A,\ \tan B,\tan C\ are\,in\ H.P\Rightarrow$

$\cot A,\ \cot B,\cot C\ are\,in\ A.P\Rightarrow$

$\Rightarrow 2CotB=CotA+CotC$

$\Rightarrow a^2,b^2,c^2\ are\,in\ A.P$

22. Given $a=2b$ and $|A-B|=\frac{\pi}{3}$

using tangent rule

$$\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}\cot\frac{c}{2}$$

$$\tan 30^o=\frac{b}{3b}\cot\frac{c}{2}$$

$$\Rightarrow\frac{1}{\sqrt{3}}=\frac{1}{3}\cot\frac{c}{2}$$

$$\cot\frac{c}{2}=\sqrt{3}=\cot\frac{\pi}{6}$$

$$\angle C=\frac{\pi}{3}$$

23. Given A,B,C are in A.P

$2B=A+c$

$3B=A+B+C=180^o$

$B=60^o$

$$\cos B=\frac{a^2+c^2-b^2}{2ac}$$

$$ac=a^2+c^2-b^2\Rightarrow b^2=a^2+c^2-ac$$

Now $\frac{a+c}{\sqrt{a^2+c^2-ac}}=\frac{a+c}{b}=\frac{\sin A+\sin C}{\sin B}$

$$=\frac{2\sin\frac{(A+C)}{2}\cos\frac{(A-C)}{2}}{\sin B}\qquad B=60^o$$

$$=2\cos\frac{(A-C)}{2}\qquad \frac{A+C}{2}=60^o$$

24. $\sum(b+c)\tan\frac{A}{2}\tan\left(\frac{B-C}{2}\right)=$

$$\sum(b+c)\tan\frac{A}{2}\times\frac{b-c}{b+c}\cot\frac{A}{2}$$

$$=\sum(b-c)=0.$$

25. $2Cos\left(\frac{B-C}{2}\right)=\frac{b+c}{a}$, Use mollweide rule

26. Given $\cot\frac{A}{2}=\frac{b+c}{a}$

$$\cot\frac{A}{2}=\frac{\sin B+\sin C}{\sin A}$$

$$\frac{\cot A/2}{\sin A/2}=\frac{2\sin\frac{(B+C)}{2}\cos\frac{(B-C)}{2}}{2\sin\frac{A}{2}\cos A/2}$$

$$\cos\frac{A}{2}=\cos\left(\frac{B-C}{2}\right)$$

$A = B - C$

$A + C = B$

$A + B + C = 2B$

$2B = 180^o$

$\angle B = 90^o$

$\frac{B+C}{2} = 90 - A/2$

$Sin\frac{(B+C)}{2} = \cos A/2$

$\therefore \Delta\, ABC$ is right angle triangle

27. Use tangent rule
28. a = bCosC+cCosB
29. Expand and use projection rule
30. $a(\cos^2 B + \cos^2 C) + \cos A(c\cos c + b\cos B)$

$= a\cos^2 B + a\cos^2 c + c\cos A\cos c + b\cos A\cos B$

$= \cos B(a\cos B + b\cos A) + \cos C(a\cos c + \cos A)$

$= c\cos B + b\cos c$

$= a$

31. $(a^2+b^2-2ab)Cos^2\frac{C}{2} + (a^2+b^2-2ab)Sin^2\frac{C}{2}$

$= a^2 + b^2 - 2abCosC$

32. a+b+c=2s, $\tan\frac{A}{2} = \frac{\Delta}{s(s-a)}$,

$\tan\frac{B}{2} = \frac{\Delta}{s(s-b)}$

33. a,b,c are in AP, s-a,s-b,s-c are in AP
34. Given a,b,c are in A.P

$2b = a + c$

$3b = 2s$

Now $\cot\frac{A}{2}\cot\frac{c}{2} = \frac{s(s-a)}{\Delta}.\frac{s(s-c)}{\Delta}$

$= \frac{s^2(s-a)(s-c)}{s(s-a)(s-b)(s-c)} = \frac{s}{s-b}$

$= \frac{2s}{2s-2b} = \frac{3b}{b} = 3$

35. $P+Q+R = \pi \Rightarrow P+Q = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\Rightarrow \frac{P}{3} + \frac{Q}{3} = \frac{\pi}{4} \Rightarrow \frac{Tan\frac{P}{3} + Tan\frac{Q}{3}}{1 - Tan\frac{P}{3}Tan\frac{Q}{3}} = 1$

$\Rightarrow \frac{-b}{a-c} = 1 \Rightarrow a + b = c$

36. b+c = 3a $\Rightarrow$ a+b+c=4a $\Rightarrow$ s=2a
37. Given $\frac{s-a}{11} = \frac{s-b}{12} = \frac{s-c}{13} = k$

$s-a = 11k,\ s-b = 12k,\ s-c = 13k$

$s-a+s-b+s-c = 36k$

$s = 36k,\qquad a = 25k,\ b = 24k,\ c = 23k$

Now

$\tan^2\left(\frac{A}{2}\right) = \frac{(s-b)(s-c)}{s(s-a)} = \frac{(12k)(13k)}{(36k)(11k)} = \frac{13}{33}$

38. $\frac{\sqrt{3}}{4}a^2$
39. $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$
40. $A = 30^0, \frac{2}{\sin 30^0} = \frac{b}{\sin 120^0}$ *and then use*

$\Delta = \frac{1}{2}ab\sin C$.

41. $\Delta = a^2 - (b-c)^2 \Rightarrow \Delta = (a+b-c)(a-b+c)$

$\Rightarrow \sqrt{s(s-a)(s-b)(s-c)} = 2(s-a)(s-b)$

$\Rightarrow \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{1}{4} \Rightarrow \tan\frac{A}{2} = \frac{1}{4}$.

42. Given $\cot\frac{A}{2} : \cot\frac{B}{2} : \cot\frac{C}{2} = 1:4:15$

$\frac{s(s-a)}{\Delta} : \frac{s(s-b)}{\Delta} : \frac{s(s-c)}{\Delta} = 1:4:15$

$s-a:s-b:s-c=1:4:15$

$\frac{s-a}{1}=\frac{s-b}{4}=\frac{s-c}{15}=k$

$s-a=k, s-b=4k, s-c=15k$

$s-a+s-b+s-c=20k$

$s=20k$

$a=19k,\ b=16k,\ c=5k$

$$\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{256k^2+25k^2-361k^2}{2(16k)(5k)}$$

$$=-\frac{80k^2}{2\times 80k^2}=\frac{-1}{2}$$

$\angle A=120^o$

43. $G.E=\frac{\Delta}{s(s-a)},\frac{\Delta}{s(s-b)},\frac{\Delta}{s(s-c)}$ are in H.P

$\Rightarrow$ a,b,c are in A.P.

44. Given $A=60^0$.

Then $\left(1+\frac{a}{c}+\frac{b}{c}\right)\left(1+\frac{c}{b}-\frac{a}{b}\right)$

$=\frac{(a+b+c)(b+c-a)}{bc}=$

$\frac{2s(2s-2a)}{bc}=4.\frac{s(s-a)}{bc}=3$

45. $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}-\frac{2ab}{(a+b+c)\Delta}\cos^2\frac{C}{2}=$

$\frac{a}{2\Delta}+\frac{b}{2\Delta}-\frac{c}{2\Delta}-\frac{2ab}{2s\Delta}\times\frac{s(s-c)}{ab}$

$\frac{a+b-c-2(s-c)}{2\Delta}=\frac{2(s-c)-2(s-c)}{2\Delta}=0.$

46. $\alpha=\beta=\gamma=\frac{\sqrt{3}}{2}$

47. $A=B=C=60^0$, $p_1=p_2=p_3=\frac{\sqrt{3}}{2}$

48. $\frac{rr_1}{r_2r_3}=\tan^2\frac{A}{2}$

49. $r_1=r_2=r_3=\frac{\sqrt{3}}{2}$, $R=\frac{1}{\sqrt{3}}$

50. $\frac{r_1-r}{r_2+r_3}=1$

51. $\frac{1}{2\sqrt{3}}:\frac{1}{\sqrt{3}}:\frac{\sqrt{3}}{2}$

.

52. Given $\gamma:R:\gamma_1=2:5:12$

$\frac{\gamma}{2}=\frac{R}{5}=\frac{\gamma_1}{12}=k(say)$

$r=2k, R=5k, \gamma_1=12k$

Now $\gamma_1=\gamma=10k=2(5k)$

$\gamma_1-\gamma=2R$

We know $\gamma_1-\gamma=4R\sin^2\frac{A}{2}$

$$\sin^2 A/2=\frac{1}{2}\Rightarrow \sin A/2=\frac{1}{\sqrt{2}}$$

$\frac{A}{2}=45^o$ $\angle A=90^o$

53. Given $\gamma_1=2\gamma_2=3\gamma_3$

$\frac{\Delta}{s-a}=2\frac{\Delta}{s-b}=\frac{3\Delta}{s-c}=\frac{1}{k}(say)$

$s-a=k, s-b=2k, s-c=3k$

$s-a+s-b+s-c=6k$

$s=6k$

$a=5k,\ b=4k,\ c=3k$

Now $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=\frac{5}{4}+\frac{4}{3}+\frac{3}{5}$

$=\frac{75+80+36}{60}=\frac{191}{60}$

54. $r=\frac{1}{2\sqrt{3}}$, $r_1=r_2=r_3=\frac{\sqrt{3}}{2}$

55. $r_1-r=4R\sin^2\frac{A}{2}$, $r_2+r_3=4R\cos^2\frac{A}{2}$

56. a, b, c are in A.P then r_1, r_2, r_3 are in H.P

57. $\Delta = a^2 - (b-c)^2 = a^2 - b^2 - c^2 + 2bc$

$$\therefore \cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{2bc-\Delta}{2bc}$$

$$\Rightarrow 1 - CosA = \frac{a}{8R} \Rightarrow 2\sin^2\frac{A}{2} = \frac{a}{8R} \Rightarrow R = \frac{a\cos ec^2 A/2}{16}$$

$$\Rightarrow R = (a/16)\cos ec^2(A/2).$$

58. $s = 21, \Delta = \sqrt{21\times 8\times 7\times 6} = 84,$

$$r_1 = \frac{\Delta}{s-a} = \frac{84}{8} = \frac{21}{2}$$

59. $r = \dfrac{1}{\frac{1}{3}+\frac{1}{10}+\frac{1}{15}} = \dfrac{30}{10+3+2} = 2,$

$c = \sqrt{(15-2)(3+10)} = 13.$

60. $\left(\dfrac{r_1-r}{rr_1}\right)\left(\dfrac{r_2-r}{rr_2}\right)\left(\dfrac{r_3-r}{rr_3}\right) =$

$$\frac{\left(4R\sin^2\frac{A}{2}\right)\left(4R\sin^2\frac{B}{2}\right)\left(4R\sin^2\frac{C}{2}\right)}{r^3 r_1 r_2 r_3} =$$

$$\frac{4Rr^2}{r^2\Delta^2} = \frac{abc/\Delta}{\Delta^2} = \frac{abc}{\Delta^3}.$$

61. Given,

$\sin^2 A + \sin^2 B = \sin^2 C \Rightarrow a^2 + b^2 = c^2$

$\Rightarrow \angle C = 90^0$.

62. $\tan B\tan C + \tan C\tan A + \tan A\tan B$

$= \sqrt{3}\tan A\tan B\tan C$

$$\Rightarrow \frac{1}{\tan A} + \frac{1}{\tan B} + \frac{1}{\tan C} = \sqrt{3}.$$

63. $A = B = C = 60^0$

EXERCISE - II

1. In ΔABC, If $\dfrac{SinA}{SinC} = \dfrac{Sin(A-B)}{Sin(B-C)}$ then a^2, b^2, c^2 are in

1) A.P. 2) G.P. 3) H.P. 4) A.G.P.

2. In ΔABC, If C = 90^o then $\dfrac{a^2+b^2}{a^2-b^2}$ sin (A-B)=

1) 1 2) 2 3) 3 4) 4

3. In ΔABC, If a:b:c = 3:4:5, then sin A : sin B : sin C =

1) 3: 4 : 5 2) 9 : 16 : 25
3) 9 : 8 : 7 4) 7 : 9 : 8

4. If the angles of a right angled triangle are in A.P., then ratio of its sides is

1) $1:\sqrt{3}:2$ 2) 1 : 1 : $\sqrt{2}$
3) 2 : 3 : $\sqrt{13}$ 4) 1 : 2 :3

5. If the perimeter of a triangle is six times the arithmetic mean of the sine angles and the side 'a' is unity, then A =

1) $\frac{\pi}{6}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{3}$ 4) $\frac{\pi}{2}$

6. In a triangle ABC, $\angle B = \dfrac{\pi}{3}, \angle C = \dfrac{\pi}{4}$ and D divides BC internally in the ratio 1:3. Then $\dfrac{\sin\angle BAD}{\sin\angle CAD}$ is equal to

1) $\frac{1}{\sqrt{6}}$ 2) $\frac{1}{3}$ 3) $\frac{1}{\sqrt{3}}$ 4) $\sqrt{\frac{2}{3}}$

7. If the sides a,b,c of a triangle are in G.P. and largest angle exceeds the smallest by 60^0, then $\cos B$ =

1) 1 2) $\frac{\sqrt{13}-1}{4}$ 3) $\frac{1}{2}$ 4) $\frac{1-\sqrt{13}}{4}$

8. If in a triangle ABC sines of angles A and B satisfy the equation

$4x^2 - 2\sqrt{6}\,x + 1 = 0$ **,then** $\cos(A-B) =$

1) 0 2) 1/2 3) $1/\sqrt{2}$ 4) $\sqrt{3}/2$

9. In $\Delta ABC, if\ \angle A = \dfrac{\pi}{4}, \angle B = \dfrac{5\pi}{12}$

$then\ a + \sqrt{2}c =$

1) b 2) 2b 3) b/2 4) $\sqrt{3}.b$

10. **If the angles A, B,C of a triangle are in A.P and sides a,b,c are in G.P, then** a^2, b^2, c^2 **are in**

1) A.P 2) H.P 3) G.P 4) A.G.P

11. **In** ΔABC**, if** $\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$**, then** $\sin^2 A + \sin^2 B + \sin^2 C$ **is equal to**

1) $\frac{4}{9}$ 2) $\frac{9}{4}$ 3) $3\sqrt{3}$ 4) 1

12. **If a, b and A are given and** c_1, c_2 **are two values of the side c in the ambiguous case, then** $c_1^{\,2} + c_2^{\,2} - 2c_1c_2.\cos 2A =$

1) $4a^2\cos^2 A$ 2) $4a^2\sin^2 A$

3) $4a^2$ 4) $4a^2\cos 2A \cdot$

13. **In triangle ABC,**

$(a^2 - b^2 - c^2)\tan A + (a^2 - b^2 + c^2)\tan B =$

1) $(a^2 + b^2 - c^2)\tan C$ 2) $(a^2 + b^2 + c^2)\tan C$

3) $(b^2 + c^2 - a^2)\tan C$ 4) 0

14. **If in a triangle** ABC**,**

$4\sin A = 4\sin B = 3\sin C, then \ \cos C =$

1) 1/3 2) 1/9 3) 1/27 4)1/18

15. **If 6, 10, 14 are the sides of a triangle then its obtuse angle is**

1) 110^0 2) 120^0 3) 135^0 4) 115^0

16. **If in a triangle ABC,**

$a^2 + b^2 + c^2 = ca + ab\sqrt{3}$**, then the triangle is**

1) equilateral

2) right angled and isosceles

3) right angled with one of the acute angles measuring $\frac{\pi}{3}$

4) obtuse angled

17. **If one angle of a triangle is 30^0 and the lengths of the sides adjacent to it are 40 and 40** $\sqrt{3}$**, then the triangle is**

1) equilateral 2) right angled

3) isosceles 4) scalene

18. **In** ΔABC**, if**

$c^4 - 2\left(a^2 + b^2\right)c^2 + a^4 + a^2b^2 + b^4 = 0$

then $\underline{|C}$ **=**

1) 30^0 2) 45^0 3) 60^0 4) 75^0

19. **If b + c : c + a : a + b = 11 : 12 : 13, then cos A : cos B : cos C =**

1) 7 : 9 : 11 2) 14 : 11 : 6

3) 7 : 19 : 25 4) 8 : 6 : 5

20. **The roots of** $x^2 - 2\sqrt{3}x + 2 = 0$ **represent two sides of a triangle. If the angle between them is** $\frac{\pi}{3}$**, then the perimeter of the triangle is**

1) $2\sqrt{3} + 6$ 2) $2\sqrt{3} + \sqrt{6}$

3) $3\sqrt{2} + 6$ 4) $3\sqrt{2} + \sqrt{6}$

21. **The base of a triangle is 80 and one of the base angles is** 60^0**.If the sum of the lenghts of the other two sides is 90, then the shortest side is**

1) 15 2) 17 3) 19 4) 21

22. **In** ΔABC**, If**

$\frac{b + c - a}{4} = \frac{c + a - b}{3} = \frac{a + b - c}{2}$**, then**

Cos A=

1) $\frac{5}{7}$ 2) $\frac{3}{7}$ 3) $\frac{2}{7}$ 4) $\frac{1}{7}$

23. **In triangle** ABC**,**

$\frac{bc\cos A + ca\cos B + ab\cos C}{\cot A + \cot B + \cot C} =$

1) Δ 2) 2Δ 3) $\frac{1}{2}\Delta$ 4) Δ^2

24. **If in** ΔABC**, A=B-C=**60^0**, then** $\frac{b - c}{b + c} =$

1)1 2) $\frac{1}{2}$ 3) $\frac{1}{3}$ 4) $\frac{1}{4}$

25. **If one side of a triangle is double that of an other and the angles opposite to these sides differ by 60^0, then the triangle is**

1) isosceles 2) right angled

3) equilateral 4) right angled isosceles

26. In triangle ABC,

$a(b^2+c^2)\cos A + b(c^2+a^2)\cos B +$

$c(a^2+b^2)\cos C =$

1) $3abc^2$ 2) $2a^2bc$ 3) $3abc$ 4) $3ab^2c$

27. In triangle ABC, a = 5, b = 4 and

$\cos(A+B) = \frac{31}{32}$**. In this triangle , c =**

1) $\sqrt{6}$ 2) 36 3) 6 4) $\frac{\sqrt{319}}{2}$

28. In a triangle ABC,

$(a+b+c)(b+c-a) = \lambda bc$ **if**

1) $\lambda < 0$ 2) $\lambda > 0$ 3) $0 < \lambda < 4$ 4) $\lambda > 4$

29. In ΔABC, **If (a-b)(s-c)=(b-c)(s-a) then r_1, r_2, r_3 are in**

1) A.P. 2) G.P. 3) H.P. 4) A.G.P.

30. **In ΔABC, If** $3Tan\frac{A}{2}.Tan\frac{C}{2} = 1$**, then a,b,c are in**

1) A.P 2) G.P 3) H.P 4) A.G.P.

31. If a:b:c=3:4:5, then $\cot\frac{A}{2} : \cot\frac{B}{2} : \cot\frac{C}{2} =$

1) 5:6:7 2) 1:2:3 3) 3:2:1 4) 4:5:6

32. In a triangle ABC, if the sides of a = 3, b = 5 and c=4, then $\sin\frac{B}{2} + \cos\frac{B}{2}$ **is equal to**

1) $\sqrt{2}$ 2) $\frac{\sqrt{3}+1}{2}$ 3) $\frac{\sqrt{3}-1}{2}$ 4) 1

33. In a triangle ABC if

$\cot\frac{A}{2}.\cot\frac{B}{2} = c$, $\cot\frac{B}{2}.\cot\frac{C}{2} = a$ **and**

$\cot\frac{C}{2}.\cot\frac{A}{2} = b$ **then** $\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} =$

1) -1 2) 0 3) 1 4) 2

34. The area Δ of a triangle ABC is given by

$\Delta = a^2 - (b-c)^2$ *then* $Tan\frac{A}{2} =$

1) -1 2) 0 3) 1/4 4) 1/2

35. In ΔABC, If cot $\frac{A}{2} : \cot\frac{B}{2} : \cot\frac{C}{2}$ **= 3 : 7 : 9, then a:b:c =**

1) 7 : 9 : 11 2) 14 : 11 : 6

3) 7 : 19 : 25 4) 8 : 6 : 5

36. If AD, BE, CF are internal bisectors of the angles of ΔABC, then

$\frac{\cos A/2}{AD} + \frac{\cos B/2}{BE} + \frac{\cos C/2}{CF} =$

1) $\frac{abc}{2s}$ 2) $\frac{2s}{abc}$

3) $\frac{ab+bc+ca}{abc}$ 4) $\frac{a+b+c}{abc}$

37. In a triangle ABC,

$\frac{\cot(A/2)+\cot(B/2)+\cot(C/2)}{\cot A + \cot B + \cot C} =$

1) $\frac{(a+b+c)^2}{a^2+b^2+c^2}$ 2) $\frac{a^2+b^2+c^2}{(a+b+c)^2}$

3) s 4) Δ

38. In ΔABC, area of incircle : area of ΔABC =

1) π : $\text{Tan}\frac{A}{2}$ $\text{Tan}\frac{B}{2}$ $\text{Tan}\frac{C}{2}$

2) π : $\text{Cot}\frac{A}{2}$ $\text{Cot}\frac{B}{2}$ $\text{Cot}\frac{C}{2}$

3) 1 : 1 4) π : r.

39. If p_1, p_2, p_3 are the lengths of the altitudes of a tringle from the vertices A, B, C, then

$\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} =$

1) $\frac{2ab\cos^2 C/2}{\Delta(a+b+c)}$ 2) $\frac{1}{R}$

3) $\frac{\cot A + \cot B + \cot C}{\Delta}$ 4) $2R$

40. If 2R + r = r_1**, then**

1) A = 90^0 2) B = 90^0

3) C = 90^0 4) A = 60^o

41. In ΔABC , if C = 90^0, then R + r =

1) a+ b 2) $\frac{a+b}{2}$ 3) b + c 4) $\frac{b+c}{2}$

42. If the diameter of any excircle of a triangle is equal to its perimeter, then the triangle is

1) equilateral 2) isosceles
3) right angled 4) scalene

43. In ΔABC, if $\sqrt{3}$ r + a = $\sqrt{3}$ r + b = s then the triangle is

1) right angled 2) isosceles
3) equilateral 4) scalene

44. In ΔABC, r = 1, R=4, Δ = 8 then the value of ab + bc + ca =

1) 18 2) 81 3) 72 4) 27

45. The equation whose roots are radii of escribed circles is

1) $x^3-2x^2(r+R)+s^2x-s^2r=0$

2) $x^3-x^2(r+4R)+s^2x-s^2r=0$

3) $x^3-x^2(r+4R)+s^2x-\Delta^2 s=0$

4) $x^3-4Rrx^2+s^2x-sr=0$

46. $\frac{\sqrt{r r_1 r_2 r_3}}{2Rr(\sin A+\sin B+\sin C)}=$

1) 1 2) 1 / 3 3) 1 / 4 4) 1 / 2

47. If r_1, r_2, r_3 are the radii of the escribed circles of a ΔABC and if r is the radius of its incircle then $r_1 r_2 r_3$ - r($r_1 r_2 + r_2 r_3 + r_3 r_1$) =

1) 0 2) 1 3) 2 4) 3

48. In ΔABC, right angled at A, $\cos^{-1}\left(\frac{R}{r_2+r_3}\right)$ **is**

1) 30^0 2) 60^0 3) 90^0 4) 45^0

49. In an ambiguous case of solving a triangle when $a=\sqrt{5}, b=2, \angle A=\frac{\pi}{6}$ and the two possible values of third side are c_1 and c_2 then

1) $|c_1-c_2|=2\sqrt{6}$ 2) $|c_1-c_2|=4\sqrt{6}$
3) $|c_1-c_2|=4$ 4) $|c_1-c_2|=6$

50. If A, A_1, A_2, A_3 are areas of excircles and incircle of a triangle, then $\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}=$

1) $\frac{2}{\sqrt{A}}$ 2) $\frac{3}{\sqrt{A}}$ 3) $\frac{1}{\sqrt{A}}$ 4) $\frac{4}{\sqrt{A}}$

51. In triangle $ABC, (r_3+r_1)\sqrt{\left[\frac{rr_2}{r_3r_1}\right]}=$

1) a 2) b 3) c 4) 0

52. In triangle ABC, If $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right)=2$, **then the triangle is**

1) isosceles 2) equilateral
3) right angled 4) scalene.

53. In a triangle ABC,

$\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}=$

1) $\frac{1}{r}-\frac{1}{2R}$ 2) $1+\frac{r}{R}$ 3) $2+\frac{r}{2R}$ 4) $1-\frac{r}{2R}$

KEY

01) 1	02) 1	03) 1	04) 1	05 1	06) 1
07) 2	08) 2	09) 2	10) 1	11) 2	12) 1
13) 4	14) 2	15) 2	16) 3	17) 3	
18) 3	19) 3	20) 3	21) 2	22 2	23) 1
24) 2	25) 3	26) 2	27) 3	28) 4	29) 3
30) 1	31) 1	32) 3	33) 1	34) 4	35) 3
36) 4	37) 3	38) 1	39) 2	40) 3	41) 1
42) 2	43) 3	44) 3	45) 2	46) 2	46) 4
47) 1	48) 2	49) 3	50) 3	51) 2	52) 3
53) 3					

SOLUTIONS

1. Given $\frac{\sin A}{\sin C}=\frac{\sin(A-B)}{\sin(B-C)}$

$$\Rightarrow \frac{\sin(180-(B+C))}{\sin(180-(A+B))}=\frac{\sin(A-B)}{\sin(B-C)}$$

$\sin(B+C)\sin(B-C)=\sin(A+B)\sin(A-B)$

$\Rightarrow \sin^2 B\sin^2 C=\sin^2 A-\sin^2 B$

$\Rightarrow \dfrac{b^2-c^2}{4R^2}=\dfrac{a^2-b^2}{4R^2}$

$\Rightarrow b^2-c^2=a^2-b^2$

$2b^2=a^2+c^2$

$\therefore a^2,b^2,c^2$ are in A.P

2. Given $\angle C=90^0$

Then $A+B=90^0$

$\Rightarrow \sin^2 B+\sin^2 A=1$

Now

$\dfrac{a^2+b^2}{a^2-b^2}\sin(A-B)=\dfrac{\sin^2 A+\sin^2 B}{\sin^2 A-\sin^2 B}\sin(A-B)$

$=\dfrac{1}{\sin(A+B)\sin(A-B)}\sin(A-B)$

$=\dfrac{1}{\sin 90^o}=1$

3. SinA : SinB : SinC = a : b : c

4. A $=30^0$, B $=60^0$, C$=90^0$

5. Given $2s=\dfrac{6(\sin A+\sin B+\sin C)}{3}$

$s=\sin A+\sin B+\sin C$

$s=\dfrac{a+b+c}{2R}=\dfrac{2S}{2R}$

$R=1$

Using sin rule $\sin A=\dfrac{a}{2R}=\dfrac{1}{2}$

$\angle A=30^o=\dfrac{\pi}{6}$

6. $\dfrac{\sin\lfloor BAD}{1}=\dfrac{\sin 60^0}{AD}$ *and* $\dfrac{\sin\lfloor CAD}{3}=\dfrac{\sin 45^0}{AD}$

7. a,b,c are in G.P

$\Rightarrow b^2=ac\Rightarrow \sin^2 B=\sin A\sin C$

$\Rightarrow 2\sin^2 B=\cos(A-C)-\cos(A+C)$

$\Rightarrow 2(1-\cos^2 B)=\cos 60^0-\cos(180^0-B)$

8. We have

$\sin A+\sin B=\sqrt{6}/2$ *and* $\sin A\sin B=1/4.$

Let A > B [$\because$ A $\neq$ B].

$\Rightarrow \sin^2 A+\sin^2 B+2\times 1/4=6/4\cdot$

$\sin A=\cos B\Rightarrow B=90^0-A$

$\Rightarrow A+B=C=90^0$

Also $\sin A\sin B=\cos B\sin B=1/4$

$\Rightarrow \sin 2B=1/2\Rightarrow 2B=30^0\ or\ 150^0$

$\Rightarrow B=15^0\ or\ 75^0\Rightarrow B=15^0\ and\ A=75^0$

9. $C=\dfrac{\pi}{3},a+\sqrt{2}c=2R(\sin A+\sqrt{2}\sin C)$.

$=2R\left(\dfrac{\sqrt{3}+1}{\sqrt{2}}\right)=2b$.

10. Given A,B,C are in A.P and a,b,c are in G.P

$2B=A+C$

$3B=A+B+C=180^o$

$\angle B=60^o$ and $b^2=ac$

Using consine rule

$b^2=a^2+c^2-2a\cos B$

$b^2=a^2+c^2-ac$

$b^2=a^2+c^2-b^2$

$2b^2=a^2+c^2$

$\therefore a^2,b^2,c^2$ are in A.P

11. $a^2+b^2+c^2-ab-bc-ac=0$

$\Rightarrow a=b=c\Rightarrow A=B=C=60^0$

Required value $=\dfrac{9}{4}$

12. G.E. $=(c_1+c_2)^2-2c_1c_2-2c_1c_2\cos 2A$

$=4b^2\cos^2 A-4(b^2-a^2)\cos^2 A$

13. The Given expression is equal to

$-2bc\cos A\tan A+2ac\cos B\tan B$

$=2abc\left(-\dfrac{\sin A}{a}+\dfrac{\sin B}{b}\right)=0$

14. $\dfrac{\text{Sin } A}{3}=\dfrac{\text{Sin } B}{3}=\dfrac{\text{Sin } C}{4}$

$\Rightarrow a=3k,b=3k,c=4k$ then find CosC

15. a = 6, b = 10, c = 14; $CosC = \frac{a^2+b^2-c^2}{2ab}$

16. $a^2+b^2+c^2 = a(c+b\sqrt{3})$

$= 2a\left(\frac{1}{2}.c + b\frac{\sqrt{3}}{2}\right)$

17. $b^2 = c^2 + a^2 - 2caCosB$

18. $(c^2)^2+(a^2)^2+(b^2)^2-2a^2c^2-2b^2c^2+2a^2b^2=a^2b^2$

$\Rightarrow (a^2+b^2-c^2)^2 = a^2b^2$

19. Given $b+c : a+c : a+b = 11:12:13$

$\frac{b+c}{11} = \frac{a+c}{12} = \frac{a+b}{13} = k\ (say)$

$b+c = 11k, a+c = 12k, a+b = 13k$

$\Rightarrow 2(a+b+c) = 36k$

$a = 7k, b = 6k, c = 5k$

$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{36k^2+25k^2-49k^2}{2\times 6k \times 5k}$

$= \frac{12k^2}{60k^2} = \frac{1}{5}$

$\cos B = \frac{c^2+a^2-b^2}{2ac} = \frac{25k^2+49k^2-36k^2}{2\times 5k\times 7k}$

$= \frac{38k^2}{10\times 7k^2} = \frac{19}{35}$

$\cos\ C = \frac{a^2+b^2-c^2}{2ab} = \frac{49k^2+36k^2-25k^2}{2\times 7k\times 6k}$

$= \frac{60k^2}{12\times 7k^2} = \frac{5}{7}$

$\cos A : \cos B : \cos\ C = 7:19:25$

20. $b = \sqrt{3}+1$, $c = \sqrt{3}-1$

$a^2 = b^2+c^2-2bcCosA$, $A = \frac{\pi}{3}$

21. $a = 80, b = 73, c = 17$

$\therefore \text{Cos}\, B = \frac{1}{2}(by \cos ine\, rule)$

smallest side is 17

22. $a = \frac{5k}{2}, b = 3k, c = \frac{7k}{2}$

23. G.E.

$= \frac{1}{2}\left[\frac{b^2+c^2-a^2+c^2+a^2-b^2+a^2+b^2-c^2}{(a^2+b^2+c^2)/4\Delta}\right]$

$= \frac{4\Delta}{2} = 2\Delta$

24. $\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)Cot\frac{A}{2}$

25. b=2a $B - A = 60^0$

$\tan\left(\frac{B-A}{2}\right) = \left(\frac{b-a}{b+a}\right)Cot\frac{C}{2}$

26. Required value

$= \sum ab(b\,\text{Cos}\,A + a\,\text{Cos}\,B) = 3abc$

27. $-\text{Cos}\,C = \frac{a^2+b^2-c^2}{2ab} = \frac{31}{32} \Rightarrow c = \frac{\sqrt{319}}{2}$

28. Given $(a+b+c)(b+c-a) = \lambda bc$

$(2s)(2s-2a) = \lambda bc$

$\frac{s(s-a)}{bc} = \frac{\lambda}{4}$

$\Rightarrow \cos^2\frac{A}{2} = \frac{\lambda}{4}$

We know $0 < \cos^2 A, < 1$

$\Rightarrow 0 < \gamma < 4$

29. $[(s-b)-(s-a)](s-c) = [(s-c)-(s-b)](s-a)$

$\Rightarrow$ Dividing on both sides (s-a)(s-b)(s-c)

30. $3\tan\frac{A}{2}\tan\frac{c}{2} = 1$

$\frac{3(s-b)(s-c)}{\Delta}.\frac{(s-a)(s-b)}{\Delta} = 1$

$\Rightarrow \frac{3(s-a)(s-b)^2(s-c)}{s(s-a)(s-b)(s-c)} = 1$

$\Rightarrow 3(s-b)=s$

$2s=3b$

$a+b+c=3b$

$a+c=2b$

a,b,c are in A.P

31. a = 3k, b = 4k, c =5k

$\therefore s=6k$ and required ratio =

$s-a:s-b:s-c\cdot$

32. Find s and use $\sin\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{ac}}$,

$\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ac}}$

33. $\frac{(s)(s-a)}{\Delta}\cdot\frac{(s)(s-b)}{\Delta}=c$

$\Rightarrow \frac{s}{s-c}=c$, $\Rightarrow \frac{1}{s-c}=\frac{c}{s}$

34. Given $\Delta=a^2-(b-c)^2$

$=(a-(b-c))(a+b-c)$

$\Delta=(2s-2b)(2s-2c)$

$\frac{1}{4}=\frac{(s-b)(s-c)}{\Delta}$

$\tan A/2=\sqrt{\frac{(s-b)(s-c)}{\Delta}}=\sqrt{\frac{1}{4}}=\frac{1}{2}$

35. $Cot\frac{A}{2}:Cot\frac{B}{2}:Cot\frac{C}{2}=1:\text{m}:\text{n}$

then a : b : c = (m+n) : (l+n) : (l+m)

36. $AD=\frac{2bc}{b+c}\cos\frac{A}{2}, BE=\frac{2ca}{c+a}\cos\frac{B}{2},$

$CF=\frac{2ab}{a+b}\cos\frac{C}{2}.$

37. $\frac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot c/2}{\cot A+\cot B+\cot C}$

N.r

$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{c}{2}=\frac{s(s-a)}{\Delta}+\frac{s(s-b)}{\Delta}+\frac{s(s-c)}{\Delta}$

$=\frac{s}{\Delta}(35-(a+b+c))$

$=\frac{s^2}{\Delta}$

Dr:

$\cot A+\cot B+\cot C=\frac{\cos A}{\sin A}+\frac{\cos B}{\sin B}+\frac{\cos C}{\sin C}$

$=\frac{\frac{b^2+c^2-a^2}{2bc}}{\frac{a}{2R}}+\frac{\frac{c^2+a^2-b^2}{2ca}}{\frac{b}{2R}}+\frac{\frac{a^2+b^2-c^2}{2ab}}{\frac{c}{2R}}$

$=R\left(\frac{b^2+c^2-a^2+c^2+a^2-b^2+a^2+b^2-c^2}{abc}\right)$

$=R\frac{(a^2+b^2+c^2)}{4R\Delta}=\frac{a^2+b^2+c^2}{4\Delta}$

L.H.S= $=\frac{Nr}{Dr}=\frac{(a+b+c)^2}{a^2+b^2+c^2}$

$\therefore s=\frac{a+b+c}{2}$

38. $\pi r^2:\frac{abc}{4R}$ and

$r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}, a=4R\sin\frac{A}{2}\cos\frac{A}{2}$

39. $p_1=\frac{2\Delta}{a}, p_2=\frac{2\Delta}{b}, p_3=\frac{2\Delta}{c}.$

$\therefore \frac{1}{p_1{}^2}+\frac{1}{p_2{}^2}+\frac{1}{p_3{}^2}=\frac{\cot A+\cot B+\cot C}{\Delta}$

40. $2R=r_1-r$

41. $C=90^0, R=\frac{c}{2}, r=(s-c)\tan\frac{C}{2}$

42. $2r_1=2s\Rightarrow \tan\frac{A}{2}=1$

43. Through verification (equilateral properties)

a = b = 1, $s=\frac{3}{2}$ $r=\frac{1}{2\sqrt{3}}$

44. $r\left(r_1+r_2+r_3\right)=ab+bc+ca-s^2$

$r_1+r_2+r_3-r=4R$.

45. We know $r_1+r_2+r_3=4R+r$

$r_1r_2+r_3r_2+r_3r_1=s^2$

$r_1r_2r_3=\frac{\Delta^2}{\gamma}$

The equation whose roots r_1,r_2,r_3 is

$x^3-x^2\left(r_1+r_2+r_3\right)+x\left(r_1r_2+r_2r_3+r_3r\right)-r_1r_2r_3=0$

$x^3-x^2\left(4R+r\right)+x\left(s^2\right)-\frac{\Delta^2}{r}=0$

$\therefore \Delta=rs$

$x^3-x^2\left(4R+r\right)+s^2x-s^2r=0$.

46. $r=\frac{1}{2\sqrt{3}}$, $r_1=r_2=r_3=\frac{\sqrt{3}}{2}$, $A=B=C=60^0$

47. $r_1=r_2=r_3=\frac{\sqrt{3}}{2}$, $r=\frac{1}{2\sqrt{3}}$

48. Given $\angle A=90^o$

We know $r_3+r_3=4R\cos^2\frac{A}{2}$

$=4R\left(\frac{1}{2}\right)=2R$

Now $\cos^{-1}\left(\frac{R}{2R}\right)=\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}=60^o$

49. Given $a=\sqrt{5}, b=2$ $\angle A=\frac{\pi}{6}=30^o$

We know $a^2=b^2+c^2-2bc\cos A$

$5=4+c^2-2c\sqrt{3}$

$\Rightarrow c^2-2c\sqrt{3}-1=0$ (1)

Let c_1,c_2 are the roots of (1)

$c_1+c_2=2\sqrt{3}$ $c_1c_2=-1$

50. $A=\pi r^2, A_1=\pi r_1^2, A_2=\pi r_2^2, A_3=\pi r_3^2$

$\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}}=\frac{1}{\sqrt{\pi r_1^2}}+\frac{1}{\sqrt{\pi r_2^2}}+\frac{1}{\sqrt{\pi r_3^2}}$

and simplify.

51. $|c_1-c_2|=\sqrt{(c_1+c_2)^2-4c_1c_2}=4$

$\left(r_3+r_1\right)\sqrt{\frac{rr_2}{r_1r_3}}=4R\cos^2\frac{B}{2}\tan\frac{B}{2}$

$=4R\sin\frac{B}{2}\cos B/2$

$=2R\sin B$

$=b$

52. $\left(1-\frac{r_1}{r_2}\right)\left(1-\frac{r_1}{r_3}\right)=2$

$\Rightarrow\left(1-\frac{s-b}{s-a}\right)\left(1-\frac{s-c}{s-a}\right)=2$

$\Rightarrow (b-a)(c-a)=2(s-a)^2$

$\Rightarrow b^2+c^2=a^2\Rightarrow \angle A=90^0$.

53. $\frac{1+\cos A}{2}+\frac{1+\cos B}{2}+\frac{1+\cos C}{2}$

$=\frac{3}{2}+\frac{1}{2}[\cos A+\cos B+\cos C]$

$=\frac{3}{2}+\frac{1}{2}[1+\frac{r}{R}]=2+\frac{r}{2R}$.

EXERCISE - III

1. In a triangle ABC , $\angle A=\frac{2\pi}{3}$, $b-c=3\sqrt{3}$ and $\Delta=\frac{9\sqrt{3}}{2}cm^2$ then a=

1) $6\sqrt{3}$ cm 2) 9cm 3) 18cm 4) 6cm

2. In Δ le ABC , if a,b and A are given then there

are two triangles are formed with third side c_1 and c_2 such that then the value of

$\left(c_1 - c_2\right)^2 + \left(c_1 + c_2\right)^2 \tan^2 A =$

1) $4a^2$ 2) $4b^2$ 3) a^2 4) b^2

3. **The number of triangles ABC that can be formed with sin A = 5/13, a =3 and b = 8 is**
1) 3 2) 2 3) 1 4) 0

4. **If AD, BE and CF are the medians of a triangle ABC, then** $AD^2 + BE^2 + CF^2 : BC^2 + CA^2 + AB^2$ **is equal to**
1) 4:3 2) 3:2 3) 3:4 4) 2:3

5. **If in a triangle ABC,** $a^2 Cos^2 A = b^2 + c^2$**, then**
1) $0 < A < \frac{\pi}{4}$ 2) $\frac{\pi}{4} < A < \frac{\pi}{2}$
3) $\frac{\pi}{2} < A < \pi$ 4) $A = \frac{\pi}{2}$

6. **In a $\triangle$ ABC, medians AD and BE are drawn. If AD = 4,** $\angle DAB = \frac{\pi}{6}$ **and** $\angle ABE = \frac{\pi}{3}$ **then the area of ABC is**
1) $\frac{64}{3\sqrt{3}}$ 2) $\frac{8}{3\sqrt{3}}$ 3) $\frac{16}{3\sqrt{3}}$ 4) $\frac{32}{3\sqrt{3}}$

7. **In a triangle ABC ,** $\angle A = \frac{2\pi}{3}$, $b - c = 3\sqrt{3}$ **and** $\Delta = \frac{9\sqrt{3}}{2} cm^2$ **then a=**
1) $6\sqrt{3}$ cm 2) 9cm 3) 18cm 4) 6cm

8. **If length of the sides of a triangle ABC are 4 cm, 5 cm and 6 cm, O is point inside the triangle ABC such that $\angle$OBC = $\angle$OCA = $\angle$OAB = θ, then value of cot θ is**
1) $\frac{11\sqrt{7}}{15}$ 2) $\frac{4\sqrt{7}}{5}$ 3) $\frac{2}{3}\sqrt{7}$ 4) $\sqrt{7}$

9. **In a $\triangle$ ABC if a^2 sin (B-C) + b^2 sin (C-A)+c^2 sin (A-B)=0, then triangle is**
1) right angled 2) obtuse angled
3) isosceles 4) None of these

10. **A triangle is inscribed in a circle. The vertices of the triangle divide the circle into three arcs of length 3, 4 and 5 units, then area of the triangle is equal to**
1) $\frac{9\sqrt{3}\left(1+\sqrt{3}\right)}{\pi^2}$ 2) $\frac{9\sqrt{3}\left(\sqrt{3}-1\right)}{\pi^2}$
3) $\frac{9\sqrt{3}\left(1+\sqrt{3}\right)}{2\pi^2}$ 4) $\frac{9\sqrt{3}\left(\sqrt{3}-1\right)}{2\pi^2}$

11. **If x, y, z are respectively perpendiculars from the circumcentre on the sides of the** ΔABC**, the value of** $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-\frac{abc}{4xyz}$ **is**
1) 0 2) 1 3) 2 4) -1

12. **If α,β,γ are lengths of internal bisectors of angles *A*, *B*, *C* respectively of Δ*ABC*, then** $\Sigma \frac{1}{\alpha}\cos\frac{A}{2}$ **is**
1) $a + b + c$ 2) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$
3) $2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ 4) $2(a+b+c)$

13. **In a triangle *ABC*, *D* and *E* are the mid-points of *BC*, *CA* respectively. If *AD*=5, *BC*=*BE*=4, then *CA* =**
1) 5 2) $\sqrt{7}$ 3) $2\sqrt{7}$ 4) $5\sqrt{5}$

14. **In Δ*ABC*, if *AD* is the altitude and *O* is the orthocentre of Δ*ABC* then *AO : OD* =**
1) Tan *A* : Tan *B* + Tan *C*
2) Tan *B* + Tan *C* : Tan *A*
3) Cos *A* : Cos *B* + Cos *C*
4) Cot *A* : Cot *B* + Cot *C*

15. **The two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60^0. If the third side is 3, the remaining fourth side is**
1) 2 2) 3 3) 4 4) 5

16. **The alternate vertices of a regular hexagon are joined to form another regular hexagon. The ratio of the sides of two hexagons is**
1) $\frac{1}{2}$ 2) $\frac{1}{\sqrt{2}}$ 3) $\frac{1}{\sqrt{3}}$ 4) $\frac{1}{3}$

17. **In Δ*ABC*, if $r_1 = 2r_2 = 3r_3$ and *D* is the mid**

point of BC then $\cos \angle ABC =$

1) 7/25   2) 24/25   3) -7/25   4) -24/25

18. The sides of triangle are in A.P. and the greatest angle exceeeds the least by90^0. The sides are in the ratio

1) $1:2:\sqrt{2}$ 2) $1:\sqrt{3}:2$

3) $\sqrt{7}+1:\sqrt{7}:\sqrt{7}-1$ 4) $\sqrt{3}+1:1:\sqrt{3}-1$

19. In ΔABC, if a=7, b=8, c=9 then the distance from the vertex B to the centroid is

1) $\frac{14}{3}$ 2) $\frac{7}{3}$ 3) 7   4) 14

20. In ΔABC, the line joining the circumcentre and incentre is parallel to BC then the value of $\cos B + \cos C$=

1) 1/2   2) 3/4   3) 1   4) 3/2

21. If for ΔABC, $\cot A \cot B \cot C > 0$ then the triangle is

1) right angled   2) acute angled

3) obtuse angled   4) all   al

22.)In Δ ABC the lengths of the sides AC and AB are 12cm & 5cm respectively. If the area of Δ ABC is $30cm^2$ and R and r are respectively the radius of circumcircle and incircle of Δ ABC then the value of 2R+r (in cm) is equal to____(Mains2021)

KEY

1) 2	2) 1	3) 4	4) 3	5) 3	6) 4
7) 2	8) 1	9) 3	10) 1	11) 1	12) 3
13) 3	14) 2	15) 1	16) 3	17) 3	18) 3
19) 1	20)3	**21) 2**	**22)15cm**		

.

SOLUTIONS

1. Find bc by using $\Delta = \frac{1}{2} bc \sin A$ now use consine rule

2. $a^2 = b^2 + c^2 - 2bc\cos A$

$c^2 - 2b\cos A + b^2 - a^2 = 0$

$c_1 + c_2 = 2b\cos A$

$c_1 c_2 = b^2 - a^2$

3. use sine rule

4. $\frac{1}{4}\frac{\left[2b^2+2c^2-a^2+2c^2+2a^2-b^2+2a^2+2b^2-c^2\right]}{a^2+b^2+c^2}$

5. $CosA = \frac{b^2+c^2-a^2}{2bc}$

$= \frac{a^2Cos^2A - a^2}{2bc} = \frac{-a^2Sin^2A}{2bc} < 0$

$\Rightarrow \frac{\pi}{2} < A < \pi$

6. AG = 8/3

$\frac{\sin\frac{\pi}{3}}{8/3} = \frac{\sin \pi/6}{BG} \Rightarrow BG = \frac{8}{3} X \frac{1}{2} X \frac{2}{\sqrt{3}} = \frac{8}{3\sqrt{3}}$

Area of $\Delta AGB = \frac{1}{2} X \frac{8}{3} X \frac{8}{3\sqrt{3}} X1 = \frac{32}{9\sqrt{3}}$

Area of $\Delta ABC = 3X\frac{32}{9\sqrt{3}} = \frac{32}{3\sqrt{3}}$

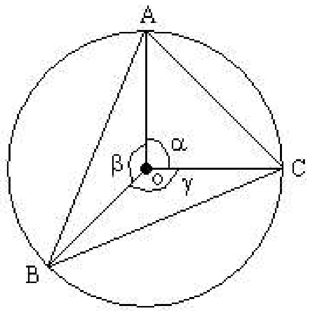
7.Find bc by using $\Delta = \frac{1}{2} bc \sin A$ now use consine rule

8. $\cot\theta = \cot A + \cot B + \cot C$,

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

9.Conceptual

10. Given, arc AC=3, arc AB=4 and arc BC=5. Let r be the radius of the circle.



Then $AC = r\alpha, BC = r\gamma, AB = r\beta$

$\Rightarrow \alpha = \frac{3}{r}, \beta = \frac{4}{r}, \gamma = \frac{5}{r}$

Now, $3+4+5 = 2\pi r \Rightarrow r = \frac{6}{\pi} \Rightarrow \frac{1}{r} = \frac{\pi}{6}$

$$\Delta ABC = \Delta OAC + \Delta OAB + \Delta OBC$$

$$= \frac{1}{2}r^2\left[\sin\left(\frac{3}{r}\right)+\sin\left(\frac{4}{r}\right)+\sin\left(\frac{5}{r}\right)\right]$$

$$= \frac{1}{2}\frac{36}{\pi^2}\left(\sin\frac{\pi}{2}+\sin\frac{2\pi}{3}+\sin\frac{5\pi}{6}\right)$$

$$= \frac{18}{\pi^2}\left(1+\frac{\sqrt{3}}{2}+\frac{1}{2}\right) = \frac{9\sqrt{3}\left(1+\sqrt{3}\right)}{\pi^2}$$

11. $x = R\cos A,\ y = r\cos B,\ \ z = r\cos C$
12. $\alpha = \frac{2bc}{b+c}\cos\frac{A}{2} \Rightarrow \frac{1}{\alpha}\cos\frac{A}{2} = \frac{b+c}{bc}$
13. Conceptutal
14. $AO = 2R\cos A, OB = 2R\cos B\cos C$
15. $c^2 = 4+25-\cancel{2}X2X5X\frac{1}{\cancel{2}} = 19$

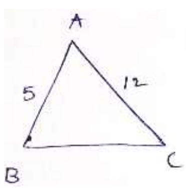
    and then $a^2 = b^2 + c^2 - 2bcCosA$

    $19 = 9 + c^2 - \cancel{2}X3Xc\left(-\frac{1}{\cancel{2}}\right)$

    $c^2 + 3c - 10 = 0 \Rightarrow c^2 + 5c - 2c - 10 = 0$

    $\Rightarrow c(c+5) - 2(c+5) = 0$

    $\Rightarrow c = 2$
16. Conceptual
17. $lr_1 = mr_2 = nr_3 \Rightarrow a:b:c = m+n:n+l:l+m$
18. Conceptual
19. Conceptual
20. $\cos A = \frac{r}{R}, \cos A + \cos B + \cos C = 1 + \frac{r}{R}$
21. Out of $\cot A, \cot B, \cot C$ two values should not be -ve
22.



$$\Delta = \frac{1}{2}\times 5\times 12\times \sin A = 30$$

$\sin A = 1$

$A = 90^o$

$BC = 13cm$

$BC = 2R = 13 \qquad R = \frac{13}{2}cm$

$r = \frac{\Delta}{s} = \frac{30}{15} = 2cm$

$2R + r = \frac{2\times 13}{2} + 2 = 15cm$

ADVANCED QUESTIONS
SINGLE ANSWER

1. **In a triangle ABC If the angles are in AP and b : C = $\sqrt{3}:\sqrt{2}$ then $\underline{|A}$ =**
(A) 60^0 (B) 75^0 (C) 15^0 (D) 90^0

2. **In a triangle ABC, AD is altitude from A, b > c, $\underline{|C}=23^0$. $AD=\dfrac{abc}{b^2-c^2}$ then $\underline{|B}$ =**
(A) 70^0 (B) 113^0 (C) 123^0 (D) 103^0

3. **If D is mid point of the side BC of a triangle ABC and AD is perpendicular to AC then**
$\dfrac{a^2-c^2}{b^2}=$
(A) 1 (B) 2 (C) 3 (D) 4

4. **If the median of a triangle ABC passing through A is perpendicular to AB then tanA + 2tanB=**
(A) 1 (B) -1 (C) 0 (D) None

5. **If the sides of a triangle are in A.P and greatest angle exceeds the least angle of the triangle by 90^0. Then ratio of the sides.**
(A) $3:4:5$ (B) $\sqrt{7}-1:\sqrt{7}:\sqrt{7}+1$
(C) $\sqrt{2}:\sqrt{3}:\sqrt{7}$ (D) $\sqrt{3}:\sqrt{7}:\sqrt{2}$

6. **Let *a*, *b*, *c* be the sides of a triangle. No two of them are equal and $\lambda \in R$. If the roots of the equations,**
$x^2 + 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$ are real, then
(A) $\lambda < 4/3$ (B) $\lambda > 5/3$
(C) $\lambda \in (1/3, 5/3)$ (D) $\lambda \in (4/3, 5/3)$

7. **Let *ABC* be a triangle such that $\angle ACB = \pi/6$ and let *a*, *b* and *c* denote the lengths of the side opposite to *A*, *B* and *C*, respectively. The values of *x* for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are)**
(A) $-(2+\sqrt{3})$ (B) $1+\sqrt{3}$
(C) $2+\sqrt{3}$ (D) $4\sqrt{3}$

8. **If *a*, *b*, *c* denote the lengths of the sides of a triangle opposite angles *A*, *B*, *C* of a triangle *ABC*, then the correct relation among *a*, *b*, *c*, *A*, *B* and *C* is given by**
(A) $(b + c)\sin((B + C)/2) = a\cos(A/2)$
(B) $(b - c)\cos(A/2) = a\sin((B - C)/2)$
(C) $(b - c)\cos(A/2) = 2a\sin((B + C)/2)$
(D) $(b - c)\sin((B - C)/2) = a\cos(A/2)$

9. **Three circular coins each of radii 1 cm are kept in an equilateral triangle so that all the three coins touch each other and also the sides of the triangle. Area of the triangle is**
(A) $(4+2\sqrt{3})\text{cm}^2$
(B) $(1/4)(12+7\sqrt{3})\text{cm}^2$
(C) $(1/4)(48+7\sqrt{3})\text{cm}^2$
(D) $(6+4\sqrt{3})\text{cm}^2$

10. **The sides of a t riangle are inAP and its area is $\frac{3}{5}$th of area of equilateral triangle of same perimeter then ratio of sides**
(A) 3 : 4 : 5 (B) $\sqrt{7}-1:\sqrt{7}:\sqrt{7}+1$
(C) $3:5:7$ (D) $\sqrt{3}:\sqrt{2}:\dfrac{\sqrt{6}+\sqrt{2}}{2}$

11. **In a triangle ABC if cosAcosB+sinAsinBsinC=1 then** a:b:c=
(A) 1:1:1 (B) $1:1:\sqrt{2}$
(C) $\sqrt{2}:1:1$ (D) $1:\sqrt{2}:1$

12. **In a triangle ABC, the median to the side BC is of length $\dfrac{1}{\sqrt{11-6\sqrt{3}}}$ and it divides A into the angles 30^0 and 45^0. then length of side BC is**
(A) 1 (B) 2 (C) 3 (D) 4

13. **In a triangle ABC, D, E are t wo points on the side BC such that BD = DE = EC.** $\underline{|BAD}=\alpha, \underline{|DAE}=\beta, \underline{|EAC}=\gamma$ **then**
$\dfrac{Sin(\alpha+\beta).Sin(\beta+\gamma)}{Sin\alpha.Sin\gamma}=$
(A) 1 (B) 2 (C) 3 (D) 4

14. **If O is the point inside the triangle ABC such that** $\underline{|OBC}=A/2$, $\underline{|OCA}=B/2$,

$\underline{|OAB} = C/2$ **then**

$$\frac{Sin(A - C/2)Sin\left(B - C/2\right).Sin\left(C - B/2\right)}{SinA/2.\,SinB/2.SinC/2} =$$

(A) 1 (B) 2 (C) 3 (D) 4

15. S is the circumcentre of Δle ABC and R_1, R_2, R_3 are radii of circum circle of $\Delta^{le}\ SBC, \Delta^{le}\ SCA, \Delta^{le}\ SAB$ then

$$\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} =$$

(A) $\frac{abc}{R}$ (B) $\frac{abc}{R^3}$ (C) $\frac{a+b+c}{R^3}$ (D) $\frac{abc}{R^2}$

KEY

01) B 02) B 03) C 04) C 05) B 06) A
07) B 08) B 09) D 10) C 11) B 12) B
13) D 14) A 15) B

SOLUTIONS

1. $\angle A, \angle B, \angle C$ are in A.P

$2\angle B = \angle A + \angle C = \pi - \angle B$

$\angle B = \frac{\pi}{3}$

Sin B: Sin C $= \frac{\sqrt{3}}{\sqrt{2}}\left(given\right)$

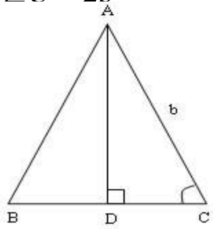
$\frac{\sqrt{3}}{2SinC} = \frac{\sqrt{3}}{\sqrt{2}}$

$\Rightarrow SinC = \frac{1}{\sqrt{2}}$

$\Rightarrow \angle C = 45^0$

$\angle A = 180^0 - 60^0 - 45^0 = 75^0$

2. $\angle C = 23^0$



$\frac{AD}{SinC} = b$

$\frac{abc}{b^2 - c^2} = b\sin c$

$\frac{SinA\ SinC}{Sin^2 B - Sin^2 C} = SinC$

$\frac{SinA\,SinC}{Sin\left(B+C\right)Sin\left(B-C\right)} = SinC$

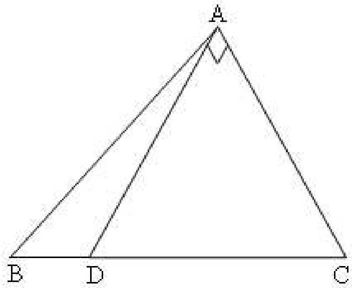
$\frac{SinA}{SinA\sin\left(B-C\right)} = 1$

$\Rightarrow Sin\left(B-C\right) = 1$

$B - C = 90^0 \quad B = 90 + 23^0 = 113^0$

3. $\underline{|BAD} = \underline{|A} - 90^0$, BD=DC

$\underline{|BAD} = 90 + C$



Apply Sinc rule in ΔABD

$\frac{a}{2\sin\left(A - 90^0\right)} = \frac{C}{Cosc}$

$\frac{a}{-2CosA} = \frac{C}{Cosc}$

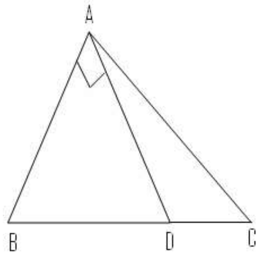
$\Rightarrow \frac{2abc}{-2\left(b^2 + c^2 - a^2\right)} = \frac{2abc}{a^2 + b^2 - c^2}$

$\Rightarrow -2b^2 + 2\left(a^2 - c^2\right) = b^2 + a^2 - c^2$

$a^2 - c^2 = 3b^2 \quad \dfrac{a^2 - c^2}{b^2} = 3$

4. $\underline{|DAC} = \underline{|A} - 90, \underline{|CDA} = 90 + B$

Apply Sinc rule in ΔADC



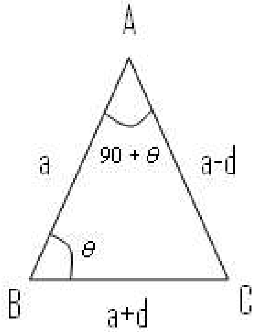
$$\frac{a}{-2\cos A} = \frac{b}{\cos B}$$

$$\Rightarrow \frac{-\tan A}{2} = \tan B$$

$$\Rightarrow -\tan A = 2\tan B$$

$$\Rightarrow \tan A + 2\tan B = O$$

5. Let $a-d, a, a+d$ are the sides of traingle let $\angle ABC = \theta$ Given $\angle BAC = 90 + \theta$



Apply sine rule : $\dfrac{a+d}{\sin(90+\theta)} = \dfrac{a-d}{\sin\theta}$

$$\frac{a+d}{\cos\theta} = \frac{a-d}{\sin\theta} \text{ --------(1)}$$

$$Cos\theta = \frac{a^2 + (a+d)^2 - (a-d)^2}{2a(a+d)} = \frac{a+4d}{2(a+d)}$$

$$Cos(90+\theta) = -\sin\theta$$

$$= \frac{a^2 + (a-d)^2 - (a+d)^2}{2a(a-d)} = \frac{a-4d}{2(a-d)} \text{ --- (2)}$$

from (1) and (2)

$$(a+d) \times \frac{2(a+d)}{a+4d} = \frac{(a-d)}{4d-a} \times 2(a-d)$$

$$\frac{a-d}{a+d} = \frac{4d^2 + 3ad - a^2}{4d^2 - 3ad - a^2}$$

Apply componendo and dividedo

$$\frac{d}{a} = \frac{a^2 - 4d^2}{3ad}$$

$$\Rightarrow 3d^2 = a^2 - 4d^2 \quad \Rightarrow d = \frac{a}{\sqrt{7}}$$

$$a-d : a : a+d = a - \frac{a}{\sqrt{7}} : a : a + \frac{a}{\sqrt{7}}$$

$$= \sqrt{7} - 1 : \sqrt{7} : \sqrt{7} + 1$$

6. a,b,c are sides of a triangle and $a \neq b \neq c.$

$\therefore |a-b| < |c| \Rightarrow a^2 + b^2 - 2ab < c^2$

Similarly, we have

$b^2 + c^2 - 2bc < a^2$ and $c^2 + a^2 - 2ca < b^2$

On adding, we get

$$a^2 + b^2 + c^2 < 2(ab + bc + ca)$$

$$\Rightarrow \frac{a^2 + b^2 + c^2}{ab + bc + ca} < 2 \qquad (1)$$

Since the roots of the given equation are real, therefore

$$(a+b+c)^2 - 3\lambda(ab + bc + ca) \geq 0 \qquad (2)$$

From (1) and (2), we get

$$3\lambda - 2 < 2 \Rightarrow \lambda < \frac{4}{3}$$

7. Using cosine rule of $\angle C$, we get

$$\frac{\sqrt{3}}{2} = \frac{(x^2+x+1)^2 + (x^2-1)^2 - (2x+1)^2}{2(x^2+x+1)\,(x^2-1)}$$

$$\Rightarrow \sqrt{3} = \frac{2x^2 + 2x - 1}{x^2 + x + 1}$$

$$\Rightarrow (\sqrt{3} - 2)x^2 + (\sqrt{3} - 2)x + (\sqrt{3} + 1) = 0$$

$$\Rightarrow x = \frac{(2 - \sqrt{3}) \pm \sqrt{3}}{2(\sqrt{3} - 2)}$$

$\Rightarrow x = -(2+\sqrt{3}),\ 1+\sqrt{3} \Rightarrow\ x = 1+\sqrt{3}$
as $(x > 0)$.

8. Let us consider $\frac{b-c}{a}$, which is involved in three of the options

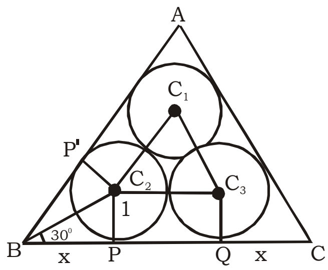
$$\frac{b-c}{a} = \frac{\sin B - \sin C}{\sin A}$$

$$= \frac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin(A/2)\cos(A/2)}$$

$$= \frac{\sin(A/2)\sin\left(\frac{B-C}{2}\right)}{\sin(A/2)\cos(A/2)}$$

$$= \frac{\sin\left(\frac{B-C}{2}\right)}{\cos(A/2)}$$

$$\therefore (b-c)\cos(A/2) = a\sin\left(\frac{B-C}{2}\right)$$

9. 

For the circle with centre C_2, BP and BP' are two tangents to the circle, therefore BC_2 must be the bisector of $\angle B$. But $\angle B = 60^0$ (as ΔABC is an equilateral triangle)

$$\therefore \angle C_2BP = 30^0$$

$$\tan 30^0 = \frac{1}{x} \quad \Rightarrow x = \sqrt{3}$$

$$BC = BP + PQ + QC = x + 2 + x = 2 + 2\sqrt{3}$$

Therefore, area of $\Delta ABC = \frac{\sqrt{3}}{4}\left(2+2\sqrt{3}\right)^2$

$$= \sqrt{3}\left(1+3+2\sqrt{3}\right) = 4\sqrt{3} + 6\,sq.units$$

10. Let the sides be $a-d, a, a+d$; Perimeter=3a

Area of Δ with same perimeter $= \frac{\sqrt{3}a^2}{4}$

Area of given tringle $=$

$$\sqrt{3\frac{a}{2}\left(\frac{a}{2}+d\right)\left(\frac{a}{2}\right)\left(\frac{a}{2}-d\right)} = \frac{3}{5}\times\frac{\sqrt{3}a^2}{4}$$

$$\frac{3a^2}{4}\left(\frac{a^2}{4}-d^2\right) = \frac{9}{25}\times\frac{3a^4}{16}$$

$$a^2 - 4d^2 = \frac{9a^2}{25}$$

$$\frac{16a^2}{25} = 4d^2 \qquad d = \frac{2a}{5}$$

$$a-d : a : a+d = \frac{3a}{5} : a : \frac{7a}{5}$$

$= 3:5:7$

11. $\text{Sinc} = \frac{1-\cos A\cos B}{\sin A\sin B} \le 1$

$$1 - \cos A\cos B \le \sin A\sin B$$
$$1 \le \cos A\cos B + \sin A\sin B$$
$$\cos(A-B) \ge 1$$
$$\Rightarrow \cos(A-B) = 1$$
$$\Rightarrow A - B = 0$$
$$\Rightarrow A = B$$

$$Sinc = \frac{1-\cos^2 A}{Sin^2 A} = \frac{Sin^2 A}{Sin^2 A} = 1$$

$$\angle C = 90^0 \Rightarrow A = B = 45^0$$

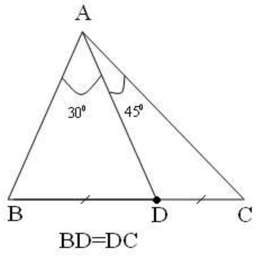
a: b: c = Sin A: Sin B: Sin C $= \frac{1}{\sqrt{2}} : \frac{1}{\sqrt{2}} : 1$

$= 1 : 1 : \sqrt{2}$

12. $\frac{AD}{Sinc} = \frac{a}{2S\text{in}30^0}$

$$\Rightarrow Sinc = \frac{AD}{a}$$

$$C = \frac{AD}{SinA}$$



$$\frac{a}{2Sin45^0}=\frac{AD}{SinB}$$

$$SinB=\frac{\sqrt{2}AD}{a}\Rightarrow b=\frac{\sqrt{2}AD}{Sin\,A}$$

But $AD=\frac{1}{2}\sqrt{2\left[b^2+c^2\right]-a^2}$

$$4\left[AD\right]^2=\frac{4\left(AD\right)^2}{Sin^2A}+\frac{2\left(AD\right)^2}{Sin^2A}-a^2$$

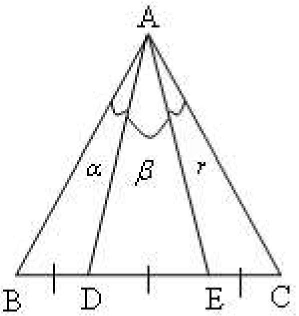
$$\Rightarrow a^2=\frac{6\left(AD\right)^2}{Sin^2A}-4\left(AD\right)^2$$

$$a=AD\sqrt{\frac{6-4Sin^2A}{Sin^2A}}\quad SinA=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$=\frac{1}{\sqrt{11-6\sqrt{3}}}\sqrt{4\left(11-6\sqrt{3}\right)}$$

$$=\sqrt{4}\ =2$$

13. Apply sine rule in ΔABE and ACE



$$\frac{AE}{SinC}=\frac{a}{Sinr};\frac{AE}{SinB}=\frac{2a}{3\sin\left(\alpha+\beta\right)}$$

$$\Rightarrow\frac{SinC}{SinB}=\frac{2\sin\gamma}{Sin(\alpha+\beta)}\text{--------}(1)$$

Apply sine rule in ΔABD and ΔADC

$$\frac{AD}{SinC}=\frac{2a}{3\sin\left(B+\gamma\right)};\frac{AD}{Sin\,B}=\frac{a}{3\sin\alpha}$$

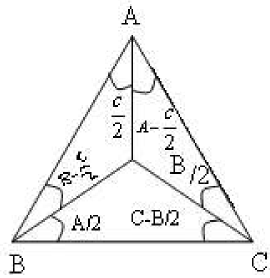
$$\Rightarrow\frac{SinB}{SinC}=\frac{2Sin\theta_1}{Sin\left(\theta_2+\theta_3\right)}\text{-------}(2)$$

from (1) and (2)

$$\frac{4Sin\theta_1Sin\theta_3}{Sin\left(\theta_1+\theta_2\right)Sin\left(\theta_2+\theta_3\right)}=1$$

$$\Rightarrow\frac{Sin\left(\theta_1+\theta_2\right)Sin\left(\theta_2+\theta_3\right)}{Sin\theta_1\ Sin\theta_3}=4$$

14. In OAB



$$\frac{OB}{Sin\frac{C}{2}}=\frac{OA}{Sin\left(B-\frac{A}{2}\right)}$$

$$\frac{OA}{OB}=\frac{Sin\left(B-\frac{A}{2}\right)}{Sin\frac{c}{2}}\text{----------}(1)$$

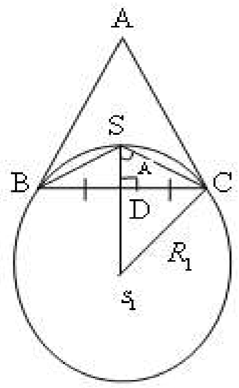
Similarly $\frac{OB}{OC}=\frac{Sin\left(C-\frac{B}{2}\right)}{Sin\frac{A}{2}}$ -------(2)

$$\frac{OC}{OA}=\frac{Sin\left(A-\frac{C}{2}\right)}{Sin\frac{B}{2}}\text{------}(3)$$

Multiplying (1), (2) and (3)

$$\frac{Sin\left(B-\frac{A}{2}\right)Sin\left(A-\frac{C}{2}\right)Sin\left(C-\frac{B}{2}\right)}{Sin\frac{A}{2}Sin\frac{B}{2}Sin\frac{C}{2}}=1$$

15. 'S' be the circumcentre of ΔABC
CD=R Sin A



MULTIPLE ANSWER

1. **If H is the orthocentre of triangle ABC, then AH is equal to**

   (A) $2R\cos A$ (B) $2R\sin A$

   (C) $a\cot A$ (D) $\frac{2abc}{\Delta}\cos A$

2. **Internal bisector of an angle $\angle A$ of triangle ABC meets side BC at D. A line drawn through D perpendicular to AD intersects the side AC at E and the side AB at F. If a, b, c represent the sides of ΔABC then**

   (A) AE is H.M. of b and c

   (B) $AD = \frac{2bc}{b+c}\cos(A/2)$

   (C) $EF = \frac{4bc}{b+c}\sin(A/2)$

   (D) The triangle AEF is isosceles

3. **If the sines of the angles A and B of a triangle ABC satisfy the equation $c^2x^2 - c(a+b)x + ab = 0$, then the triangle**

   a) is acute angled b) is right angled

   c) is obtuse angled

   d) satisfy $\sin A + \cos A = \frac{(a+b)}{c}$

4. **In a triangle ABC with fixed base BC, the vertex A moves such that $\cos B + \cos C = 4\sin^2(A/2)$. If a, b, c denote the lengths of the sides of the triangle opposite to the angles A, B and C, respectively, then**

   (A) $b + c = 4a$ (B) $b + c = 2a$

   (C) $\cos\frac{B-C}{2} = 2\sin\frac{A}{2}$

   (D) locus of point A is a pair of straight lines

5. **If the tangents of the angles A and B of triangle ABC satisfy the equation $abx^2 - c^2x + ab = 0$, then**

   (A) $\tan A = a/b$ (B) $\tan B = b/a$

   (C) $\cos C = 0$

   (D) $\sin^2 A + \sin^2 B + \sin^2 C = 2$

KEY

01) A, C 02) A, B, C, D
03) B, D 04) B, C 05) A, B, C, D

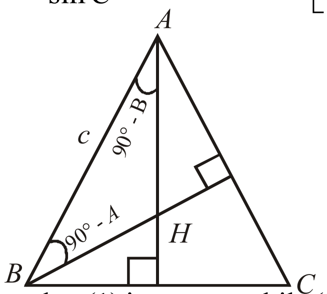
SOLUTIONS

1. Referring to ΔAHB, we have

$$\frac{AH}{\sin(90°-A)} = \frac{c}{\sin(A+B)}$$

$$\Rightarrow AH = \frac{c\cos A}{\sin(180°-C)}$$

$$= \frac{c\cos A}{\sin C} = 2R\cos A \quad \left[\because \frac{c}{\sin C} = 2R\right]$$



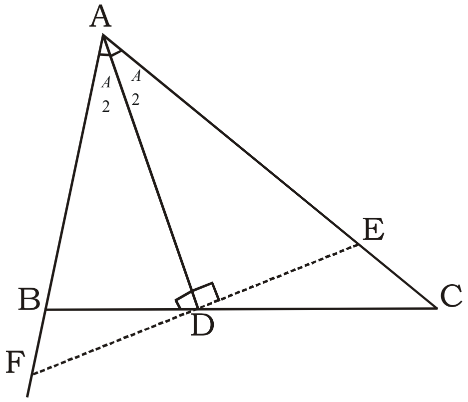
so that (1) is correct while (2) is not correct. Also,

$$AH = 2R\cos A = 2.\frac{abc}{4\Delta}\cos A = \frac{abc}{2\Delta}\cos A$$

and $AH = 2R\cos A = \frac{a}{\sin A}.\cos A = a\cot A$

so that (3) is correct, while (4) is not correct.

2. By simple geometry, in ΔAFE, we get AF = AE.



Therefore, ΔAFE, is an isosceles triangle

Now area (ΔABC) = are (ΔABD) + area $(\Delta ADC) \Rightarrow \frac{1}{2}bc\sin A = \frac{1}{2}cAD\sin\frac{A}{2} + \frac{1}{2}b$

$AD\sin\frac{A}{2}$

$$\Rightarrow AD = \frac{2bc\cos\frac{A}{2}}{b+c}$$

Also, $AD = AE\cos\frac{A}{2}$

$\Rightarrow AE = \frac{2bc}{b+c} = H.M.$ of b and c

Again $EF = 2DE = 2\,AD\tan\frac{A}{2} = \frac{4bc\sin\frac{A}{2}}{b+c}$

3. $\because$ sinA and sinB are the roots of

$c^2x^2 - c(a+b)x + ab = 0$

Then $\sin A + \sin B = \frac{a+b}{c}$

And $\sin A\sin B = \frac{ab}{c^2}$

$$\Rightarrow \frac{a}{2R} + \frac{b}{2R} = \frac{a+b}{c}$$

And $\frac{a}{2R} \times \frac{b}{2R} = \frac{ab}{c^2}$

$\therefore c = 2R \Rightarrow 2R\sin C = 2R$

$\therefore \sin C = 1$

$\therefore \angle C = 90^o$

$A + B = 90^o$

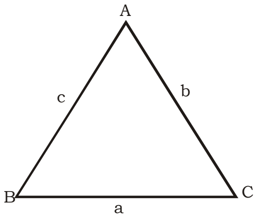
$B = 90^o - A$

$\because \sin A + \sin B = \frac{a+b}{c}$

$\Rightarrow \sin A + \sin\left(90^o - A\right) = \frac{a+b}{c}$

$\Rightarrow \sin A + \cos A = \frac{a+b}{c}$

4. $\cos B+\cos C=4\sin^2\frac{A}{2}$



$$\Rightarrow 2\cos\frac{B+C}{2}\cos\frac{B-C}{2}=4\sin^2\frac{A}{2}$$

$$\Rightarrow \cos\frac{B-C}{2}=2\cos\frac{B+C}{2}\quad\left[\because \sin\frac{A}{2}\cos\frac{B+C}{2}\right]$$

$$\Rightarrow \frac{\cos\frac{B-C}{2}}{\cos\frac{B+C}{2}}=2$$

By componendo and divideno, we get

$$\frac{\cos\frac{B-C}{2}+\cos\frac{B+C}{2}}{\cos\frac{B-C}{2}-\cos\frac{B+C}{2}}=\frac{3}{1}$$

$$\Rightarrow \frac{2\cos\frac{B}{2}\cos\frac{C}{2}}{2\sin\frac{B}{2}\sin\frac{C}{2}}=3$$

$$\Rightarrow \tan\frac{B}{2}\tan\frac{C}{2}=\frac{1}{3}\Rightarrow\frac{s-a}{s}=\frac{1}{3}$$

$$\Rightarrow 3s=s+3a$$

$$\Rightarrow b+c=2a$$

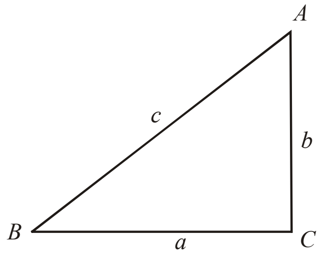
$$AB+AC>BC$$

Therefore, locus of A is ellipse

5.From the given equation, we get
$\tan A+\tan B=(c^2/ab)$ and $\tan A\tan B=1$

Since $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\ \tan B}$, we have

$A+B=\frac{\pi}{2}$ and hence $C=\frac{\pi}{2}$



Therefore, triangle ABC is right angled at C.
Hence, $\tan A=a/b$, $\tan B=b/a$, $\cos C=0$
$\sin A=a/c$, $\sin B=b/c$ and $\sin C=1$.
So,

$$\sin^2 A+\sin^2 B+\sin^2 C$$

$$=\frac{a^2}{c^2}+\frac{b^2}{c^2}+1=\frac{a^2+b^2}{c^2}+1=1+1=2$$

$[\because\ a^2+b^2=c^2]$

COMPHERENSION TYPE

Passage - 1

If $r=1, R=3, \Delta=7$ **then**

1. $a^2+b^2+c^2=$

A) 72 B) 144 C) 60 D) 21

2. $a^3+b^3+c^3=$

A) 192 B) 392 C) 288 D) 144

3. $a^4+b^4+c^4=$

A) 392 B) 192 C) 2200 D) 1982

Passage - 2

Circum circle of a Δ^{le} ABC is a circle passing through the vertices, Nine point circle is circum circle of pedal triangle and its centre N is midpoint of circum centre and ortho centre of Δ^{le} ABC then

4. If the circum circle and nine point circle cut orthogonally then $\sum(\cos 2A)=$

A) 0 2) $\frac{1}{2}$ 3) -1 4) 1

P

5. If nine point circle touches circum circle then $\sum(Cos2A) =$

A) 0 B) $\frac{1}{2}$ C)-1 D)1

6. If circum circle passes through N then $\sum(Cos2A) =$

A) 0 B) $\frac{1}{2}$ C) -1 D) 1

KEY

01) A 02) B 03) C 04) D 05) C 06) B

SOLUTIONS

assage - 1

$r = 1$, R=3, $\Delta = 7$

$r = \frac{\Delta}{S} \Rightarrow S = \frac{\Delta}{r} = 7 \Rightarrow a+b+c = 14$ .....(1)

Now $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$7 = \sqrt{7(7-a)(7-b)(7-C)}$.

$7^3 - 49(a+b+c) + 7(ab+bc+ca) - abc = 7$

$\Rightarrow ab+bc+ca = 62$

1. $a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca)$

$= 196 - 24 = 72 \quad \therefore$ (A) is correct.

2. $a^3+b^3+c^3 = 3abc + (a+b+c)$

$(a^2+b^2+c^2-ab-bc-ca)$

$= 3(84) + 14(72-62)$

$= 392 \Rightarrow$ (B) is correct.

3. $a^4+b^4+c^4 = (a^2+b^2+c^2)^2 -$

$2\{(ab+bc+ca)^2 - 2abc(ab-bc+ca)\}$

$= (72)^2 - 2\{(62)^2 - 2(84)(14)\}$

$5184 - 2984 = 2200$

4. $SN^2 = R^2 + \frac{R^2}{4}$

$\left(\frac{SH}{2}\right)^2 = \frac{5R^2}{4}$

$\frac{R^2(1-8\sin A\sin B\sin C)}{4} = \frac{5R^2}{4}$

$1 - 8\cos A\cos B\cos C = 5$

$8\cos A\cos B\cos C = -4$ ....(1)

$\sum(\cos 2A) = \cos 2A + \cos 2B + \cos 2C$

$= 2\cos(A+B)\cos(A-B) + 2\cos C - 1$

$-2\cos C\{\cos(A-B) - \cos C\} - 1$

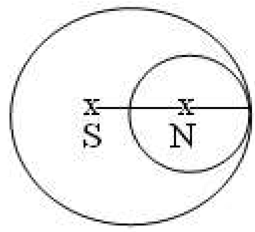
$2\cos C\{\cos(A+B) + \cos s(A+B)\} - 1$

$= -4\cos A\cos B\cos C - 1$

$-4\left(-\frac{1}{2}\right) - 1$

$= 2 - 1 = 1$

5.



Nine point circle touches circumcircle.

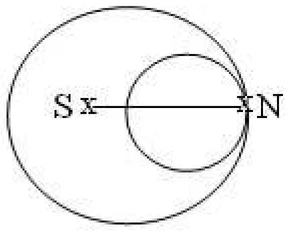
$SN = R - R/2 = R/2$

$\left(\frac{SH}{2}\right)^2 = \frac{R^2}{4} \quad R^2(1 - 8\cos A\cos B\cos C) = R^2$

$8\sin A\sin B\sin C = 1$

$\therefore \sum\cos 2A = -\cancel{4}\left(\frac{1}{\cancel{8}\,2}\right) - 1 = -3/2$

6.



$SN = R$

$\frac{SH}{2} = R$

$SH^2 = 4R^2$

$R^2(1-8\cos A\cos B\cos C) = 4R^2$

$8\cos A\cos B\cos C = -3$

$2\cos 2A = -4(\cos A\cos B\cos C)-1$

$= -\cancel{4}\left(\frac{-3}{\cancel{8}2}\right)-1$

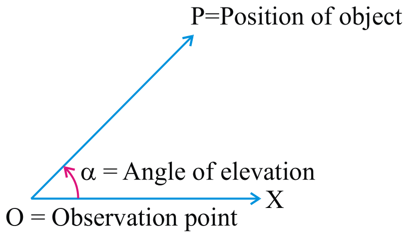
$= \frac{3}{2}-1 = \frac{1}{2}$ (B) is correct.

****

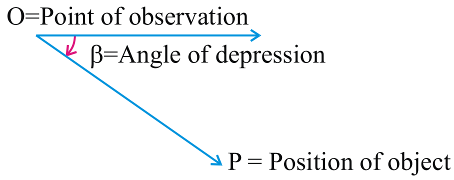
HIGHTS & DISTANCES

SYNOPSIS

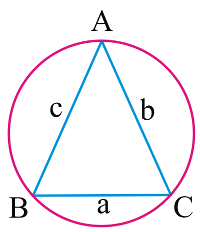
- **Angle of Elevation** :
If the position of the object is above the position of the observation then the angle made by the line joining object and observation point with the horizontal line drawn at the observation point is called angle of elevation.



- **Angle of Depression :**
If the position of the object is below the position of the observation then the angle made by the line joining the object and observation point with the horizontal line drawn at the observation point is called angle of depression.



- **Note:** (i)Angle of elevation and depression are always acute angles.
(ii)Any line perpendicular to a plane is perpendicular to all lines lying in the plane.
- In a triangle ABC, the angles are denoted by the capital letters A,B,C and length of the sides opposite to these angles are denoted by a,b,c respectively.



- **The Law of sines :**

In $\Delta ABC, \frac{a}{SinA} = \frac{b}{SinB} = \frac{c}{SinC} = 2R$ where R is the radius of the circumcircle of the ΔABC.
R is called circum radius of the triangle.
Note: In any triangle other than right angled triangle we can use the sine rule.

- **The law of cosines :**

$$In\ \Delta ABC,\ CosA = \frac{b^2 + c^2 - a^2}{2bc},$$

$$(or)\, a^2 = b^2 + c^2 - 2bc\ CosA$$

- **Length of the median :**

In ΔABC, let AD,BE,CF be the medians then

(i) $AD = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

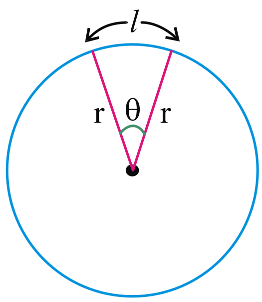
(ii) $BE = \frac{1}{2}\sqrt{2a^2 + 2c^2 - b^2}$

(iii) $CF = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$

- (i) Area of the sector $= \frac{\theta}{360}\pi r^2 = \frac{1}{2} lr$

(ii) Length of the arc $(l) = r\theta$

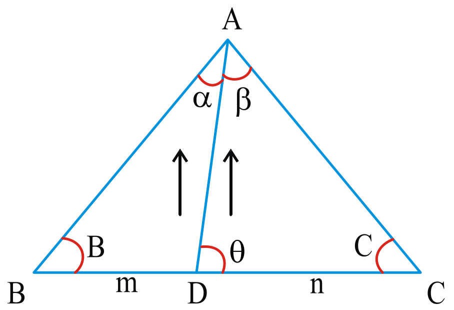
(iii) perimeter $= P = r(\theta + 2) = l + 2r$



✈ The m - n Theorem:

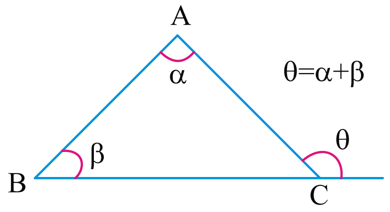
(i) $(m+n)\cot\theta = n\cot B - m\cot C$

(ii) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

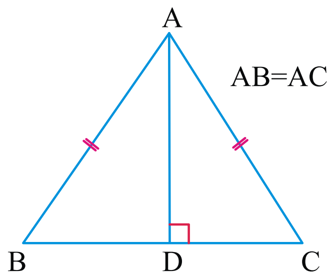


✈ Some geometrical properties of triangle :

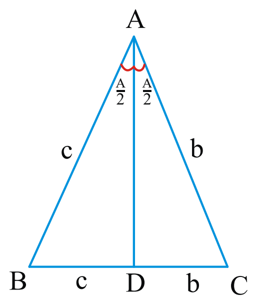
(i) In a triangle ABC, the exterior angle is equal to the sum of interior opposite angles



(ii) In an Isosceles triangle the median is perpendicular to the base.



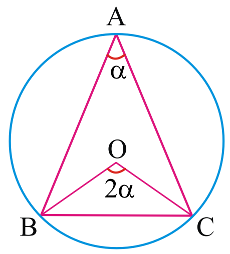
(iii) In a triangle the internal bisector of an angle divides the opposite side in the ratio of the arms of the angle.



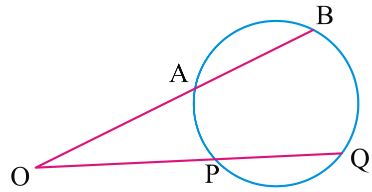
(iv) In similar triangles the corresponding sides are proportional.

✈ Some results in a circle :

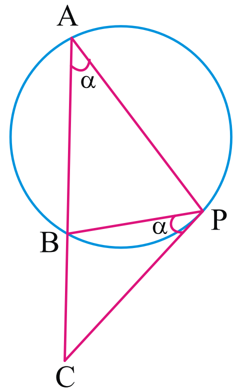
(i) The angle subtended by any chord at the centre is twice the angle subtended by the same on any point on the circumference of the circle.



(ii) If two secants AB and PQ of a circle meet at point 'O', then OA.OB=OP.OQ



(iii) Angles in the alternate segments of a circle are equal.

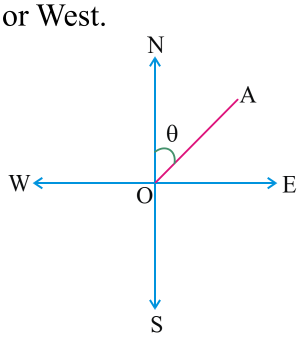


BP is a chord and CP is a tangent. A is any point on major arc.

✈ Bearings of a point :

The acute angle which OA makes with NS is called the bearing of the point A from O.

The bearing of a point indicated by giving the size of the acute angle and specifying whether it is measured from ON or OS and whether to the East or West.

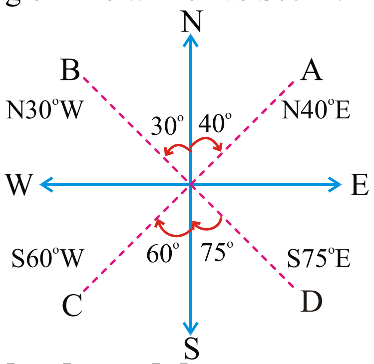


(i) OA is in t he direction 40^0 East of North and the bearing of A is written as N40^0E.

(ii) OB is in the direction 30^0 west of North and bearing of B and is written as N30^0W.

(iii) OC is in the direction 60^0 West of South and bearing of C is written as S60^0W.

(iv) OD is the direction 75^0 east of south and bearing of D is written as S75^0E.

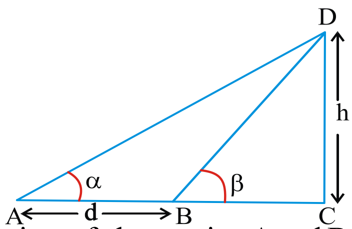


✈ **Standard models :**

(i) The angle of elevation of the top of a tower, standing on a horizontal plane, from a point A is α. After walking a distance 'd' metres towards the foot of the tower, the angle of elevation is found to be β.

The height of the tower h = $\dfrac{d\sin\beta\sin\alpha}{\sin(\beta-\alpha)}$

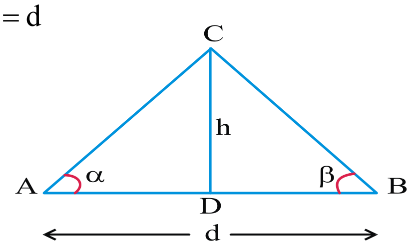
(or) $h = \dfrac{d}{\cot\alpha - \cot\beta}$ Where $AB = d$



(ii) If the Points of observation A and B lie on either side of the tower, then height of the tower

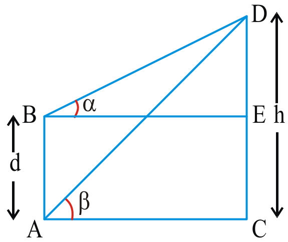
$$h = \frac{d\sin\alpha\sin\beta}{\sin(\alpha+\beta)} = \frac{d}{\cot\alpha+\cot\beta}$$ Where

$AB = d$



(iii) The angles of elevation of the top of a tower from the bottom and top of a building of height 'd' metres are β and α respectively. The height of the tower is

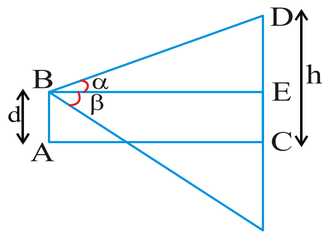
$h = \dfrac{d\sin\beta\cos\alpha}{\sin(\beta-\alpha)}$ metres (or) $h = \dfrac{d\cot\alpha}{\cot\alpha - \cot\beta}$



(iv) The angle of elevation of a cloud from a height 'd' metres above the level of water in a lake is 'α' and the angle of depression of its image in the lake is β. The height of the cloud from the water level in metres is

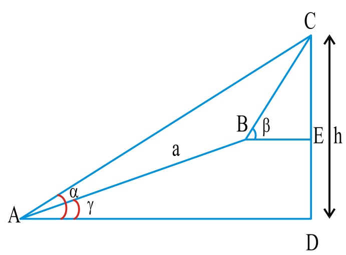
$h = \dfrac{d\sin(\beta+\alpha)}{\sin(\beta-\alpha)}$ (or) $h = \left[\dfrac{d(\tan\beta+\tan\alpha)}{(\tan\beta-\tan\alpha)}\right]$

or $h = d\left[\dfrac{Cot\alpha + Cot\beta}{Cot\alpha - Cot\beta}\right]$



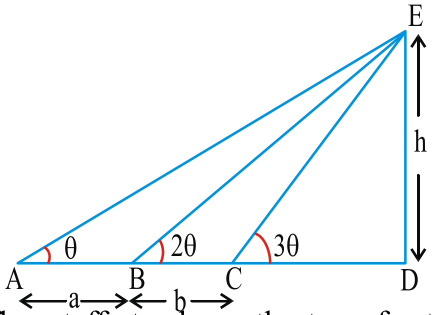
(v) The angle of elevation of a hill from a point A is 'α'. After walking to some point B at a distance 'a' metres from A on a slope inclined at 'γ' to the horizon, the angle of elevation was found to be β

Height of the hill $h = \dfrac{a\sin\alpha\ \sin(\beta-\gamma)}{\sin(\beta-\alpha)}$

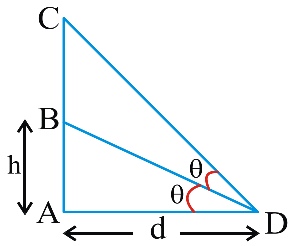


(vi) A balloon is observed simultaneously from the three points A, B, C on a straight road directly beneath it. The angular elevation at B is twice that at A and the angular elevation at 'C' is thrice that at A. If AB=a and BC=b then the height of the balloon 'h' in terms of a and b is,

$$h = \frac{a}{2b}.\sqrt{(3b-a)\ (a+b)}$$



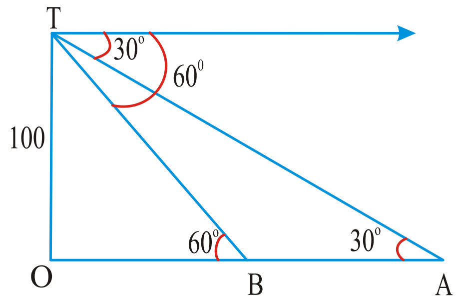
(vii) A flag staff stands on the top of a tower of height h metres. If the tower and flag staff subtend equal angles at a distance 'd' metres from the foot of the tower, then the height the flag - staff in metres is $h\left[\dfrac{d^2+h^2}{d^2-h^2}\right]$



Eg 1:

A man from the top of 100 metres high tower sees a car moving towords the tower at an angle of depression of 30^0. After some time the angle of depression becomes 60^0. The distance travelled by the car during this time is _____ metres

Sol: Let OT be the tower and A, B be the positions of the car.



From ΔOAT, $\text{Cot}30^0 = \dfrac{OA}{OT} \Rightarrow OA = 100\sqrt{3}$

From ΔOBT, $\text{Cot}60^0 = \dfrac{OB}{OT} \Rightarrow OB = \dfrac{100}{\sqrt{3}}$

Distance travelled by a car = AB = OA-OB =

$$100\sqrt{3} - \frac{100}{\sqrt{3}} = \frac{200}{\sqrt{3}}\,mts$$

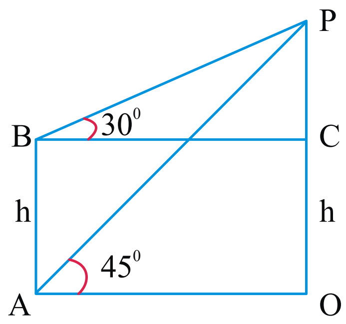
Eg 2:

The angle of elevation of the top of a tower from the top and bottom of a building of hight 'h' are 30^0 and 45^0 respectively if the tower and building stand at the same level then the height of the tower is

Sol: In triangles PCB and POA

$$\text{Cot}30^0 = \frac{BC}{PC}, \text{Cot}45^0 = \frac{OA}{OP}$$

$BC = OA \Rightarrow PC\,\text{Cot}30^0 = OP\,\text{Cot}45^0$



$$PC\sqrt{3} = (OC+CP) \Rightarrow PC\sqrt{3} = h + PC \Rightarrow PC = \frac{h}{\sqrt{3}-1}$$

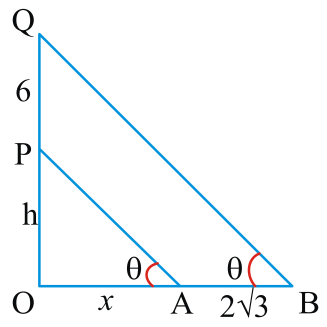
Thus $OP = h + \dfrac{h}{\sqrt{3}-1} = \dfrac{\sqrt{3}h}{\sqrt{3}-1} = \dfrac{h\left(3+\sqrt{3}\right)}{2}$

Eg 3:

If a flag of 6 metres high placed on the top of a tower throws a shadow of $2\sqrt{3}$ metres along the ground then the angle (in degrees) that the sun makes with the ground is

Sol: Let OA and AB be the shadows of tower OP and flag staff PQ respectively on the ground. Suppose the sun makes an angle 'θ' with the ground.

Let OA = x



In triangles OAP and OBQ, we have $\tan\theta = \frac{h}{x}$

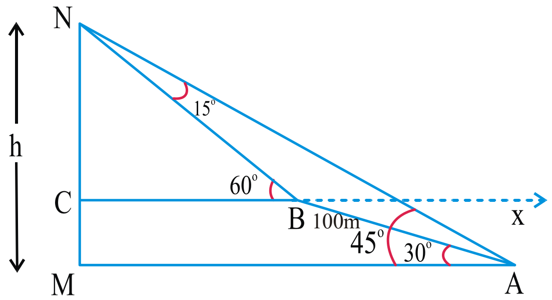
$$\tan\theta = \frac{h+6}{x+2\sqrt{3}} \Rightarrow \frac{h}{x} = \frac{h+6}{x+2\sqrt{3}}$$

$$\Rightarrow 2\sqrt{3}h = 6x \Rightarrow \frac{h}{x} = \frac{6}{2\sqrt{3}} = \sqrt{3} \Rightarrow \tan\theta = \sqrt{3}$$

Thus $\theta = 60^0$

Eg 4:

At the foot of the mountain the elevation of its summit is 45^0 After ascending 100 mt towards the mountain up a slope of 30^0 inclination is found to be 60^0. The height of the mountain is



Sol: Height of mountain MN = h mts A,B are points of observation. Angle of elevation at A=45^0 and at B=60^0. Let AB = 100 mt

$\underline{|XBA} = \underline{|BAM} = 30^0$, $\underline{|NBX} = 180^0 - 60^0 = 120^0$

$\therefore \underline{|NBA} = 120^0 + 30^0 = 150^0$ and

$\underline{|BNA} = 180^0 - (150^0 + 15^0) = 15^0$

From $\Delta MNA, \text{Sin}45^0 = \frac{h}{AN} \Rightarrow AN = \sqrt{2}h$

From ΔNBA, $\frac{AN}{\sin 150^0} = \frac{AB}{\sin 15^0} \Rightarrow$

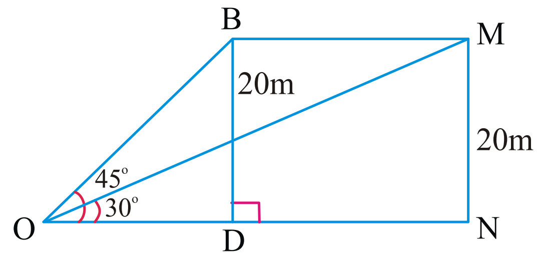
$$\sqrt{2}h = \frac{100 \times \sin 150^0}{\sin 15^0} \Rightarrow h = \frac{100}{\sqrt{3}-1}$$

$$= 50\left(\sqrt{3}+1\right) mts$$

Eg 5:

A bird is perched on the top of a tree 20 mts high and its elevation from the point on the ground is 45^0. It flies off horizontally straight away from the observer and in one second the elevation of the bird is reduced to 30^0. The speed of the bird is (JEE-MAINS 2014)

Sol:



Let the bird flight at B, the top of the tree BD, and 'O' be the observer. Then $\underline{|BOD} = 45^0$ and BD = 20 mts. Now the bird flying horizontally reaches M in 1 second.

$\underline{|MON} = 30^0$ where MN perpendicular to ON

Now BD = MN = 20 mts. From ΔBOD,

$\text{Tan}45^0 = \frac{BD}{OD} = \frac{20}{OD} \Rightarrow OD = 20mts$ from

ΔMON, $Tan30^0 = \frac{MN}{ON} = \frac{20}{20+DN}$

$\Rightarrow DN = 20\left(\sqrt{3}-1\right)$

= 20(0.732) = 14.64 mts = BM

Speed of bird = Distance/Time = 14.64 m/s

EXERCISE - I

1. The angle of elevation of the top of a flag - staff when observed from a point, distance 60 metres from its foot is 30^0 the height of the flag-staff in metres is

1) $20\sqrt{3}$ 2) $10\sqrt{3}$ 3) $60\sqrt{3}$ 4) $30\sqrt{3}$

2. From the top of a tree, a man observes the angle of depression of a point which is at a distance of 40 metres from the foot is 75^0. The height of the tree is

1) $40\sqrt{3}\,mts$ 2) $21\sqrt{3}\,mts$

3) $40\left(2+\sqrt{3}\right)mts$ 4) $3\sqrt{21}\,mts$

3. The angle of elevation of the top of a tower from a point on the same level as foot of tower is 15^0 if the point is at a distance of $6\left(2+\sqrt{3}\right)$ metres from the foot of tower then height of tower is ---

1) 6 mts 2) $6\left(2-\sqrt{3}\right)mts$

3) $12\left(2-\sqrt{3}\right)mts$ 4) $10\left(2-\sqrt{3}\right)mts$

4. The tops of two poles of heights 24 mt and 20 mt are connected by wire. If the wire makes an angle 45^0 with the horizontal, then the length of wire is

1) $8\sqrt{3}\,mt$ 2) $8\sqrt{2}\,mt$

3) $8\sqrt{5}\,mt$ 4) $4\sqrt{2}\,mts$

5. From a point on the level ground, the angle of elevation of the top of a pole is 30^0 on moving 20 metres nearer, the angle of elevation is 45^0. Height of the pole in metres is ----

1) $10\left(\sqrt{3}+1\right)$ 2) $10\left(\sqrt{3}-1\right)$

3) 20 4) 15

6. The angle of elevation of an electric pole from a point A on the ground is 60^0 and from a Point B towards the pole on the line joining the foot of the pole to the point A, is 75^0. If the distance AB = a, then the height of the pole is

1) $\frac{a(3+2\sqrt{3})}{2}$ 2) $a(4+2\sqrt{3})$

3) $\frac{a(2+\sqrt{3})}{2}$ 4) $\frac{a(2\sqrt{3}-3)}{2}$

7. A man standing on a level plane observes the anlge of elevation of top of pole to be $'\alpha'$. He walks a distance equal to double the height of pole towards it and finds that the elevation is 2α then $\alpha =$

1) $\frac{\pi}{12}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{6}$ 4) $\frac{\pi}{3}$

8. The shadow of a tower standing on a level plane is found to be 60 mt longer when the sun's altitude is 30^0 than when it is 45^0. Then the height of the tower in metres.

1) 30m 2) $60\sqrt{3}\,m$

3) $30\left(\sqrt{3}+1\right)m$ 4) 60 m

9. From the top of a light house the angle of depression of boats on opposite sides of the light house observed to be 30^0 and 45^0 if the distance between the boats is 20 metres then the height of the light house is

1) $10\left(\sqrt{3}-1\right)m$ 2) $20\left(\sqrt{3}-1\right)m$

3) $20\left(\sqrt{3}+1\right)m$ 4) $10\left(\sqrt{3}+1\right)m$

10. If the angle of elevation of the top of a tower from a point is 60^0 and 40 metres vertically above this point the angle of elevation is 45^0. The height of the tower in metres is

1) 64.64 2) 94.64 3) 54.64 4) 74.64

11. Two pillars of equal height stand at a distance of 100 metres. At a point between them the elevation of their tops are found to be 30^0 and 60^0. Then height of the each pillar in metres is

1) $25\sqrt{3}$ 2) $20\sqrt{3}$ 3) $50\sqrt{3}$ 4) $35\sqrt{3}$

12. There are two stations P,Q due north, due south of a tower of height 15 metres. The angle of depression of P and Q as seen from top a tower are $\cot^{-1}\frac{12}{5}, Sin^{-1}\frac{3}{5}$. The distance between P and Q is -----

1) 48 2) 56 3) 65 4) 25

13. If from the top of a tower of 60 metre heigh, the angles of depression of the top and floor of a house are α and β respectively and if the height of the house is $\frac{60\sin(\beta-\alpha)}{x}$, then $x =$

1) $\sin\alpha\sin\beta$ 2) $\cos\alpha\cos\beta$
3) $\sin\alpha\cos\beta$ 4) $\cos\alpha\sin\beta$

14. Two towers are standing on a level ground. From a point on the ground mid-way between them, the angles of elevation of their tops are 60^0 and 30^0 respectively. If the height of the first tower is 100 metres, the height of the second tower is

1) 5/3 2) 100/3 3) 80/3 4) 135/3

15. The upper 3/4th portion of a vertical pole subtends an angle $\tan^{-1}$(3/5) at a point in the horizontal plane through its foot and at a distance 40m from the foot. A possible height of the vertical pole is

1) 40 m 2) 60 m 3) 80 m 4) 20 m

16. A flag staff of height 10 metres is placed on the top of a tower of height 30 metres. At the top of a tower of height 40 metres, the flag staff and the tower subtend equal angles then the distance between the two towers in metres is

1) $10\sqrt{2}$ 2) $20\sqrt{2}$ 3) $30\sqrt{2}$ 4) $40\sqrt{2}$

17. From a point at a height of 27 metres above a lake the angle of elevation of the top of a tree on opposite side is 30^0 and the angle of depression of the image is 45^0. The height of the tree from water level is -- mts

1) $10\left(2+\sqrt{3}\right)$ 2) $10\left(2-\sqrt{3}\right)$
3) $27\left(2-\sqrt{3}\right)$ 4) $27\left(2+\sqrt{3}\right)$

18. The angle of elevation of the top of the tower is 45^0 on walking up a slope inclined at an angle of 30^0 to the horizontal a distance 20 metres, the angle of elevation of top of tower is observed to be 60^0. The height of the tower is

1) $10\left(\sqrt{3}+1\right)m$ 2) $20\left(\sqrt{3}+1\right)m$
3) $100\sqrt{3}\,m$ 4) $50\left(3+\sqrt{3}\right)m$

19. An aeroplane flying horizontally 1 km above the ground is observed at an elevation of 60^0. After 10 seconds the elevation is observed to be 30^0 then the uniform speed per hour of the aeroplane is

1) $235\sqrt{5}\,km$ 2) $235\sqrt{3}\,km$
3) $240\sqrt{3}\,km$ 4) $240\sqrt{2}\,km$

20. Two pillars are 120 ft a part and the height of one is double that of the other. From the middle point of the line joining their feet, an observer finds that the angular elevations of their tops are complementary. The height of the longer tower is ------- feet.

1) $35\sqrt{2}$ 2) $60\sqrt{2}$ 3) $50\sqrt{2}$ 4) $40\sqrt{2}$

21. A ballon is observed simultaneously from the 3 points A,B,C on a straight road directly beneath it. The angles elevation at B is twice that at A and the angular elevation on at C is thrice that at A. If AB = 20, BC = 40 then the height of the ballon is

1) $3\sqrt{15}$ 2) $5\sqrt{15}$ 3) $8\sqrt{15}$ 4) $15\sqrt{5}$

22. The height of a hill is 3300 mts. From the point P on the ground the angle of elevation of the top of the hill is 60^0. A balloon is moving with constant speed vertically upwards from P. After 5 minutes of its movement a person sitting in it observes the angle of elevation of the top of hill as 30^0. The speed of balloon is

1) 2.64 km/hr 2) 26.4 km/hr
3) 22.4 km/hr 4) 2.24 km/hr

KEY

01) 1	02) 3	03) 1	04) 4	05) 1	06) 1
07) 1	08) 3	09) 1	10) 2	11) 1	12) 2
13) 4	14) 2	15) 1	16) 1	17) 4	18) 1
19) 3	20) 2	21) 2	22) 2		

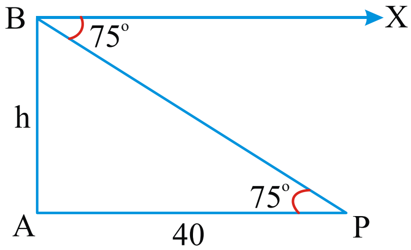
SOLUTIONS

1. Height of tower AB=h mts $\tan 30^0 = \dfrac{h}{60}$

   $\Rightarrow h = 60 \times \tan 30$

2. B is the top of tree AB of height 'h' mts

   From $\Delta APB, Tan75^0 = \dfrac{h}{40}$

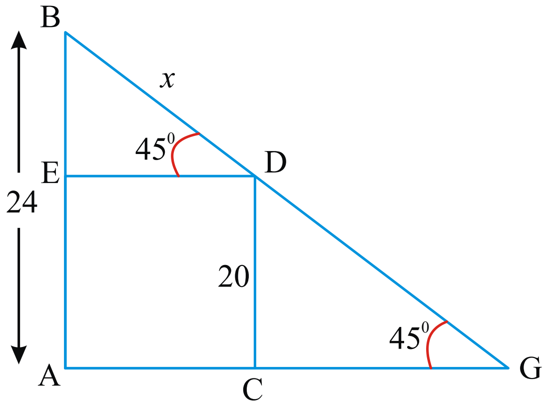


3. Height of the tower AB = h mts

   $Tan15^0 = 2 - \sqrt{3}$

4. BE = 24- 20 = 4 mts

   from ΔBED, $Sin45^0 = \dfrac{4}{BD} = \dfrac{4}{x}$

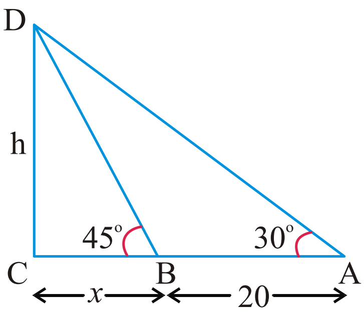


5. As in standard model (1) d = 20,

   $\alpha = 30^0, \beta = 45^0$

   Height of pole CD =

   $\dfrac{d}{\cot\alpha - \cot\beta} = \dfrac{20}{\cot 30^0 - \cot 45^0} = 10\left(\sqrt{3}+1\right)$

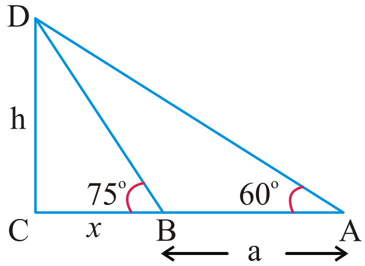


6. Height of the pole CD= h mts, AB=a mts

   From ΔACD, $Cot60^0 = \dfrac{x+a}{h}$

   From ΔBCD, $Cot75^0 = \dfrac{x}{h}$

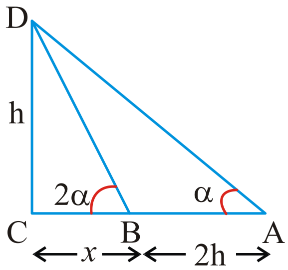
   $\Rightarrow h(Cot60^0 - Cot75^0) = a$



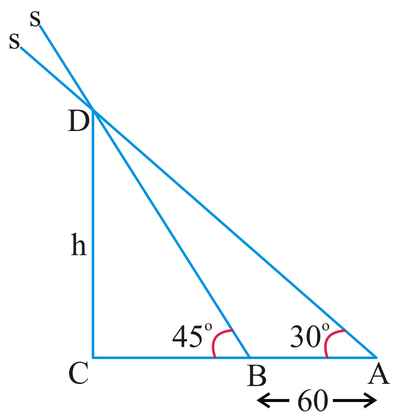
7. Height of pole CD = h mts, $\alpha = \alpha, \beta = 2\alpha$

   d = AB = 2h mts

   height of pole $h = \dfrac{d \sin\alpha \sin\beta}{\sin(\beta - \alpha)}$



8. Let S be position of sun length shadow BC=x mts, when sun altitude is 45^0. Length of shadow CA=(x+60) mts ,when sun altitude is 30^0. d=AB=60 mts

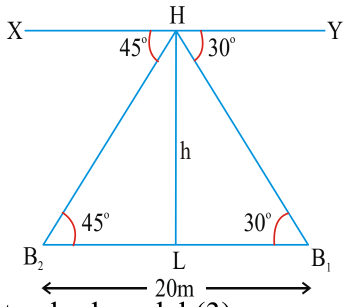


$\Rightarrow h = \dfrac{d}{Cot\alpha - Cot\beta} = \dfrac{60}{\sqrt{3}-1} = 30(\sqrt{3}+1) mts$

9. From $\Delta HB_2L, Cot45^0 = \dfrac{B_2L}{h} \Rightarrow B_2L = h(1) = h$

From $\Delta HB_1L, Cot30^0 = \dfrac{B_1L}{h} \Rightarrow B_1L = h\sqrt{3}$

$$B_1B_2 = 20 = h\left(\sqrt{3}+1\right) \Rightarrow h = \frac{20}{\sqrt{3}+1} = 10\left(\sqrt{3}-1\right) mts$$



10. As in standard model (3)

d = 40 mts , $\alpha = 45^0, \beta = 60^0$

Height of tower = $h = \dfrac{d\cot\alpha}{\cot\alpha - \cot\beta}$

$$\Rightarrow \frac{40\cot 45^0}{\cot 45^0 - \cot 60^0} = \frac{40\sqrt{3}}{\sqrt{3}-1} = 20\sqrt{3}\left(\sqrt{3}+1\right)$$

$$= 20\left(30+\sqrt{3}\right) = 94.64 \text{ mts}$$

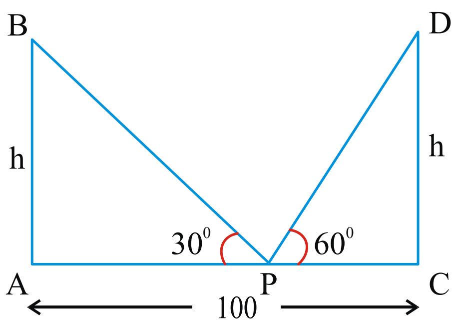
$$\Rightarrow 100 = h\left(\sqrt{3}+\frac{1}{\sqrt{3}}\right) = h\left(\frac{4}{\sqrt{3}}\right)$$

11. Let B and D be tops of pillars AB and CD of eqaul height h mts. AC = 100 mts.

$$AC = AP + PC = h\cot 30^0 + h\cot 60^0$$

$$\Rightarrow 100 = h\left(\sqrt{3}+\frac{1}{\sqrt{3}}\right) = h\left(\frac{4}{\sqrt{3}}\right)$$

$$\Rightarrow h = 25\sqrt{3}\, mts$$



12. If height of tower AT=15 mts . Angles of depression of P and Q from T are respectively α and β.

$\underline{|XTP} = \alpha = \underline{|TPA}$

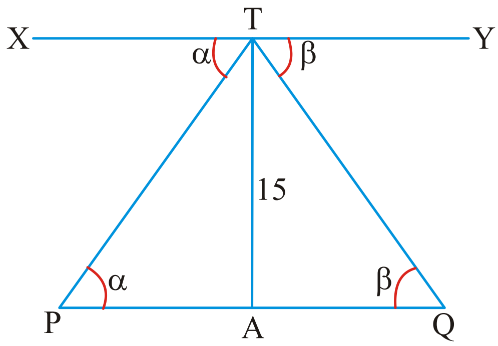
$\underline{|YTQ} = \beta = \underline{|PQT}$

$$\alpha = Cot^{-1}\frac{12}{15} \Rightarrow Cot\alpha = \frac{12}{5},$$

$$\beta = Sin^{-1}\frac{3}{5} \Rightarrow Sin\beta = \frac{3}{5}$$

$$Cot\alpha = \frac{PA}{15} \Rightarrow PA = 36\, mts$$

$$Sin\beta = \frac{3}{5} \Rightarrow Cot\beta = \frac{4}{3} = \frac{AQ}{15} \Rightarrow AQ = 20\, mts$$



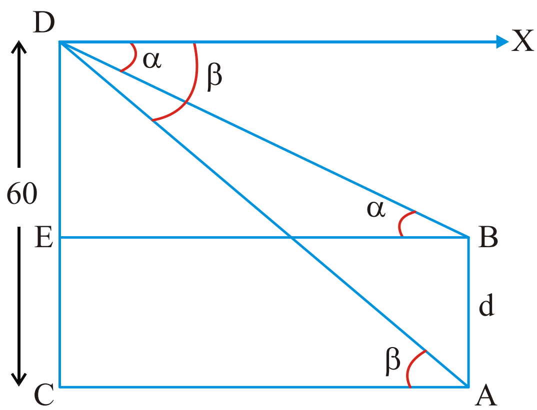
Distance PQ =PA+AQ=36+20=56 mts

13. D is the point of observation

Height of house=AB=x mts =d

Height of tower CD=60=$\dfrac{dCot\alpha}{Cot\alpha - Cot\beta}$

$$\Rightarrow d = \frac{60(Cot\alpha - Cot\beta)}{Cot\alpha} = \frac{60\,Sin(\beta-\alpha)}{Sin\beta Cos\alpha}$$



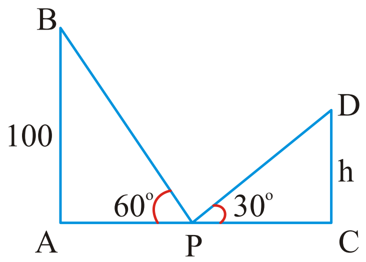
Thus $x = Sin\beta Cos\alpha$

14. P is mid point of BC Height of first tower=100 mts,

Height of second tower = h mts

From $\Delta APD; Cot60^0 = \dfrac{AP}{100}$

From $\Delta PCD; Cot30^0 = \dfrac{PC}{h}$



$AP = PC \Rightarrow 100Cot60^0 = hCot30^0 \Rightarrow \dfrac{100}{3} = h$

15. From ΔAPC; $Tan(\alpha+\beta) = \dfrac{h}{40} \to (1)$

From $\Delta APB; Tan\alpha = \dfrac{h/4}{40} = \dfrac{h}{160} \to (2)$

given $\beta = Tan^{-1}\dfrac{3}{5} \Rightarrow Tan\beta = \dfrac{3}{5} \to (3)$

From (1),(2)&(3) $Tan\beta = Tan((\alpha+\beta)-\alpha)$

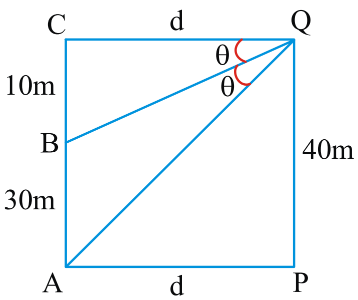
$\Rightarrow \dfrac{3}{5} = \dfrac{\dfrac{h}{40} - \dfrac{h}{160}}{1+\dfrac{h}{40}\cdot\dfrac{h}{160}} \Rightarrow \dfrac{3}{5} = \dfrac{120h}{6400+h^2}$, find h.

16. Length of flag staff BC = 10 mts

Height of tower AB = 30 mts

Height of tower PQ = 40 mts.

Distance between towers AP = d



$\underline{|AQB} = \underline{|BQC} = \theta$

$Tan\theta = \dfrac{10}{d}, \tan 2\theta = \dfrac{40}{d}$ *and* $tan2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$d = 10\sqrt{2}\ mts$

17. As in standard formula (4)

d = 27 mts, $\alpha = 30^0, \beta = 45^0$

Height of the tree above water level =

$$h = d\left(\frac{\cot\alpha+\cot\beta}{\cot\alpha-\cot\beta}\right)$$

$$\Rightarrow h = 27\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) = \frac{27(4+2\sqrt{3})}{2} = 27(2+\sqrt{3})m$$

18. As in standard formula (5)

a = 20 mts, $\alpha = 45^0, \beta = 60^0, \gamma = 30^0$

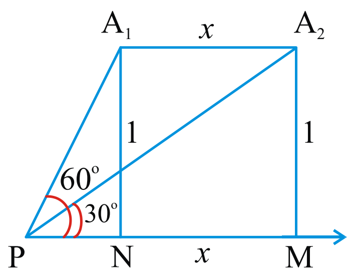
Height of the tower

$$h = \frac{a\sin\alpha\sin(\beta-\gamma)}{\sin(\beta-\alpha)} = \frac{20\sin 45^0.\sin 30^0}{\sin 15^0}$$

$$h = 10(\sqrt{3}+1)mts$$

19. From $\Delta A_1PN; Cot60^0 = \dfrac{PN}{1} \Rightarrow PN = \dfrac{1}{\sqrt{3}}kms$

From $\Delta A_2PM; Cot30^0 = \dfrac{PM}{1} \Rightarrow PM = \sqrt{3}kms$



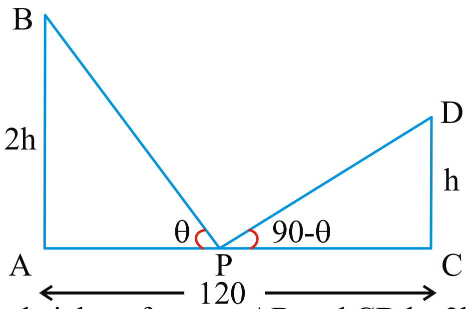
Distance travelled in 10 sec

$$= \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}kms$$

Speed of aeroplane =

$$\frac{2}{\sqrt{3}} \times 360 = 2\times 120\sqrt{3} = 240\sqrt{3}kmph$$

20.



Let heights of towers AB and CD be 2h and h respectively.

Let P be midpoint of AC AP = PC = 60 ft

At P, $\underline{|APB} = \theta \text{ and } \underline{|CPD} = 90-\theta$

$Cot\theta = \frac{AP}{2h}, Cot(90-\theta) = \frac{PC}{h} = \frac{AP}{h}$

$\Rightarrow Cot\theta.Cot(90-\theta) = \frac{AP}{2h}.\frac{AP}{h}$

$\Rightarrow 1 = \frac{60.60}{2h^2} \Rightarrow h^2 = 1800 \Rightarrow h = 30\sqrt{2}$

$\Rightarrow$ Height of longer tower = 2h = $60\sqrt{2}$ ft

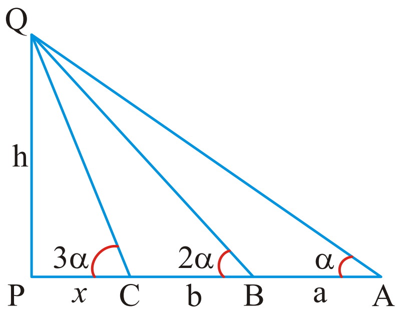
21. As in standard formula (6)

AB = a = 20, BC = b = 40

Height of the ballon is

$h = \frac{a}{2b}\sqrt{(3b-a)(a+b)}$

$h = \frac{1}{4}\sqrt{100\times 60} = \frac{10}{4}(2\sqrt{15}) = 5\sqrt{15}$ mts



22. Distance travelled in 5 minutes = PQ = 5 mts

Height of hill AB = 3,300 mts

from $\Delta ABP, \cot 60^0 = \frac{AP}{3300}$

$\Rightarrow AP = 3300\times\frac{1}{\sqrt{3}} = 1100\sqrt{3}$

$\Delta BPX, \cot 30^0 = \sqrt{3}$

$= \frac{BX}{3300-x} = (3300-x)\sqrt{3} = BX$

$\therefore 1100\sqrt{3} = (3300-x)\sqrt{3} \Rightarrow x = 2200\,mts$

speed of the ballon

$= \frac{2200}{5}\times 60 = 2200\times 12\,mt/hour$

$= \frac{2200\times 12}{1000} = 26.4\,km/hr$

EXERCISE - II

1. At a particular instant the height of the tower is equl to the length of its shadow after some time the length of the shadow is $\sqrt{3}$ times of the height of the tower, then the time lapsed between the two observations in hours is

1)1 2) $\frac{1}{2}$ 3) $\frac{1}{4}$ 4) 24

2. A tower of height 50 metres stands on a level ground. A flag-staff standing on the tower subtends an angle of $\text{Tan}^{-1}\left(\frac{1}{3}\right)$ at a point 100 metres away from the tower on the ground. The length of the flag-staff in metres is

1) 50 2) 75 3) 100 4) 125

3. At a certain point the angle of elevation of a tower is found to be $Cot^{-1}\left(\frac{3}{5}\right)$. On walking 32 metres directly towards the tower its angle of elevation is $Cot^{-1}\left(\frac{2}{5}\right)$. The height of the tower in metres is

1) 32 2) 160 3) 320 4) 340

4. A pole of height h stands at one corner of a park in the shape of an equilateral triangle. If α is the angle which the pole subtends at the midpoint of the opposite side, the length of each side of the park is

1) $\left(\frac{\sqrt{3}}{2}\right)h\cot\alpha$ 2) $\left(\frac{2}{\sqrt{3}}\right)h\cot\alpha$

3) $\left(\frac{\sqrt{3}}{2}\right)h\tan\alpha$ 4) $\left(\frac{2}{\sqrt{3}}\right)h\tan\alpha$

5. **Three vertical poles of heights h_1, h_2 and h_3 at the vertices A, B and C of a ΔABC subtend angles α, β and γ respectively at the cicumcentre of triangle. If $\cot\alpha, \cot\beta$ and $\cot\gamma$ are in A.P. then h_1, h_2, h_3 are in**

1) A.P. 2) G.P. 3) H.P. 4) A.G.P.

6. **At a point, the angle of elevation of top of a tower is found to be $\tan^{-1}\left(\frac{5}{12}\right)$ on walking 240 metres nearer the tower, the elevation is found to be $\tan^{-1}\left(\frac{3}{4}\right)$. The height of the tower in metres is**

1) 175 2) 225 3) 275 4) 300

7. **A flag-staff 20 metres long standing on a wall 10 metres high subtends an angle whose tangent is 0.5 at a point on the ground. If θ is the angle subtended by wall at that point then $\tan\theta =$**

1) 1 only 2) 1/3 only 3) 1 or 1/3 4) 2

8. **ABC is a triangular park with AB=AC=100cm A clock tower is situated at the midpoint of BC. The angles of elevation of the top of the tower at A and B are $\mathrm{Cot}^{-1}3.2$ and $\mathrm{Cosec}^{-1}2.6$ The height of the tower is**

1) 25 mt 2) 50 mt 3) 100 mt 4) $50\sqrt{2}mt$

9. **A tower stands at the top of a hill whose height is three times the height of the tower. The tower is found to subtend an angle of $\mathrm{Tan}^{-1}\left(\frac{1}{7}\right)$ at a point 2 km away on the horizontal throught the foot of the hill. Then the height of the tower is**

1) $\frac{1}{2}km$ or $\frac{1}{3}km$ 2) $\frac{1}{3}km$ or $\frac{2}{3}km$

3) $\frac{2}{3}km$ or $\frac{1}{2}km$ 4) $\frac{3}{4}km$ or $\frac{1}{2}km$

10. **A sphere of radius 'a' subtends an angle 60^0 at a point P. Then the distance of P from the centre of the sphere is.**

1) $\frac{a}{\sqrt{3}}$ 2) 2a 3) $\frac{a\sqrt{3}}{2}$ 4) $\frac{2a}{\sqrt{3}}$

11. **A person in a balloon, who has ascended vertically from flat land at the sea level, observes the angle of depression of a ship at anchor to be 30^0. After descending vertically 600 metres, he finds the angle of depression to be 15^0. The horizontal distance of the ship from the foot of ascent in metres is**

1) $300\left(3+\sqrt{3}\right)$ 2) $300\left(3+2\sqrt{3}\right)$

3) $150\left(3+2\sqrt{3}\right)$ 4) $150\left(3-2\sqrt{3}\right)$

12. **The upper $\frac{3}{4}$th portion of a vertical pole subtends an angle $\tan^{-1}\left(\frac{3}{5}\right)$ at the point in the horizontal plane through its foot. The tangent of the angle subtended by the pole at the same point is**

1) 1 or 2 2) 2 or 3 3) 3 or 4 4) 4 or 1

13. **A vertical tower stands on a declivity which is inclined at 15^0 to the horizon. From the foot of the tower a man ascends the declivity from 80feet and then finds that the tower subtends an angle of 30^0. The height of the tower is**

1) $40\left(\sqrt{6}+\sqrt{2}\right)$ 2) $20\left(\sqrt{6}-\sqrt{2}\right)$

3) $40\left(\sqrt{6}-\sqrt{2}\right)$ 4) $10\left(\sqrt{6}-\sqrt{2}\right)$

14. **A tower leans towards west making an angle α with the vertical. The angular elevation of B, the top most point of the tower is β as observed from a point C due east of A at a distance d from A. If the angular elevation of B from a point due east of C at a distance 2d from C is γ, then $2\tan\alpha$ can be written as**

1) $3\cot\beta - 2\cot\gamma$ 2) $3\cot\gamma - 2\cot\beta$

3) $3\cot\beta - \cot\gamma$ 4) $\cot\beta - 3\cot\gamma$

15. A flag-staff stands on a tower which is on level ground. The total height of the flag-staff and tower taken together is 300 metres. The flag-staff subtends an angle of $\mathrm{Tan}^{-1}\left(\frac{1}{5}\right)$ at a point P on the level ground at a distance 300 metres from the foot of the tower. The heightof the tower is

1) 100 metres 2) 200 metres
3) 250 metres 4) 300 metres

KEY

01) 1	02) 1	03) 2	04) 2	05) 3	06) 2
07) 3	08) 1	09) 3	10) 2	11) 1	12) 4
13) 3	14) 3	15) 2			

SOLUTIONS

1. The difference in altitudes of sun $= 15^0$

$360^0 = 24$H there fore for a lapse of 15^0 time taken = 1 Hr

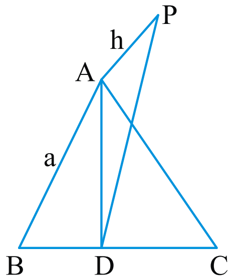
2. $Tan\alpha = \frac{1}{3}$, $\quad Tan\beta = \frac{50}{100} = \frac{1}{2}$,

$Tan(\alpha+\beta) = \frac{h+50}{100}$ apply $Tan(\alpha+\beta)$

3. $\alpha = Cot^{-1}\frac{3}{5} \Rightarrow Cot\alpha = \frac{3}{5}$, $\quad \beta = Cot^{-1}\frac{2}{5}$

As in standard model (1) $\quad h = \dfrac{d}{\cot\alpha - \cot\beta}$

4.



Let ABC be the triangular park, AP be the pole at A,D be the midpoint of BC, Let each side of the equalilateral triangle ABC be 'a' then

$AD^2 = AB^2 - BD^2 = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}$

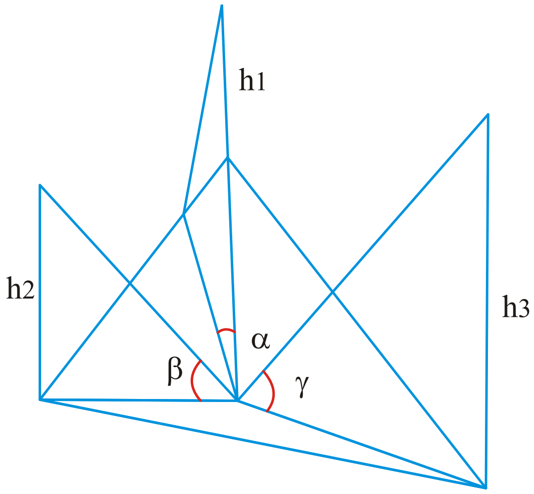
$\Rightarrow AD = \frac{\sqrt{3}}{2}a$

and since AP = h and $\underline{|ADP} = \alpha$ we have

$AD = h\cot\alpha$

$\therefore \frac{\sqrt{3}}{2}a = h\cot\alpha \Rightarrow a = \frac{2}{\sqrt{3}}h\cot\alpha$

5.



Since $\frac{R}{h_1} = \cot\alpha$ *similerly* $\frac{R}{h_2} = \cot\beta$

and $\frac{R}{h_3} = \cot\gamma$ where 'R' is the circumradius

since $\cot\alpha, \cot\beta, \cot\gamma$ are in A.P

$\Rightarrow \frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}$ are in A.P and h_1, h_2, h_3 are in H.P

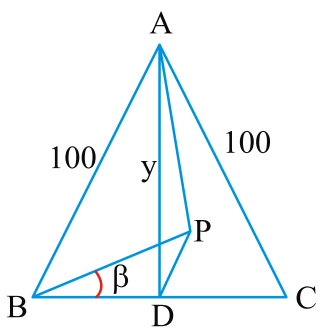
6. $\alpha = \mathrm{Tan}^{-1}\frac{5}{12}$, $\quad \beta = \mathrm{Tan}^{-1}\frac{3}{4}$

d = 240. Height of tower $h = \dfrac{d}{\cot\alpha - \cot\beta}$

7. $Tan\alpha = 0.5 = \frac{1}{2}$.........(1), $\quad Tan\theta = \frac{10}{d}$........(2)

$Tan(\alpha+\theta) = \frac{30}{d}$, Apply $Tan(\alpha+\theta)$ formula

8.



$y^2 = AD^2 = 100^2 - BD^2 = 10000 - x^2$

$\cot^{-1}(3.2)=\alpha, \operatorname{cosec}^{-1}(2.6)=\beta$

$\cot\alpha=(3.2)\ \&\ \operatorname{cosec}\beta=(2.6)$

$\cot\alpha=\dfrac{y}{h}\Rightarrow y=h\cot\alpha=(3.2)h$

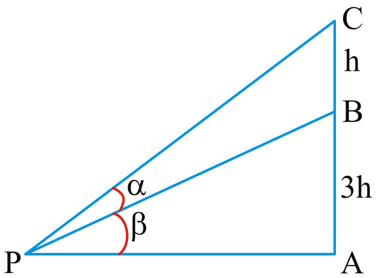
$\cot\beta=\dfrac{x}{h}\Rightarrow x=h\cot\beta$

$x=h\sqrt{(2.6)^2-1}=2.4h$

$x^2+y^2=100^2\Rightarrow h^2(2.4)^2+h^2(3.2)^2=100^2$

$\Rightarrow h^2(16)=100^2\Rightarrow h=25$

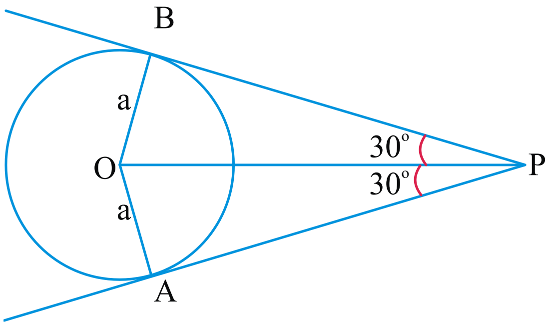
9.



$\tan\alpha=\dfrac{1}{7};\tan\beta=\dfrac{3h}{2},\ \tan(\alpha+\beta)=\dfrac{4h}{2}=2h$

$\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=2h$, find 'h'

10.



$\operatorname{cosec}30^0=\dfrac{OP}{OB}$, OP = 2a

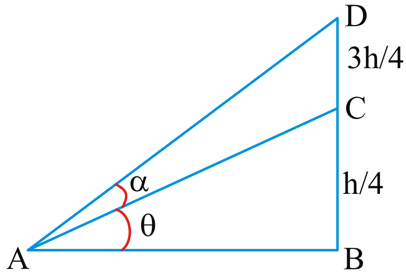
11. $Tan30^0=\dfrac{600+x}{d}$

$x=d(\tan 30)-600----(1)$

$\tan 15^0=\dfrac{x}{d},\quad x=d\tan 15^0----(2)$

from (1) and (2) find 'd'

12.



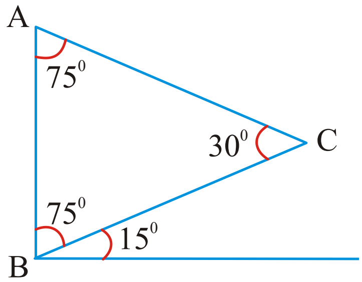
$\alpha=Tan^{-1}\left(\dfrac{3}{5}\right),\quad Tan\theta=\dfrac{h}{4x}$

$\tan(\theta+\alpha)=\dfrac{h}{x}$

$\dfrac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}=\dfrac{4x\tan\theta}{x}=4\tan\theta$

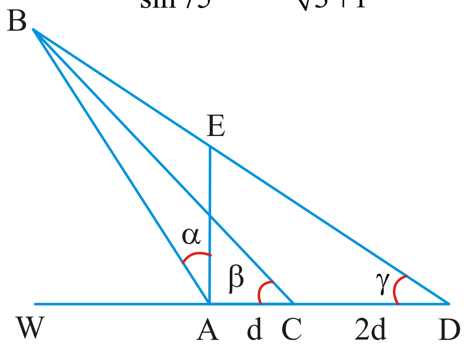
13. Let BC be the declivity and BA be the tower. Using sine rule in ΔABC,

we have $\dfrac{BC}{\sin 75^0}=\dfrac{AB}{\sin 30^0}$



$\Rightarrow AB=\dfrac{80\sin 30^0}{\sin 75^0}=\dfrac{40\times 2\sqrt{2}}{\sqrt{3}+1}=40\left(\sqrt{6}-\sqrt{2}\right)$

14.



$(d+2d)\cot\beta=d\cot\gamma-2d\cot\left(\dfrac{\pi}{2}+\alpha\right)$

$\Rightarrow 3\cot\beta=\cot\gamma+2\tan\alpha$

$\therefore 2\tan\alpha=3\cot\beta-\cot\gamma$

15. $Tan\alpha=\dfrac{x}{300};Tan(\alpha+\beta)=1$

$\dfrac{\dfrac{x}{300}+\dfrac{1}{5}}{1-\dfrac{x}{1500}}=1; x=200\,mt$

EXERCISE - III

1. AB is a vertical pole, The end A is on the level ground. C is the mid point of AB. P is the point on the level ground. The portion CB subtends an angle β at P. If AP = n.AB then tan β =

1) $\frac{n}{n^2+1}$ 2) $\frac{n}{2n^2+1}$

3) $\frac{n}{n^2-1}$ 4) $\frac{n}{2n^2-1}$

2. A ladder rests against wall at an angle α to the horizontal. Its foot is pulled away from the wall through a distance 'a' so that it slides a distance 'b' down the wall making an angle β with the horizontal, then tan

$\left(\frac{\alpha+\beta}{2}\right)$ =

1) b/a 2) a/b 3) 2/ab 4) 2a/b

3. The angle of elevation of the top of a tower standing on a horizontal plane from the two points lying on a line passing through the foot of the tower at distances a and b respectively are complementary angles. if the line joining the two points sutends an angle θ at the top of the tower, then sin θ is

1) $\frac{a+b}{a-b}$ 2) $\frac{a-b}{a+b}$ 3) $\frac{b-a}{a+b}$ 4) $\frac{a}{b}$

4. A flag staff of the height (a-b) stands on the top of a tower subtends the same angle at the point on the horizontal plane through the foot of the tower which are at distant a and b from the tower. The height of the tower is

1)b 2)a+b 3)a 4)a-b

5. The angle of elevation of a cloud from a point h meters above a lake is θ. The angle of depression of its reflection in a lake is 45^0. The height of the cloud is

1)h tan (45+θ) 2)h cot (45+θ)

3)h tan (45-θ) 4)h tan θ

6. An obsercver finds that the angular elevation of a tower is θ. On advancing a metres towards the tower the elevation is 45^0 and an advancing 'b' metres nearer the elevation is 90^0-θ then the height of the tower in metres is

1) $\frac{ab}{a+b}$ 2) $\frac{ab}{a-b}$ 3) $\frac{2ab}{a+b}$ 4) $\frac{2ab}{a-b}$

7. A vertical pole subtends an angle $\tan^{-1}\frac{1}{2}$ at a point P on the ground. The angle subtended by the upper half the pole at p is

1)$\tan^{-1}\frac{1}{4}$ 2) $\tan^{-1}\frac{1}{8}$ 3)$\tan^{-1}\frac{2}{3}$ 4)$\tan^{-1}\frac{2}{9}$

8. A tower MPQ surmounted by a spiral QR stands on a horizontal plane. At the extremity 'A' of a horizontal line AM it is found that MP and QR subtend equal angles. if MP = 3 m ,PQ = 28 m and QR = 5 m then MA =

1) $\sqrt{36\times93}$ 2) $\sqrt{18\times93}$

3) $\sqrt{34\times36}$ 4) $\sqrt{34\times93}$

9. The length of the shadow of vertical pole of height h, thrown by the sun's rays at three different movements are h,2h,3h. The sum of the angles of elevation of rays at these three moments is equal to

1) $\frac{\pi}{2}$ 2) $\frac{\pi}{3}$ 3) $\frac{\pi}{4}$ 4) $\frac{\pi}{6}$

10. A vertical tower stands on a triangular field . Angle of elevation of the top of the tower from each of the vertex is θ. If the length of the sides of the field are 30 m, 50 m and 70 m. The height of the tower is

1) $70\sqrt{3}\ \tan\theta$ 2) $\frac{70}{\sqrt{3}}\tan\theta$

3) $\frac{50}{\sqrt{3}}\ \tan\theta$ 4) $75\sqrt{3}\ \tan\theta$

11. **The angle of elevation of a top of a tower from a point A due south of it is $\tan^{-1}(6)$ and that from B due to west of it is $\tan^{-1}(7.5)$. If h is the height of the tower and AB = λ h then $\lambda^2 =$**

1) $\frac{21}{700}$ 2) $\frac{42}{1300}$ 3) $\frac{41}{900}$ 4) $\frac{52}{1100}$

12. **The angular elevation of a tower OP at a point A due south of it is 60^0 and at a point B due to west of A, the evaluation is 30^0. If AB = 3 m , the height of the tower is**

1) $2\sqrt{3}m$ 2) $2\sqrt{6}m$ 3) $\frac{3\sqrt{3}}{2}m$ 4) $\frac{3\sqrt{6}}{4}m$

13. **A lamp post is sittuated at the middle point M of the side AC of a triangular plot ABC with BC = 7 m CA = 8 m and AB = 9 m. Lamp post subtends an angle 15^0 at the point B. The height of the lamp post is**

1) $7(2+\sqrt{3})$ m 2) $7(2-\sqrt{3})$ m
3) $14(2-\sqrt{3})$ m 4) $14(2+\sqrt{3})$ m

14. **A tower ABCD stands on a level ground with foot A. At a point P on the ground the portion AB,AC and AD subtends angles α, β, γ respectively. If AB=a, AC = b, AD= c, AP = x and $\alpha+\beta+\gamma=180^0$ then $(a+b+c)x^2=$**

1) abc 2) a + b + c
3) a + b – c 4) a - b - c

15. **A pole is slightly inclined towords the east. At two points due west of it at distance 'a' and b, the angles of elevation of the top of the pole are α and β respectively. The inclination of the pole to the horizon is**

1) $Tan^{-1}\left[\frac{a+b}{b\cot\alpha - a\cot\beta}\right]$

2) $Tan^{-1}\left[\frac{b-a}{b\cot\alpha - a\cot\beta}\right]$

3) $\cos^{-1}\left[\frac{a-b}{b\cot\alpha - a\cot\beta}\right]$

4) $\sin^{-1}\left[\frac{a-b}{b\cot\alpha - a\cot\beta}\right]$

16. **PQ is a vertical tower, P is the foot, Q the top of the tower, A, B, C are three points in the horizontal plane through P. The angles of elevation of Q from A, B, C are equal and each is equal to θ. The sides of the triangle ABC are a, b, c and the area of the triangle ABC is Δ, the height of the tower is**

1) $(abc)\tan\theta/4\Delta$ 2) $(abc)\cot\theta/4\Delta$
3) $(abc)\sin\theta/4\Delta$ 4) $(abc)\tan\theta/2\Delta$

17. **Two ships leave a port at the same time. One goes 24 Km per hour in the direction N 45^0 E and other travels 32 Km per hour in the direction S 75^0 E. The distance between the ships at the end of 3 hours is _______Km**

1) 86.4 2) 96.4 3) 66.8 4) 98.4

18. **On one side of a road of width 'd' meters there is a point of observation P at a height 'h' meters from ground. If a tree on the other side of the road, makes a right angle at P, height of the tree in meters is**

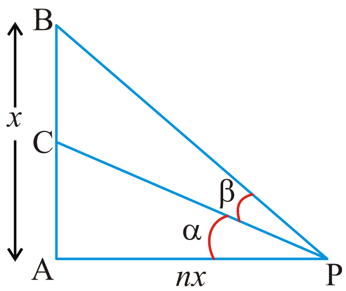
1) $\frac{h^2-d^2}{h}$ 2) $\frac{h^2+d^2}{h}$
3) $\frac{d^2-h^2}{h}$ 4) $\frac{2d^2+h^2}{h}$

KEY

01) 2	02) 2	03) 2	04) 1	05) 1	06) 2
07) 4	08) 2	09) 1	10) 2	11) 3	12) 4
13) 2	14) 1	15) 2	16) 1	17) 1	18) 2

SOLUTIONS

1.



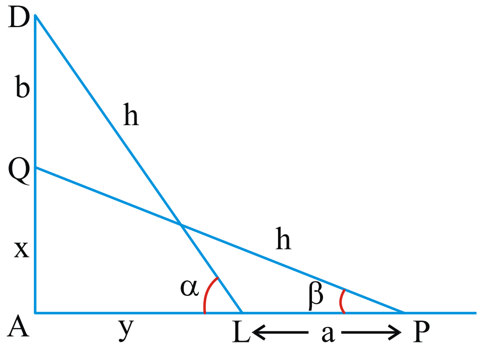
Let $AB = x \Rightarrow AP = n.AB = nx$

$\tan\alpha = \frac{x}{2nx} = \frac{1}{2n}$; $\tan(\alpha+\beta) = \frac{x}{nx} = \frac{1}{n}$

$\tan\beta = \tan((\alpha+\beta)-\alpha)$

$= \frac{\tan(\alpha+\beta)-\tan\alpha}{1+\tan(\alpha+\beta)\tan\alpha}$; simplify

2.



Length of ladder $LD = PQ = h$

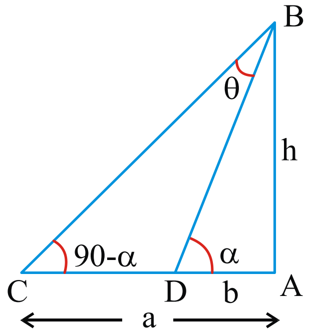
Let $AQ = x, AL = y, LP = a$ and $QD = b$

from ΔALD; $\sin\alpha = \dfrac{b+x}{h}, \cos\alpha = \dfrac{y}{h}$

from ΔAPQ; $\sin\beta = \dfrac{x}{h}, \cos\beta = \dfrac{y+a}{h}$

$$\frac{\cos\alpha - \cos\beta}{\sin\alpha - \sin\beta} = -\frac{a}{b} \Rightarrow \tan\left(\frac{\alpha+\beta}{2}\right) = \frac{a}{b}$$

3.



$$\tan\alpha = \frac{h}{b}, \tan(90-\alpha) = \cot\alpha = \frac{h}{a}$$

$$\tan\alpha\cot\alpha = 1 = \frac{h^2}{ab} \Rightarrow h = \sqrt{ab}$$

Thus $\tan\alpha = \dfrac{\sqrt{ab}}{b} = \sqrt{\dfrac{a}{b}}$

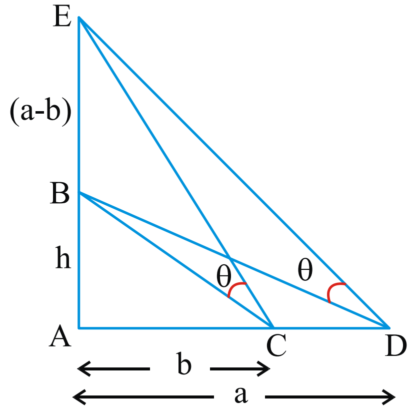
$$\alpha = (90-\alpha) + \theta \Rightarrow 2\alpha = 90^0 + \theta$$

$$\therefore \cos 2\alpha = \cos(90^0 + \theta) = -\sin\theta$$

$$\sin\theta = -\cos 2\alpha = -\left(\frac{1-\tan^2\alpha}{1+\tan^2\alpha}\right)$$

$$= -\left(\frac{1-\frac{a}{b}}{1+\frac{a}{b}}\right) = \frac{a-b}{a+b}$$

4.



$\underline{|BCE} = \theta = \underline{|BDE} \Rightarrow B, C, D, E$ are concyclic

$AB.BE = AC.CD$

$$\Rightarrow h(a-b) = b(a-b) \Rightarrow h = b$$

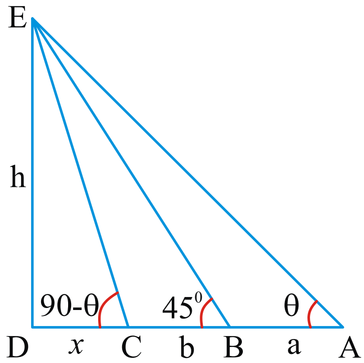
5. As in standard model (4):

$d = h, \alpha = \theta, \beta = 45^0$

$\Rightarrow$ height of cloud

$$= d\left(\frac{\tan\beta + \tan\alpha}{\tan\beta - \tan\alpha}\right) = h\tan(45^0 + \theta)$$

6.



Height of tower $DE = h$ mtrs.

Let $CD = x$ mtrs.

$$\tan\theta = \frac{h}{a+b+x} \Rightarrow a+b+x = h\cot\theta \text{....(1)}$$

$$\tan 45^0 = \frac{h}{b+x} \Rightarrow b+x = h \text{....(2)}$$

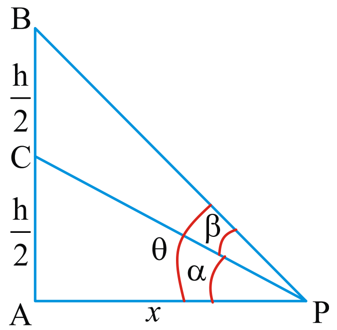
$$\tan(90-\theta) = \cot\theta = \frac{h}{x} \text{....(3)} \quad \Rightarrow x = h\tan\theta$$

from (1) and (2) $a + h = h\cot\theta$

from (2) and (3) $b = h - h\tan\theta$

$$\Rightarrow b = h - h\left(\frac{h}{a+h}\right) \qquad \Rightarrow b = \frac{ah + h^2 - h^2}{a+h}$$

$$\Rightarrow ab + bh = ah \quad \Rightarrow h = \frac{ab}{a-b}$$

7. 

Height of the pole $AB = h$

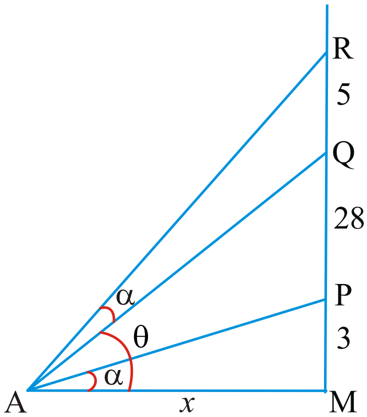
Let $\underline{|APB} = \theta$ given $\tan\theta = \frac{1}{2} \Rightarrow \frac{h}{x} = \frac{1}{2}$

$\underline{|CPA} = \alpha \Rightarrow \tan\alpha = \frac{h}{2x} = \frac{1}{4}$

$\theta = \alpha + \beta \Rightarrow \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{1}{2}$

$\Rightarrow 2\tan\alpha + 2\tan\beta = 1 - \tan\alpha\tan\beta$

$\frac{1}{2} + 2\tan\beta = 1 - \frac{1}{4}\tan\beta \quad \Rightarrow \tan\beta = \frac{2}{9}$

8. 

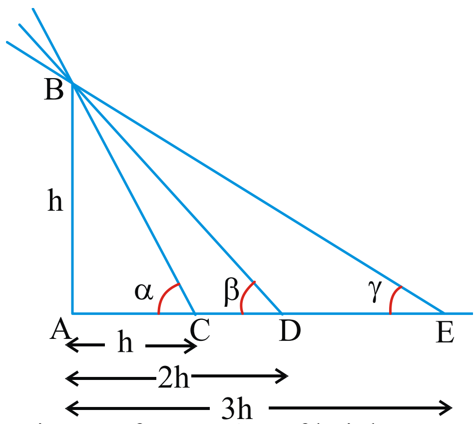
Let $\underline{|MAP} = \underline{|QAR} = \alpha$ and $\underline{|QAM} = \theta$

Let $AM = x$ from ΔAMR, $\tan(\theta + \alpha) = \frac{36}{x}$

$\tan\alpha = \frac{3}{x}, \tan\theta = \frac{31}{x}$

$\tan(\theta + \alpha) = \frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\theta}$

$\frac{36}{x} = \frac{\frac{3}{x} + \frac{31}{x}}{1 - \frac{3}{x}\frac{31}{x}} \Rightarrow x^2 = 18 \times 93$

9. 

B is top of tower AB of height h mtrs.

lengths of shadows are $h, 2h$ and $3h$ when sun altitude is α, β, γ

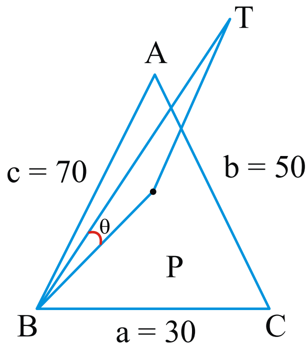
from ΔABC, $\tan\alpha = \frac{h}{h} = 1 \Rightarrow \alpha = \frac{\pi}{4}$

from ΔABD, $\tan\beta = \frac{h}{2h} = \frac{1}{2} \Rightarrow \beta = \tan^{-1}\frac{1}{2}$

from ΔABE, $\tan\gamma = \frac{h}{3h} = \frac{1}{3} \Rightarrow \gamma = \tan^{-1}\frac{1}{3}$

$$\alpha + \beta + \gamma = \frac{\pi}{4} + \tan^{-1}\left(\frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2}.\frac{1}{3}}\right)$$

$$= \frac{\pi}{4} + \tan^{-1}(1) = \frac{\pi}{2}$$

10. 

In ΔABC, $a = BC = 30$ mtrs.

$b = CA = 50$ mtrs. $C = AB = 70$ mtrs.

$s = \frac{30 + 50 + 70}{2} = 75$

$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{75(45)(25)(5)} = \sqrt{(25)^2(15)^2 3} = 375\sqrt{3}$

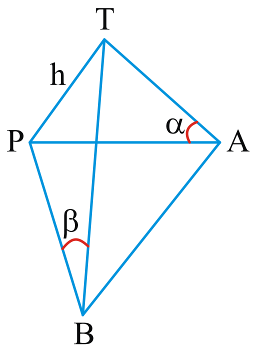
angle of elevation of top of tower T from each vertex $= \theta$

So, tower is at circumcentre f triangle

$BP = h\cot\theta =$ circum radius

$$= \frac{abc}{4\Delta} = \frac{30\times 50\times 70}{4\left(375\sqrt{3}\right)} = \frac{70}{\sqrt{3}}$$

height of tower $h = \frac{70}{\sqrt{3}}\frac{1}{\cot\theta} = \frac{70}{\sqrt{3}}\tan\theta$ mts.

11. 

Height of the tower $PT = h$ mtrs

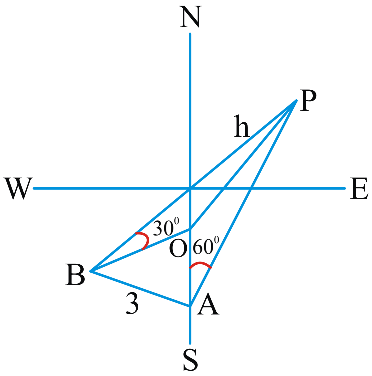
angle of elevation at $A = \alpha = \tan^{-1} 6$

$$\Rightarrow \tan\alpha = 6 = \frac{h}{AP} \Rightarrow AP = \frac{h}{6}$$

angle of elevatin at $B = \beta = \tan^{-1}(7.5)$

$$\Rightarrow \tan\beta = \frac{15}{2} = \frac{h}{BP} \Rightarrow BP = \frac{2h}{15}$$

$$AB = \lambda h \Rightarrow \lambda^2 h^2 = \frac{h^2}{36} + \frac{4h^2}{225}$$

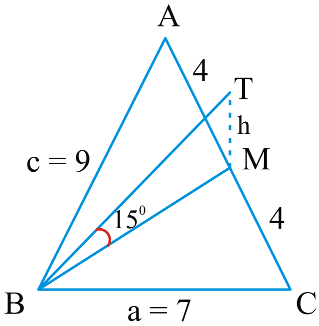
12. 

$$OA = h\cot 60^0 = \frac{h}{\sqrt{3}},$$

$$OB = h\cot 30^0 = \sqrt{3}h$$

$$OB^2 = OA^2 + AB^2 \Rightarrow 3h^2 = \frac{h^2}{3} + 9$$

$$\Rightarrow h^2 = \frac{27}{8}, \quad \Rightarrow h = \frac{3\sqrt{3}}{2\sqrt{2}} = \frac{3\sqrt{6}}{4}$$

13. 

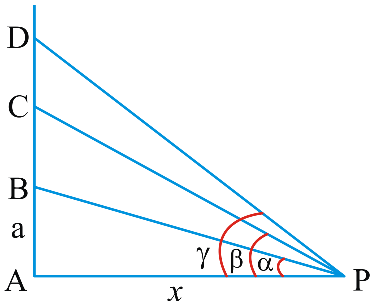
Let M be foot of the tower MT of height h mtrs.

$BC = a = 7$ mtrs., $AC = b = 8$ mtrs., $AB = c = 9$ mtrs.

$$BM = h\cot 15^0 \Rightarrow h = \frac{BM}{\cot 15^0}$$

$$= \frac{\sqrt{2\left(a^2+c^2\right)-b^2}}{2}\left(2-\sqrt{3}\right)$$

$$\Rightarrow h = \frac{\sqrt{2(49+81)-64}}{2}\left(2-\sqrt{3}\right)$$

14. 

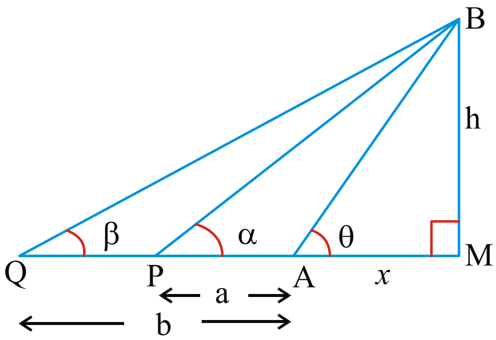
$AB = a, AC = b, AD = c, AP = x$ mtrs.

$$\tan\alpha = \frac{a}{x}, \tan\beta = \frac{b}{x}, \tan\gamma = \frac{c}{x}$$

$$\alpha + \beta + \gamma = 180 \Rightarrow \tan\alpha + \tan\beta + \tan\gamma$$

$$= \tan\alpha\tan\beta\tan\gamma \Rightarrow \frac{a}{x} + \frac{b}{x} + \frac{c}{x} = \frac{abc}{x^3}$$

$$\Rightarrow (a+b+c) = \frac{abc}{x^2}, \Rightarrow (a+b+c)x^2 = abc$$

15. 

$$\cot\theta = \frac{x}{h} \text{....(1)}$$

$$\cot\alpha = \frac{a+x}{h} \Rightarrow a+x = h\cot\alpha \;....(2)$$

$$\cot\beta = \frac{b+x}{h} \Rightarrow b+x = h\cot\beta \;....(3)$$

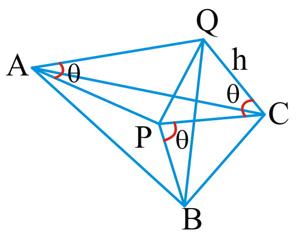
(2) $\times$ b - (3) $\times$ a

$$= (b-a)x = h(b\cot\alpha - a\cot\beta)$$

$$\Rightarrow \frac{b-a}{b\cot\alpha - a\cot\beta} = \frac{h}{x} = \tan\theta$$

16. $AP = BP = CP = h\cot\theta$

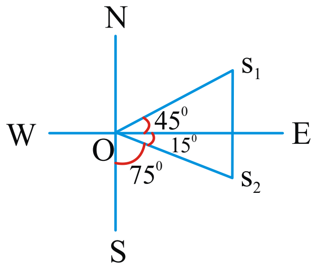
P is circumcentre of ΔABC.



$$\therefore AP = BP = CP = R = \frac{abc}{4\Delta}$$

$$\therefore h = \frac{abc\tan\theta}{4\Delta}$$

17.

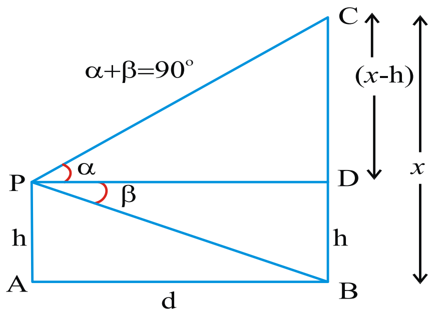


$$(s_1s_2)^2 = (os_1)^2 + (os_2)^2 - 2(os_1)(os_2)\cos 60^0$$

$$= (72)^2 + (96)^2 - (72)(96)$$

$$= (72)^2 + (96)^2 - (72)96, \quad \text{find } s_1s_2$$

18.



from ΔAPB, $PB^2 = h^2 + d^2$

from ΔPDC, $PC^2 = (x-h)^2 + d^2$

from ΔPBC, $BC^2 = PB^2 + PC^2$

$$\Rightarrow x^2 = h^2 + d^2 + (x-h)^2 + d^2$$

JEE MAIN , EAMCET QUESTIONS

1. A tower stands at the center of a circular park. A and B are two points on the boundary of the park such that $AB(=a)$ subtends an angle of 60^0 at the foot of the tower, and the angle of elevation of the top of the tower from A or B is 30^0. The height of the tower is (JEE MAIN 2007)

1) $\frac{2a}{\sqrt{3}}$ 2) $2a\sqrt{3}$ 3) $\frac{a}{\sqrt{3}}$ 4) $a\sqrt{3}$

2. The angle of elevation of the top of a vertical tower from a point A, due east of it is 45^0. The angle of elevation of the top of the same tower from a point B, due south of A is 30^0. If the distance between A and B is $54\sqrt{2}$ m, then the height of the tower (in metres), is : (JEE MAIN 2016)

(1) $36\sqrt{3}$ (2) 54 (3) $54\sqrt{3}$ (4) 108

3. Let a vertical tower AB has its end A on the level ground let C be be the midpoint of AB and P be a point on the ground such that $AP = 2AB$. If $\angle BPC = \beta$ then $\tan\beta$ is equal to (JEE MAIN-2017)

1) $\frac{1}{4}$ 2) $\frac{2}{9}$ 3) $\frac{4}{9}$ 4) $\frac{6}{7}$

4. A man on the top of a vertical tower observes a car moving at a uniform speed towards the tower on a horizontal road. If it takes 18 min, for the angle of depression of the car to change from 30^0 to 45^0; then after this , the time taken (in min) by the car to reach the foot of the tower , is:

1) $9(1+\sqrt{3})$ 2) $18(1+\sqrt{3})$

3) $18(\sqrt{3}-1)$ 4) $\frac{9}{2}(\sqrt{3}-1)$

5. If the angle of elevation of a cloud from a point P which is 25 m above a lake be 30^0 and the angle of depression of reflection of the cloud in the lake from P be 60^0, then the height of the cloud (in meters) from the surface of the lake is:(MAINS-2019)

1) 45 2) 42 3) 50 4) 60

6. **A man is walking towards a vertical pillar in a straight path, at a uniform speed. At a certain point 'A' on the path, he observes that the angle of elevation of the top of the pillar is 30^0. After walking 10 minutes from A in the same direction, at a point B, he observes that the angle of elevation of the top of the pillar is 60^0. Then the time taken (in minutes) by him, from 'B' to reach the pillar is___**

(1) 10 (2) 20 (3) 5 (4) 6

7. **If the angles of elevation of the top of a tower from three collinear points A, B and C on a line leading to the foot of the tower, are $30^0, 45^0 \; \& \; 60^0$ respectively, then the ratio $AB : BC$ is_____**

(1) $\sqrt{3}:1$ (2) $\sqrt{3}:\sqrt{2}$ 3) $1:\sqrt{3}$ (4) $2:3$

8. **ABCD is a trapezium such that AB and CD are parallel and $BC \perp CD$. If $\angle ADB = \theta$, $BC = p$ and $CD = q$, then AB is equal to**

1) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$ 2) $\frac{p^2+q^2\cos\theta}{p\cos\theta + q\sin\theta}$

3) $\frac{p^2+q^2}{p\cos\theta + q\sin\theta}$ 4) $\frac{(p^2+q^2)\sin\theta}{(p\sin\theta + q\cos\theta)}$

9. **PQ is a vertical tower, P is the foot, Q the top of the tower, A, B, C are three points in the horizontal plane through P. The angles of elevation of Q from A, B, C are equal and each is equal to θ. The sides of the triangle ABC are a, b, c and the area of the triangle ABC is Δ, the height of the tower is**

1) $(abc)\tan\theta/4\Delta$ 2) $(abc)\cot\theta/4\Delta$

3) $(abc)\sin\theta/4\Delta$ 4) $(abc)\tan\theta/2\Delta$

10. **Consider a triangular plot ABC with sides AB= 7 m, BC= 5m and CA =6m. A vertical lamp-post at the mid point D of AC subtends an angle 30^0 at B. The height (in m) of the lamp-post is:(MAINS-2019)**

1) $2\sqrt{21}$ 2) $\frac{3}{2}\sqrt{21}$ 3) $7\sqrt{3}$ 4) $\frac{2}{3}\sqrt{21}$

11. **AB is a vertical pole, The end A is on the level ground. C is the mid point of AB. P is the point on the level ground. The portion CB subtends an angle β at P. If AP = n.AB then tan β = (JEE MAINS-2017)**

1) $\frac{n}{n^2+1}$ 2) $\frac{n}{2n^2+1}$ 3) $\frac{n}{n^2-1}$ 4) $\frac{n}{2n^2-1}$

12. **AB is vertical pole with B at ground level and A at the top. A man finds that the angle of elevation of the point A from a certain point** C on the ground is 60^0. **He moves away from the pole along the line BC to a point D such that CD = 7 m . From D the angle of elevation of the point A is 45^0 then the height of the pole is (MAIN-2008)**

1) $\frac{7\sqrt{3}}{2}(\sqrt{3}+1)$ 2) $\frac{7\sqrt{3}}{2}(\sqrt{3}-1)$

3) $\frac{7\sqrt{3}}{2(\sqrt{3}+1)}$ 4) $\frac{7\sqrt{3}}{2(\sqrt{3}-1)}$

13. **Let 10 vertical poles standing at equal distances on a straight line, subtend the same angle of elevation α at a point O on this line and all the poles are on the same side of O. If the height of the longest pole is 'h' and the distance of the foot of the smallest pole from O is 'a'; then the distance between two consecutive poles, is : (JEE MAIN- 2016)**

(1) $\frac{h\sin\alpha + a\cos\alpha}{9\sin\alpha}$ (2) $\frac{h\cos\alpha - a\sin\alpha}{9\cos\alpha}$

(3) $\frac{h\cos\alpha - a\sin\alpha}{9\sin\alpha}$ (4) $\frac{h\sin\alpha + a\cos\alpha}{9\cos\alpha}$

14. **An aeroplane flying at a constant speed parallel to the horizontal ground, $\sqrt{3}$ km above it, is observed at an elevation of 60^0 from a point on the ground. If, after five seconds, its elevation from the same point, is 30^0, then the speed(in km / hr) of the aeroplane, is: (JEE MAINS-2018)**

1) 1500 2) 1440 3) 750 4)720

15. A tower T_1 **of height 60 m is located exactly opposite to a tower T_2 of height 80 m on a straight road. From the top of T_1, if the angle of depression of the foot of T_2 is twice the angle of elevation of the top of T_2, then the width (in m) of the road between the feet of the towers T_1 and T_2 is: (MAINS-2018)**

1) $10\sqrt{2}$ 2) $10\sqrt{3}$ 3) $20\sqrt{3}$ 4) $20\sqrt{2}$

16. PQR is a triangular park with PQ=PR=200m. A T.V. tower stands at the mid-point of QR. If the angles of elevation of the top of the tower at P,Q and R are respectively 45^0, 30^0 and 30^0, then the height of the tower (in m) is(MAINS-2018)

1) $50\sqrt{2}$ 2) 100 3) 50 4) $100\sqrt{3}$

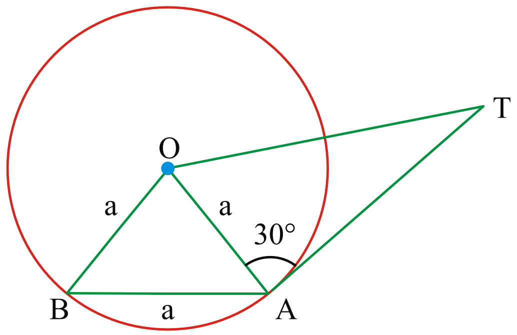
17. A bird is perched on the top of a tree 20 mts high and its elevation from the point on the ground is 45⁰. It flies off horizontally straight away from the observer and in one second the elevation of the bird is reduced to 30⁰. The speed of the bird is -----
(JEE MAINS-2014)

KEY

01) 3	02) 2	03) 2	04) 1	05) 3
06) 3	07)10.00	08) 1	09) 1	10) 4
11) 2	12)1	13) 3	14) 2	15) 3
16) 2	17)14.64			

SOLUTIONS

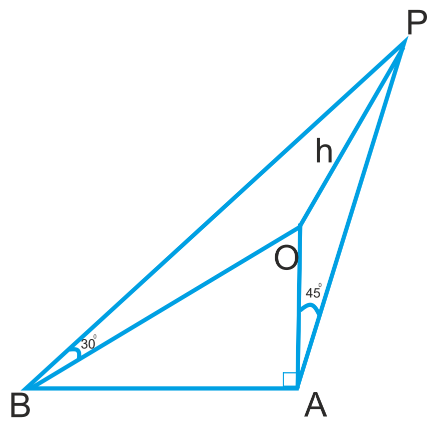
1. ΔOAB is equilateral.



$\therefore OA = OB = AB = a$

$\tan 30^0 = \frac{h}{a} \Rightarrow h = \frac{a}{\sqrt{3}}$

2.

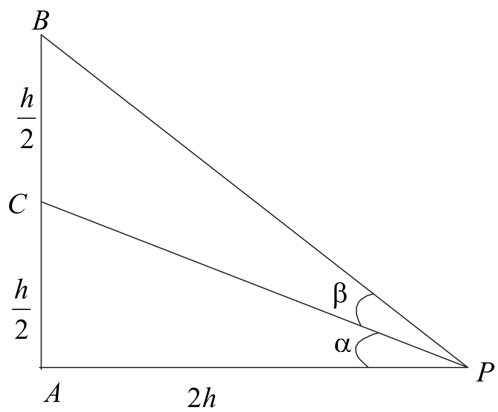


$Tan45^0 = \frac{h}{oA}$, $Tan30^0 = \frac{h}{oB}$

$OA^2 + AB^2 = OB^2$

$\left(54\sqrt{2}\right)^2 + h^2 = \left(h\sqrt{3}\right)^2 \Rightarrow h^2 = \left(54\right)^2 \Rightarrow h = 54$

3.



From ΔAPC; $\tan\alpha = \frac{1}{4}$

from $\Delta\, APB$; $\tan(\alpha+\beta)=\dfrac{1}{2}$

$$\frac{\frac{1}{4}+\tan\beta}{1-\frac{1}{4}\tan\beta}=\frac{1}{2} \Rightarrow 2+8\tan\beta=4-\tan\beta$$

$9\tan\beta = 2$

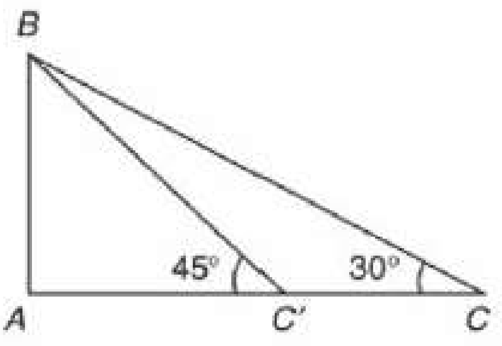
4. Let length of tower $= h$

$AC' = AB = h \Rightarrow AC = AB\cot 30^0 = \sqrt{3}h$

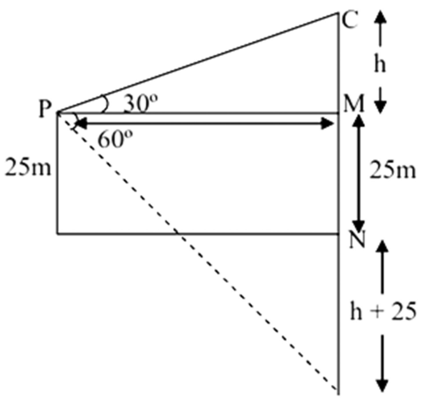
$CC' = \left(\sqrt{3}-1\right)h$

Time taken by car to C to $C' = 18$

So, time taken by car to reach the foot of the tower $= \dfrac{18}{\sqrt{3}-1}$ min $= 9\left(\sqrt{3}+1\right)$ min



5.

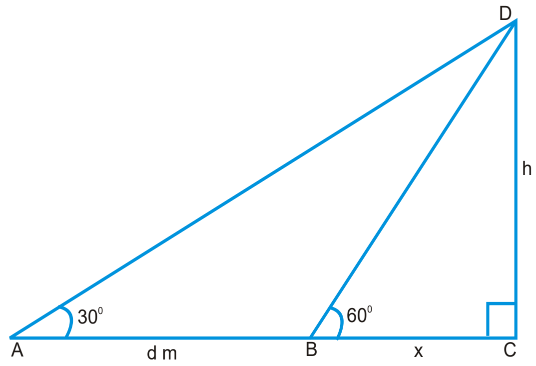


$x=\sqrt{3}h \Rightarrow \tan 60^0 = \dfrac{h+50}{x}$

$\sqrt{3}x = h+5 \Rightarrow \sqrt{3}\left(\sqrt{3}h\right)=h+50$

$\Rightarrow h = 25$

6.



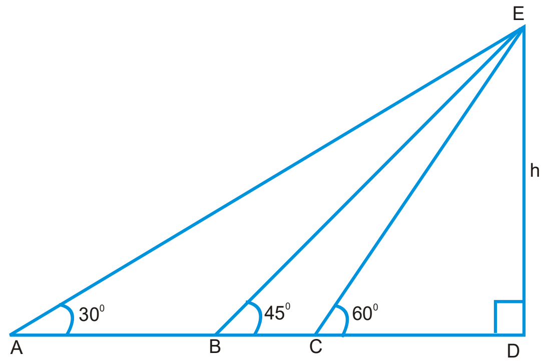
$h = d\sin 60^0 = \dfrac{d\sqrt{3}}{2}$

$x = h\cot 60^0 = \dfrac{d\sqrt{3}}{2} = \dfrac{d}{2}$

for d: Time taken = 10 mts

for d/2 : time taken = 5 mts

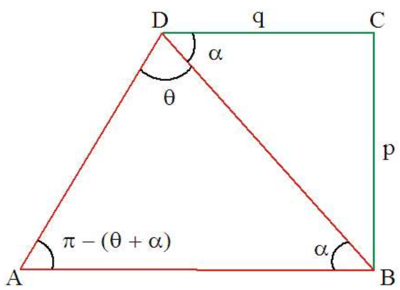
7.



$$AB = h\left(\cot 30^0 - \cot 45^0\right) = h\left(\sqrt{3}-1\right)$$

8. $BD = \sqrt{p^2+q^2}$, $\angle ABD = \angle BDC = \alpha$

$\Rightarrow \angle DAB = \pi - (\theta+\alpha)$



$\tan\alpha = \dfrac{p}{q}$, ΔABD

$$\frac{AB}{\sin\theta}=\frac{BD}{\sin\left(\pi-(\theta-\alpha)\right)}=\frac{BD}{\sin(\theta+\alpha)}$$

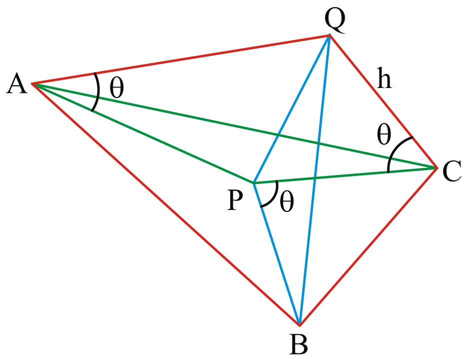
$$\therefore AB=\frac{BD\sin\theta}{\sin(\theta+\alpha)}=\frac{BD^2\sin\theta}{BD\sin(\theta+\alpha)}$$

$$=\frac{BD^2\sin\theta}{BD\sin\theta\cos\alpha+BD\cos\theta\sin\alpha}$$

$$=\frac{(p^2+q^2)\sin\theta}{q\sin\theta+p\cos\theta}$$

9. $AP=BP=CP=h\cot\theta$

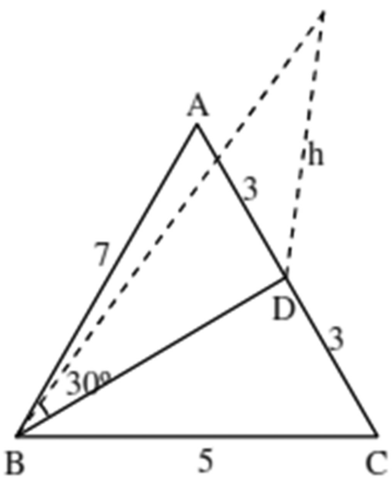
P is circumcentre of ΔABC.



$$\therefore AP=BP=CP=R=\frac{abc}{4\Delta}$$

$$\therefore h=\frac{abc\tan\theta}{4\Delta}$$

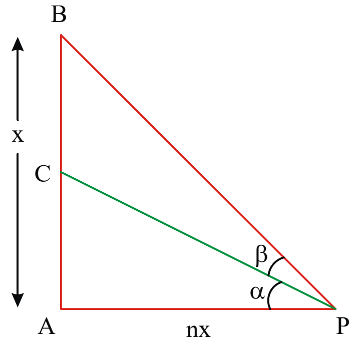
10.



Median mB= $\frac{1}{2}\sqrt{2a^2+2c^2-b^2}=\sqrt{28}$

$$\tan 30^0=\frac{h}{BD}=\frac{1}{\sqrt{3}}\Rightarrow h=\frac{\sqrt{28}}{\sqrt{3}}=\frac{2\sqrt{21}}{3}cm$$

11.



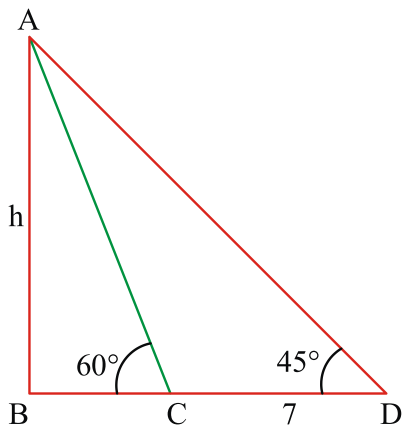
Let $AB=x\Rightarrow AP=n.AB=nx$

$$\tan\alpha=\frac{x}{2nx}=\frac{1}{2n};\ \tan(\alpha+\beta)=\frac{x}{nx}=\frac{1}{n}$$

$$\tan\beta=\tan\left((\alpha+\beta)-\alpha\right)=\frac{\tan(\alpha+\beta)-\tan\alpha}{1+\tan(\alpha+\beta)\tan\alpha};$$

simplify

12.

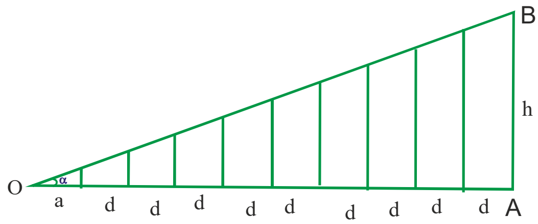


$d=7$ mtrs, $\alpha=45,\beta=60$

height of the pole $h=\dfrac{d}{\cot\alpha-\cot\beta}$

$$=\frac{7}{1-\frac{1}{\sqrt{3}}}=\frac{7\sqrt{3}}{\sqrt{3}-1}$$

13.



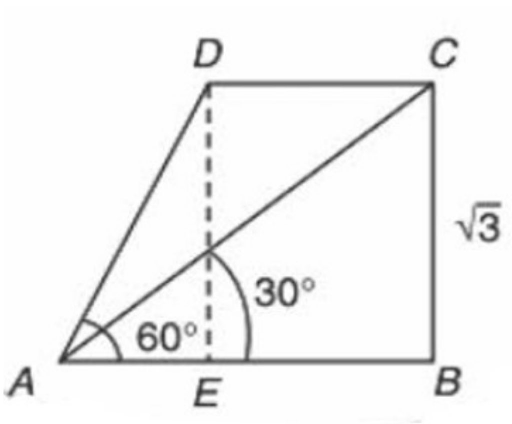
$$Tan\alpha = \frac{h}{a+9d}$$

$$\frac{Sin\alpha}{Cos\alpha} = \frac{h}{a+9d}$$

$$a sin\alpha + 9d\sin\alpha = h\cos\alpha$$

$$d = \frac{h\cos\alpha - a\sin\alpha}{9\sin\alpha}$$

14. From given data we draw the figure below



Height of plane is DE=BC=$\sqrt{3}$

$$\tan 60^0 = \frac{\sqrt{3}}{AE} \Rightarrow AE = 1$$

Now In ΔACB

$$\tan 30^0 = \frac{\sqrt{3}}{AB} \Rightarrow AB = 3$$

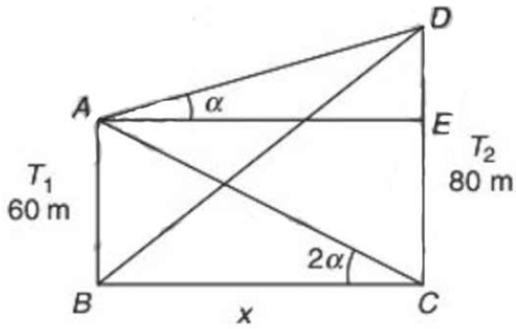
Distance travelled by plane in 5 seconds is :

CD=CE=3-1=2km

Therefore, 2=5 × v $\Rightarrow v = \frac{2}{5} kms^{-1}$

$$= \frac{2}{5} \times 3600 kmh^{-1} = 2 \times 720 = 1440 kmh^{-1}$$

15.



Now, ΔADE

$$\tan\alpha = \frac{DE}{AE} = \frac{20}{x}, \tan 2\alpha = \frac{60}{x}$$

$$\Rightarrow \frac{2\tan\alpha}{(1-\tan^2\alpha)\tan\alpha} = \frac{60}{20} = 3$$

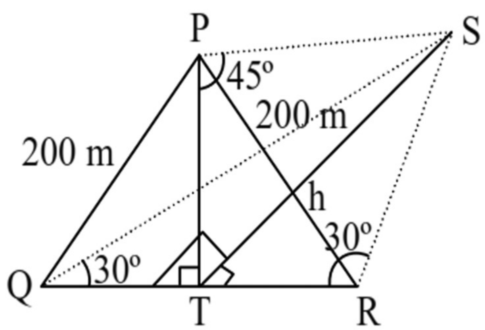
$$\Rightarrow \frac{2}{3} = 1 - \tan^2\alpha \Rightarrow \tan^2\alpha = \frac{1}{3}$$

$$\Rightarrow \tan\alpha = \frac{1}{\sqrt{3}}$$

Therefore,

$$\tan\alpha = \frac{20}{x} \Rightarrow x = 20\cot\alpha = 20\sqrt{3}$$

16. Let ST=h(height of tower)

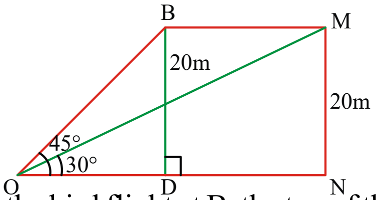


PT=ST=h$\Rightarrow \frac{ST}{QT} = \tan 30^0$

$QT = h\sqrt{3} \Rightarrow$ Now $PT^2 + QT^2 = 200^2$

$4h^2 = 200^2 \Rightarrow \quad h = 100$

17.



Let the bird flight at B, the top of the tree BD, and 'O' be the observer. Then $\underline{|BOD} = 45^0$ and BD = 20 mts. Now the bird flying horizontally reaches M in 1 second.

$\underline{|MON} = 30^0$ where MN perpendicular to ON

Now BD = MN = 20 mts. From ΔBOD ,

$\text{Tan}45^0 = \dfrac{BD}{OD} = \dfrac{20}{OD} \Rightarrow OD = 20mts$ from

$\Delta MON, \ Tan30^0 = \dfrac{MN}{ON} = \dfrac{20}{20+DN}$

$\Rightarrow DN = 20\left(\sqrt{3}-1\right)$

= 20(0.732) = 14.64 mts = BM

Speed of bird = Distance/Time = 14.64 m/s

******

HYPEBOLIC FUNCTIONS

SYNOPSIS

- If x is any real number then

  $e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots + \frac{x^n}{\lfloor n} + \ldots\infty$ is called the exponential series .

- $e^{-x} = 1 - \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} + \ldots\ldots\ldots\ldots + (-1)^n \frac{x^n}{\lfloor n} + \ldots\infty$.

- The function e^x can be written as

  $e^x = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}$ for all $x \in \mathbb{R}$;

- (i) $\sinh x = \frac{e^x - e^{-x}}{2}$

  $= \frac{x}{\lfloor 1} + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \ldots, x \in \mathbb{R}$

  (ii) $\cosh x = \frac{e^x + e^{-x}}{2}$

  $= 1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \ldots, x \in \mathbb{R}$

  (iii) $\tanh x = \frac{\sinh x}{\cosh x}, x \in \mathbb{R}$

  (iv) $\sec hx = \frac{1}{\cosh x}, x \in \mathbb{R}$

  (v) $\cos echx = \frac{1}{\sinh x}, x \in \mathbb{R} - \{0\}$

  (vi) $\coth x = \frac{\cosh x}{\sinh x}, x \in \mathbb{R} - \{0\}$

- Hyperbolic functions are not circular functions and hence are not meant to use trigonometric identities.

- **Hyperbolic Identities :**

  (i) $\cosh^2 x - \sinh^2 x = 1, \ \forall x \in \mathbb{R}$

  (ii) $\operatorname{sech}^2 x + \tanh^2 x = 1, \ \forall x \in \mathbb{R}$

  (iii) $\coth^2 x - \cos ech^2 x = 1, \ \forall x \in \mathbb{R} - \{0\}$

- **Properties of Hyperbolic Functions :**

  (i) $\sin h(-x) =$ -$\sin h\, x$

  (ii) $\cosh(-x) = \cosh x$

  (iii) $\tanh(-x) =$ -$\tanh x$

  (iv) $\coth(-x) =$ -$\coth x$

  (v) $\operatorname{sech}(-x) = \operatorname{sech} x$

  (vi) $\operatorname{cosech}(-x) =$ -$\operatorname{cosech} x$

- (i) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

  (ii) $\cos h(x \pm y) = \cos h x \cos h y \pm \sin h x \sin h y$

  (iii) $\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

  (iv) $\coth(x \pm y) = \frac{\coth x \coth y \pm 1}{\coth y \pm \coth x}$

- (i) $\sinh 2x = 2\sinh x \cosh x = \frac{2\tanh x}{1 - \tanh^2 x}$

  (ii) $\cosh 2x = \cosh^2 x + \sinh^2 x = \frac{1 + \tanh^2 x}{1 - \tanh^2 x}$

  (iii) $\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}$

  (iv) $\sinh 2x + \cosh 2x = \frac{1 + \tanh x}{1 - \tanh x}$

- (i) $\sinh 3x = 3\sinh x + 4\sinh^3 x$

  (ii) $\cosh 3x = 4\cosh^3 x - 3\cosh x$

  (iii) $\tanh(3x) = \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$

- $\sinh(x+y)\sinh(x-y)=\sinh^2 x-\sinh^2 y$
- $\cosh(x+y)\cosh(x-y)=\cosh^2 x+\sinh^2 y$
- $(\cosh x+\sinh x)^n=(\cosh(nx)+\sinh(nx))=e^{nx}$
- $(\cosh x-\sinh x)^n=(\cosh(nx)-\sinh(nx))=e^{-nx}$
- cosh 2nx + sinh 2nx $=\left(\dfrac{1+\tanh x}{1-\tanh x}\right)^n$
- cosh x, sech x are even functions.
- sinh x, cosech x, tanh x and coth x are odd functions.
- None of the six hyperbolic functions are periodic.
- sinh x, tanh x ,coth x and cosech x are one - one functions but cosh x and sech x are not one one functions as

  $\cosh(-x)=\cosh x$ *and* $\operatorname{sech}(-x)=\operatorname{sech} x$ *for all* $x\in\mathbb{R}$
- sinhx + coshx, sinhx - coshx are bijective functions
- **Domain and range of hyperbolic functions :**

Function	**Domain**	**Range**
Sinh x	R	R
Cosh x	R	$[1,\infty)$
Tanh x	R	$(-1,1)$
Coth x	R - {0}	R-[-1,1]
Cosech x	R - {0}	R - {0}
Sech x	R	(0,1]

- Since hyperbolic functions are defined in terms of exponential functions. Therefore Inverse hyperbolic functions can be expressed in terms of logarithmic functions.
- **Inverse Hyperbolic Functions in Terms of Logarithmic Functions:**

i) $\sinh^{-1}x=\log_e\left(x+\sqrt{x^2+1}\right)\ \forall\, x\in\mathbb{R}$

ii) $\cosh^{-1}(x)=\log_e\left(x+\sqrt{x^2-1}\right)$ *for* $x\ge 1$

iii) $\tanh^{-1}(x)=\dfrac{1}{2}\log\left(\dfrac{1+x}{1-x}\right)$ *for* $x\in(-1,1)$

Similarly we have

iv) $\coth^{-1}(x)=\dfrac{1}{2}\log\left(\dfrac{x+1}{x-1}\right)$ *for* $|x|>1$

v) $\operatorname{sech}^{-1}(x)=\log_e\left(\dfrac{1+\sqrt{1-x^2}}{x}\right)$ for $x\in(0,1]$

vi) $\operatorname{cosech}^{-1}(x)=\log_e\left(\dfrac{1+\sqrt{1+x^2}}{x}\right)$ for $x>0$

$=\log_e\left[\dfrac{1-\sqrt{1+x^2}}{x}\right]$ for $x<0$

- **Domain and Range of inverse Hyperbolic Functions:**

Function	**Domain**	**Range**
$\text{Sinh}^{-1} x$	R	**R**
$\text{Cosh}^{-1} x$	$[1,\infty)$	$[0,\infty)$
$\text{Tanh}^{-1} x$	(-1,1)	R
$\text{Coth}^{-1} x$	$R-[-1,1]$	R - {0}
$\text{Sech}^{-1} x$	(0,1]	$[0,\infty)$
$\text{Cosech}^{-1} x$	R - {0}	R - {0}

- $\sinh^{-1}(x)=\cosh^{-1}\left(\sqrt{x^2+1}\right)$

  $=\operatorname{cosech}^{-1}\left(\dfrac{1}{x}\right)$

  $=\tanh^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right)$
- $\cosh^{-1}(x)=\sinh^{-1}\left(\sqrt{x^2-1}\right)$

  $=\operatorname{sech}^{-1}\left(\dfrac{1}{x}\right)=\tanh^{-1}\left(\dfrac{\sqrt{x^2-1}}{x}\right)$
- **Euler's Formula :**

  $e^{ix}=\cos x+i\sin x\ \forall x\in R$

  $e^{-ix}=\cos x-i\sin x\ \forall x\in R$
- **Hyperbolic Functions Using Euler's Formula :**

  sinh (ix) = i sin x
  cosh (ix) = cos x
  tanh (ix) = i tan x

coth (i x) = -i cot x
cosech (i x) = -i cosec x
sech (i x) = sec x

✈ $\sin ix = i\sinh x$, $\cos ix = \cosh x$

$\tan ix = i\tanh x$

$\cot ix = -i\coth x$, $\sec ix = \sec hx$

$\cos ecix = -i \operatorname{cosec} h\, x$

✈ $\sinh^{-1} x = -i\sin^{-1} ix$

$\cosh^{-1} x = -i\cos^{-1} x$

$\tanh^{-1} x = -i\tan^{-1} ix$

Eg 1:

$\sinh x = 3/4$ **then** $\cosh x =$

Sol: $\cosh^2 x - \sinh^2 x = 1 \Rightarrow \cosh^2 x = 1 + \sinh^2 x$

$= 1 + \frac{9}{16} \Rightarrow \cosh x = \frac{5}{4}$

Eg 2:

$\tanh x = 3/5$ **then** $\cosh(2x) =$

Sol: $\cosh 2x = \frac{1+\tanh^2 x}{1-\tanh^2 x} = \frac{34}{16} = \frac{17}{8}$

Eg 3:

$\tanh x = \frac{1}{2}$ **then** $\tanh(3x) =$

Sol: $\tanh(3x) = \frac{3\tanh(x) + \tanh^3(x)}{1 + 3\tanh^2(x)} = \frac{13}{14}$

Eg 4:

Show that $f(x) = \cosh x$ **is an even function**

Sol: $f(-x) = \cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^{x}}{2} = f(x)$

Eg 5:

If $\sinh x = 3$**. Then x = ...**

Sol: $\sinh x = 3 \Rightarrow x = \sinh^{-1}(3) = \log\left(3 + \sqrt{3^2+1}\right)$

$= \log\left(3 + \sqrt{10}\right)$

Eg 6:

If $\cosh x = 3$ **then x = ...**

Sol: $\cosh x = 3 \Rightarrow x = \cosh^{-1}(3) = \log\left(3 + \sqrt{3^2 - 1}\right) = \log\left(3 + \sqrt{8}\right)$

EXERCISE - I

1. $sinh(3) - cosh(3) =$

1) e^{-3} 2) $-e^{-3}$ 3) e^3 4) $-e^3$

2. $(\cosh x - \sinh x)^n =$

1) cos h n x - sin h n x 2) 2cos h n x
3) 2sinhn x 4) cos h n x + sin h n x

3. If $sinh\,x = \frac{3}{4}$ **then** $sinh(2x) =$

1) $\frac{5}{8}$ 2) $\frac{15}{8}$ 3) $\frac{7}{8}$ 4) $\frac{17}{8}$

4. If $tanh\,x = \frac{3}{5}$ **then** $sinh\,(2x) =$

1) $\frac{15}{8}$ 2) $\frac{15}{17}$ 3) $\frac{18}{17}$ 4) $\frac{17}{8}$

5. $\cosh^4(x) - \sinh^4(x) =$

1) $\cosh x$ 2) $\cosh 2x$ 3) $\sinh x$ 4) $\sinh 2x$

6. If $cosh\,x = \frac{5}{4}$ **then** $\cosh(3x) =$

1) $\frac{61}{16}$ 2) $\frac{63}{16}$ 3) $\frac{61}{63}$ 4) $\frac{65}{16}$

7. The domain of $\cos ech\ x$ **is**

1) $(-\infty, \infty)$ 2) $(-\infty, 0) \cup (0, \infty)$

3) $(0, \infty)$ 4) $(-\infty, 0)$

8. The range of $coth\,x$ **is**

1) $(-\infty, -1) \cup (1, \infty)$ 2) $[-1, 1]$

3) $(-\infty, \infty)$ 4) $(-\infty, -1)$

9. $\sinh^{-1}(2) =$

1) $\log_e\left(2 + \sqrt{5}\right)$ 2) $\log_e\left(2 + \sqrt{7}\right)$

3) $\log_e\left(3 + \sqrt{10}\right)$ 4) $\log_e\left(5 + \sqrt{26}\right)$

10. $cosh^{-1}(1) =$

1) 2 2) 3 3) $\log_e\left(1+\sqrt{2}\right)$ 4) 0

11. $tanh^{-1}\left(\frac{1}{2}\right)=$

1) $\frac{1}{2}\log_e 3$ 2) $\frac{1}{2}\log_e 2$

3) $\log_e 3$ 4) $\log_e 5$

12. $tanh^{-1}\left(\frac{1}{4}\right)+coth^{-1}(4)=$

1) $\frac{1}{2}\log_e\left(\frac{5}{3}\right)$ 2) $\log_e\left(\frac{5}{3}\right)$

3) $\log_e(5)$ 4) $\log_e(3)$

13. $sech^{-1}\left(\frac{1}{\sqrt{2}}\right)=$

1) $\log_e\left(\sqrt{2}-1\right)$ 2) $\log_e\left(\sqrt{2}+1\right)$

3) $\log_e\left(\sqrt{3}+1\right)$ 4) $\log_e\left(\sqrt{3}-1\right)$

14. $2\,\text{Tanh}^{-1}\left(\frac{1}{2}\right)=$

1) 0 2) $\log_e 2$ 3) $\log_e 3$ 4) $\log_e 4$

15. If $\sinh^{-1}(x)=\log_e\left(5+\sqrt{26}\right)$ **then** $x=$

1) 1 2) 2 3) 3 4)5

16. If $x=\tanh^{-1}(y)$ *then* $\log_e\left(\frac{1+y}{1-y}\right)$

(EAM-1998)

1) x 2) $4x$ 3) $2x$ 4) $3x$

17. If $\cosh^{-1}(k)=\log_e\left(3+2\sqrt{2}\right)$ *then* $k=$

1)1 2) 2 3) 3 4) 4

18. $e^{\sinh^{-1}(\cot\theta)}=$

1) $cot\theta+cosec\theta$ 2) $-\cot\theta+\operatorname{cos}ec\theta$

3) $sec\theta-tan\theta$ 4) $sec\theta+tan\theta$

19. If $x=log_e\left[tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$ **then** $coshx=$

1) $sec\theta$ 2) $cosec\theta$ 3) $sin\theta$ 4) $cos\theta$

20. $sinh(ix)=$

1) isinx 2) $\sin(ix)$ 3) -isinx 4) isin(ix)

KEY

01) 2	02) 1	03) 2	04) 1	05) 2	06) 4
07) 2	08) 1	09) 1	10) 4	11) 1	12) 2
13) 2	14) 3	15) 4	16) 3	17) 3	18) 1
19) 1	20) 1				

SOLUTIONS

1. Given $x=\log_e\left(\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right)$

$$e^x=\tan\left(\frac{\pi}{4}+\theta/2\right)=\frac{1+\tan\theta/2}{1-\tan\theta\,\theta/2}$$

$$\Rightarrow e^{-x}=\frac{1-\tan\theta/2}{1+\tan\theta/2}$$

Now

$$\cos hx=\frac{e^x+e^{-x}}{2}=\frac{1}{2}\left[\frac{1+\tan\theta/2}{1-\tan\theta/2}+\frac{1-\tan\theta/2}{1+\tan\theta/2}\right]$$

$$=\frac{1}{2}\left[\frac{\left(1+\tan\theta/2\right)^2+\left(1-\tan\theta/2\right)^2}{1-\tan^2\theta/2}\right]$$

$$=\frac{1}{2}\left[\frac{2\left(1+\tan^2\theta/2\right)}{1-\tan^2\theta/2}\right]=\sec\theta$$

$$\left(\cos\text{h}\,x-\sin\text{h}\,x\right)^n=\left(\frac{e^x+e^{-x}}{2}-\left(\frac{e^x+e^{-x}}{2}\right)\right)^n$$

2. $$=\left(\frac{e^x+e^{-x}-e^x+e^{-x}}{2}\right)=\left(e^{-x}\right)^n$$

$$=e^{-nx}$$

$$\cos hx-\sin h\,nx=\frac{e^{nx}+e^{-nx}+e^{nx}+e^{nx}}{2}=e^{-nx}$$

3. Given $\sin h\,x = \frac{3}{4}$

we know $\cos h^2 x - \sin h^2 x = 1$

$\cos \mathrm{h}^2\,x = 1 + \sin h^2 x$

$\cos h^2 x = 1 + \frac{9}{16} = \frac{25}{16}$

$\cos hx \frac{5}{4}$

$\sin h(2x) = 2\sin h\,x \cos hx$

$= 2\left(\frac{3}{4}\right)\left(\frac{5}{4}\right) = \frac{15}{8}$

4. $\operatorname{Tanh} x = \frac{3}{5} \Rightarrow \sinh 2x = \frac{2\tanh x}{1-\tanh^2 x}$

5. $\cosh^4 x - \sinh^4 x = \cosh 2x$

6. $\cosh 3x = 4\cosh^3 x - 3\cosh x$

7. The domain of cosechx is $(-\infty, 0) \cup (0, \infty)$

8. The range of cothx is $(-\infty, -1) \cup (1, \infty)$

9. We know $\sin h^{-1}(x) = \log_e\left(x + \sqrt{1+x^2}\right)$

$\sin h^{-1}(2) = \log_e\left(2 + \sqrt{1+4}\right)$

$= \log_e\left(2 + \sqrt{5}\right)$

10. $cosh^{-1}(1) = log(1) = 0$

11. We know $\tan\ h^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

$\tan h^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\log\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)$

$= \frac{1}{2}\log_e(3)$

12. $\tan h^{-1}\left(\frac{1}{4}\right) + \cot h^{-1}(4) = \frac{1}{2}\left[\log\left(\frac{1+\frac{1}{4}}{1-\frac{1}{4}}\right) + \log\left(\frac{4+1}{4-1}\right)\right]$

$= \frac{1}{2}\left[\log_e\left(\frac{5}{3}\right) + \log_e\left(\frac{5}{3}\right)\right] = \log_e\left(\frac{5}{3}\right)$

13. $\operatorname{sech}^{-1} x = \log\left(\frac{1+\sqrt{1-x^2}}{x}\right)$

14. $2\operatorname{Tanh}^{-1}\left(\frac{1}{2}\right) = \log(3)$

15. $\sinh^{-1} x = \log\left(5 + \sqrt{5^2+1}\right)$

$\therefore x = 5$

16. $x = \operatorname{Tanh}^{-1} y$

$\Rightarrow x = \frac{1}{2}\log\left(\frac{1+y}{1-y}\right)$

17. $\cosh^{-1}(k) = \log\left(k + \sqrt{k^2-1}\right)$

18. $e^{\log\left(\cot\theta + \sqrt{\cot^2\theta+1}\right)}$

$= cot\theta + cosec\theta$

19. $e^x = \frac{1+\operatorname{Tan}\theta/2}{1-\operatorname{Tan}\theta/2} \Rightarrow \cosh x = \sec\theta$

20. $Sinh(ix) = isinx$

EXERCISE - II

1. $cosh2x + sinh2x =$

1) $\frac{1+\tanh(x)}{1-\tanh(x)}$ 2) $\frac{1-\tanh x}{1+\tanh x}$

3) $\frac{\tanh x - 1}{\tanh x + 1}$ 4) $\frac{1-\tanh x}{\tanh x - 1}$

2. The value of $\frac{\cosh 2\theta - 1}{\sinh 2\theta}$ **is equal to**

1) $\cosh\theta$ 2) $\tanh\theta$ 3) $\cos ech\theta$ 4) $\operatorname{sech}\theta$

3. If $\tanh^2 x = \cos\theta\ then \cosh 2x =$

1) $\tan^2\left(\frac{\theta}{2}\right)$ 2) $\cot^2\left(\frac{\theta}{2}\right)$

3) $\sec^2\left(\frac{\theta}{2}\right)$ 4) $\sin^2\left(\frac{\theta}{2}\right)$

4. **If** $\cosh x = \sec\theta$ *then* $\tanh^2\left(\frac{x}{2}\right) =$

1) $\tan^2\frac{\theta}{2}$ 2) $\cot^2\frac{\theta}{2}$

3) $-\tan^2\frac{\theta}{2}$ 4) $-\cot^2\frac{\theta}{2}$

5. $\frac{\tanh x}{\operatorname{sech} x - 1} + \frac{\tanh x}{\operatorname{sech} x + 1} =$

1) $\cos echx$ 2) 2cosechx

3) -cosechx 4) -2cosechx

6. $\frac{\cosh(x)}{1-\tanh x} + \frac{\sinh(x)}{1-\coth(x)} =$

1) $\sinh(x) + \coth(x)$

2) $\tanh(x) + \coth(x)$

3) $\tanh(x) - \coth x$

4) $\sinh(x) + \cosh(x)$

7. **If** $\sin x \sinh y = \cos\theta$ **and**

$\cos x \cosh y = \sin\theta$ **then** $\cosh^2 y + \cos^2 x =$

1) -1 2) 0 3) 1 4) 2

8. $\sec h^{-1}\left(\frac{2}{3}\right) = \log_e\left(\frac{(k+1)+\sqrt{k^2+1}}{k}\right)$ **then** k =

1) 2 2) 3 3) 5 4) 6

9. **If** *sinx coshy* = $\cos\theta, \cos x \sinh y = \sin\theta$

then $\sinh^2 y =$

1) $\cosh^2 x$ 2) $\cos^2 x$ 3) $\cosh^3 x$ 4) $\sinh^2 x$

10. **If** x **is an acute angle and**

$y = \log_e\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]$ **then** *cosx. coshy* =

1) 0 2) -1 3) 1 4) 2

11. $\sinh\left(\cosh^{-1} x\right) =$

1) $\sqrt{x^2+1}$ 2) $\frac{1}{\sqrt{x^2+1}}$

3) $\sqrt{x^2-1}$ 4) $\frac{1}{\sqrt{x^2-1}}$

12. $\log_e\left[(x-1) + \sqrt{x^2-2x}\right], x \geq 2$ **is equal to**

1) $\sinh^{-1}(x-1)$ 2) $\cosh^{-1}(x-1)$

3) $\sinh^{-1}(x+1)$ 4) $\cosh^{-1}(x+1)$

KEY

01) 1	02) 2	03) 2	04) 1	05) 4
06) 4	07) 4	08) 1	09) 2	10) 3
11) 3	12) 2			

SOLUTIONS

1. $\cosh 2x + \sinh 2x$

$= \frac{1+\text{Tanh}^2 x}{1-\text{Tanh}^2 x} + \frac{2\,\text{Tanh}\, x}{1-\text{Tanh}^2 x}$

2. $\frac{\cosh 2\theta - 1}{\sinh 2\theta} = \text{Tanh}\,\theta$

3. $\cosh 2x = \frac{1+\text{Tanh}^2 x}{1-\text{Tanh}^2 x}$

4. Given $\cos h\, x = \sec\theta$

We know $\cos h\, x = \frac{1+\tan h^2 \,{}^{x}\!/_{2}}{1-\tan h^2 \frac{x}{2}}$

$\sec\theta = \frac{1+\tan\ h^2 \frac{x}{\varepsilon}}{1-\tan\ h^2 \frac{x}{\varepsilon}}$

$\sec c\theta - \sec\theta \tan h^2 \frac{x}{2} = 1 + \tan h^2 \frac{x}{2}$

$$\sec c\theta - 1 = \tan h^2 \frac{x}{2}(1+\sec\theta)$$

$$\tan\ h^2 \frac{x}{2} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{2\sin^{2\theta}_{\varepsilon}}{2\cos^{2\theta}_{\varepsilon}} = \tan^2\frac{\theta}{2}$$

5. $\frac{\tan\ hx}{\sec x-1} + \frac{\tan\ hx}{\sec h x+1} = \frac{\tan hx(\sec h x+1+\sec hx-1)}{\sec h^2 x-1}$

$$= \frac{\tan hx\, 2\sec hx}{-\tan h^2 x} = -2\cos ec\, hx$$

6. $\frac{\cosh^2 x}{\cosh x - \sinh x} + \frac{\sinh^2 x}{\sinh x - \cosh x}$

7. $\sin^2 x \sinh^2 y + \cos^2 x \cosh^2 y = 1$

$\Rightarrow (1-\cos^2 x)(-1+\cosh^2 y) + \cos^2 x \cosh^2 y = 1$

8.

$$\text{sech}^{-1} x = \log\left(\frac{1+\sqrt{1-x^2}}{x}\right)$$

9. Given $\sin x \cos hy = \cos\theta$

$\cos x \sin hy = \sin\theta$

Squaring and adding

$$\sin^2 x \cos h^2 y + \cos^2 x \sin h^2 y = \cos^2\theta + \sin^2\theta$$

$$(1-\cos^2 x)\cos h^2 y + \cos^2 x(\cos h^2 - 1) = 1$$

$$\Rightarrow \cos h^2 y - \cos^2 x \cos h^2 y + \cos^2 x \cos h^2 y - \cos^2 x = 1$$

$$\Rightarrow \cos h^2 y - 1 = \cos^2 x$$

$$\sin h^2 y = \cos^2 x$$

10. $e^y = \dfrac{1+\text{Tan}\frac{x}{2}}{1-\text{Tan}\frac{x}{2}}$

11. $Sinh(cosh^{-1}x) = \sqrt{x^2-1}$

12. $\log\left[(x-1)+\sqrt{(x-1)^2-1}\right] = \cosh^{-1}(x-1)$

EXERCISE - III

1. $2(\sinh 2 + \sinh 3 + \sinh 5)\frac{e^5}{e^5-1}$

$= a + be^2 + ce^3 + de^5$ **then** a+c+b+d=

1) 1 2) 2 3) 3 4) 4

2. $\sinh^{-1}(x) + \cosh^{-1}(x) = y$ **then** $\sinh(y) =$

1) $x^2+\sqrt{x^4-1}$ 2) $x^2-\sqrt{x^4-1}$

3) $x^2+\sqrt{x^4+1}$ 4) $x^2-\sqrt{x^4+1}$

3. $\sinh^{-1}(2\alpha) = 2\cosh^{-1}(\beta)$ **then**

1) $\alpha^2+\beta^2=\alpha^4$ 2) $\alpha^2+\beta^2=4$

3) $\alpha^2+\beta^2=\beta^4$ 4) $\alpha^2=\beta^2$

4. If $\sinh^3 x - \cosh^3 x = \frac{ke^x - e^{kx}}{1-k}$ **then** $k =$

1) -1 2) 0 3) -3 4) 2

5. If $\tan\frac{x}{2}\coth\left(\frac{x}{2}\right) = 1$ **then** $\cos x \cosh x =$

1) 1 2) 2 3) 0 4) -1

6. $\sec h^2\left[\tanh^{-1}\left(\frac{1}{2}\right)\right] + \cos ec h^2\left(\coth^{-1}(3)\right) =$

1) $\frac{35}{9}$ 2) $\frac{43}{4}$ 3) $\frac{35}{4}$ 4) $\frac{43}{9}$

7. The value of $\tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right]$ **is**

1) $\frac{ab}{a^2+b^2}$ 2) $\frac{ab}{a^2-b^2}$ 3) $\frac{2ab}{a^2-b^2}$ 4) $\frac{2ab}{a^2+b^2}$

8. $\sin h^{-1}\left(2^{3/2}\right) =$

1) $\log_e\left(2+\sqrt{18}\right)$ 2) $\log_e\left(3+\sqrt{8}\right)$

3) $\log_e\left(3-\sqrt{8}\right)$ 4) $\log_e\left(\sqrt{8}+\sqrt{27}\right)$

KEY

01) 4 02) 1 03) 3 04) 3 05) 1 06) 3
07) 3 08) 2

HINTS

1. $$2\left[\frac{e^2-e^{-2}+e^3-e^{-3}+e^5-e^{-5}}{2}\right]\frac{e^5}{e^5-1}$$

$$=e^2+e^3+e^5+1$$

$\therefore a=1, b=1, c=1, d=1$

2. $\sinh^{-1}x+\cosh^{-1}x=y$

$$y=\log\left(x+\sqrt{x^2+1}\right)\left(x+\sqrt{x^2-1}\right)$$

$$\Rightarrow e^y=\left(x+\sqrt{x^2+1}\right)\left(x+\sqrt{x^2-1}\right)$$

3. $$\log\left(2\alpha+\sqrt{4\alpha^2+1}\right)=2\log\left(\beta+\sqrt{\beta^2-1}\right)$$

$$\Rightarrow 2\alpha+\sqrt{4\alpha^2+1}=\left(\beta+\sqrt{\beta^2-1}\right)^2$$

after simplification $\alpha^2+\beta^2=\beta^4$

4. $sinh^3x\text{-}cosh^3x=$

$$\left(sinhx\text{-}coshx\right)\left(cosh^2x+sinh^2x+coshxsinhx\right)$$

5. $$\operatorname{Tan}\frac{x}{2}=\operatorname{Tanh}\frac{x}{2}$$

$$\Rightarrow \cos x\cosh x=\left(\frac{1-\operatorname{Tan}^2\frac{x}{2}}{1+\operatorname{Tan}^2\frac{x}{2}}\right)(\cosh x)$$

$= \text{sechx.coshx}$

6. $$\operatorname{sech}^2\left(\operatorname{Tanh}^{-1}\frac{1}{2}\right)+\cos ech^2\left(\coth^{-1}3\right)$$

$$=1-\operatorname{Tanh}^2\left(\operatorname{Tanh}^{-1}\frac{1}{2}\right)+\coth^2\left(\coth^{-1}3\right)-1$$

7. By using the formula $\operatorname{Tan} ix=i\operatorname{Tanh}x$

8. $$\sin h^{-1}\left(2^{3/\varepsilon}\right)=\log\left(2^{3/\varepsilon}+\sqrt{2^3+1}\right)$$

$$=\log_e\left(\sqrt{8}+3\right)$$

EAMCET QUESTIONS

1. $\tanh^{-1}\frac{1}{2}+\coth^{-1}3=$ **(AP EAM-2018)**

1) $\log\sqrt{6}$ 2) $\log 6$ 3) $-\log\sqrt{6}$ 4) $-\log 6$

2. **If** $\sec\theta\cosh y=\operatorname{cosec}x$ **and** $\operatorname{cosec}\theta$ **sinh y = sec x, then** $\sinh^2 y=$ **(AP EAM-2018)**

1) $\cos^2 x$ 2) $\cos x$ 3) $\sin^2 x$ 4) $\sin x$

3. **If** $x=\log_e\left[\cot\left(\frac{\pi}{4}+\theta\right)\right]$ **and** $\theta\in\left(\frac{-\pi}{4},\frac{\pi}{4}\right)$ **then consider the following statements**

I. cosh x = sec 2θ **II. sin h x = –tan** 2θ

(AP EAM-2018)

1) I is true and II is false
2) I is false and II is true
3) Both I and II are true
4) Both I and II are false

4. $\operatorname{sech}^2\left(\tanh^{-1}\frac{1}{2}\right)+\operatorname{cosech}^2\left(\coth^{-1}3\right)=$ **(AP EAM-2018)**

1) $\frac{35}{9}$ 2) $\frac{3}{2}$ 3) $\frac{25}{4}$ 4) $\frac{35}{4}$

5. **If** $x=-\frac{1}{2}$, $\sinh^{-1}x+\operatorname{cosech}^{-1}x=$ **(AP EAM-2018)**

1) $\log_e\left(\frac{7-3\sqrt{5}}{2}\right)$ 2) $\log_e\left(\frac{3+\sqrt{5}}{2}\right)$

3) $\log_e\left[\frac{(\sqrt{5}-1)(2+\sqrt{3})}{2}\right]$

4) $\log_e\left[\frac{(\sqrt{5}+1)(2+\sqrt{3})}{2}\right]$

6. **If** $\cosh x=\frac{\sqrt{14}}{3}$, **sinh x =** $\cos\theta$ **and** $-\pi<\theta<-\frac{\pi}{2}$, **then** $\sin\theta$ **=(TS EAM-2018)**

1) $\frac{1}{3}$ 2) $\frac{2}{3}$ 3) $-\frac{1}{3}$ 4) $-\frac{2}{3}$

7. **If cosh** β **= sec**α **cos**θ, **sin h** β = $\operatorname{cosec}\alpha\sin\theta$, **then** $\sinh^2\beta=$ **(TS EAM-2018)**

1) $\sin\alpha\cos\alpha$ 2) $\cos^2\alpha$

3) $\sin^2\alpha$ 4) $\sin\alpha+\cos\alpha$

8. **If** $\sinh x=\frac{3}{4}$ **and** $\cosh y=\frac{5}{3}$, **then x + y = (TS EAM-2018)**

1) log2 2) log6 3) log3 4) log5

9. $\coth^{-1}3+\tanh^{-1}\frac{1}{3}-\operatorname{cosech}^{-1}\left(-\sqrt{3}\right)=$ **(TS EAM-2018)**

1) $\log_e\left(\frac{2}{\sqrt{3}}\right)$ 2) $\log_e 2\sqrt{3}$

3) 0 4) $\log_e 3\sqrt{3}$

10. $\sinh^{-1}2+\cosh^{-1}2-\tanh^{-1}\frac{2}{3}+\coth^{-1}(-2)$

1) $\log\left(\frac{4+2\sqrt{3}+2\sqrt{5}+\sqrt{15}}{\sqrt{15}}\right)$

2) $\log\left(\frac{4+\sqrt{3}+\sqrt{5}+\sqrt{15}}{\sqrt{15}}\right)$

3) $\log\frac{(2+\sqrt{3})(2+\sqrt{5})\sqrt{5}}{\sqrt{3}}$

4) $\log\frac{(2+\sqrt{3})(2+\sqrt{5})\sqrt{3}}{\sqrt{5}}$

11. **The solution of the equation** $2\cosh 2x+10\sinh 2x=5$ **is(TS EAM-2019)**

1) $\frac{1}{2}\log\left(\frac{3}{5}\right)$ 2) $\frac{1}{2}\log\left(\frac{4}{3}\right)$

3) $\frac{1}{2}\log\left(\frac{5}{4}\right)$ 4) $\frac{1}{2}\log\left(\frac{5}{3}\right)$

12. **If** $\sinh^{-1}\left(\sqrt{8}\right)+\sinh^{-1}\left(\sqrt{24}\right)=\alpha$, **then** $\sinh\alpha=$ **(TS EAM-2019)**

1) $6\sqrt{6}-10\sqrt{2}$ 2) $6\sqrt{6}+10\sqrt{2}$

3) $16\sqrt{6}$ 4) $16\sqrt{6}+4\sqrt{2}$

13. **If** $y=\log_e\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$ **then** $\tanh\left(\frac{y}{2}\right)=$

1) $\cot\frac{x}{2}$ 2) $\tan x$ 3) $\coth x$ 4) $\tan\frac{x}{2}$

INVERSE TRIGONOMETRIC FUNCTIONS

SYNOPSIS

Inverse of a function :

$f : A \rightarrow B$ is bijective $\Leftrightarrow f^{-1} : B \rightarrow A$ exists and it is also bijective. All trigonometric functions are not bijective functions. By restricting the domains of the functions,we make them bijective

- The function $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1,1]$ defined by $f(x) = \sin x$ is a bijection.

  Then $f^{-1} : [-1,1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is also a bijection. This function is called **inverse sine** function and it is denoted by **Arc sine** x or $Sin^{-1} x$

- The function $f : [0, \pi] \rightarrow [-1,1]$ defined by $f(x) = \cos x$ is a bijection.

  Then $f^{-1} : [-1,1] \rightarrow [0, \pi]$ is also a bijection. This function is called **inverse cosine** function and it is denoted by **Arc cos** x or $Cos^{-1} x$

- The function $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow R$ defined by $f(x) = \tan x$ is a bijection.

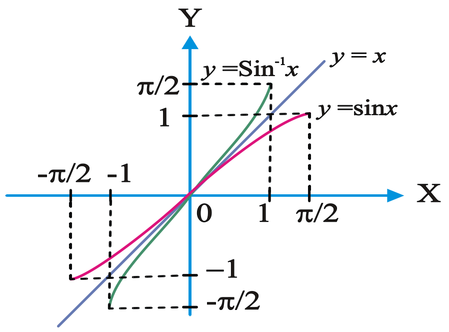
  Then $f^{-1} : \mathrm{R} \rightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is also a bijection. This function is called **inverse tangent** function and it is denoted by **Arc tan** x or $Tan^{-1} x$

Domains and Ranges of Inverse trigonometric functions:

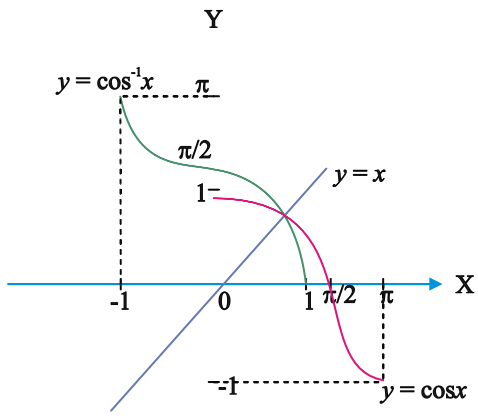
Function	Domain	Range
$Sin^{-1}x$	$[-1,1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$Cos^{-1}x$	$[-1,1]$	$[0, \pi]$
$Tan^{-1}x$	R	$\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
$Cot^{-1}x$	R	$(0, \pi)$
$Sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$Cosec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$\left[\frac{-\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

Graphs of inverse circular functions :

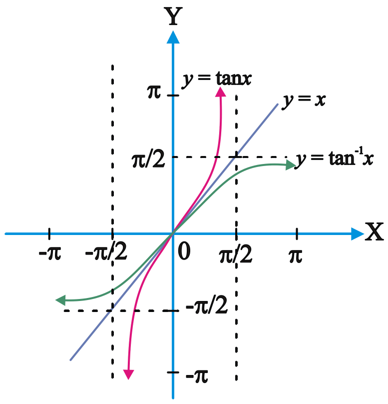
1. $y = Sin^{-1} x, |x| \le 1, \ y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



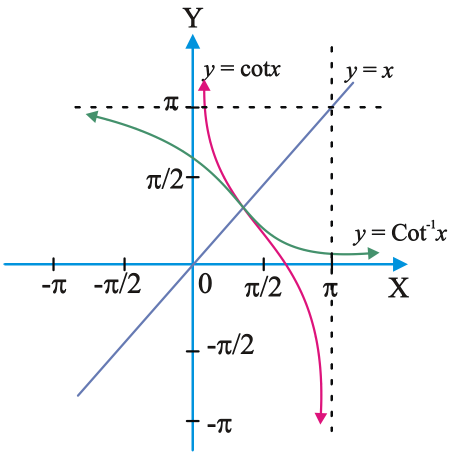
2. $y = Cos^{-1} x, |x| \le 1, y \in [0, \pi]$



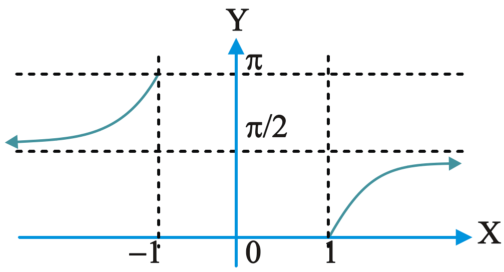
3. $y = \text{Tan}^{-1} x,\ x \in R,\ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



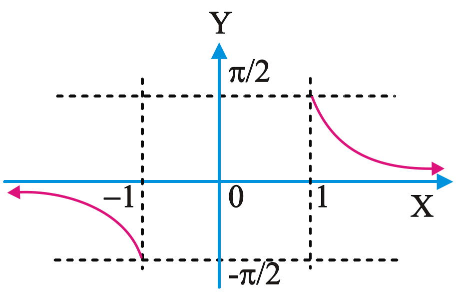
4. $y = \text{Cot}^{-1} x,\ x \in R, \qquad y \in (0, \pi)$



5. $y = \text{Sec}^{-1} x, |x| \geq 1, \quad y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



6. $y = \text{Cosec}^{-1} x, |x| \geq 1, \quad y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$



Properties of inverse trigonometric functions:

✈ i) $Sin^{-1}x = Cosec^{-1}\frac{1}{x}, \quad \forall x \in [-1,1], x \neq 0$

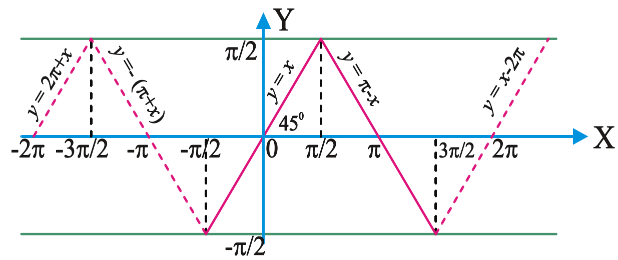
ii) $Cos^{-1}x = Sec^{-1}\frac{1}{x}, \quad \forall x \in [-1,1], x \neq 0$

iii) $Tan^{-1}x = \begin{cases} Cot^{-1}\frac{1}{x}, & \forall x > 0 \\ -\pi + Cot^{-1}\frac{1}{x}, & \forall x < 0 \end{cases}$

✈ Some useful periodic graphs :

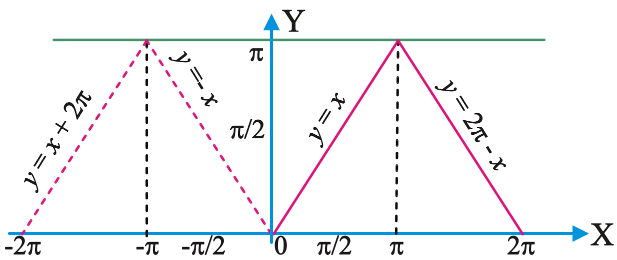
1. $y = Sin^{-1}(\sin x) = \begin{cases} -\pi - x, & -\frac{3\pi}{2} \leq x \leq -\frac{\pi}{2} \\ x, & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} \leq x \leq \frac{3\pi}{2} \\ -2\pi + x, & \frac{3\pi}{2} \leq x \leq \frac{5\pi}{2} \end{cases}$

y is Periodic with period 2π and $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$



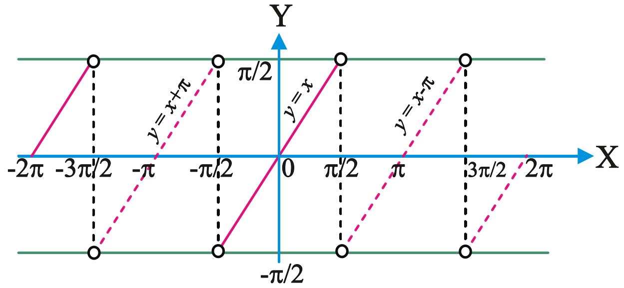
2. $y = Cos^{-1}(\cos x) = \begin{cases} -x, & -\pi \leq x \leq 0 \\ x, & 0 \leq x \leq \pi \\ 2\pi - x, & \pi \leq x \leq 2\pi \\ -2\pi + x, & 2\pi \leq x \leq 3\pi \end{cases}$

y is periodic with period 2π and $y \in [0, \pi]$



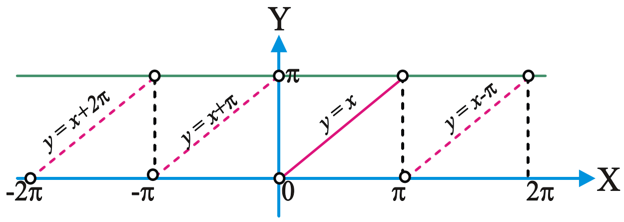
3. $y = Tan^{-1}(\tan x) = \begin{cases} x+\pi, & -\dfrac{3\pi}{2} < x < -\dfrac{\pi}{2} \\ x, & -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ -\pi + x, & \dfrac{\pi}{2} < x < \dfrac{3\pi}{2} \\ x - 2\pi, & \dfrac{3\pi}{2} < x < \dfrac{5\pi}{2} \end{cases}$

y is periodic with period π and $y \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$



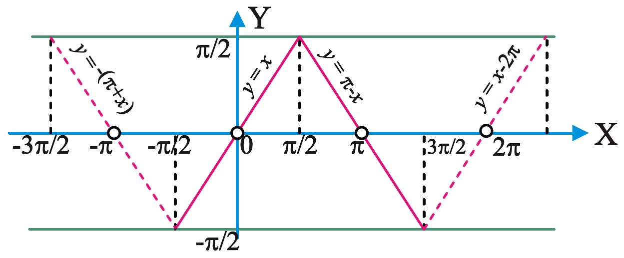
4. y = Cot⁻¹(Cot x) = x, $x \in (0, \pi)$ and so on.

y is periodic with period π and $y \in (0, \pi)$



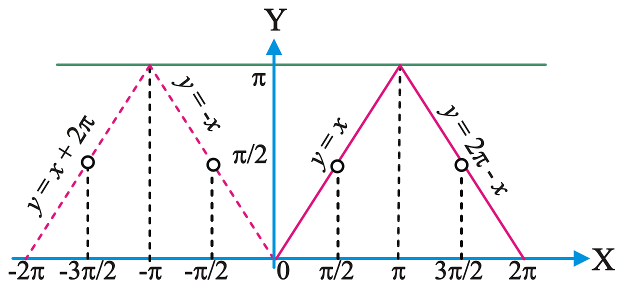
5. $y = Cosec^{-1}(\operatorname{cosec} x) = \begin{cases} x, & x \in \left[\dfrac{-\pi}{2}, 0\right) \cup \left(0, \dfrac{\pi}{2}\right] \\ \pi - x, & x \in \left[\dfrac{\pi}{2}, \pi\right) \cup \left(\pi, \dfrac{3\pi}{2}\right] \\ \text{and so on} & \end{cases}$

y is periodic with period 2π *and* $y \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$



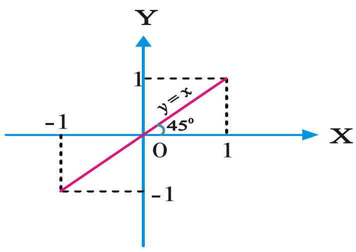
6. $y = Sec^{-1}(\sec x) = \begin{cases} x, & x \in \left[0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right] \\ 2\pi - x, & x \in \left[\pi, \dfrac{3\pi}{2}\right) \cup \left(\dfrac{3\pi}{2}, 2\pi\right] \\ \text{and so on} & \end{cases}$

y is periodic with period 2π and $y \in [0, \pi]$

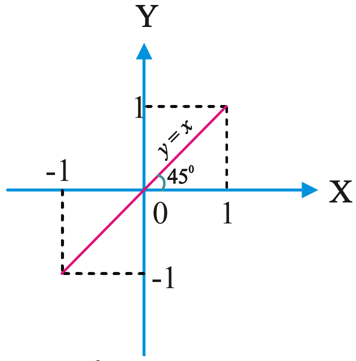


✈ Some useful non-periodic graphs :

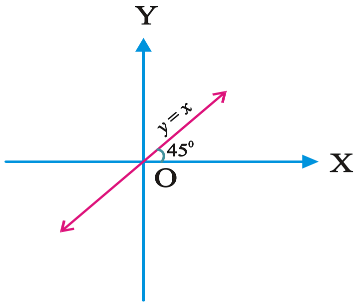
1. $y = Sin(Sin^{-1}x) = x,\ x \in [-1, 1],\ y \in [-1, 1]$



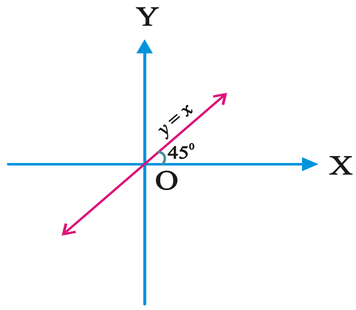
2. $y = Cos(Cos^{-1}x) = x,\ x \in [-1, 1],\ y \in [-1, 1]$



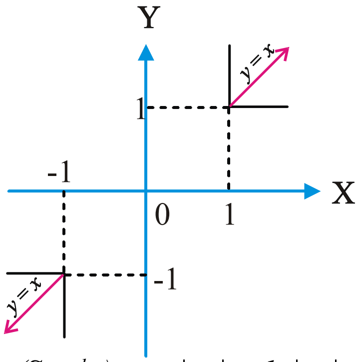
3. $y = Tan(Tan^{-1}x) = x,\ x \in R,\ y \in R$



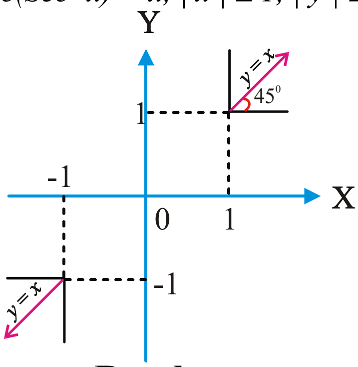
4. $y = Cot\,(Cot^{-1}x) = x,\ x \in R,\ y \in R$



5. $y = \operatorname{cosec}\left(\operatorname{cosec}^{-1}x\right) = x,\ |\,x\,| \geq 1,\ |\,y\,| \geq 1,$



6. $y = Sec(Sec^{-1}x) = x,\ |\,x\,| \geq 1,\ |\,y\,| \geq 1$



Important Results :

i) $Sin^{-1}(-x) = -Sin^{-1}x\ ,\ \forall\ x \in [-1,1]$

ii) $Cos^{-1}(-x) = \pi - Cos^{-1}x\ ,\forall\ \ x \in [-1,1]$

iii) $Tan^{-1}(-x) = -Tan^{-1}x\ ,\forall\ \ x \in R$

iv) $Cosec^{-1}(-x) = -Cosec^{-1}x\ ,\forall\ x \in R - (-1,1)$

v) $Cot^{-1}(-x) = \pi - Cot^{-1}x\ ,\forall\ x \in R$

vi) $Sec^{-1}(-x) = \pi - Sec^{-1}x\ ,\forall\ x \in (-\infty,-1] \cup [1,\infty)$

i) $\theta \in [0,\pi]$ then $Sin^{-1}(\cos\theta) = \dfrac{\pi}{2} - \theta$

ii) $\theta \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ then $Cos^{-1}(\sin\theta) = \dfrac{\pi}{2} - \theta$

iii) $\theta \in (0,\pi)$ then $Tan^{-1}(\cot\theta) = \dfrac{\pi}{2} - \theta$

iv) $\theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ then $Cot^{-1}(\tan\theta) = \dfrac{\pi}{2} - \theta$

v) $\theta \in \left[-\dfrac{\pi}{2}, 0\right) \cup \left(0, \dfrac{\pi}{2}\right]$ then

$$Sec^{-1}(\operatorname{cosec}\theta) = \frac{\pi}{2} - \theta$$

vi) $\theta \in \left[0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right]$ then

$$Cosec^{-1}(\sec\theta) = \frac{\pi}{2} - \theta$$

i) $$Sin^{-1}x = \begin{cases} Cos^{-1}\sqrt{1-x^2} & if\ \ 0 \leq x \leq 1 \\ -Cos^{-1}\sqrt{1-x^2} & if\ \ -1 \leq x < 0 \\ Tan^{-1}\dfrac{x}{\sqrt{1-x^2}} & if\ \ x \in (-1,1) \end{cases}$$

ii) $$Cos^{-1}x = \begin{cases} Sin^{-1}\sqrt{1-x^2} & if\ \ x \in [0,1] \\ \pi - Sin^{-1}\sqrt{1-x^2} & if\ \ x \in [-1,0) \end{cases}$$

iii) $$Tan^{-1}x = \begin{cases} Sin^{-1}\dfrac{x}{\sqrt{1+x^2}} & for\ \ x > 0 \\ Cos^{-1}\dfrac{1}{\sqrt{1+x^2}} & for\ \ x > 0 \end{cases}$$

i) $Sin^{-1}x + Cos^{-1}x = \dfrac{\pi}{2},\ \forall\ x \in [-1,1]$

ii) $Tan^{-1}x + Cot^{-1}x = \pi/2, \forall\ x \in R$

iii)

$$Sec^{-1}x + Cosec^{-1}x = \frac{\pi}{2}, \forall\ x \in (-\infty,-1] \cup [1,\infty)$$

$$Sin^{-1}x+Sin^{-1}y=\begin{cases} Sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\} \\ 0\le x,y\ ,x^2+y^2\le 1 \text{ or} \\ -1\le x,y\le 1, xy<0 \ and\ x^2+y^2>1 \\ \pi-Sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\} \\ if\ 0<x,y\le 1 \ \text{ and } \ x^2+y^2>1 \\ -\pi-Sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\} \\ \text{if } -1\le x,y<0 \text{ and } x^2+y^2>1 \end{cases}$$

$$Sin^{-1}x-Sin^{-1}y=\begin{cases} Sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\} \\ 0\le x,y<1 \ \text{ and } x^2+y^2>1 \text{ or} \\ -1\le x,y\le 1, xy<0 \text{ and} x^2+y^2\le 1 \\ \pi-Sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\} \\ if\ 0<x\le 1,-1\le y<1 \text{and} x^2+y^2>1 \\ -\pi-Sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\} \\ \text{if } 0<y\le 1,-1\le x<0 \text{ and} x^2+y^2>1 \end{cases}$$

$$Cos^{-1}x+Cos^{-1}y=\begin{cases} Cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\} \\ if\ -1\le x,y\le 1 \text{and} x+y\ge 0 \\ 2\pi-Cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\} \\ if\ -1\le x,y\le 1 \text{and} x+y<0 \end{cases}$$

$$Cos^{-1}x-Cos^{-1}y=\begin{cases} Cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\} \\ if\ -1\le x,y\le 1, x\le y \\ -Cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\} \\ if\ -1\le y\le 0, 0<x\le 1 \text{and} x\ge y \end{cases}$$

✈ (i) If x,y,z have same sign and $xy+yz+zx<1$ then

$$Tan^{-1}x+Tan^{-1}y+Tan^{-1}z=Tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

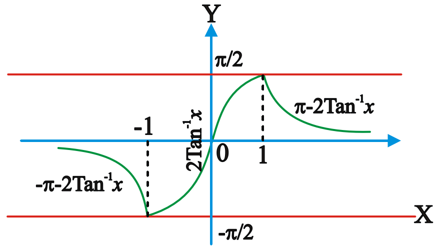
(ii) $\mathrm{Tan}^{-1}x_1+\mathrm{Tan}^{-1}x_2+......\mathrm{Tan}^{-1}x_n=$

$$\mathrm{Tan}^{-1}\left(\frac{S_1-S_3+S_5-....}{1-S_2+S_4-S_6....}\right)$$

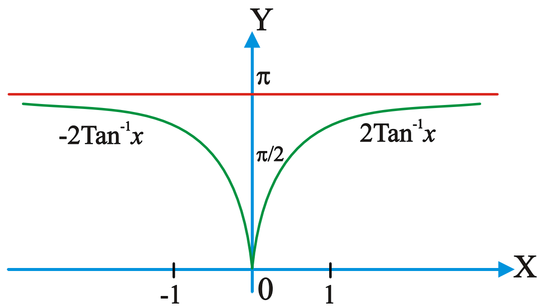
where S_1 = sum of values , S_2 = sum of product of taken two elements at a time and so on..,
S_n = product of values.

✈ Transformation of Inverse functions by elementry substitution and their graphs :

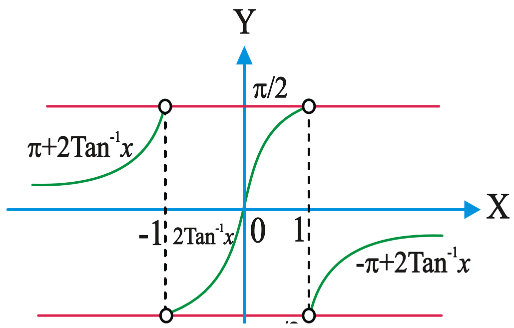
i) $$\mathrm{Sin}^{-1}\frac{2x}{1+x^2}=\begin{cases} -\pi-2\,\mathrm{Tan}^{-1}x & x\le -1 \\ 2\,\mathrm{Tan}^{-1}x & -1\le x\le 1 \\ \pi-2\,\mathrm{Tan}^{-1}x & x\ge 1 \end{cases}$$



ii) $$\mathrm{Cos}^{-1}\frac{1-x^2}{1+x^2}=\begin{cases} 2\,\mathrm{Tan}^{-1}x & x\ge 0 \\ -2\,\mathrm{Tan}^{-1}x & x<0 \end{cases}$$

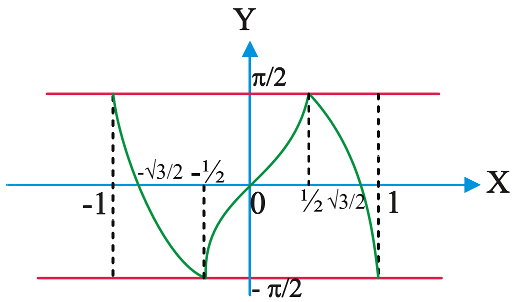


iii) $$\mathrm{Tan}^{-1}\frac{2x}{1-x^2}=\begin{cases} \pi+2\,\mathrm{Tan}^{-1}x & x<-1 \\ 2\,\mathrm{Tan}^{-1}x & -1<x<1 \\ -\pi+2\,\mathrm{Tan}^{-1}x & x>1 \end{cases}$$

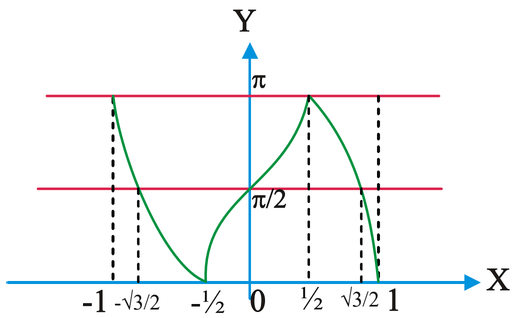


✈ i)

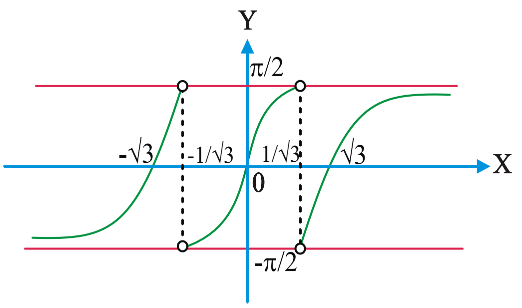
$$Sin^{-1}\left(3x-4x^3\right)=\begin{cases}-\left(\pi+3Sin^{-1}x\right) & if\ -1\le x\le -1/2\\ 3Sin^{-1}x & if\ -1/2\le x\le 1/2\\ \pi-3Sin^{-1}x & if\ 1/2\le x\le 1\end{cases}$$



ii) $$Cos^{-1}\left(4x^3-3x\right)=\begin{cases}3\cos^{-1}x-2\pi & if\ -1\le x\le -1/2\\ 2\pi-3Cos^{-1}x & if\ -1/2\le x\le 1/2;\\ 3Cos^{-1}x & if\ 1/2\le x\le 1\end{cases}$$



iii) $$Tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)=\begin{cases}3Tan^{-1}x & if\ -\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}\\ \pi+3Tan^{-1}x & if\ x>\frac{1}{\sqrt{3}}\\ -\pi+3Tan^{-1}x & if\ x<-\frac{1}{\sqrt{3}}\end{cases}$$



✈ i) $$2Sin^{-1}x=\begin{cases}-\pi-Sin^{-1}\left(2x\sqrt{1-x^2}\right) & if\ -1\le x\le -\frac{1}{\sqrt{2}}\\ Sin^{-1}\left(2x\sqrt{1-x^2}\right) & if\ -\frac{1}{\sqrt{2}}\le x\le \frac{1}{\sqrt{2}}\\ \pi-Sin^{-1}\left(2x\sqrt{1-x^2}\right) & if\ \frac{1}{\sqrt{2}}\le x\le 1\end{cases}$$

ii) $$2Cos^{-1}x=\begin{cases}Cos^{-1}\left(2x^2-1\right) & if\ 0\le x\le 1\\ 2\pi-Cos^{-1}\left(2x^2-1\right) & if\ -1\le x\le 0\end{cases}$$

iii)

$$2Tan^{-1}x=\begin{cases}Tan^{-1}\left(\frac{2x}{1-x^2}\right) & if\ -1<x<1\\ \pi+Tan^{-1}\left(\frac{2x}{1-x^2}\right) & if\ x>1\\ -\pi+Tan^{-1}\left(\frac{2x}{1-x^2}\right) & if\ x<-1\end{cases}$$

iv) $$2Tan^{-1}x=\begin{cases}-\pi-Sin^{-1}\left(\frac{2x}{1+x^2}\right) & if\ x<-1\\ Sin^{-1}\left(\frac{2x}{1+x^2}\right) & if\ -1\le x\le 1\\ \pi-Sin^{-1}\left(\frac{2x}{1+x^2}\right) & if\ x>1\end{cases}$$

v) $$2Tan^{-1}x=\begin{cases}-Cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) & if\ -\infty<x\le 0\\ Cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) & if\ 0\le x<\infty\end{cases}$$

✈ Some important facts :

1) $Tan^{-1}x+Tan^{-1}y+Tan^{-1}z=\frac{\pi}{2}$, if $xy+yz+zx=1$

2) $Tan^{-1}x+Tan^{-1}y+Tan^{-1}z=\pi$,

   if $x+y+z=xyz$ **(EAM-2014)**

3) $Tan^{-1}\frac{a}{x}+Tan^{-1}\frac{b}{x}=\frac{\pi}{2}$,then $x=\sqrt{ab}$

4) $Sin^{-1}\frac{a}{x}+Sin^{-1}\frac{b}{x}=\frac{\pi}{2}$, then $x=\sqrt{a^2+b^2}$

5) $Tan^{-1}\left(\frac{p}{q}\right)+Tan^{-1}\left(\frac{q-p}{q+p}\right)=\frac{\pi}{4}$

6) $Tan^{-1}\left(\frac{1+x}{1-x}\right)=\frac{\pi}{4}+\tan^{-1}x \quad if \ x<1$

7) $Tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{\pi}{4}-\tan^{-1}x \quad if \ x>-1$

8) If $Tan^{-1}x+Tan^{-1}y=\frac{\pi}{2}$ then xy=1.

9) $Cot^{-1}x+Cot^{-1}y=\frac{\pi}{2}$ then xy=1

10) If $\mathrm{Cos}^{-1}x+\mathrm{Cos}^{-1}y+\mathrm{Cos}^{-1}z=3\pi$ then $xy+yx+zx=3$

11) If $\mathrm{Sin}^{-1}x+\mathrm{Sin}^{-1}y+\mathrm{Sin}^{-1}z=\frac{3\pi}{2}$ then $xy+yx+zx=3$

12) If $\mathrm{Sin}^{-1}x+\mathrm{Sin}^{-1}y=\theta$ then $\mathrm{Cos}^{-1}x+\mathrm{Cos}^{-1}y=\pi-\theta$

13) If $\mathrm{Cos}^{-1}x+\mathrm{Cos}^{-1}y=\theta$ then $\mathrm{Sin}^{-1}x+\mathrm{Sin}^{-1}y=\pi-\theta$

14) If $a\mathrm{Sin}^{-1}x-b\mathrm{Cos}^{-1}x=c$ then

$$a\mathrm{Sin}^{-1}x+b\mathrm{Cos}^{-1}x=\frac{\pi ab+c(a-b)}{a+b}$$

15) If $Cos^{-1}\frac{x}{a}+Cos^{-1}\frac{y}{b}=\theta$ then

$$\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\theta+\frac{y^2}{b^2}=Sin^2\theta$$

16) If $Sin^{-1}\frac{x}{a}+Sin^{-1}\frac{y}{b}=\theta$ *then*

$$\frac{x^2}{a^2}+\frac{2xy}{ab}\cos\theta+\frac{y^2}{b^2}=Sin^2\theta$$

17) $Tan^{-1}\left(\frac{1}{1+x(x+1)}\right)+Tan^{-1}\left(\frac{1}{1+(x+1)(x+2)}\right)+....+$

$$Tan^{-1}\left(\frac{1}{1+(x+n-1)(x+n)}\right)=Tan^{-1}(x+n)-Tan^{-1}x, n\in N$$

✈ If an expression contains

i) $\sqrt{a^2-x^2}$, *put* $x=a\sin\theta, \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

ii) $\sqrt{a^2+x^2}$, *put* $x=a\tan\theta, \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

iii) $\sqrt{x^2-a^2}$, *put* $x=a\sec\theta, \theta\in(0,\pi)$

✈ **Range of some special inverse Trigonometric Functions :**

i) $\frac{\pi^3}{32}\le\left(\mathrm{Sin}^{-1}x\right)^3+\left(\mathrm{Cos}^{-1}x\right)^3\le\frac{7\pi^3}{8}$

ii) $\frac{\pi^2}{8}\le\left(\mathrm{Sin}^{-1}x\right)^2+\left(\mathrm{Cos}^{-1}x\right)^2\le\frac{5\pi^2}{4}$

iii) $-\frac{\pi^2}{4}\le\left(\mathrm{Cos}^{-1}x\right)^2-\left(\mathrm{Sin}^{-1}x\right)^2\le\frac{3\pi^2}{4}$

EXAMPLES

1. $Cos^{-1}\left(\frac{-1}{2}\right)-2Sin^{-1}\left(\frac{1}{2}\right)+3Cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)-4Tan^{-1}(-1)=$ ***(EAM-2009)***

Sol: $Cos^{-1}\left(\frac{-1}{2}\right)-2Sin^{-1}\left(\frac{1}{2}\right)+3Cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

$$-4Tan^{-1}(-1)=\pi-Cos^{-1}\left(\frac{1}{2}\right)-2\left(\frac{\pi}{6}\right)$$

$$+3\left(\pi-Cos^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)+4Tan^{-1}(1)$$

$$=\pi-\frac{\pi}{3}-\frac{\pi}{3}+3\left(\pi-\frac{\pi}{4}\right)+4\left(\frac{\pi}{4}\right)$$

$$=\frac{\pi}{3}+3\left(\frac{3\pi}{4}\right)+\pi=\frac{43\pi}{12}$$

2:

The value of x, where $x>0$ and $Tan\left(Sec^{-1}\frac{1}{x}\right)=Sin\left(Tan^{-1}2\right)$ is (EAM-2007)

Sol: $Tan\left(Sec^{-1}\frac{1}{x}\right)=Sin\left(Tan^{-1}2\right)$

$$Tan\left(Tan^{-1}\frac{\sqrt{1-x^2}}{x}\right)=Sin\left(Sin^{-1}\frac{2}{\sqrt{1+2^2}}\right)$$

$$\left(\because Tan^{-1}x=Sin^{-1}\frac{x}{\sqrt{1+x^2}}\right)$$

$$\frac{\sqrt{1-x^2}}{x}=\frac{2}{\sqrt{5}}\Rightarrow 5\left(1-x^2\right)=4x^2$$

$$\Rightarrow x^2=\frac{5}{9}\Rightarrow x=\frac{\sqrt{5}}{3}$$

3:

If $\frac{1}{2}\le x\le 1$ **then**

$Cos^{-1}\left(\frac{x}{2}+\frac{1}{2}\sqrt{3-3x^2}\right)+Cos^{-1}x$ **is equal to**

(EAM-2012)

Sol: $Cos^{-1}x+Cos^{-1}\left(\frac{x}{2}+\frac{1}{2}\sqrt{3-3x^2}\right)$

$$=Cos^{-1}x+Cos^{-1}\left(x.\frac{1}{2}+\frac{\sqrt{3}}{2}\sqrt{1-x^2}\right)$$

$$=Cos^{-1}x+Cos^{-1}\left(\frac{1}{2}\right)-Cos^{-1}x=Cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$$

4:

The value of $Cot\left(Co\sec^{-1}\frac{5}{3}+Tan^{-1}\frac{2}{3}\right)=$

(AIE-2008)

Sol: $Cot\left(Co\sec^{-1}\frac{5}{3}+Tan^{-1}\frac{2}{3}\right)=Cot\left(Sin^{-1}\frac{3}{5}+Tan^{-1}\frac{2}{3}\right)$

$$=Cot\left(Tan^{-1}\frac{3}{4}+Tan^{-1}\frac{2}{3}\right)\quad\left[\because Sin^{-1}x=Tan^{-1}\frac{x}{\sqrt{1-x^2}}\right]$$

$$=Cot\left[Tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\left(\frac{3}{4}\right)\left(\frac{2}{3}\right)}\right)\right]=Cot\left[Tan^{-1}\left(\frac{\frac{17}{12}}{\frac{1}{2}}\right)\right]$$

$$=\cot\cot^{-1}\left(\frac{17}{6}\right)=\cot\tan^{-1}\left(\frac{17}{6}\right)=\frac{6}{17}$$

5:

$Sin^{-1}\frac{4}{5}+2Tan^{-1}\frac{1}{3}=$ **(EAM-05)**

Sol: $Sin^{-1}\frac{4}{5}+2Tan^{-1}\frac{1}{3}$

$$=Sin^{-1}\frac{4}{5}+Tan^{-1}\left(\frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2}\right)\left[\because 2Tan^{-1}x=Tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$=Sin^{-1}\frac{4}{5}+Tan^{-1}\frac{3}{4}$$

$$=Sin^{-1}\frac{4}{5}+Cos^{-1}\frac{4}{5}\left[\because Tan^{-1}x=Cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right]$$

$$=\frac{\pi}{2}\qquad\left[\because Sin^{-1}x+Cos^{-1}x=\frac{\pi}{2}\right]$$

6:

The set of values of x such that $\mathrm{Sin}^{-1}x-\mathrm{Cos}^{-1}x>0$ **are**

Sol: $\mathrm{Sin}^{-1}x-\mathrm{Cos}^{-1}x>0\Rightarrow \mathrm{Sin}^{-1}x>\frac{\pi}{4}$

$$\therefore\ x\in\left(\frac{1}{\sqrt{2}},1\right]as\ |x|\le 1$$

7:

The sum to the n terms of the series

$Co\sec^{-1}\sqrt{10}+Co\sec^{-1}\sqrt{50}+Co\sec^{-1}\sqrt{70}+.....$

$...+Co\sec^{-1}\sqrt{\left(n^2+1\right)\left(n^2+2n+2\right)}$ **is**

Sol: Let $y=Co\sec^{-1}\sqrt{\left(n^2+1\right)\left(n^2+2n+2\right)}$

$$Co\sec^2 y=\left(n^2+1\right)\left[\left(n^2+1\right)+2n+1\right]$$

$$=\left(n^2+1\right)^2+2n\left(n^2+1\right)+n^2+1$$

$$=\left(n^2+n+1\right)^2+1$$

$$\cot^2 y=\left(n^2+n+1\right)^2\quad\left[\because \cos ec^2\theta-\cot^2\theta=1\right]$$

$$Tany=\frac{1}{n^2+n+1}=\frac{(n+1)-n}{1+(n+1)n}$$

$$y = Tan^{-1}\left[\frac{(n+1)-n}{1+(n+1)n}\right]$$

$$y = Tan^{-1}(n+1) - Tan^{-1}n$$

Thus sum of n terms of the given series

$$y = \left(Tan^{-1}2 - Tan^{-1}1\right) + \left(Tan^{-1}3 - Tan^{-1}2\right) + \left(Tan^{-1}4 - Tan^{-1}3\right) + ... + \left(Tan^{-1}(n+1) - Tan^{-1}n\right)$$

$$= Tan^{-1}(n+1) - Tan^{-1}1 \quad = Tan^{-1}(n+1) - \frac{\pi}{4}$$

8:

$Sin^{-1}(\sin 5) =$

Sol: Here $\theta = 5$ rad , Clearly it does not lie between $\frac{-\pi}{2}\, and\, \frac{\pi}{2}$. But $2\pi - 5$ and $5 - 2\pi$ both lies between $\frac{-\pi}{2}\, and\, \frac{\pi}{2}$.

$$Sin(5-2\pi) = Sin\left[-(2\pi-5)\right] = -Sin(2\pi-5) = Sin5$$

$$\therefore Sin^{-1}(Sin5) = Sin^{-1}\left(Sin(5-2\pi)\right) = 5 - 2\pi$$

9:

$Cos^{-1}(\cos 10)$ **is equal to....**

Sol: We know that $Cos^{-1}(\cos\theta) = \theta$ if $0 \le \theta \le \pi$

Here $\theta = 10\, rad$, clearly, it does not lie between 0 and π . But $4\pi - 10$ lies between 0 and π

$$\therefore Cos^{-1}(\cos 10) = Cos^{-1}\left(\cos(4\pi - 10)\right) = 4\pi - 10$$

10:

$Tan^{-1}\left(Tan(-6)\right)$ **is equal to.....**

Sol: We know that $Tan^{-1}(Tan\theta) = \theta$ if $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$

Here $\theta = -6\, rad$, does not lie between $\frac{-\pi}{2}\, and\, \frac{\pi}{2}$

But $2\pi - 6$ lies between $\frac{-\pi}{2}\, and\, \frac{\pi}{2}$

Now $Tan(2\pi - 6) = -Tan6 = Tan(-6)$

$$\therefore Tan^{-1}\left(Tan(-6)\right) = Tan^{-1}\left(Tan(2\pi-6)\right) = 2\pi - 6$$

EXERCISE - I

1. The domain of $Sin^{-1}\frac{2x+1}{3}$ **is**

1. (-2, 1] 2. [-2, 1] 3. R 4. [-1, 1]

2. The domain of $\mathrm{Sin}^{-1}x + \mathrm{Cos}^{-1}x$ **is**

1) $(-\pi, \pi)$ 2) $[-1,1]$ 3) $(0, 2\pi)$ 4) $(-\infty, \infty)$

3. The range of $\mathrm{Tan}^{-1}x$ **is**

1) R 2) $(0, \pi)$ 3) $[0, \pi]$ 4) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

4. The principal value of $Sin^{-1}\left(Sin\left(\frac{2\pi}{3}\right)\right)$ **is**

1) $\frac{2\pi}{3}$ 2) $\frac{\pi}{3}$ 3) $\frac{-\pi}{3}$ 4) $\frac{-2\pi}{3}$

5. The principal value of $Sin^{-1}(Tan(\frac{-5\pi}{4}))$ **is**

1) $\frac{\pi}{4}$ 2) $\frac{-\pi}{4}$ 3) $\frac{\pi}{2}$ 4) $\frac{-\pi}{2}$

6. The principal value of $\mathrm{Cos}^{-1}\left(\cos\frac{7\pi}{6}\right)$ **is**

1) $\frac{7\pi}{6}$ 2) $\frac{5\pi}{3}$ 3) $\frac{5\pi}{6}$ 4) $\frac{13\pi}{6}$

7. $Sin^{-1}(\frac{\sqrt{3}}{2}) - Tan^{-1}(-\sqrt{3})$ **is**

1) $\frac{-\pi}{3}$ 2) $\frac{2\pi}{3}$ 3) $\frac{\pi}{6}$ 4) 0

8. $\cot\left[\mathrm{Sin}^{-1}\sqrt{\frac{13}{17}}\right] - \sin\left[\mathrm{Tan}^{-1}\frac{2}{3}\right] =$

1) $-\frac{2}{\sqrt{13}}$ 2) 0 3) $\frac{2}{\sqrt{13}}$ 4) $\frac{2}{3\sqrt{13}}$

9. $\sec^2(\cot^{-1} ½) + \mathrm{cosec}^2(\tan^{-1} 1/3) =$

1) 5 2) 10 3) 15 4) 50

10. Find the value of $\mathrm{Sin}\left(\mathrm{Sin}^{-1}\frac{4}{5}+\mathrm{Sin}^{-1}\frac{7}{25}\right)$

1) $\frac{119}{125}$ 2) $\frac{117}{125}$ 3) $\frac{118}{125}$ 4) $\frac{113}{125}$

11. $\mathrm{Tan}^{-1}(2) + \mathrm{Tan}^{-1}(3) =$

1) $-\frac{\pi}{4}$ 2) $\frac{\pi}{4}$ 3) $\frac{3\pi}{4}$ 4) $\frac{5\pi}{4}$

12. $Tan^{-1}(\frac{m}{n}) - Tan^{-1}(\frac{m-n}{m+n}) =$

1) $\frac{\pi}{2}$ 2) $\frac{\pi}{3}$ 3) $\frac{\pi}{4}$ 4) $\frac{3\pi}{4}$

13. $2\,\mathrm{Tan}^{-1}\frac{1}{3} + \mathrm{Tan}^{-1}\frac{1}{7} =$

1) π 2) $\frac{\pi}{2}$ 3) $\frac{\pi}{4}$ 4) $\frac{3\pi}{4}$

14. The numerical value of tan $\left(2\,\mathrm{Tan}^{-1}\frac{1}{5}-\frac{\pi}{4}\right)$ **is**

1) 1 2) 0 3) $\frac{7}{17}$ 4) $-\frac{7}{17}$

15. $\sec[\mathrm{Tan}^{-1} 5 + \mathrm{Tan}^{-1}\frac{1}{5} - \mathrm{Tan}^{-1}\frac{3}{4}] =$

1) $3/5$ 2) $5/3$ 3) $4/5$ 4) $\sqrt{2}$

16. If $\mathrm{Sin}^{-1}(\frac{12}{13}) + \mathrm{Sec}^{-1}(\frac{13}{x}) = \frac{\pi}{2}$ **then x =**

1) 12 2) 13 3) 11 4) 5

17. If $\mathrm{Sin}^{-1}(\frac{3}{x}) + \mathrm{Sin}^{-1}(\frac{4}{x}) = \frac{\pi}{2}$ **then x =**

(EAM-2008)

1) 6 2) 7 3) 8 4) 5

18. If $\mathrm{Sin}^{-1}(x-\frac{x^2}{2}+\frac{x^3}{4}-...)+\mathrm{Cos}^{-1}(x^2-\frac{x^4}{2}+\frac{x^6}{4}-...)$

$= \frac{\pi}{2}$ **for** $0<|x|<\sqrt{2}$ **then x =**

1) 1/2 2) 1 3) - 1/2 4) -1

19. $Tan^{-1}(\frac{a}{x}) + Tan^{-1}(\frac{b}{x}) = \frac{\pi}{2}$ *then* $x =$

1) ab 2) $\sqrt{ab}$ 3) $\sqrt{a^2+b^2}$ 4) a^2+b^2

20. If $\mathrm{Tan}^{-1}(\frac{3a^2x - x^3}{a^3 - 3ax^2}) = k\,\mathrm{Tan}^{-1}\left(\frac{x}{a}\right)$ **then k =**

1) 2 2) 3 3) - 2 4) 4

21. If $\mathrm{Tan}^{-1}(\frac{2x}{x^2 - 1}) + \mathrm{Cos}^{-1}\frac{x^2-1}{x^2+1} = \frac{2\pi}{3}$ **then x =**

1) $2-\sqrt{3}$ 2) $\sqrt{3}-\sqrt{2}$ 3) $2+\sqrt{3}$ 4) $+\sqrt{2}$

22. If $\mathrm{Tan}^{-1}(\frac{x-1}{x-2}) + \mathrm{Tan}^{-1}(\frac{x+1}{x+2}) = \frac{\pi}{4}$ **then x=**

1) $\frac{1}{\sqrt{2}}$ 2) $\pm\frac{1}{\sqrt{2}}$ 3) $\pm\frac{1}{\sqrt{3}}$ 4) $\frac{1}{\sqrt{3}}$

23. A value of $\mathrm{Tan}^{-1}\left\{\sin\left(\mathrm{Cos}^{-1}\sqrt{\frac{2}{3}}\right)\right\}$ **is**

1) $\frac{\pi}{4}$ 2) $\frac{\pi}{2}$ 3) $\frac{\pi}{3}$ 4) $\frac{\pi}{6}$

24. The equation $2\,\mathrm{Cos}^{-1}x = \mathrm{Cos}^{-1}(2x^2-1)$ **is satisfied by**

1) $-1\le x\le 1$ 2) $0\le x\le 1$
3) $x\ge 1$ 4) $x\le 1$

25. $\mathrm{Sin}^{-1}\left[x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right] =$

1) $\sin^{-1}x+\sin^{-1}\sqrt{x}$ 2) $\sin^{-1}x-\sin^{-1}\sqrt{x}$
3) $\sin^{-1}\sqrt{x}-\sin^{-1}x$ 4) 0

26. If n $\in$ **N,** $\sum_{k=1}^{n} Sin^{-1}(x_k) = \frac{n\pi}{2}$ **then** $\sum_{k=1}^{n} x_k =$

1) n 2) k 3) $\dfrac{k(k+1)}{2}$ 4) $\dfrac{n(n+1)}{2}$

27. If $\sum_{r=1}^{n} \text{Cos}^{-1} x_r = 0$, then $\sum_{r=1}^{n} x_r$ equals to

1) 0 2) n 3) $\dfrac{n(n+1)}{2}$ 4) $\dfrac{n}{2}$

28. If Tan^{-1}x, Tan^{-1}y, Tan^{-1}z are in A.P. then $\dfrac{2y}{1-y^2}$=

1) $\dfrac{x-z}{1+xz}$ 2) $\dfrac{x+z}{1-xz}$ 3) x + z 4) xz

29. If $\theta = \text{Tan}^{-1} a,\ \phi = \text{Tan}^{-1} b$ and ab=-1, then $\theta - \phi$ is equal to :

1) 0 2) $\dfrac{\pi}{4}$ 3) $\dfrac{\pi}{2}$ 4) π

30. If $\alpha = \sin^{-1}\dfrac{\sqrt{3}}{2} + \sin^{-1}\dfrac{1}{3}$ and

$\beta = \cos^{-1}\dfrac{\sqrt{3}}{2} + \cos^{-1}\dfrac{1}{3}$ then

1) $\alpha > \beta$ 2) $\alpha = \beta$

3) $\alpha < \beta$ 4) $\alpha + \beta = 2\pi$

KEY

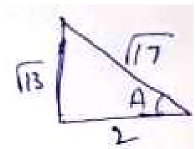
01) 2	02) 2	03) 4	04) 2	05) 4	06) 3
07) 2	08) 2	09) 3	10) 2	11) 3	12) 3
13) 3	14) 4	15) 2	16) 1	17) 4	18) 2
19) 2	20) 2	21) 1	22) 2	23) 4	24) 2
25) 2	26) 1	27) 2	28) 2	29) 3	30) 3

SOLUTIONS

1. $Sin^{-1}x$ domain [-1, 1] $\Rightarrow -1 \le \dfrac{2x+1}{3} \le 1$
2. Domain =[-1, 1]
3. Range of Tan^{-1}x=$(\dfrac{-\pi}{2}, \dfrac{\pi}{2})$
4. $Sin^{-1}\left(Sin\left(\dfrac{2\pi}{3}\right)\right) = Sin^{-1}[Sin(\pi - \dfrac{\pi}{3})]$
5. $Sin^{-1}\left[Tan\left(\dfrac{-5\pi}{4}\right)\right] = Sin^{-1}(-1)$
6. $Cos^{-1}[Cos(\dfrac{7\pi}{6})] = Cos^{-1}[\cos(2\pi - \dfrac{5\pi}{6})] = \dfrac{5\pi}{6}$
7. $Sin^{-1}(\dfrac{\sqrt{3}}{2}) - Tan^{-1}(-\sqrt{3}) = \dfrac{\pi}{3} + \dfrac{\pi}{3} = \dfrac{2\pi}{3}$
8. $\cot\left(\sin^{-1}\sqrt{\dfrac{13}{17}}\right] - \sin\left(\tan^{-1}\dfrac{2}{3}\right) =$

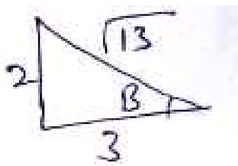
Let $\sin^{-1}\sqrt{\dfrac{13}{17}} = A$

$\Rightarrow \sin A = \dfrac{\sqrt{13}}{\sqrt{17}}$



$\cot A = \dfrac{2}{\sqrt{13}}$

Let $\tan^{-1}\left(\dfrac{2}{3}\right) = B \Rightarrow \tan B = \dfrac{2}{3}$



$\sin B = \dfrac{2}{\sqrt{13}}$

Now $\cot A - \sin B = 0$

9. $\sec^2\left(\tan^{-1} 2\right) + \cos ec^2\left(\cot^{-1} 3\right) =$

$1 + \tan^2\left(\tan^{-1} 2\right) + 1 + \cot^2\left(\cot^{-1} 3\right)$

10. $\sin\left(\sin^{-1}\dfrac{4}{5} + \sin^{-1}\dfrac{7}{25}\right)$

We know

$\sin^{-1}(x) + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + \sqrt{1-x^2}\,y\right)$

$= \sin(\sin^{-1}\left(\dfrac{4}{5}\sqrt{1-\dfrac{49}{625}} + \sqrt{1-\dfrac{16}{25}}\,\dfrac{7}{25}\right)$

$= \dfrac{4}{5}\dfrac{24}{25} + \dfrac{3}{5}\dfrac{7}{25} = \dfrac{96+21}{125} = \dfrac{117}{125}$

11. $Tan^{-1}x + Tan^{-1}y = \pi + Tan^{-1}(\dfrac{x+y}{1-xy}),$

x>0,y>0 xy>1 formula

12. Apply $Tan^{-1}x - Tan^{-1}y = Tan^{-1}(\frac{x-y}{1+xy})$, formula

13. $2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)$

we know $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$$= \tan^{-1}\left(\frac{2.\,1/3}{1-1/9}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{2/3}{8/9_3}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}.\frac{1}{7}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{21+4}{28}}{\frac{28-3}{28}}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

14. $\tan\left(2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right) = \tan\left(\tan^{-1}\left(\frac{2\,1/5}{1-1/25}\right)-\tan^{-1}(1)\right)$

$$\left(\therefore 2\tan^{-1}x = \tan^{-1}(\frac{2k}{1-x^2}\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{2/5}{24/25}\right)-\tan^{-1}(1)\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{5}{12}\right)-\tan^{-1}(1)\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{\frac{5}{12}-1}{1+\frac{5}{21}1}\right)\right)$$

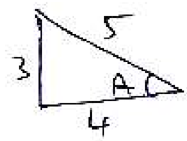
$$= \frac{5-12}{12+5} = \frac{-7}{17}$$

15. $\text{see}\left[\tan^{-1}5+\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{3}{4}\right]$

$$= \text{see}\left[\frac{\pi}{2}-\tan^{-1}\frac{3}{4}\right] = \text{cosec}\left(\tan^{-1}\frac{3}{4}\right)$$

Let $\tan^{-1}\left(\frac{3}{4}\right) = A$

$\tan A = \frac{3}{4}$



$\text{cosec}\,A = \frac{5}{3}$

16. $Sec^{-1}x = Cos^{-1}\frac{1}{x}$ and $Sin^{-1}x + Cos^{-1}x = \frac{\pi}{2}$

17. Apply $Sin^{-1}\frac{a}{x}+Sin^{-1}\frac{b}{x}=\frac{\pi}{2}$ then $x=\sqrt{a^2+b^2}$

18. Put $x = x^2 \Rightarrow x(x-1)=0 \Rightarrow x = 0\ (or)\ 1$

19. Apply $Tan^{-1}x + Cot^{-1}x = \frac{\pi}{2} \Rightarrow \frac{a}{x}=\frac{x}{b} \Rightarrow x=\sqrt{ab}$

20. Given $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right) = k\ \tan^{-1}\left(\frac{x}{a}\right)$

put $x = a\tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{x}{a}\right)$

$$\Rightarrow \tan^{-1}\left(\frac{3a^3\tan\theta - a^3\tan^3\theta}{a^3-3a^3\tan^2\theta}\right) = K\tan^{-1}\left(\frac{x}{a}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right) = K\tan^{-1}\left(\frac{x}{a}\right)$$

$$\Rightarrow \tan^{-1}(\tan 3\theta) = K\tan^{-1}\left(\frac{x}{a}\right)$$

$$\Rightarrow 3\theta = K\tan^{-1}\left(\frac{x}{a}\right)$$

$$\Rightarrow 3\tan^{-1}\left(\frac{x}{a}\right) = k\tan^{-1}\left(\frac{x}{a}\right)$$

$$k = 3$$

21. Given $\tan^{-1}\left(\frac{2x}{x^2-1}\right)+\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)=\frac{2\pi}{3}$

put $x=\tan\theta$

$$\Rightarrow \theta = \tan^{-1}x$$

$$\Rightarrow \tan^{-1}\left(\frac{2\tan\theta}{\tan^2\theta-1}\right)+\cos^{-1}\left(\frac{\tan^2\theta-1}{\tan^2\theta+1}\right)=\frac{2\pi}{3}$$

$$\Rightarrow \tan^{-1}(-\tan 2\theta)+\cos^{-1}(-\cos 2\theta)=\frac{2\pi}{3}$$

$$\Rightarrow -\tan^{-1}(\tan 2\theta)+\pi-\cos^{-1}(\cos 2\theta)=\frac{2\pi}{3}$$

$$\Rightarrow -2\theta+\pi-2\theta=\frac{2\pi}{3}$$

$$\Rightarrow \pi-4\theta=\frac{2\pi}{3}$$

$$4\theta = \pi/3$$

$$\Rightarrow \theta=\frac{\pi}{12}$$

$$\tan^{-1}x=\frac{\pi}{12}$$

$$x=\tan\frac{\pi}{12}=2-\sqrt{3}$$

22. $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}.\frac{x+1}{x+2}}\right)=\frac{\pi}{4}$$

$$\Rightarrow \frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)-(x-1)(x+1)}=1$$

$$\Rightarrow x^2+x-2+x^2-x-2=x^2-4-x^2+1$$

$$2x^2-4=-3$$

$$2x^2=1$$

$$\Rightarrow x^2 = 1/2 \Rightarrow x \pm \frac{1}{\sqrt{2}}$$

23. $Tan^{-1}[Sin(Sin^{-1}\sqrt{\frac{1}{3}})]=Tan^{-1}(\frac{1}{\sqrt{3}})=\frac{\pi}{6}$

24. Given equation satisfied for $0\le x\le 1$

25. $Sin^{-1}[x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}]=Sin^{-1}[x\sqrt{1-(\sqrt{x})^2}-\sqrt{x}\sqrt{1-x^2}]$

$= Sin^{-1}x - Sin^{-1}\sqrt{x}$

26. Put $x_1=x_2=x_3=........=1$

$$\sum_{k=1}^{n} x_k = 1+1+1+........+1(n\ time)=n$$

27. $Cos^{-1}x_1=Cos^{-1}x_2=.........=Cos^{-1}x_n=0$

$$\Rightarrow x_1=x_2=............=x_n=+1$$

$$\therefore \sum_{r-1}^{n} x_r = n$$

28. $2Tan^{-1}y=Tan^{-1}x+Tan^{-1}z$

$$Tan^{-1}(\frac{2y}{1-y^2})=Tan^{-1}(\frac{x+z}{1-xz})$$

29. $Tan(\theta-\Phi)=\frac{Tan\theta-Tan\Phi}{1+Tan\theta Tan\Phi}=\frac{a-b}{1-1}=\infty$

$$\theta-\Phi=\frac{\pi}{2}$$

30. Given

$$\alpha=\sin^{-1}\frac{\sqrt{3}}{2}+\sin^{-1}\frac{1}{3},\beta=\cos^{-1}\frac{\sqrt{3}}{2}+\cos^{-1}\frac{1}{3}$$

$$\alpha+\beta=\sin^{-1}\frac{\sqrt{3}}{2}+\sin^{-1}\frac{1}{3}+\cos^{-1}\frac{\sqrt{3}}{2}+\cos^{-1}\frac{1}{3}$$

$$\alpha+\beta=\frac{\pi}{2}+\frac{\pi}{2}=\pi$$

Also $\alpha=\frac{\pi}{3}+\sin^{-1}\frac{1}{3}<\frac{\pi}{3}+\sin^{-1}\frac{1}{2}$

As $\sin\theta$ is increasing in $\left[0,\pi/2\right]$

$$\therefore \alpha<\frac{\pi}{3}+\frac{\pi}{6}=\pi/2$$

$$\alpha<\pi/2<\beta \qquad (\therefore \alpha+\beta=\pi)$$

$$\therefore -\alpha<\beta$$

* For $x\in\left(0,\frac{\pi}{2}\right)$,If

$$\cos^{-1}\left(\frac{7}{2}(1+\cos 2x)+\sqrt{\sin^2 x-48\cos^2 x}\sin x\right)$$

$=x-\cos^{-1}(k\cos x)$ then k is

1)1 2)5 3)7 4)14

Key:- 3

Sol:- Let

$$y=\cos^{-1}\left(\frac{7}{2}(1+\cos 2x)+\sqrt{1-49\cos^2 x}\sqrt{\sin^2 x}\right)$$

$$y=\cos^{-1}\left((7\cos x)(\cos x)+\sqrt{1-(7\cos x)^2}\sqrt{1-(\cos x)^2}\right)$$

$$y=\cos^{-1}\cos(x)-\cos^{-1}(7\cos x)$$

$$=x-\cos^{-1}(7\cos x)$$

$$\therefore k=7$$

* The value of the expression

$\cot^{-1}\frac{1}{2}+\cot^{-1}\frac{9}{2}+\cot^{-1}\frac{25}{2}+----$ upto n term

1) $\tan^{-1}2n$ 2) $\tan^{-1}(2n-1)$

3) $\tan^{-1}n$ 4) $\tan^{-1}2n-\tan^{-1}1$

Key:- 1

Sol:- Given expression

$$=\tan^{-1}2+\tan^{-1}\frac{2}{9}+\tan^{-1}\frac{2}{25}+\tan^{-1}\frac{2}{49}---$$

General term

$$\frac{2}{(2n-1)^2}=\frac{2}{4n^2-4n+1}=\frac{2}{1+4n(n-1)}$$

$$=\frac{2n-(2n-2)}{1+2n(2n-2)}$$

$$T_n=\tan^{-1}\left(\frac{2n-(2n-2)}{1+2n(2n-2)}\right)=\tan^{-1}2n-\tan^{-1}(2n-2)$$

$\therefore$ Sum of the series

$$=\tan^{-1}2-\tan^{-1}0+\tan^{-1}4-\tan^{-1}2$$
$$+\tan^{-1}6-\tan^{-1}4---\tan^{-1}2n-\tan^{-1}(2n-2)$$

$$=\tan^{-1}2n-\tan^{-1}0$$
$$=\tan^{-1}2n$$

EXERCISE - II

1. Range of $Sin^{-1}x-Cos^{-1}x$ is

1) $[\frac{-3\pi}{2},\frac{\pi}{2}]$ 2) $[\frac{-5\pi}{3},\frac{\pi}{3}]$

3) $[\frac{-3\pi}{2},\pi]$ 4) $[0,\pi]$

2. Range of $f(x)=\text{Sin}^{-1}x+\text{Tan}^{-1}x+\text{Sec}^{-1}x$ is

1) $\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$ 2) $\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$

3) $\left\{\frac{\pi}{4},\frac{3\pi}{4}\right\}$ 4) $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$

3. Range of $\text{Sin}^{-1}x+\text{Cos}^{-1}x+\text{Tan}^{-1}x$ is

1) $\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$ 2) $(0,\pi]$ 3) $\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$ 4) $[0,\pi]$

4. The domain of $\text{Sin}^{-1}[\log_2(x^2/2)]$ is

1) $[-2,-1]$ 2) $[1,2]$

3) $[-2,-1]\cup[1,2]$ 4) $[-2,0]$

5. **The ascending order of** $A=\mathrm{Sin}^{-1}\left(\log_3{}^2\right)$, $B=\mathrm{Cos}^{-1}\left(\log_3\left(\frac{1}{2}\right)\right)$, $C=\mathrm{Tan}^{-1}\left(\log_{1/3}2\right)$ **is**

1) C,B,A 2) B,A,C 3) C, A,B 4) B,C,A

6. **The decreasing order of ,**

$A=\left(\mathrm{Sin}^{-1}\left(\frac{1-\sqrt{3}}{2\sqrt{2}}\right)\right)$ $B=\mathrm{Cos}^{-1}\left(\frac{-1}{\sqrt{2}}\right)$,

$C=\mathrm{Tan}^{-1}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$ **is**

1)B,A,C 2)B,C,A 3)C,A,B 4)C,B,A

7. $\mathrm{Cos}^{-1}\left(\cos\frac{10\pi}{7}\right)+\mathrm{Sin}^{-1}\left(\sin\frac{35\pi}{11}\right)+$ $\mathrm{Tan}^{-1}\left(\tan\frac{24\pi}{13}\right)+Cot^{-1}\left(\cot\frac{26\pi}{5}\right)$ **is**

1) $\frac{4\pi}{7}+\frac{\pi}{55}+\frac{2\pi}{13}$ 2) $\frac{4\pi}{7}-\frac{\pi}{55}-\frac{2\pi}{13}$

3) $\frac{4\pi}{7}-\frac{\pi}{55}+\frac{2\pi}{13}$ 4) $-\frac{4\pi}{7}+\frac{\pi}{55}+\frac{2\pi}{13}$

8. If $\mathrm{Sin}^{-1}(6x)+\mathrm{Sin}^{-1}(6\sqrt{3}\,x)=-\frac{\pi}{2}$ **then the value of x is** (Eamcet- 2017)

1) $\frac{1}{12}$ 2) $-\frac{1}{12}$ 3) $-\frac{1}{4\sqrt{3}}$ 4) $\frac{1}{4\sqrt{3}}$

9. **If** $\mathrm{Tan}^{-1}(2x)+\mathrm{Tan}^{-1}(3x)=\frac{\pi}{4}$ **then x =**

1) 1/2 2) 1/4 3) 1/6 4) 6

10. **If** $\mathrm{Cot}^{-1}\left(\frac{1-x^2}{2x}\right)+\mathrm{Cos}^{-1}\left(\frac{1-x^2}{1+x^2}\right)=\frac{2\pi}{3}$; **x > 0 ; x $\neq$ 1 then x =**

1) $\frac{1}{\sqrt{2}}$ 2) $\pm\frac{1}{\sqrt{2}}$ 3) $\pm\frac{1}{\sqrt{3}}$ 4) $\frac{1}{\sqrt{3}}$

11. **If** $Tan^{-1}\frac{1}{1+2x}+Tan^{-1}\frac{1}{4x+1}=Tan^{-1}\frac{2}{x^2}$ **then x =**

1) 1 2) 0 3) -3 4) $\frac{2}{3}$

12. $6Sin^{-1}(x^2-6x+8.5)=\pi$ **, then x =**

1) 1 2) 2 3) 3 4) 0

13. If Tan^{-1} **(x+1) + Tan^{-1}(x -1) = Tan^{-1}** $\frac{8}{31}$ **then x =**

1) 1 2) 1/2 3) -1/2 4) 1/4

14. $Tan\left(\frac{\pi}{4}+\frac{1}{2}Cos^{-1}\frac{a}{b}\right)+Tan\left(\frac{\pi}{4}-\frac{1}{2}Cos^{-1}\frac{a}{b}\right)$

1) b/a 2) a/b 3) 2a/b 4) 2b/a

15. $\mathrm{Tan}\left[\mathrm{Tan}^{-1}\left(\frac{1}{a+b}\right)+\mathrm{Tan}^{-1}\left(\frac{b}{a^2+ab+1}\right)\right]=$

1) a 2) 1/a 3) b 4) 1/b

16. **In a ΔABC, If C is a right angle then**

$Tan^{-1}\left(\frac{a}{b+c}\right)+Tan^{-1}\left(\frac{b}{c+a}\right)=$

1) $\frac{\pi}{3}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{6}$ 4) $\frac{5\pi}{2}$

17. **If** $\mathrm{Tan}^{-1}\left(\frac{x+1}{x-1}\right)+\mathrm{Tan}^{-1}\left(\frac{x-1}{x}\right)$ $=\pi+\mathrm{Tan}^{-1}(-7)$ **then** x = (Eamcet- 2008)

1) 2 2) -2 3) 1
4) no solution

18. **The number of real solutions of**

$\mathrm{Tan}^{-1}\left(\sqrt{x(x+1)}\right)+\mathrm{Sin}^{-1}\sqrt{(x^2+x+1)}=\frac{\pi}{2}$ **is**

1) 0 2) 1 3) 2 4) ∞

19. **The number of solutions of the equation** $2\left(Sin^{-1}x\right)^2-5Sin^{-1}x+2=0$ **is**

1) 0 2) 1 3) 2 4) 3

20. **If** $\frac{1}{\sqrt{2}}<$ **x < 1 then**

$Cos^{-1}x+Cos^{-1}\left(\frac{x+\sqrt{1-x^2}}{\sqrt{2}}\right)=$

1) $2Cos^{-1}x-\frac{\pi}{4}$ 2) $2Cos^{-1}x$ 3) $\frac{\pi}{4}$ 4) 0

21. **Let a,b,c be a positive real numbers**

$\theta=\mathrm{Tan}^{-1}\sqrt{\dfrac{a(a+b+c)}{bc}}+\mathrm{Tan}^{-1}\sqrt{\dfrac{b(a+b+c)}{ca}}+$

$\mathrm{Tan}^{-1}\sqrt{\dfrac{c(a+b+c)}{ab}}$ **then** $\tan\theta=$

1) 0 2) 3π 3) 1 4) 4π

22. If $\left[\sin^{-1}\left(\cos^{-1}\left(\sin^{-1}\left(\tan^{-1}x\right)\right)\right)\right]=1$**, where [•] denotes the greatest integer function, then** $x\in$

1) $[\tan\sin\cos 1,\ \tan\sin\cos\sin 1]$

2) $(\tan\sin\cos 1,\ \tan\sin\cos\sin 1)$

3) $[-1,\ 1]$

4) $[\sin\cos\tan 1,\ \sin\cos\sin\tan 1]$

23. If $\left[Cot^{-1}x\right]+\left[\mathrm{Cos}^{-1}x\right]=0$ **where** x **is non-negative real number and** $[\bullet]$ **denotes the greatest integer function, then complete set of values of** x **is**

1) $(\cos 1,\ 1]$ 2) $(\cos 1,\ \cot 1)$

3) $(\cot 1\ \ 1]$ 4) $(0,\ \cos 1)$

24. If $\alpha=2\,\mathrm{Tan}^{-1}\left(\sqrt{2}-1\right)$,

$\beta=3\,\mathrm{Sin}^{-1}\left(\dfrac{1}{\sqrt{2}}\right)+\mathrm{Sin}^{-1}(-\dfrac{1}{2})$ **and**

$\gamma=\mathrm{Cos}^{-1}\dfrac{1}{3}$ **then** (Eamcet-2019)

1) $\alpha<\beta<\gamma$ 2) $\alpha<\gamma<\beta$

3) $\beta<\gamma<\alpha$ 4) $\gamma<\beta<\alpha$

25. The value of $'a'$ **for which**

$ax^2+\mathrm{Sin}^{-1}(x^2-2x+2)+\mathrm{Cos}^{-1}(x^2-2x+2)=0$

has a real solution, is

1) $\dfrac{\pi}{2}$ 2) $-\dfrac{\pi}{2}$ 3) $\dfrac{2}{\pi}$ 4) $-\dfrac{2}{\pi}$

26. For the equation $Cos^{-1}x+\mathrm{Cos}^{-1}2x+\pi=0$ **the number of real solution is**

1) 0 2) 1 3) 2 4) 3

27. The equation $2\,\mathrm{Cos}^{-1}x+\mathrm{Sin}^{-1}x=\dfrac{11\pi}{6}$ **has**

1) No solution 2) One solution

3) Two solutions 3) Three solutions

28. If $\mathrm{Cos}^{-1}x>\mathrm{Sin}^{-1}x$ **, then** x **belongs to the interval**

1) $(-\infty,\ 0)$ 2) $(-1,\ 0)$

3) $[0,\ \dfrac{1}{\sqrt{2}})$ 4) $[-1,\ \dfrac{1}{\sqrt{2}})$

29. The least integral value of x **for which** $Tan^{-1}x>Cot^{-1}x$ **is**

1) 1 2) 2 3) 3 4) 4

30. If x **(3 - x)** ≥ 2 **then**

$\mathrm{Sin}^{-1}(x)+\mathrm{Sin}^{-1}(x^2)+\ldots\ldots\ldots+\mathrm{Sin}^{-1}(x^{10})=$

1) $\dfrac{\pi}{2}$ 2) 2π 3) 5π 4) 10π

31. If $\mathrm{Cos}^{-1}x+\mathrm{Cos}^{-1}y=\dfrac{\pi}{2}$ **and**

$\mathrm{Tan}^{-1}x-\mathrm{Tan}^{-1}y=0$ **then** $x^2+xy+y^2=$

1) 0 2) $-\dfrac{1}{2}$ 3) $\dfrac{1}{2}$ 4) $\dfrac{3}{2}$

32. If $0<x<1$**, then**

$\sqrt{1+x^2}\left[\left\{x\cos(\cot^{-1}x)+\sin(\cot^{-1}x\right\}^2-1\right]=$

1) $\dfrac{x}{\sqrt{1+x^2}}$ 2) x

3) $x\sqrt{1+x^2}$ 4) $\sqrt{1+x^2}$

33. If $Sin^{-1}x+Sin^{-1}y+Sin^{-1}z=\dfrac{3\pi}{2}$ **then**

$\sum(\dfrac{x^{201}+y^{201}}{x^{603}+y^{603}})(\dfrac{x^{402}+y^{402}}{x^{804}+y^{804}})=$ (Eamcet-2017)

1) 0 2) 1 3) 2 4) 3

34. $\mathrm{Tan}^{-1}\left(\dfrac{1}{1+(1)(2)}\right)+\mathrm{Tan}^{-1}\left(\dfrac{1}{1+(2)(3)}\right)+$

$...............+\text{Tan}^{-1}\left(\frac{1}{1+(n-1)(n)}\right)=$

1) 0 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{2}$ 4) $\text{Tan}^{-1}\left(\frac{n-1}{n+1}\right)$

35. $\sum_{m=1}^{n}\text{Tan}^{-1}\left(\frac{2m}{m^4+m^2+2}\right)=$

1) $\text{Tan}^{-1}\left(\frac{n^2+n}{n^2+n+2}\right)$ 2) $\text{Tan}^{-1}\left(\frac{n^2-n}{n^2-n+2}\right)$

3) $\text{Tan}^{-1}\left(\frac{n^2+n+2}{n^2+n}\right)$ 4) $\frac{\pi}{4}$

36. If $Tan^{-1}x+Tan^{-1}y+Tan^{-1}z=\frac{\pi}{2}$ **and** $(x-y)^2+(y-z)^2+(z-x)^2=0$ **then** $x^2+y^2+z^2=$

1) 0 2) 4 3) 1 4) 2

37. If α,β,γ **are the roots of the equation** $x^3+mx^2+3x+m=0$**, then the general value of** $\text{Tan}^{-1}\alpha+\text{Tan}^{-1}\beta+\text{Tan}^{-1}\gamma$ **is**

1) $(2n+1)\frac{\pi}{2}$ 2) $n\pi$ 3) $\frac{n\pi}{2}$

4) dependent upon the value of m

38. If $\log_2^{\frac{x}{2}}\geq 0$ **then**

$\log_{\frac{1}{\pi}}\{\text{Sin}^{-1}\frac{2x}{1+x^2}+2Tan^{-1}x\}=$ (Eamcet-2016)

1) $\log_{\frac{1}{\pi}}(4\text{Tan}^{-1}x)$ 2) 0 3) -1 4) -2

39. The value of $\text{Sin}^{-1}(\sin 12)+\text{Sin}^{-1}(\cos 12)=$

1) 0 2) $24-2\pi$ 3) $4\pi-24$ 4) 8π

40. If $Cos^{-1}\frac{x}{a}+Cos^{-1}\frac{y}{b}=\frac{5\pi}{12}$ **and** $Sin^{-1}\frac{x}{a}-Sin^{-1}\frac{y}{b}=\frac{\pi}{12}$

then $\frac{x^2}{a^2}+\frac{y^2}{b^2}=$

1) 1 2) $\frac{1}{4}$ 3) $\frac{3}{4}$ 4) $\frac{5}{4}$

41. If 'a' is twice the tangent of the arithmetic mean of $\text{Sin}^{-1}x$ **and** $\text{Cos}^{-1}x$ **, 'b' is the geometric mean of** $\tan x$ **and** $\cot x$**. Then** $x^2-ax+b=0\Rightarrow x=$

1) 2 2) 3 3) 1 4) 0

KEY

01) 1	02) 3	03) 3	04) 3	05) 3	06) 2
07) 1	08) 2	09) 3	10) 4	11) 2	12) 2
13) 4	14) 4	15) 2	16) 2	17) 1	18) 3
19) 2	20) 3	21) 1	22) 1	23) 3	24) 2
25) 4	26) 1	27) 1	28) 4	29) 2	30) 3
31) 4	32) 3	33) 4	34) 4	35) 1	36) 3
37) 2	38) 3	39) 1	40) 4	41) 3	

SOLUTIONS

1. Let $y=Sin^{-1}x-Cos^{-1}x=\frac{\pi}{2}-2Cos^{-1}x$

$$0\leq Cos^{-1}x\leq\pi\Rightarrow-\frac{3\pi}{2}\leq y\leq\frac{\pi}{2}$$

2. Common domain $=\{-1,1\}$

range $=\{f(-1),f(1)\}$

3. Domain of $Sin^{-1}x+Cos^{-1}x+Tan^{-1}x$ is [-1,1]

Range is $[\frac{\pi}{2}+Tan^{-1}(-1),\frac{\pi}{2}+Tan^{-1}(1)]$

4. $-1\leq\log_2^{(\frac{x^2}{2})}\leq 1\Rightarrow 2^{-1}\leq\frac{x^2}{2}\leq 2\Rightarrow 1\leq x^2\leq 4$

5. $0<A<90^0,\ B>90^0, C<0$

6. $A<0, B>90^0, 0<C<90^0$

7. $\cos^{-1}\cos\left(\frac{10\pi}{7}\right)=$

$\cos^{-1}\cos\left(2\pi-\frac{10\pi}{7}\right)=\cos^{-1}\cos\left(\frac{4\pi}{7}\right)=\left(\frac{4\pi}{7}\right)$

$\sin^{-1}\sin\left(\frac{35\pi}{11}\right) = \sin^{-1}\sin\left(2\pi+\frac{13\pi}{11}\right)$ +

$\sin^{-1}\sin\left(\frac{13\pi}{11}\right) = \sin^{-1}\sin\left(\pi+\frac{2\pi}{11}\right)$

$= \sin^{-1}\sin\left(-\frac{2\pi}{11}\right) = -\frac{2\pi}{11}$

$\tan^{-1}\tan\left(\frac{24\pi}{13}\right) = \tan^{-1}\tan\left(\frac{2\pi}{13}-2\pi\right) =$

$\tan^{-1}\tan\left(\frac{2\pi}{13}\right) = \frac{2\pi}{13}$]

$\cot^{-1}\left(\cot\frac{26\pi}{5}\right) = \cot^{-1}\left(\cot\left(5\pi+\frac{\pi}{5}\right)\right) = \frac{\pi}{5}$

8. Apply $\sin^{-1}x + \sin^{-1}y$ formula
9. Apply $Tan^{-1}x + Tan^{-1}y$ formula
10. Put $x = Tan\theta$
11. Given $\tan^{-1}\frac{1}{1+2x} + \tan^{-1}\left(\frac{1}{4x+1}\right) = \tan^{-1}\frac{2}{x^2}$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{1}{1+2x}+\frac{1}{4x+1}}{1-\frac{1}{1+2x}\frac{1}{4x+1}}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \frac{4x+1+1+2x}{(1+2x)(4x+1)-1} = \frac{2}{x^2}$$

$$\Rightarrow \frac{3x+1}{8x^2+6x} = \frac{1}{x^2}$$

$$3x^3+x^2 = 8x^2+6x$$

$$\Rightarrow 3x^3-7x^2-6x = 0$$

$$x\left(3x^2-7x-6\right) = 0$$

$$x = 0$$

12. $6\sin^{-1}\left(x^2-6x+\frac{17}{2}\right) = \pi$

$\sin^{-1}\left(x^2-6x+\frac{17}{2}\right) = \frac{\pi}{6}$

$x^2-6x+\frac{17}{2} = \frac{1}{2}$ solve the equation. It has two values.

13. Apply $Tan^{-1}x + Tan^{-1}y$ formula
14. Put $\frac{1}{2}Cos^{-1}\frac{a}{b} = \theta \Rightarrow Cos2\theta = \frac{a}{b}$

Apply $Tan(\frac{\pi}{4}+\theta)+Tan(\frac{\pi}{4}-\theta) = \frac{2}{Cos2\theta}$

15. $Tan[Tan^{-1}(\frac{1}{a+b})+Tan^{-1}(\frac{(a+b)-a}{1+(a+b)a})]$

$= Tan[Tan^{-1}(\frac{1}{a+b})+Tan^{-1}(a+b)-Tan^{-1}a]$

$= Tan[\frac{\pi}{2}-Tan^{-1}(a)] = \frac{1}{a}$

16. Applying $Tan^{-1}x+Tan^{-1}y$ formula and substitude $c^2 = a^2+b^2$
17. By verification x=2 satisfied
18. $x(x+1) \geq 0$ and $x^2 + x + 1 \geq 0$

But $\sin^{-1}\left(\sqrt{x}\right)$ domain [0, 1]

$\therefore$ $x(x+1) = 0 \Rightarrow x = 0$ or -1

19. Put $\sin^{-1}x = a$
20. $Cos^{-1}x + Cos^{-1}(x.\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\sqrt{1-x^2})$

$= Cos^{-1}x + Cos^{-1}(\frac{1}{\sqrt{2}}) - Cos^{-1}x = \frac{\pi}{4}$

21. use $Tan^{-1}x+Tan^{-1}y+Tan^{-1}z = Tan^{-1}\left(\frac{\sum x-\Pi x}{1-\sum xy}\right)$

(or) Put a=b=c=1 verify option

22. $1 \leq Sin^{-1}Cos^{-1}Sin^{-1}Tan^{-1}x \leq \frac{\pi}{2}$

$\Rightarrow Sin1 \leq Cos^{-1}Sin^{-1}Tan^{-1}x \leq 1$

$\Rightarrow CosSin1 \geq Sin^{-1}Tan^{-1}x \geq Cos1$

$\Rightarrow SinCosSin1 \geq Tan^{-1}x \geq SinCos1$

$\Rightarrow TanSinCosSin1 \geq x \geq TanSinCos1$

23. $0 < Cot^{-1}x < \pi$ and $0 \leq Cos^{-1}x \leq \pi$

$Cot \quad 1 < x < \infty$ and $Cos \quad 1 \le x \le 1$

$x \in (Cot 1, 1] \qquad \because Cos \quad 1 < Cot 1$

24. $\alpha = 2Tan^{-1}(\sqrt{2}-1) = \dfrac{\pi}{4}, \ \beta = \dfrac{7\pi}{12},$

$\gamma = Cos^{-1}(\dfrac{1}{3}) > \dfrac{\pi}{4}$

25. $x^2 - 2x + 2 = (x-1)^2 + 1 \ge 1 \therefore x = 1$

$a + \dfrac{\pi}{2} = 0 \Rightarrow a = -\dfrac{\pi}{2}$

26. We have $\text{Cos}^{-1} x + \text{Cos}^{-1} 2x = -\pi$ which is not possible as $\text{Cos}^{-1} x \, and \, \text{Cos}^{-1} 2x$ never take negative values

27. $Cos^{-1}x + (Cos^{-1}x + Sin^{-1}x) = \dfrac{11\pi}{6} \Rightarrow Cos^{-1}x + \dfrac{\pi}{2} = \dfrac{11\pi}{6}$

$\Rightarrow Cos^{-1}x = \dfrac{4\pi}{3}$ which is not possible as

$Cos^{-1}x \in [0, \pi]$

28. $If \ -1 \le x < \dfrac{1}{\sqrt{2}} \Rightarrow Cos^{-1}x > Sin^{-1}x$

29. $2 \ Tan^{-1}x > \dfrac{\pi}{2}.$

30. Given $3x - x^2 \ge 2 \Rightarrow x^2 - 3x + 2 \le 0 \Rightarrow x = 1 (or) 2$

At x=2 sine is not defined

So $Sin^{-1}(1) + Sin^{-1}(1) + ... + 10$ times

$= \dfrac{\pi}{2} + \dfrac{\pi}{2} + ... + 10 \text{ times} = 5\pi$

31. $Tan^{-1}x - Tan^{-1}y = 0 \Rightarrow x = y$

$Cos^{-1}x + Cos^{-1}y = \dfrac{\pi}{2} \Rightarrow x = \dfrac{1}{\sqrt{2}}$

32. Apply $Cot^{-1}x = Cos^{-1}(\dfrac{x}{\sqrt{1+x^2}})$ and

$Cot^{-1}x = Sin^{-1}(\dfrac{1}{\sqrt{1+x^2}})$ formulas

33. $Sin^{-1}x + Sin^{-1}y + Sin^{-1}z = \dfrac{3\pi}{2} \Rightarrow x = y = z = 1$

34. $\sum_{n=1}^{n} \tan^{-1}\left(\dfrac{1}{1+(n-1)n}\right) = \tan^{-1}(n) - \tan^{-1}(1)$

35. $\sum_{m=1}^{n} \tan^{-1}(\dfrac{2m}{m^4+m^2+2})$

$= \sum_{m=1}^{n} \tan^{-1}(\dfrac{(m^2+m+1)-(m^2-m+1)}{1+(m^2+m+1)(m^2-m+1)})$

$= \sum_{m=1}^{n} [\tan^{-1}(m^2+m+1) - \tan^{-1}((m^2-m+1)]$

36. Put $x = y = z = \dfrac{1}{\sqrt{3}}$

$x^2 + y^2 + z^2 = \dfrac{1}{3} + \dfrac{1}{3} + \dfrac{1}{3} = 1$

37. Let $\alpha = Tan\theta_1, \quad \beta = Tan\theta_2, \quad \gamma = Tan\theta_3$

$Tan(\theta_1 + \theta_2 + \theta_3) \Rightarrow \theta_1 + \theta_2 + \theta_3 = n\pi, \ n \in z$

38. $\log^{x}{}_{2} \ge 0 \Rightarrow 1,$ for

$x \ge 1, Sin^{-1}(\dfrac{2x}{1+x^2}) = \pi - 2Tan^{-1}x$

39. $Sin^{-1}(Sin12) + Cos^{-1}(Cos12) = 12 - 4\pi + 4\pi - 12 = 0$

40. $Cos^{-1}\dfrac{x}{a} + Cos^{-1}\dfrac{y}{b} = \dfrac{5\pi}{2} \rightarrow (1)$ and

$Cos^{-1}\dfrac{y}{b} - Cos^{-1}\dfrac{x}{a} = \dfrac{\pi}{2} \rightarrow (2)$

(1)+(2) $\Rightarrow Cos^{-1}\dfrac{y}{b} = \dfrac{\pi}{4} \Rightarrow \dfrac{y}{b} = \dfrac{1}{\sqrt{2}}$

(1)-(2) $\Rightarrow Cos^{-1}\dfrac{x}{a} = \dfrac{\pi}{6} \Rightarrow \dfrac{x}{a} = \dfrac{\sqrt{3}}{2}$

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{5}{4}$

41. $a = 2Tan(\dfrac{Sin^{-1}x + Cos^{-1}x}{2}) = 2(1) = 2$

$b = \sqrt{Tanx.Cotx} = 1$

$\therefore x^2 - ax + b = 0 \Rightarrow x^2 - 2x + 1 = 0 \Rightarrow x = 1$

EXERCISE - III

1. The value of

$\mathrm{Sin}^{-1}\left(\cos\left\{\mathrm{Cos}^{-1}(\cos x)+\mathrm{Sin}^{-1}(\sin x)\right\}\right)$ **where**

$x \in \left(\frac{\pi}{2}, \pi\right)$ **is**

1) $\frac{\pi}{2}$ 2) $-\pi$ 3) π 4) $-\frac{\pi}{2}$

2. $\mathrm{Sin}^{-1}(\sin 3) + \mathrm{Cos}^{-1}(\cos 7) - \mathrm{Tan}^{-1}(\tan 5) =$

1) $\pi-1$ 2) π 3) $3\pi-1$ 4) $2\pi-1$

3. The number of solutions of

$\mathrm{Sin}^{-1}\left(1+b+b^2+....\infty\right)+\mathrm{Cos}^{-1}\left(a-\frac{a^2}{3}+\frac{a^3}{9}+...\infty\right)=\frac{\pi}{2}$

1) 1 2) 2 3) 3 4) ∞

4. If *x* takes negative permissible value, then $\mathrm{Sin}^{-1}x =$

1) $\mathrm{Cos}^{-1}\sqrt{1-x^2}$ 2) $-\mathrm{Cos}^{-1}\sqrt{1-x^2}$

3) $\mathrm{Cos}^{-1}\sqrt{x^2-1}$ 4) $\pi-\mathrm{Cos}^{-1}\sqrt{1-x^2}$

5. $\mathrm{Cos}^{-1}\left\{\frac{1}{2}x^2+\sqrt{1-x^2}\sqrt{1-\frac{x^2}{4}}\right\}=\mathrm{Cos}^{-1}\frac{x}{2}-\mathrm{Cos}^{-1}x$

holds for :

1) $|x|\le 1$ 2) $x\in R$

3) $0\le x\le 1$ 4) $-1\le x\le 0$

6. The set of values of x from which the identity $\mathrm{Cos}^{-1}x+\mathrm{Cos}^{-1}\left(\frac{x}{2}+\frac{1}{2}\sqrt{3-3x^2}\right)=\frac{\pi}{3}$ **holds good is**

1) $[0,1]$ 2) $[0,\frac{1}{2}]$ 3) $[\frac{1}{2},1]$ 4) $\{-1,0,1\}$

7. The number of solutions of the equation $\mathrm{Tan}^{-1}(x-1) + \mathrm{Tan}^{-1}(x)+\mathrm{Tan}^{-1}(x+1)=\mathrm{Tan}^{-1}(3x)$ **is**

1) 1 2) 2 3) 3 4) 4

8. $Cot^{-1}\left(\sqrt{\cos\alpha}\right)-\mathrm{Tan}^{-1}\left(\sqrt{\cos\alpha}\right)=x\ge 0$, **then sin x = (AIE-2002)**

1) $tan^2\frac{\alpha}{2}$ 2) $\cot^2\left(\frac{\alpha}{2}\right)$ 3) $\tan\alpha$ 4) $\cot\frac{\alpha}{2}$

9. If $y=(Sin^{-1}x)^3+(Cos^{-1}x)^3$ **then**

1) $\min y=\frac{\pi^3}{8}$ 2) $\min y=\frac{\pi^3}{32}$

3) $\max y=\frac{\pi^3}{8}$ 4) $\max y=\frac{7\pi^3}{32}$

10. $Sin^{-1}(\frac{1}{\sqrt{2}})+Sin^{-1}(\frac{\sqrt{2}-1}{\sqrt{6}})+....+Sin^{-1}(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}})+....\infty=$

1) π 2) $\frac{\pi}{2}$ 3) $\frac{\pi}{4}$ 4) $\frac{3\pi}{2}$

11. $\mathrm{Tan}^{-1}\left(\frac{1}{3}\right)+\mathrm{Tan}^{-1}\left(\frac{2}{9}\right)+\mathrm{Tan}^{-1}\left(\frac{4}{33}\right)+......\infty$

1) $\frac{\pi}{4}$ 2) $\frac{\pi}{2}$ 3) π 4) 2π

12. $Tan^{-1}(\frac{c_1x-y}{c_1y+x})+Tan^{-1}(\frac{c_2-c_1}{1+c_2c_1})+Tan^{-1}(\frac{c_3-c_2}{1+c_3c_2})$

$+....+Tan^{-1}(\frac{1}{c_n})=$

1) $Tan^{-1}\left(\frac{2x}{y}\right)$ 2) $Tan^{-1}(xy)$

3) $Tan^{-1}\left(\frac{x}{y}\right)$ 4) $Tan^{-1}\left(\frac{y}{x}\right)$

13. Value of $\mathrm{Sec}^{-1}\left(\frac{1}{1-2x^2}\right)+4\mathrm{Cos}^{-1}\sqrt{\frac{1+x}{2}}=$

1) $2\mathrm{Tan}^{-1}x$ 2) $\mathrm{Tan}^{-1}\left(\frac{\sqrt{1-x^2}}{1-x}\right)$

3) $Cot^{-1}\left(\frac{\sqrt{1-x^2}}{1-x}\right)$ 4) constant for all x

14. The solution set of the equation

$\text{Tan}^{-1}x - Cot^{-1}x = \text{Cos}^{-1}(2-x)$ **is :**

1) $[0,1]$ 2) $[-1,1]$ 3) $[1,3)$ 4) $(1,3)$

15. The number of solutions of the equation

$3\,\text{Cos}^{-1}x - \pi x - \frac{\pi}{2} = 0$

1) 0 2) 1 3) 2 4) infinitely many

16. If the equation

$\text{Sin}^{-1}(x^2+x+1) + \text{Cos}^{-1}(\lambda x + 1) = \frac{\pi}{2}$

has exactly two solutions, then λ can have the integral value.

1) -1 2) 0 3) 1 4) 2

17. If $Cot^{-1}\frac{n}{\pi} > \frac{\pi}{6}, n \in N$ then the maximum value of 'n' is

1) 6 2) 7 3) 5 4) 10

18. If $(Cot^{-1}x)^2 - 7(Cot^{-1}x) + 10 > 0$, then x lies in the interval

1) $(\cot 5, \cot 2)$ 2) $(-\infty, \cot 5) \cup (\cot 2, \infty)$

3) $(-\infty, \cot 5)$ 4) $(\cot 2, \infty)$

19. The value of x for which

$\text{Cos}^{-1}(\cos 4) > 3x^2 - 4x$ **is**

1) $\left(0, \frac{2+\sqrt{6\pi-8}}{3}\right)$ 2) $\left(\frac{2+\sqrt{6\pi-8}}{3}, 0\right)$

3) $(-2,2)$ 4) $\left(\frac{2-\sqrt{6\pi-8}}{3}, \frac{2+\sqrt{6\pi-8}}{3}\right)$

20. If α is the only real root of the equation $x^3+bx^2+cx+1=0 (b<c)$ then the value of

$\text{Tan}^{-1}\alpha + \text{Tan}^{-1}\left(\frac{1}{\alpha}\right) =$

1) $\frac{\pi}{2}$ 2) $-\frac{\pi}{2}$ 3) 0 4) π

21. $x = n\pi - \text{Tan}^{-1}3$ is a solution of the equation $12\tan 2x + \frac{\sqrt{10}}{\cos x} + 1 = 0$ **if**

1) n is any integer
2) n is an even integer
3) n is a positive integer
4) n is an odd integer

22. Point P(x, y) satisfying the equation

$\text{Sin}^{-1}x + \text{Cos}^{-1}y + \text{Cos}^{-1}(2xy) = \frac{\pi}{2}$ **lies on**

1) the bisector of the first and third quadrant
2) bisector of the second and fourth quadrant.
3) the rectangle formed by the lines x = ± 1 and $y = \pm 1$.
4) a unit circle with centre at the origin.

23. The least integral value of k for which $(k-2)x^2 + 8x + k + 4 > \text{Sin}^{-1}(\sin 12) + \text{Cos}^{-1}(\cos 12)$ for all $x \in R$, is

1) – 7 2) – 5 3) – 3 4) 5

24. $\frac{\alpha^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\text{Tan}^{-1}\frac{\alpha}{\beta}\right) +$

$\frac{\beta^3}{2}\sec^2\left(\frac{1}{2}\text{Tan}^{-1}\left(\frac{\beta}{\alpha}\right)\right) =$

1) $(\alpha-\beta)(\alpha^2+\beta^2)$ 2) $(\alpha+\beta)(\alpha^2-\beta^2)$

3) $(\alpha+\beta)(\alpha^2+\beta^2)$ 4) 0

25 .

$\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right) - \frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right] =$

1) 0 2) 1 3) $\frac{x-y}{1+xy}$ 4) $\frac{2x}{1-x^2}$

KEY

01) 4	02) 1	03) 4	04) 2	05) 3	06) 3
07) 3	08) 1	09) 2	10) 2	11) 1	12) 3
13) 4	14) 3	15) 2	16) 2	17) 3	18) 4
19) 4	20) 2	21) 4	22) 4	23) 4	24) 3
25) 3					

SOLUTIONS

1. $\sin^{-1}\left[\cos\{x+\pi-x\}\right]=-\pi/2$

2. $\pi-3+7-2\pi-(5-2\pi) = \pi-1$

3. The given equation is valid if

$-1\le 1+b+b^2+....\infty\le 1,$ *and* $a-\frac{a^2}{3}+\frac{a^3}{9}-....\infty\in[-1,1]$

also $1+b+b^2+....\infty=a-\frac{a^2}{3}+\frac{a^3}{9}-....\infty$

4. Let $\sin^{-1}x=y\Rightarrow x=\sin y$

since $-1\le x\le 0$ therefore $-\frac{\pi}{2}\le\sin^{-1}x\le 0$

$\Rightarrow \frac{\pi}{2}\ge -y\ge 0$

$\Rightarrow(-y)=\cos^{-1}\sqrt{1-x^2}\Rightarrow y=-\cos^{-1}\sqrt{1-x^2}$

5. $\mathrm{Cos}^{-1}\left(\frac{1}{2}x^2+\sqrt{1-x^2}\sqrt{1-\frac{x^2}{4}}\right)$

$=\cos^{-1}\left(\frac{x}{2}x+\sqrt{1-x^2}\sqrt{1-\left(\frac{x}{2}\right)^2}\right)$

For $\mathrm{Cos}^{-1}\left(\frac{1}{2}x^2+\sqrt{1-x^2}\sqrt{1-\frac{x^2}{4}}\right)$

$=\mathrm{Cos}^{-1}\frac{x}{2}-\cos^{-1}x$

L.H.S > 0, hence R.H.S>0

$\Rightarrow \mathrm{Cos}^{-1}\frac{x}{2}-\mathrm{Cos}^{-1}x>0$

Since $\mathrm{Cos}^{-1}x$ is decreasing function

$\frac{x}{2}\le x\Rightarrow\frac{x}{2}\ge 0\Rightarrow x\in[0,1]$

6. by verification $x=\frac{1}{2}$ *and* 1

7. $\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}(3x)$

$\Rightarrow \tan^{-1}(x-1)+\tan^{-1}(x+1)=\tan^{-1}3x-\tan^{-1}x$

Apply $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$\Rightarrow \tan^{-1}\left(\frac{x-1+x+1}{1-(x-1)(x+1)}\right)=\tan^{-1}\left(\frac{3x-x}{1+(3x)x}\right)$

$\Rightarrow\frac{2x}{1-x^2+1}=\frac{2x}{1+3x^2}$

$x=0$ $\quad$ $1+3x^2=2-x^2$

$4x^2=1$

$x^2=\frac{1}{4}\Rightarrow x=\pm\frac{1}{2}$

Number of solutions 3

8. $\cot^{-1}\sqrt{\cos\alpha}-\tan^{-1}\sqrt{\cos\alpha}=x$

$\Rightarrow\frac{\pi}{2}-2\tan^{-1}\sqrt{\cos\alpha}=x$

$\Rightarrow\frac{\pi}{2}-\tan^{-1}\frac{2\sqrt{\cos\alpha}}{1-\cos\alpha}=x$

$\Rightarrow\cot^{-1}\left(\frac{2\sqrt{\cos\alpha}}{1-\cos\alpha}\right)=x\Rightarrow\cot x=\frac{2\sqrt{\cos\alpha}}{1-\cos\alpha}$

$\therefore \sin x=\frac{1-\cos\alpha}{1+\cos\alpha}=\tan^2\alpha/2$

9. By standedformula

$\frac{\pi^3}{32}\le\left(Sin^{-1}x\right)^3+\left(Cos^{-1}x\right)^3\le\frac{7\pi^3}{8}$

10. $T_n=Sin^{-1}\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}}=Tan^{-1}\frac{\sqrt{n}-\sqrt{n-1}}{1+\sqrt{n}\sqrt{n-1}}$

$=Tan^{-1}\sqrt{n}-Tan^{-1}\sqrt{n-1}$

$S_\infty=\left\{Tan^{-1}1-Tan^{-1}0\right\}+\left\{Tan^{-1}\sqrt{2}-Tan^{-1}1\right\}+.......+Tan^{-1}\infty$

$=Tan^{-1}\infty-Tan^{-1}0=\frac{\pi}{2}$

11.

$\mathrm{T_n}=\tan^{-1}\left(\frac{2^{n-1}}{1+2^{2n-1}}\right)=\tan^{-1}\left(\frac{2^n-2^{n-1}}{1+2^n.2^{n-1}}\right)$

$=\tan^{-1}2^n-\tan^{-1}2^{n-1}$

$S_\infty = \tan^{-1}\infty - \tan^{-1}1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

12. $Tan^{-1}\left(\frac{\frac{x}{y}-\frac{1}{c_1}}{1+\frac{x}{y}.\frac{1}{c_1}}\right) + Tan^{-1}\left(\frac{\frac{1}{c_1}-\frac{1}{c_2}}{1+\frac{1}{c_1}.\frac{1}{c_2}}\right) + ...+ Tan^{-1}\left(\frac{1}{c_n}\right)$

$\left\{Tan^{-1}\left(\frac{x}{y}\right) - Tan^{-1}\left(\frac{1}{c_1}\right)\right\} + \left\{Tan^{-1}\left(\frac{1}{c_1}\right) - Tan^{-1}\left(\frac{1}{c_2}\right)\right\} ..... + Tan^{-1}\left(\frac{1}{c_n}\right)$

$= Tan^{-1}\left(\frac{x}{y}\right)$

13. $\sec^{-1}\left(\frac{1}{1-2x^2}\right) + 4\cos^{-1}\sqrt{\frac{1+x}{2}}$

$= \cos^{-1}\left(1-2x^2\right) + 4\cos^{-1}\sqrt{\frac{1+x}{2}}$

$\therefore \pi - \cos^{-1}\left(2x^2-1\right) + 4\cos^{-1}\sqrt{\frac{1+x}{2}}$ -- (1)

put $x = \cos\theta$ then (1) can be written as

$\pi - \cos^{-1}\left(\cos 2\theta\right) + 4\cos^{-1}\cos\left(\frac{\theta}{2}\right)$

$= \pi - 2\theta + 2\theta = \pi$ which is constant for all x.

14. Since, $\tan^{-1}x \; and \; \cot^{-1}x$ exists for all $x \in R$ and $\cos^{-1}\left(2-x\right)$ exists, if

$-1 \le 2-x \le 1$

$\therefore \tan^{-1}x - \cot^{-1}x = \cos^{-1}\left(2-x\right)$ is possible only, if $1 \le x \le 3$, thus the solution of given equation is [1,3] but at x = 3 given equation not satisifed therefore x $\in [1,3)$.

15. $\cos^{-1}x = \frac{\pi}{3}x + \frac{\pi}{6}$ has only one real root.

16. $x^2 + x + 1 = \lambda x + 1$

$\Rightarrow x = 0, \lambda - 1 \; and \; -1 \le x \le 0$

$\Rightarrow 0 \le \lambda < 1 \Rightarrow \lambda = 0$

17. $\cot^{-1}\left(\frac{n}{\pi}\right) > \frac{\pi}{6} \Rightarrow \frac{n}{\pi} < \sqrt{3} \quad n < \pi\sqrt{3}$

$n < 5.46$

$\therefore$ Maximum value of 'n' is 5.

18. $\left(\cot^{-1}x\right)^2 - 7\cot^{-1}x + 10 > 0$

$\Rightarrow \left(\cot^{-1}x - 2\right)\left(\cot^{-1}x - 5\right) > 0$

Since $Cot^{-1}x - 5$ is negative and does not exits

$\therefore \cot^{-1}x < 2 \quad \Rightarrow x \in \left(\cot 2, \infty\right)$

19. $\cos^{-1}\left(\cos 4\right) = \cos^{-1}\left\{\cos\left(2\pi - 4\right)\right\} = 2\pi - 4$

$\Rightarrow 2\pi - 4 > 3x^2 - 4x$

$\Rightarrow 3x^2 - 4x - \left(2\pi - 4\right) < 0$

$\Rightarrow \frac{2-\sqrt{6\pi-8}}{3} < x < \frac{2+\sqrt{6\pi-8}}{3}$

20. Let $f(x) = x^3 + bx^2 + cx + 1$

$f(0) = 1 > 0, f(-1) = b - c < 0$

$\Rightarrow \alpha$ lies between -1 and 0

$\Rightarrow \alpha < 0 \;\; \Rightarrow \tan(\frac{1}{\alpha}) = -\pi + \cot^{-1}\alpha$

$\tan^{-1}\alpha + \tan^{-1}\frac{1}{\alpha} = -\pi + \tan^{-1}\alpha + \cot^{-1}\alpha$

$= -\pi + \frac{\pi}{2} = -\frac{\pi}{2}$

21. $x = n\pi - \tan^{-1}3$

$\Rightarrow \tan^{-1}3 = n\pi - x \quad \Rightarrow \tan(n\pi - x) = 3$

$\Rightarrow -\tan x = 3 \Rightarrow \tan x = -3$

$\Rightarrow \tan 2x = \frac{2\tan x}{1-\tan^2 x} = \frac{3}{4}$ and

$\cos x = \pm\frac{1}{\sqrt{1+\tan^2 x}} = \pm\frac{1}{\sqrt{10}}$

On substituting these value in the given equation we find only $\cos x = -\frac{1}{\sqrt{10}}$ satisfies the equation. So that the given equation holds for values of

x for which $\tan x = -3$ and $\cos x = -\frac{1}{\sqrt{10}}$

Which is possible if x lies in the second quadrand only and so n must be odd integer.

22. $\cos^{-1}y + \cos^{-1}(2xy) = \frac{\pi}{2} - \sin^{-1}x = \cos^{-1}x$

$\Rightarrow \cos\left(\cos^{-1}(2xy)\right) = \cos\left(\cos^{-1}x - \cos^{-1}y\right)$

$\Rightarrow 2xy = xy + \sqrt{1-x^2}\ \sqrt{1-y^2} \Rightarrow x^2 + y^2 = 1$

23. $\sin^{-1}(\sin 12) = \sin^{-1}\sin(12 - 4\pi) = 12 - 4\pi$

$\cos^{-1}(\cos 12) = \cos^{-1}\cos(4\pi - 12) = 4\pi - 12$

$(k-2)x^2 + 8x + k + 4 > 0$

if $k = 2$ then $8x + 4 > 0$ (not possible)

if $k > 2$ then $k - 2 > 0$

and $64 - 4(k-2)(k+4) < 0$

$16 < k^2 + 2k - 8,\ k^2 + 2k - 24 > 0$

$(k+6)(k-4) > 0 \Rightarrow k = 5$

24. $\frac{\alpha^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right) + \frac{\beta^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right)$

$$= \alpha^3\left(\frac{1}{1-\cos\left(\tan^{-1}\frac{\alpha}{\beta}\right)}\right) + \beta^3\left(\frac{1}{1+\cos\left(\tan^{-1}\frac{\beta}{\alpha}\right)}\right)$$

$$= \alpha^3\left(\frac{1}{1-\frac{\beta}{\sqrt{\alpha^2+\beta^2}}}\right) + \beta^3\left(\frac{1}{1+\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}}\right)$$

$$= \sqrt{\alpha^2+\beta^2}\left\{\frac{\alpha^3}{\sqrt{\alpha^2+\beta^2}-\beta} + \frac{\beta^3}{\sqrt{\alpha^2+\beta^2}+\alpha}\right\}$$

$$= \sqrt{\alpha^2+\beta^2}\,[(\alpha+\beta)\sqrt{\alpha^2+\beta^2}\,]$$

$$= (\alpha+\beta)(\alpha^2+\beta^2)$$

25. $\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right) - \frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right]$

put $x = \tan A, \quad y = \tan B$

$A = \tan^{-1}x \quad B = \tan^{-1}y$

$$= \tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2\tan A}{1+\tan^2 A}\right) - \frac{1}{2}\left(\frac{1-\tan^2 B}{1+\tan^2 B}\right)\right]$$

$$= \tan\left[\frac{1}{2}\sin^{-1}(\sin 2A) - \frac{1}{2}\cos^{-1}(\cos 2B)\right]$$

$$= \tan\left[\frac{1}{2}(2A) - \frac{1}{2}(2B)\right]$$

$$= \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{x-y}{1+xy}$$

JEE MAINS QUESTIONS

1. **If the domain of the function** $f(x)=\sqrt{3\cos^{-1}(4x)-\pi}$ **is** $[a,b]$**, then the value of** $(4a+64b)$ **is _____**

2. **If range of the function** $f(x)=\tan^{-1}(3x^2+bx+3), x\in R$ **is** $\left[0,\frac{\pi}{2}\right]$**, then square of sum of all possible values of b will be:**

   1) 0 2) 18 3) 72
   4) None of these

3. **Range of the function** $f(x)=\tan^{-1}\sqrt{[x]+[-x]}+\sqrt{2-|x|}+\frac{1}{x^2}$ **is:**

   1) $\left[\frac{1}{4},\infty\right)$ 2) $\left\{\frac{1}{4}\right\}\cup[2,\infty)$
   3) $\left\{\frac{1}{4},2\right\}$ 4) $\left[\frac{1}{4},2\right]$

4. **Solution set of** $[\sin^{-1}x]>[\cos^{-1}x]$**, where** $[.]$ **denotes greatest intger function:**

   1) $[\sin 1,1]$ 2) $\left[\frac{1}{\sqrt{2}},1\right]$
   3) $(\cos 1,\sin 1)$ 4) None of these

5. **Range of** $y=(\cot^{-1}x)(\cot^{-1}(-x))$**.**

   1) $\left(0,\frac{\pi}{4}\right)$ 2) $\left(0,\frac{\pi^2}{4}\right)$ 3) $\left[0,\frac{\pi^2}{4}\right]$ 4) $\left(0,\frac{\pi^2}{4}\right]$

6. **If** $x_i\in[0,1]\,\forall i=1,2,3,...,28$ **then find the maximum value of**

   $$\sqrt{\sin^{-1}x_1}\sqrt{\cos^{-1}x_2}+\sqrt{\sin^{-1}x_2}\sqrt{\cos^{-1}x_3}+\sqrt{\sin^{-1}x_3}\sqrt{\cos^{-1}x_4}+...+\sqrt{\sin^{-1}x_{28}}\sqrt{\cos^{-1}x_1}$$

   1) 7π 2) 6π 3) 2π 4) 4π

7. **If the equation** $x^3+bx^2+cx+1=0,(b<c)$**, has only one real root** α**, then the value of**

   $2\tan^{-1}(\operatorname{cosec}\alpha)+\tan^{-1}(2\tan\alpha\sec^2\alpha)$ **is**

   1) $-\pi$ 2) $-\frac{\pi}{2}$ 3) $\frac{\pi}{2}$ 4) π

8. **If** $2^{2\pi/\sin^{-1}x}-2(a+2)2^{\pi/\sin^{-1}x}+8a<0$ **for at least one real** x**, then**

   1) $\frac{1}{8}\le a<2$ 2) $a<2$
   3) $a\in R-\{2\}$ 4) $a\in\left[0,\frac{1}{8}\right)\cup(2,\infty)$

9. **If x, y, z are in A.P. And** $\tan^{-1}x,\tan^{-1}y,\tan^{-1}z$ **are also in A.P. Then** **[JEE MAINS 2013]**

   1) $2x=3y=6z$ 2) $6x=3y=2z$
   3) $6x=4y=3z$ 4) $x=y=z$

10. **The principal value of** $\tan^{-1}\left(\cot\frac{43\pi}{4}\right)$ **is** **[JEE MAINS 2014]**

    1) $-\frac{3\pi}{4}$ 2) $\frac{3\pi}{4}$ 3) $-\frac{\pi}{4}$ 4) $\frac{\pi}{4}$

11. **If** $f(x)=2\tan^{-1}x+\sin^{-1}\left(\frac{2x}{1+x^2}\right), x>1$**, then** $f(5)$ **is equal to** **[JEE MAINS 2015]**

    1) $\frac{\pi}{2}$ 2) π 3) $4\tan^{-1}(5)$ 4) $\tan^{-1}(5)$

12. **Let** $\tan^{-1}y=\tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)$**, where** $|x|<\frac{1}{\sqrt{3}}$. **Then, a value of y** **[JEE MAINS 2015]**

    1) $\frac{3x-x^3}{1-3x^2}$ 2) $\frac{3x+x^3}{1-3x^2}$
    3) $\frac{3x-x^3}{1+3x^2}$ 4) $\frac{3x+x^3}{1+3x^2}$

13. **The value of** $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right),|x|<\frac{1}{2},x\ne 0$, **is equal to** **[JEE MAINS 2017]**

    1) $\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x^2$ 2) $\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$
    3) $\frac{\pi}{4}-\cos^{-1}x^2$ 4) $\frac{\pi}{4}+\cos^{-1}x^2$

14. A value of x **satisfying the equation** $\sin[\cot^{-1}(1+x)]=\cos[\tan^{-1}x]$**, is**

    1) $\frac{1}{2}$ 2) 0 3) -1 4) -$\frac{1}{2}$

15. **If** $\cos^{-1}\left(\frac{2}{3x}\right)+\cos^{-1}\left(\frac{3}{4x}\right)=\frac{\pi}{2}\left(x>\frac{3}{4}\right)$, **then x =__________** **[JEE MAINS 2019]**

1) $\frac{\sqrt{145}}{10}$ 2) $\frac{\sqrt{146}}{12}$ 3) $\frac{\sqrt{145}}{12}$ 4) $\frac{\sqrt{145}}{11}$

16. **If** $x=\sin^{-1}(\sin 10)$ **and** $y=\cos^{-1}(\cos 10)$, **then y-x equals to** **[JEE MAINS-2019]**

1) 0 2) 10 3) 7π 4) π

17. **The value of** $\cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)\right)$ **is**

1) $\frac{23}{22}$ 2) $\frac{21}{19}$ 3) $\frac{19}{21}$ 4) $\frac{22}{23}$

18. **All** ***x*** **satisfying the inequility** $\left(\cot^{-1}x\right)^2-7\left(\cot^{-1}x\right)+10>0$, **lie in the interval** **[JEE MAINS 2019]**

1) $(-\infty,\cot 5)\cup(\cot 2,\infty)$ 2) $(\cot 5,\cot 4)$

3) $(\cot 2,\infty)$ 4) $(-\infty,\cot 5)\cup(\cot 4,\cot 2)$

19. **Considering only the principal values of inverse functions, the set**

$A=\left\{x\geq 0:\tan^{-1}(2x)+\tan^{-1}(3x)=\frac{\pi}{4}\right\}$

[JEE MAINS 2019]

1) is an empty set 2) Is a singleton set
3) Contains more than two elements
4) Contains two elements

20. **If** $\alpha=\cos^{-1}\left(\frac{3}{5}\right)$, $\beta=\tan^{-1}\left(\frac{1}{3}\right)$, **where** $0<\alpha,\beta<\frac{\pi}{2}$, **then** $\alpha-\beta$ **is equal to**

[JEE MAINS 2019]

1) $\tan^{-1}\left(\frac{9}{5\sqrt{10}}\right)$ 2) $\cos^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

3) $\tan^{-1}\left(\frac{9}{14}\right)$ 4) $\sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

21. **If** $\cos^{-1}x-\cos^{-1}\frac{y}{2}=\alpha$, **where** $-1\leq x\leq 1,\ -2\leq y\leq 2, x\leq\frac{y}{2}$, **then for all x,y**

$4x^2-4xy\cos\alpha+y^2=$ **[JEE MAINS 2019]**

1) $2\sin^2\alpha$ 2) $4\cos^2\alpha+2x^2y^2$

3) $4\sin^2\alpha$ 4) $4\sin^2\alpha-2x^2y^2$

22. **The value of** $\sin^{-1}\left(\frac{12}{13}\right)-\sin^{-1}\left(\frac{3}{5}\right)=$

[JEE MAINS 2019]

1) $\pi-\sin^{-1}\left(\frac{63}{65}\right)$ 2) $\frac{\pi}{2}-\sin^{-1}\left(\frac{56}{65}\right)$

3) $\frac{\pi}{2}-\cos^{-1}\left(\frac{9}{65}\right)$ 4) $\pi-\cos^{-1}\left(\frac{33}{65}\right)$

23. **If** $f'(x)=\tan^{-1}(\sec x+\tan x),-\frac{\pi}{2}<x<\frac{\pi}{2}$, **and** $f(0)=0$, **then** $f(1)$ **is equal to:**

[JEE MAINS 2020]

1) $\frac{\pi-1}{4}$ 2) $\frac{\pi+2}{4}$ 3) $\frac{\pi+1}{4}$ 4) $\frac{1}{4}$

24. **Let** $f(x)=x\cos^{-1}(-\sin|x|), x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, **them which of the following is true?**

[JEE MAIN 2020]

1) f' is decreasing in $\left(-\frac{\pi}{2},0\right)$ and increasing in $\left(0,\frac{\pi}{2}\right)$

2) f is not differntiable at $x=0$

3) $f'(0)=-\frac{\pi}{2}$

4) f' is increasing in $\left(-\frac{\pi}{2},0\right)$ and decreasing in $\left(0,\frac{\pi}{2}\right)$

25. **Let** $f(x)=\left(\sin\left(\tan^{-1}x\right)+\sin\left(\cot^{-1}x\right)\right)^2-1, |x|>1$. **If** $\frac{dy}{dx}=\frac{1}{2}\frac{d}{dx}\left(\sin^{-1}(f(x))\right)$ **and** $y\left(\sqrt{3}\right)=\frac{\pi}{6}$, **then** $y\left(-\sqrt{3}\right)$ **is equal to**

[JEE MAINS 2020]

1) $\frac{5\pi}{6}$ 2) $-\frac{\pi}{6}$ 3) $\frac{\pi}{3}$ 4) $\frac{2\pi}{3}$

KEY

01) 7	02) 1	03) 3	04) 1	05) 4
06) 1	07) 1	08) 4	09) 4	10) 3
11) 2	12) 1	13) 2	14) 4	15) 3
16) 4	17) 2	18) 3	19) 2	20) 4
21) 3	22) 2	23) 3	24) 1	25) 1

SOLUTIONS

1. $f(x)=\sqrt{3\cos^{-1}(4x)-\pi} \geq 0$

$\Rightarrow x \leq \frac{1}{8}, -\frac{1}{4} \leq x \leq \frac{1}{4}$

$\left[-\frac{1}{4}, \frac{1}{8}\right] \Rightarrow 4a+64b=7$

2. $3x^2+bx+3 \geq 0$

$b^2-4ac \leq 0 \Rightarrow b^2-36 \leq 0$

$b=-6,-5,-4,..,0,..,4,5,6$

sum of all the values $=0$

3. $f(x)=\tan^{-1}\sqrt{[x]+[-x]}+\sqrt{2-|x|}+\frac{1}{x^2}$

Common domain $=\{-2,-1,1,2\}$

Range $=\left\{\frac{1}{4}, 2\right\}$

4. $\left[\sin^{-1}x\right]>\left[\cos^{-1}x\right]$

$\Rightarrow[\sin 1, 1]\cap[\cos 1, 1]=[\sin 1, 1]$

5. $y=\left(\cot^{-1}x\right)\left(\cot^{-1}(-x)\right)$

$A.M \geq G.M$

6. $E=\sqrt{\sin^{-1}x_1}\sqrt{\cos^{-1}x_2}+\sqrt{\sin^{-1}x_2}\sqrt{\cos^{-1}x_3}+$
$\sqrt{\sin^{-1}x_3}\sqrt{\cos^{-1}x_4}+...+\sqrt{\sin^{-1}x_{28}}\sqrt{\cos^{-1}x_1}$

$\sin^{-1}x_i>0$ $\quad A.M \geq G.M$

$\frac{a^2+b^2}{2} \geq ab$

$\sqrt{\sin^{-1}x_1}\sqrt{\cos^{-1}x_2} \leq \frac{\left(\sin^{-1}x_1+\cos^{-1}x_2\right)}{2}$

..... and so on

$\sqrt{\sin^{-1}x_{28}}\sqrt{\cos^{-1}x_1} \leq \frac{\left(\sin^{-1}x_{28}+\cos^{-1}x_1\right)}{2}$

adding all, we get

$E \leq \sum_{i=1}^{28}\frac{\sin^{-1}x_i+\cos^{-1}x_i}{2}$ $\quad E \leq \frac{28\left(\frac{\pi}{2}\right)}{2}$

$E_{\max}=7\pi$

7. *Let* $f(x)=x^3+bx^2+cx+1$

$f(0)=1>0, f(-1)=b-c<0$

So, $\alpha \in(-1,0)$

So, $2\tan^{-1}(\operatorname{cosec}\alpha)+\tan^{-1}\left(2\sin\alpha\sec^2\alpha\right)$

$=2\tan^{-1}\left(\frac{1}{\sin\alpha}\right)+\tan^{-1}\left(\frac{2\sin\alpha}{1-\sin^2\alpha}\right)$

$=2\left[\tan^{-1}\left(\frac{1}{\sin\alpha}\right)+\tan^{-1}(\sin\alpha)\right]$

$=2\left(-\frac{\pi}{2}\right)=-\pi$

8. $2^{2\pi/\sin^{-1}x}-2(a+2)2^{\pi/\sin^{-1}x}+8a<0$

$\left(2^{\pi/\sin^{-1}x}-4\right)\left(2^{\pi/\sin^{-1}x}-2a\right)<0$

Now $2^{\pi/\sin^{-1}x} \in\left(0, \frac{1}{4}\right]\cup[4,\infty)$

for $2^{\pi/\sin^{-1}x} \in\left(0, \frac{1}{4}\right]$. We have

$\left(2^{\pi/\sin^{-1}x}-4\right)<0$ $\qquad \therefore 2^{\pi/\sin^{-1}x}-2a>0$

or $0 \leq a<\frac{1}{8}$

Similarly, for $2^{\pi/\sin^{-1}x} \in[4,\infty), a>2$, we get

So, $a \in\left[0, \frac{1}{8}\right)\cup(2,\infty)$

9. x, y, z are in A.P. $\Rightarrow 2y=x+z \rightarrow (1)$

$\tan^{-1}x, \tan^{-1}y, \tan^{-1}z$ are in A.P.

$=2\tan^{-1}y=\tan^{-1}\left(\frac{x+z}{1-xz}\right)$

$y^2=xz \rightarrow (2)$ $\qquad$ From (1) and (2)

$\therefore x=y=z$

10. $\tan^{-1}\left(\cot\frac{43\pi}{4}\right)\tan^{-1}\left(\cot\left(10\pi+\frac{3\pi}{4}\right)\right)$

$\tan^{-1}(-1)=-\frac{\pi}{4}$

11. $f(x)=2\tan^{-1}x+\sin^{-1}\left(\frac{2x}{1+x^2}\right), x>1=$

$2\tan^{-1}x+\pi-2\tan^{-1}x$

$f(x)=\pi$, So $f(5)=\pi$

12. $\tan^{-1} y = \tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right), |x| < \frac{1}{\sqrt{3}}.$

$\tan^{-1} y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

$\Rightarrow y = \frac{3x-x^3}{1-3x^2}$

13. $\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$ Put

$x^2 = \cos 2\theta$

$\tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right) = \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2$

14. $\sin[\cot^{-1}(1+x)] = \cos[\tan^{-1} x]$

$= \frac{1}{\sqrt{x^2+2x+2}} = \frac{1}{\sqrt{1+x^2}}$ $\quad x = -\frac{1}{2}$

15. $\cos^{-1}\left(\frac{2}{3x}\right) + \cos^{-1}\left(\frac{3}{4x}\right) = \frac{\pi}{2}$

$\cos^{-1}\left(\frac{1}{2x^2} - \frac{\sqrt{9x^2-4}\sqrt{16x^2-9}}{12x^2}\right) = \frac{\pi}{2}$

$\sqrt{9x^2-4}\sqrt{16x^2-9} = 6$ But $\left(x > \frac{3}{4}\right)$

x=$\frac{\sqrt{145}}{12}$

16. $x = \sin^{-1}(\sin 10)$ = -10 +3π

$y = \cos^{-1}(\cos 10)$ = -10+4π

y-x = π

17. $\cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n} 2p\right)\right)$

$= \cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+n(n+1)\right)\right)$

$= \cot\left(\sum_{n=1}^{19}\tan^{-1}\frac{1}{(1+n(n+1))}\right)$

$= \cot\left(\sum_{n=1}^{19}\tan^{-1}\frac{(n+1)-n}{(1+n(n+1))}\right) = \frac{21}{19}$

18. $\left(\cot^{-1} x\right)^2 - 7\left(\cot^{-1} x\right) + 10 > 0$

$= \left(\cot^{-1} x - 2\right)\left(\cot^{-1} x - 5\right) > 0$

$= \cot^{-1} x < 2 \ or \ \cot^{-1} x > 5$

$= \cot^{-1} x \in (0,2) \quad \therefore x \in (\cot 2, \infty)$

19. $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}, x \geq 0$

$= \frac{5x}{1-6x^2} = 1 \Rightarrow x = \frac{1}{6}$ So A is a singleton set.

20. $\alpha = \cos^{-1}\left(\frac{3}{5}\right), \quad \beta = \tan^{-1}\left(\frac{1}{3}\right)$, where

$0 < \alpha, \beta < \frac{\pi}{2}$ So, $\alpha - \beta = \tan^{-1}\frac{4}{3} - \tan^{-1}\frac{1}{3}$

$= \tan^{-1}\frac{9}{13} = \sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

21. $\cos^{-1} x - \cos^{-1}\frac{y}{2} = \alpha$, where

$-1 \leq x \leq 1, -2 \leq y \leq 2, x \leq \frac{y}{2}$

$\cos^{-1}\left(x\frac{y}{2} + \sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}}\right) = \alpha$

$\sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}} = \cos\alpha - \frac{xy}{2}$

On squaring both sides

$4x^2 - 4xy\cos\alpha + y^2 = 4\sin^2\alpha$

22. $\sin^{-1}\left(\frac{12}{13}\right) - \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{12}{13}\sqrt{1-\frac{9}{25}} - \frac{3}{5}\sqrt{1-\frac{144}{169}}\right)$

$\sin^{-1}\left(\frac{33}{65}\right) = \cos^{-1}\sqrt{1-\left(\frac{33}{65}\right)} = \cos^{-1}\left(\frac{56}{65}\right) =$

$\frac{\pi}{2} - \sin^{-1}\left(\frac{56}{65}\right)$

23. $f'(x) = \tan^{-1}(\sec x + \tan x)$

$f'(x) = \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right)$

$\Rightarrow f'(x) = \frac{\pi}{4} + \frac{x}{2}$

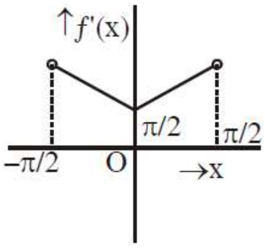
$f(x) = \frac{\pi}{4}x + \frac{x^2}{4} + c \Rightarrow f(1) = \frac{\pi+1}{4} \quad (c=0)$

24. $f(x)$ is an odd function

Now, if $x \geq 0$, then $f(x) = x\cos^{-1}(-\sin x)$

$= x\left(\frac{\pi}{2} - \sin^{-1}(-\sin x)\right) = x\left(\frac{\pi}{2} + x\right)$

Hence, $f(x) = \begin{cases} x\left(\frac{x}{2} + x\right); & x \in \left[0, \frac{\pi}{2}\right] \\ x\left(\frac{\pi}{2} - x\right); & x \in \left[-\frac{\pi}{2}, 0\right) \end{cases}$



25. $f(x)=\left(\sin\left(\tan^{-1}x\right)+\sin\left(\cot^{-1}x\right)\right)^2-1$

$f(x)=\frac{2x}{1+x^2}, |x|\geq 1 \quad \frac{dy}{dx}=\frac{1}{2}\frac{d}{dx}\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

for $|x|>1 \quad \frac{dy}{dx}=\frac{-1}{1+x^2}$

$\Rightarrow y=\cot^{-1}x+c$ given $y\left(\sqrt{3}\right)=\frac{\pi}{6}$

$\therefore c=0 \quad y\left(-\sqrt{3}\right)=\cot^{-1}\left(-\sqrt{3}\right)=\frac{5\pi}{6}$

ADVANCED LEVEL QUESTIONS

SINGLE ANSWER TYPE QUESTIONS

1. ***The value of*** $\cot\left(\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right)$ ***is***

(ADV-2013)

1) $\frac{23}{25}$ 2) $\frac{25}{23}$ 3) $\frac{23}{24}$ 4) $\frac{24}{23}$

2. ***Let*** $f:[0,4\pi]\to[0,\pi]$ ***be defined by*** $f(x)=\cos^{-1}(\cos x)$. ***The number of points*** $x\in[0,4\pi]$ ***satisfying the equation*** $f(x)=\frac{10-x}{10}$ *is* _____ **(ADV-2014)**

3. **The number of real solutions of the equation**

$\sin^{-1}\left(\sum_{i=1}^{\infty}x^{i+1}-x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^i\right)=$

$\frac{\pi}{2}-\cos^{-1}\left(\sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^i-\sum_{i=1}^{\infty}(-x)^i\right)$

lying in the interval $\left(-\frac{1}{2},\frac{1}{2}\right)$ **is** _____

(ADV 2018)

4. **The value of**

$\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)\right)$ **in the interval** $\left[-\frac{\pi}{4},\frac{3\pi}{4}\right]$ **equals** _____

(ADV 2019)

KEY

1) 2 2) 3 3) 2 4) 0

SOLUTIONS

1. $\cot\left(\sum_{n=1}^{23}\cot^{-1}\left(n^2+n+1\right)\right)$

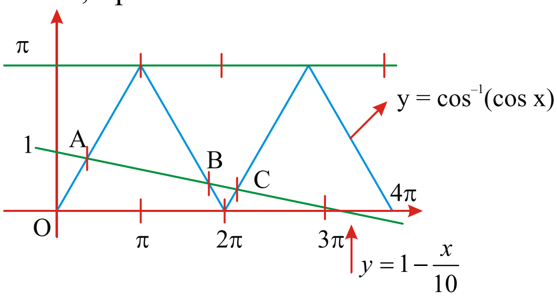
$\cot\left(\sum_{n=1}^{23}\tan^{-1}\left(\frac{n+1-n}{1+n(n+1)}\right)\right)$

$\Rightarrow \cot\left(\tan^{-1}\left(\frac{23}{25}\right)\right)=\frac{25}{23}$

2. $f:[0,4\pi]\to[0,\pi], f(x)=\cos^{-1}(\cos x)$

$\Rightarrow$ points A, B, C satisfy $f(x)=\frac{10-x}{10}$

Hence, 3 points



3. $\sin^{-1}\left(\frac{x^2}{x-1}-\frac{x\left(\frac{x}{2}\right)}{1-\frac{x}{2}}\right)=\frac{\pi}{2}-\cos^{-1}\left(\frac{-x/2}{1+\frac{x}{2}}-\frac{(-x)}{1+x}\right)$

$\Rightarrow \sin^{-1}\left(x^2\left(\frac{1}{x-1}-\frac{1}{2-x}\right)\right)=$

$\frac{\pi}{2}-\cos^{-1}\left(x\left(\frac{1}{x+1}-\frac{1}{2+x}\right)\right)$

$\sin^{-1}\left(\frac{x^2}{(x-1)(2-x)}\right)=$

$\frac{\pi}{2}-\cos^{-1}\left[\frac{x}{(1+x)(2+x)}\right]$

$=\sin^{-1}\left[\frac{x}{(1+x)(2+x)}\right]$

$\Rightarrow x\left[\frac{x}{(1-x)(2-x)}-\frac{1}{(1+x)(2+x)}\right]=0$

$x=0$ or $x^3+3x^2+2x=x^2-3x+2$

$\Rightarrow x^3+2x^2+x-2=0$

Increasing function $\forall x$

$f(0)=-2, f(1/2)>0$

$\Rightarrow$ one root between $\left(0,\frac{1}{2}\right)$

$\Rightarrow$ total number of solutions $=2$

4. $\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)\right)$

$=\sec^{-1}\left(-\frac{1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)co\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\right)$

$=\sec^{-1}\left(-\frac{1}{2}\sum_{k=0}^{10}\frac{1}{\sin\left(\frac{7\pi}{6}\right).(1)^k}\right)$

$=\sec^{-1}\left(-\frac{1}{2}\frac{1}{\sin\left(\frac{7\pi}{6}\right)}\right)=\sec^{-1}(1)=0$

MULTIPLE ANSWER TYPE QUESTIONS

1. Let f(x) = $\sin^{-1} x + \cos^{-1}x$. **Then $\pi/2$ is equal to**

A) $f\left(-\frac{1}{2}\right)$ B) $f(k^2-2k+3), k \in R$

C) $f\left(\frac{1}{1+k^2}\right), k \in R$ D) f(–2)

2. **Let** $f(x) = e^{\cos^{-1}\sin(x+\pi/3)}$ **then**

A) $f\left(\frac{8\pi}{9}\right) = e^{5\pi/18}$ B) $f\left(\frac{8\pi}{9}\right) = e^{13\pi/18}$

C) $f\left(-\frac{7\pi}{4}\right) = e^{\pi/12}$ D) $f\left(-\frac{7\pi}{4}\right) = e^{11\pi/12}$

3. **If** $(\sin^{-1} x)^2 + (\sin^{-1} y)^2 + (\sin^{-1} z)^2 = \frac{3\pi^2}{4}$, **then the value of $(x-y+z)$ can be**

A) 1 B) – 1 C) 3 D) – 3

4. **Which of the following is/are true**

A) $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right), when |x| \le 1$

B) $2\sin^{-1} x = \sin^{-1}\{[2x\sqrt{(1-x^2)}]$, when $\frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

C) $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, when $x > 1$

D) $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, when $x < -1$

5. **Which of the following is/are true**

A) $\tan^{-1}\frac{1}{3} = \frac{1}{2}\sin^{-1}\frac{3}{5}$

B) $\tan^{-1}\frac{1}{3} = \frac{\pi}{4} - \cot^{-1} 2$

C) $\tan^{-1}\frac{1}{3} = \frac{\pi}{4} - \frac{1}{2}\sin^{-1}\frac{4}{5}$

D) $\tan^{-1}\frac{1}{3} = \frac{\pi}{2} - \cot^{-1} 3$

KEY

01) A,C	02) B,C	03) A,B,C,D
04) A,B	05) A,B,C	

SOLUTIONS

1. $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \quad \forall\, x \in [-1, 1]$

2. $f\left(\frac{8\pi}{9}\right) = e^{\cos^{-1}\left(\sin\left(\frac{11\pi}{9}\right)\right)} = e^{\cos^{-1}\cos\frac{13\pi}{18}} = e^{\frac{13\pi}{18}}$

$f\left(\frac{-7\pi}{4}\right) = e^{\cos^{-1}\sin\left(\frac{-7\pi}{4}+\frac{\pi}{3}\right)} = e^{\cos^{-1}\cos\frac{\pi}{12}} = e^{\frac{\pi}{12}}$

3. As $\frac{-\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2} \quad \forall -1 \le x \le 1$

$0 \le \left(\sin^{-1} x\right) \le \frac{\pi^2}{4}$

$\therefore 0 \le \left(\sin^{-1} x\right)^2 + \left(\sin^{-1} y\right)^2 + \left(\sin^{-1} z\right)^2 \le \frac{3\pi^2}{4}$

$\therefore (\sin^{-1} x)^2 + (\sin^{-1} y)^2 + (\sin^{-1} z)^2 = \frac{3\pi^2}{4}$ is possible if $x, y, z \in \{-1, 1\}$

$\therefore$ Possible values of $x - y + z$ from the ordered triplet (x, y, z) are as follows :

(x, y, z)	x – y + z
(– 1, –1, –1)	– 1
(– 1, 1, 1)	– 1
(1, – 1, 1)	3
(1, 1, – 1)	– 1
(1, 1, 1)	1
(1, – 1, – 1)	1
(– 1, 1, – 1)	–3
(– 1, – 1, 1)	1

Hence set of values of x – y + z is {±1, ± 3)

$-1 \le x \le 1$

4. (A) $\frac{-\pi}{4} \le \tan^{-1} x \le \frac{\pi}{4} \Rightarrow \frac{-\pi}{2} \le 2\tan^{-1} x \le \frac{\pi}{2}$

$\frac{-1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}} \Rightarrow \frac{-\pi}{4} < \sin^{-1} x < \frac{\pi}{4}$

B) $\Rightarrow \frac{-\pi}{2} < 2\sin^{-1} x < \frac{\pi}{2}$

5. $2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{3}{4} = \sin^{-1}\frac{3}{5} = \frac{\pi}{2} - \cos^{-1}\frac{3}{5} = \frac{\pi}{2} - \sin^{-1}\frac{4}{5}$

$\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2} = \frac{\pi}{4} \Rightarrow \tan^{-1}\frac{1}{3} = \frac{\pi}{4} - \tan^{-1}\frac{1}{2} = \frac{\pi}{4} - \cot^{-1} 2$

COMPREHENSION TYPE QUESTIONS

Passage-1

Let $\cos^{-1}\left(4x^3 - 3x\right) = a + b\cos^{-1} x$

1. If $x \in \left[-1, -\frac{1}{2}\right)$**, then the value of** $a + b\pi$ **is**

A) 2π B) 3π C) π D) -2π

2. If $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$**, then the principal value of** $\sin^{-1}\left(\sin\frac{a}{b}\right)$ **is**

A) $-\frac{\pi}{3}$ B) $\frac{\pi}{3}$ C) $-\frac{\pi}{6}$ D) $\frac{\pi}{6}$

3. If $x \in \left(\frac{1}{2}, 1\right]$**, then the value of** $\lim_{y \to a} b\cos(y)$ **is**

A) -1/3 B) -3 C) 1/3 D) 3

Passage-2

While defining inverse trigonometric functions, a new system is followed where domains and ranges have been redefined as follows.

Function	**Domain**	**Range**
$\sin^{-1} x$	$[-1,1]$	$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$
$\tan^{1} x$	R	$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$
$\cos^{-1} x$	$[-1,1]$	$[\pi, 2\pi]$
$\cot^{-1} x$	R	$[\pi, 2\pi]$

4. $\sin^{-1}(-x)$ **is equal to**

A) $-\sin^{-1} x$ B) $\pi + \sin^{-1} x$

C) $2\pi - \sin^{-1} x$ D) $3\pi - \cos^{-1}\sqrt{1-x^2},\ x > 0$

5. If $f(x) = 3\sin^{-1}x - 2\cos^{-1}x$ then $f(x)$ is

A) even function B) odd function

C) Neither even nor odd

D) even as well as odd function

6. The minimum of $(\sin^{-1}x)^3 - (\cos^{-1}x)^3$ is equal to

A) $-\frac{63}{8}\pi^3$ B) $\frac{63}{8}\pi^3$ C) $\frac{125\pi^3}{32}$ D) $\frac{125^3}{32}$

7. If in another such system range of $\sin^{-1}x$ is $\left[-\frac{5\pi}{2}, -\frac{3\pi}{2}\right]$ and $\sin^{-1}x + \cos^{-1}x = 5\pi/2$ then the range of $\cos^{-1}x$ will be

A) $[4\pi, 5\pi]$ B) $[3\pi, 4\pi]$

C) $[6\pi, 7\pi]$ D) $[5\pi, 6\pi]$

KEY

01) C 02) A 03) D 04) C 05) B 06) A 07) A

SOLUTIONS

1-3. For

$$x \in \left[-1, -\frac{1}{2}\right), \cos^{-1}\left(4x^3 - 3x\right) = 3\cos^{-1}x - 2\pi$$

$\Rightarrow a = -2\pi$ *and* $b = 3 \Rightarrow a + b\pi = \pi$

For

$$x \in \left[-\frac{1}{2}, \frac{1}{2}\right], \cos^{-1}\left(4x^3 - 3x\right) = 2\pi - 3\cos^{-1}x$$

$\Rightarrow a = 2\pi$ and

$$b = -3 \Rightarrow \sin^{-1}\left(\sin\frac{a}{b}\right) = \sin^{-1}\left(\sin\frac{2\pi}{-3}\right)$$

$$= \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

For $x \in \left(\frac{1}{2}, 1\right]$, $\cos^{-1}\left(4x^3 - 3x\right) = 3\cos^{-1}x$

$\Rightarrow a = 0$ and $b = 3$

$\therefore \lim\limits_{y \to a} b\cos(y) = \lim\limits_{y \to 0} 3\cos(y) = 3$.

4.Let $\sin^{-1}(-x) = \theta$

$x = -\sin\theta$, $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ $= \sin(2\pi - \theta)$

$2\pi - \theta = \sin^{-1}x$

$2\pi - \sin^{-1}x = \theta = \sin^{-1}(-x)$

5. Let $\cos^{-1}(-x) = \theta$, $\pi \le \theta \le 2\pi$

$\therefore -x = \cos\theta$

$x = -\cos\theta = \cos(3\pi - \theta)$

$\therefore 3\pi - \theta = \cos^{-1}x$

$\therefore 3\pi - \cos^{-1}x = \theta$ $= \cos^{-1}(-x)$

Calculate $f(-x)$.

6. $f = (\sin^{-1}x)^3 - (\cos^{-1}x)^3$, $-1 \le x \le 1$

$$f^1 = 3(\sin^{-1}x)^2 \frac{1}{\sqrt{1-x^2}} - 3(\cos^{-1}x)^2 . \left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$f^1 < 0$

$\therefore$ Minimum value $= \left(\frac{\pi}{2}\right)^3 - (2\pi)^3$

$$= \frac{\pi^3}{8} - 8\pi^3 = \frac{-63\pi^3}{8}$$

7. $\frac{-5\pi}{2} \le \sin^{-1}x \le \frac{-3\pi}{2}$

$\therefore \frac{3\pi}{2} \le -\sin^{-1}x \le \frac{5\pi}{2}$

$$\frac{3\pi}{2} + \frac{5\pi}{2} \le \frac{5\pi}{2} - \sin^{-1}x \le \frac{5\pi}{2} + \frac{5\pi}{2}$$

$4\pi \le \cos^{-1}x \le 5\pi$

TRIGONOMETRIC EQUATIONS

SYNOPSIS

- If trigonometric functions are involving in the equation then the equation is called trigonometric equation.

  Ex. $a\cos\theta + b\sin\theta = c$

- **Solution of a trigonometric equation**:

  A value of the unknown angle satisfying the given equation is called the solution of the given equation. Every other coterminal angle of a solution is also solution of the given trigonometric equation.

- The trigonometric equation may have infinite number of solutions and can be classified as

  (a) Principal solution **(b) General solution**

  (a) Principal solution:

  If θ has infinitely many solutions among them, the least value with sign is called principal value of θ. It is denoted by 'α'.

(i) There exists a unique value of 'θ' in $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ satisfying $\sin\theta = k, k \in R, |k| \le 1$. This value of '$\theta$' is called principal value of 'θ' or prinicpal solution of $\sin\theta = k$

(ii) There exists a unique value of 'θ' in $[0, \pi]$ satisfying $\cos\theta = k, k \in R, |k| \le 1$. This value of '$\theta$' is called principal value of 'θ' or principal solution of $\cos\theta = k$

(iii) There exists a unique value of 'θ' in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ satisfying $\tan\theta = k, k \in R$. This value of 'θ' is called principal value of 'θ' or principal solution of $\tan\theta = k$.

(b) General solution: Since trigonometric functions are periodic, a solution generalised by means of periodicity of trigonometric functions. The solution consisting of all possible solutions of a trigonometric equation is called its general solution.

S no	Equation	Interval in which principal solution α	General Solution
1	$\sin\theta = k$, $(-1 \le k \le 1)$	$\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$	$\theta = n\pi + (-1)^n \alpha$, $n \in Z$
2	$\cos\theta = k$, $(-1 \le k \le 1)$	$[0, \pi]$	$\theta = 2n\pi \pm \alpha$, $n \in Z$
3	$\tan\theta = k$, $(k \in R)$	$\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$	$\theta = n\pi + \alpha$, $n \in Z$
4	$\operatorname{cosec}\theta = k$, $k \in (-\infty, -1] \cup [1, \infty)$	$\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right] - \{0\}$	$\theta = n\pi + (-1)^n \alpha$, $n \in Z$
5	$\sec\theta = k$, $k \in (-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\dfrac{\pi}{2}\right\}$	$\theta = 2n\pi \pm \alpha$, $n \in Z$
6	$\cot\theta = k$, $k \in R$	$(0, \pi)$	$\theta = n\pi + \alpha$, $n \in Z$

- The general solution of $\sin^2\theta = \sin^2\alpha$ (or) $\cos^2\theta = \cos^2\alpha$ (or) $\tan^2\theta = \tan^2\alpha$ is $\theta = n\pi \pm \alpha, n \in Z$

- **Common Solution of Two Equations:**

  The general solution for the equations of the type $\sin\theta = \sin\alpha$, $\cos\theta = \cos\alpha$ is $\theta = 2n\pi + \alpha$ where, α is common angle lies between 0 and 2π

Ex 1:

The solution of $\sin\theta = -\dfrac{1}{2}$, $\tan\theta = \dfrac{1}{\sqrt{3}}$ **is**

Sol: $\because \sin\theta < 0$ and $\tan\theta > 0 \Rightarrow \theta \in Q_3$,

$\therefore$ principal common solution is $\pi + \pi/6 = 7\pi/6$

$\therefore$ general solution $\theta = 2n\pi + \dfrac{7\pi}{6}, n \in Z$

✈ **Some important values to remember:**

Angle θ	Sin θ	Cos θ
15^0	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$
18^0	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$
$22\frac{1}{2}^0$	$\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$	$\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$
36^0	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{\sqrt{5}+1}{4}$
54^0	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$
$67\frac{1}{2}^0$	$\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$	$\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$
72^0	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{5}-1}{4}$
75^0	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$

✈ i) $\tan 15^0 = 2-\sqrt{3}$, $\tan 22\frac{1}{2}^0 = \sqrt{2}-1$,

ii) $\cot 15^0 = 2+\sqrt{3}$, $\cot 22\frac{1}{2}^0 = \sqrt{2}+1$

✈ i) The equation $a\cos\theta + b\sin\theta = c$ has a solution if $|c| \le \sqrt{a^2+b^2}$.

ii) The equation $a\cos\theta + b\sin\theta = c$ has no solution if $|c| > \sqrt{a^2+b^2}$

✈ **Important Points to Remember:**

(i) For equations of the type $\sin\theta = k$ or $\cos\theta = k$, one must check that $|k| \le 1$.

(ii) Avoid squaring the equation, if possible, because it may lead to extraneous solutions.

(iii) Do not cancel the common variable factor from the two sides of the equations, because we may loose some solutions.

(iv) The answer should not contain such values of θ which make any of the terms undefined or infinite.

(v) Check that denominator is not zero at any stage while solving equations.

(vi) (a) If $\tan\theta$ or $\sec\theta$ is involved then θ should not be an odd multiple of $\pi/2$.

(b) If $\cot\theta$ or cosec θ is involved then θ should not be a multiple of π or 0

(vii) If two different trigonometric ratios, such as, $\tan\theta$ and $\sec\theta$ are involved then after solving we cannot apply the usual formulae for general solution, because periodicity of the functions are not same.

(viii) If L.H.S.of the given trigonometric equation is always less than or equal to k and R.H.S. is always greater than or equal to k.If both the sides are equal to k for same value of θ,then solution exist and if they are equal for different value of θ,then solution does not exist.

✈ **Solution of Inequations:**

If $a < b$, then

(i) $(x-a)(x-b) < 0 \Rightarrow a < x < b$

(ii) $(x-a)(x-b) \le 0 \Rightarrow a \le x \le b$

(iii) $(x-a)(x-b) > 0 \Rightarrow x < a$ or $x > b$

(iv) $(x-a)(x-b) \ge 0 \Rightarrow x \le a$ or $x \ge b$.

EXERCISE - I

1. If $4Cos^2\theta = 3$ **then** $\theta = ----$

1) $\frac{\pi}{6}, \frac{5\pi}{6}$ 2) $\frac{\pi}{4}, \frac{3\pi}{4}$ 3) $\frac{\pi}{3}, \frac{2\pi}{3}$ 4) $\pm\frac{\pi}{2}$

2. If $\cos 2\theta = 2\sin^2\theta$ **then** $\theta =$

1) $\pm 30^0$ 2) $\pm 60^0$ 3) $\pm 45^0$ 4) $\pm 90^0$

3. The principal value of $\left(\theta + \frac{\pi}{4}\right)$ **where** $\sin\theta + \cos\theta = 1$ **is**

1) 0^0 2) $\frac{\pi}{3}$ 3) $\frac{\pi}{4}$ 4) $\frac{\pi}{2}$

4. If $Tan\theta + Sec\theta = \sqrt{3}$**, then the principal value of** $\left(\theta + \frac{\pi}{6}\right)$ **is**

1) $\frac{\pi}{3}$ 2) $\frac{\pi}{4}$ 3) $\frac{2\pi}{3}$ 4) $\frac{\pi}{2}$

5. **If** $3Tan^4\alpha-10Tan^2\alpha+3=0$ **then principal values of 'α' are**

1) $\pm45^0,\pm36^0$ 2) $\pm30^0,\pm60^0$

3) $\pm75^0,\pm36^0$ 4) $\pm60^0,\pm15^0$

6. **If** $\cot\theta-\tan\theta=2$ **then principal value of** θ

1) $\frac{\pi}{4}$ 2) $\frac{\pi}{2}$ 3) $\frac{\pi}{8}$ 4) $\frac{3\pi}{4}$

7. **If** $Sin\left(x+28^0\right)=Cos\left(3x-78^0\right)$ **then x =**

1) 37^0, 8^0 2) 39^0 3) 35^0, 8^0 4) 47^0

8. **The smallest value of** θ **satisfying the equation** $\sqrt{3}(Cot\theta+Tan\theta)=4$ **is**

1) $\frac{2\pi}{3}$ 2) $\frac{\pi}{3}$ 3) $\frac{\pi}{6}$ 4) $\frac{\pi}{12}$

9. **The most general value of** θ **satisfying the equations** $Sin\theta=\frac{1}{\sqrt{2}}$, $Cos\theta=-\frac{1}{\sqrt{2}}$ **is**

1) $2n\pi+\frac{\pi}{4},\forall n\in Z$ 2) $2n\pi+\frac{3\pi}{4},\forall n\in Z$

3) $n\pi+\frac{\pi}{6},\forall n\in Z$ 4) $n\pi+\frac{\pi}{3},\forall n\in Z$

10. **Solution of** $Cot^2\theta+\left(\sqrt{3}+\frac{1}{\sqrt{3}}\right)Cot\theta+1=0$ **is**

1) $n\pi-\frac{\pi}{6}, n\pi-\frac{\pi}{3},\forall n\in Z$

2) $n\pi+\frac{\pi}{6}, n\pi+\frac{\pi}{3},\forall n\in Z$

3) $n\pi+\frac{\pi}{12}$ $,\forall n\in Z$ 4) $n\pi+\frac{\pi}{4},\forall n\in Z$

11. **The general solution of** $\frac{\tan 5x-\tan 4x}{1+\tan 5x\tan 4x}=1$ **is**

1) $n\pi+\frac{\pi}{4};\forall n\in Z$ 2) $n\pi\pm\frac{\pi}{4},\forall n\in Z$

3) ϕ 4) $n\pi+\frac{\pi}{6},\forall n\in Z$

12. **3Sinx+4Cosx-6=0 then the general solution of x=**

1) $n\pi+(-1)^n\frac{\pi}{6},\forall n\in Z$

2) $n\pi+(-1)^n\frac{\pi}{4},\forall n\in Z$

3) $n\pi+(-1)^n\frac{\pi}{3},\forall n\in Z$

4) Empty Set

13. $\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)-\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)=2$ **then the general solution of x =**

1) $n\pi\pm\frac{\pi}{4},\forall n\in Z$ 2) $n\pi+\frac{\pi}{4},\forall n\in Z$

3) $n\pi-\frac{\pi}{4},\forall n\in Z$ 4) $n\pi,\forall n\in Z$

14. **If TanA+Tan2A+**$\sqrt{3}$ **TanATan2A=**$\sqrt{3}$ **then the general solution of** $\frac{A}{2}$ **=**

1) $\frac{n\pi}{3}+\frac{\pi}{9},\forall n\in Z$ 2) $n\pi+\frac{\pi}{9},\forall n\in Z$

3) $\frac{n\pi}{6}+\frac{\pi}{18},\forall n\in Z$ 4) $\frac{n\pi}{3}+\frac{\pi}{4},\forall n\in Z$

15. **The general value of** θ **satisfies the equation** $Tan\theta Tan\left(120^0+\theta\right)Tan\left(120^0-\theta\right)=\frac{1}{\sqrt{3}}$ **is**

1) $(6n+1)\frac{\pi}{18},\forall n\in Z$ 2) $(3n+1)\frac{\pi}{3},\forall n\in Z$

3) $(6n+1)\frac{\pi}{6},\forall n\in Z$ 4) $(3n+1)\frac{\pi}{6},\forall n\in Z$

16. **If** $Tanm\theta=Cotn\theta$ **then the G.S of** θ **=**

1) $\frac{(k+1)\pi}{2(m+n)},\forall k\in Z$ 2) $\frac{(2k+1)\pi}{2(m+n)},\forall k\in Z$

3) $\frac{(2k+1)\pi}{m+n},\forall k\in Z$ 4) $\frac{(2k+1)\pi}{m-n},\forall k\in Z$

17. **If** $Sin\, x.Sin\left(60^0+x\right).Sin\left(60^0-x\right)=\frac{1}{8}$**,then x=**

1) $n\pi+(-1)^n\frac{\pi}{6}$ 2) $\frac{n\pi}{3}+(-1)^n\frac{\pi}{18}$

3) $n\pi+(-1)^n\frac{\pi}{3}$ 4) $\frac{n\pi}{3}+(-1)^n\frac{\pi}{9}$

18. **The equation** $\sqrt{3}Sinx+\cos x=4$ **has**

1) Only one solution 2) Two solutions

3) Infinitely many solutions 4) No solution

19. If $y + Cos\theta = Sin\theta$ has a real solution then

1) $-\sqrt{2} \le y \le \sqrt{2}$ 2) $y > \sqrt{2}$

3) $y \le -\sqrt{2}$ 4) $y \le 1$

20. If $1 + Sin^2\theta = 3Sin\theta Cos\theta$ then the solution set in $\left(0, \frac{\pi}{2}\right)$ is

1) $\left\{\frac{\pi}{4}, Cos^{-1}\frac{1}{3}\right\}$ 2) $\left\{\frac{\pi}{4}, Tan^{-1}\frac{1}{2}\right\}$

3) $\left\{\frac{\pi}{3}, Tan^{-1}\frac{1}{3}\right\}$ 4) $\left\{\frac{\pi}{6}, Sin^{-1}\frac{1}{3}\right\}$

21. The value of θ satisfying

$Sin7\theta = Sin4\theta - Sin\theta$ **in** $0 < \theta < \frac{\pi}{2}$ **are**

1) $\frac{\pi}{9}, \frac{\pi}{4}$ 2) $\frac{\pi}{3}, \frac{\pi}{9}$ 3) $\frac{\pi}{6}, \frac{\pi}{4}$ 4) 0,1

22. The values of θ satisfying

$Sin5\theta = Sin3\theta - Sin\theta$ **and** $0 < \theta < \frac{\pi}{2}$ **are**

1) $\frac{\pi}{6}, \frac{\pi}{3}$ 2) $\frac{\pi}{6}, \frac{\pi}{4}$ 3) $\frac{\pi}{4}, \frac{\pi}{3}$ 4) $\frac{\pi}{4}, \frac{\pi}{2}$

23. The number of solutions of the equation

$$Sin3\alpha = 4Sin\alpha.Sin(x+\alpha).Sin(x-\alpha)$$

where $0 < \alpha < \pi$ for x in the interval $[0, \pi]$ is

1) 1 2) 2 3) 4 4) 5

24. General solution of $2^{\sin x} + 2^{\cos x} = 2^{1-\frac{1}{\sqrt{2}}}$ is

1) $n\pi - \frac{\pi}{4}$ 2) $2n\pi + \frac{5\pi}{4}$

3) $n\pi + (-1)^n \frac{\pi}{4}$ 4) $2n\pi \pm \frac{\pi}{4}$

KEY

01) 1	02) 1	03) 3	04) 1	05) 2
06) 3	07) 3	08) 3	09) 2	10) 1
11) 1	12) 4	13) 2	14) 3	15) 1
16) 2	17) 2	18) 4	19) 1	20) 2
21) 1	22) 1	22) 2	23) 2	

SOLUTIONS

1. $4\cos^2\theta = 3$

$\Rightarrow \cos^2\theta = \left(\frac{\sqrt{3}}{2}\right)^2 \Rightarrow \theta = n\pi \pm \frac{\pi}{6}$

2. $1 - 2\sin^2\theta = 2\sin^2\theta$, $4\sin^2\theta = 1$

$\sin\theta = \pm\frac{1}{2}$, $\theta = \pm 30^0$

3. $\sin\theta + \cos\theta = 1$

$\Rightarrow \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta = \frac{1}{\sqrt{2}}$

$\Rightarrow \sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, $\therefore$ P.S of $\theta + \frac{\pi}{4} = \frac{\pi}{4}$

4. $\tan\theta + \sec\theta = \sqrt{3} \Rightarrow \sin\theta + 1 = \sqrt{3}\cos\theta$

$\Rightarrow \sqrt{3}\cos\theta - \sin\theta = 1 \Rightarrow \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta = \frac{1}{2}$

$\Rightarrow \cos\theta\cos\frac{\pi}{6} - \sin\theta\sin\frac{\pi}{6} = \frac{1}{2} \Rightarrow \cos\left(\theta + \frac{\pi}{6}\right) = \frac{1}{2}$

$\therefore$ P.S. of $\theta + \frac{\pi}{6} = \frac{\pi}{3}$

5. $3\tan^4\alpha - 10\tan^2\alpha + 3 = 0$

$\Rightarrow 3\tan^4\alpha - 9\tan^2\alpha - \tan^2\alpha + 3 = 0$

$\Rightarrow (3\tan^2\alpha - 1)(\tan^2\alpha - 3) = 0$

$\Rightarrow \tan^2\alpha = \frac{1}{3} = \left(\frac{1}{\sqrt{3}}\right)^2$; $\tan^2\alpha = 3 = (\sqrt{3})^2$

$\therefore \alpha = \pm 30^0, \pm 60^0$

6. $\cot\theta - \tan\theta = 2$ on verification $\theta = \frac{\pi}{8}$ satisfies

$\left(\cot\frac{\pi}{8} = \sqrt{2} + 1; \tan\frac{\pi}{8} = \sqrt{2} - 1\right)$

7. $3x - 78^0 = 90^0 - (x + 28^0)$ or $90+x+28^0$

$4x = 90^0 - 28^0 + 78^0 = 140^0 \Rightarrow x = \frac{140^0}{4} = 35^0$

$x = 8^0$

8. $2\text{cosec}2\theta = \frac{4}{\sqrt{3}}$

$$\sin 2\theta = \frac{\sqrt{3}}{2} \Rightarrow 2\theta = \frac{\pi}{3} \ ; \ \theta = \frac{\pi}{6}$$

9. $\sin\theta = \frac{1}{\sqrt{2}}; \cos\theta = -\frac{1}{\sqrt{2}}$

$\therefore \theta \in$ II quadrant

$\therefore$ P.S. of $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

G.S. of $\theta = 2n\pi + \frac{3\pi}{4}, \forall\ n \in Z$

10. $\cot^2\theta + \left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)\cot\theta + 1 = 0$

$\Rightarrow \cot^2\theta + \sqrt{3}\cot\theta + \frac{1}{\sqrt{3}}\cot\theta + 1 = 0$

$\Rightarrow \cot\theta\left(\cot\theta + \sqrt{3}\right) + \frac{1}{\sqrt{3}}\left(\cot\theta + \sqrt{3}\right) = 0$

$\Rightarrow \left(\cot\theta + \sqrt{3}\right)\left(\cot\theta + \frac{1}{\sqrt{3}}\right) = 0$

$\Rightarrow \cot\theta = -\sqrt{3}; \cot\theta = -\frac{1}{\sqrt{3}}$

$\Rightarrow \tan\theta = -\frac{1}{\sqrt{3}}; \tan\theta = -\sqrt{3}$

$\theta = n\pi - \frac{\pi}{6}; n\pi - \frac{\pi}{3}$

11. $\tan(5x - 4x) = 1$

$\Rightarrow \tan x = 1 \Rightarrow x = n\pi + \frac{\pi}{4},\ n \in Z$

12. $3\sin x + 4\cos x = 6$, Here $|c| > \sqrt{a^2 + b^2}$

$\therefore$ There exists no solution

13. $\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) - \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) = 2\tan x$

$\therefore 2\tan x = 2 \Rightarrow \tan x = 1$

$\Rightarrow x = n\pi + \frac{\pi}{4}, n \in Z$

14. $\tan A + \tan 2A = \sqrt{3}(1 - \tan A \tan 2A)$

$\Rightarrow \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} = \sqrt{3} \Rightarrow \tan 3A = \sqrt{3}$

$\Rightarrow 3A = n\pi + \frac{\pi}{3} \Rightarrow A = \frac{n\pi}{3} + \frac{\pi}{9} \Rightarrow \frac{A}{2} = \frac{n\pi}{6} + \frac{\pi}{18}$

15. $\tan\theta\tan\left(120^0 + \theta\right)\tan\left(120^0 - \theta\right) = \frac{1}{\sqrt{3}}$

$\Rightarrow \tan 3\theta = \frac{1}{\sqrt{3}} \Rightarrow 3\theta = n\pi + \frac{\pi}{6} \Rightarrow \theta = \frac{n\pi}{3} + \frac{\pi}{18}$

$= \left(n + \frac{1}{6}\right)\frac{\pi}{3} = (6n+1)\frac{\pi}{18}$

16. $\tan m\theta = \cot n\theta = \tan\left(\frac{\pi}{2} - n\theta\right)$

$m\theta = K\pi + \frac{\pi}{2} - n\theta \Rightarrow \theta = \frac{K\pi + \frac{\pi}{2}}{m+n} = \frac{(2K+1)\pi}{2(m+n)}$

17. $\frac{1}{4}\sin 3x = \frac{1}{8} \Rightarrow 3x = n\pi + (-1)^n\frac{\pi}{6}$

18. $\sqrt{3}\sin x + \cos x = 4$ (no solution)

$[a\cos x + b\sin x = c$ has a solution

if $-\sqrt{a^2+b^2} \le c \le \sqrt{a^2+b^2}\,]$

19. $y = \sin\theta - \cos\theta$ It has solutions if

$|c| \le \sqrt{a^2+b^2} \Rightarrow -\sqrt{2} \le y \le \sqrt{2}$

20. Divide both sides with $\cos^2\theta$

$\sec^2\theta + \tan^2\theta = 3\tan\theta$

$\Rightarrow 2\tan^2\theta - 3\tan\theta + 1 = 0$

$\Rightarrow (2\tan\theta - 1)(\tan\theta - 1) = 0$

$\Rightarrow \tan\theta = 1$ or $\tan\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{4}$ or $\theta = \tan^{-1}\left(\frac{1}{2}\right)$

21. $\sin 7\theta + \sin\theta = \sin 4\theta$

$2\sin 4\theta.\cos 3\theta = \sin 4\theta$

$\sin 4\theta = 0, \quad \cos 3\theta = \frac{1}{2}$

$\theta=\frac{\pi}{4}$ $\quad 3\theta=\frac{\pi}{3}\Rightarrow\theta=\frac{\pi}{9}$

22. $\sin 5\theta+\sin\theta=\sin 3\theta$

$\Rightarrow 2\sin 3\theta.\cos 2\theta=\sin 3\theta$

$\Rightarrow \sin 3\theta=0$ or $\cos 2\theta=\frac{1}{2},\frac{\pi}{6},\frac{\pi}{3}$ satisfies

23. $3\sin\alpha-4\sin^3\alpha=4\sin\alpha\left(\sin^2 x-\sin^2\alpha\right)$

$\Rightarrow 3\sin\alpha-4\sin^3\alpha=4\sin\alpha\sin^2 x-4\sin^3\alpha$

$\Rightarrow 4\sin\alpha\sin^2 x=3\sin\alpha\Rightarrow\sin^2 x=\frac{3}{4}=\left(\frac{\sqrt{3}}{2}\right)^2$

$\Rightarrow x=n\pi\pm\frac{\pi}{6}, n\in z \quad x=\left\{\frac{\pi}{6},\frac{5\pi}{6}\right\}$

24. The given equality holds for minmum of $\sin x+\cos x$ i.e., $-\sqrt{2}$

$\therefore x=\frac{5\pi}{4}$ G.S is $2n\pi+\frac{5\pi}{4}$

EXERCISE - II

1. If $Cos6\theta+Cos4\theta+Cos2\theta+1=0$ *for* $0\le\theta\le\pi$, **then** $\theta=$

1) $\frac{\pi}{7},\frac{5\pi}{7},\pi$ 2) $\frac{\pi}{2},\frac{\pi}{4},\frac{3\pi}{4},\frac{\pi}{6},\frac{5\pi}{6}$

3) $\frac{\pi}{8},\frac{3\pi}{8},\frac{5\pi}{8},\frac{7\pi}{8},\frac{\pi}{3},\frac{2\pi}{3}$ 4) $\frac{\pi}{9},\frac{4\pi}{9},\frac{5\pi}{9}$

2. If $Cos2\theta.Cos3\theta.Cos\theta=\frac{1}{4}$ **for** $0<\theta<\pi$, *then* $\theta=$ **(Eamcet-2018)**

1) $\frac{\pi}{7},\frac{5\pi}{7},\pi$ 2) $\frac{\pi}{2},\frac{\pi}{4},\frac{3\pi}{4},\frac{\pi}{6},\frac{5\pi}{6}$

3) $\frac{\pi}{8},\frac{3\pi}{8},\frac{5\pi}{8},\frac{7\pi}{8},\frac{\pi}{3},\frac{2\pi}{3}$ 4) $\frac{\pi}{9},\frac{4\pi}{9},\frac{5\pi}{9}$

3. If $\alpha,\beta,\gamma,\delta$ **are the four solutions of the equation Tan**$\left(\theta+\frac{\pi}{4}\right)$**=3Tan 3**$\theta$**. No two of which have equal tangents, then the value of** $Tan\alpha+Tan\beta+Tan\gamma+Tan\delta=$

1) 1 2) 0 3) -1 4) 4

4. If 0 < x, y <$\frac{\pi}{2}$ **then the system of equations Sinx.Siny = 3/4 and Tanx.Tany = 3 has a solution at (Eamcet-2017)**

1) $x=\frac{\pi}{6},y=\frac{\pi}{6}$ 2) $x=\frac{\pi}{3},y=\frac{\pi}{3}$

3) $x=\frac{\pi}{12},y=\frac{\pi}{12}$ 4) $x=\frac{\pi}{4},y=\frac{\pi}{4}$

5. If $32Tan^8\theta=2Cos^2\alpha-3Cos\alpha$ **and** $3Cos2\theta=1$ **then the general value of '**α**' is**

1) $n\pi\pm\frac{\pi}{3}$ 2) $2n\pi\pm\frac{2\pi}{3}$ 3) $n\pi\pm\pi$ 4) $2n\pi\pm\frac{\pi}{2}$

6. If $Cos20^0=k$ **and Cosx = 2k²-1, then the possible values of x between** 0^0 **and** 360^0 **are**

1) 140^0 2) 40^0 and 140^0

3) 40^0 and 320^0 4) 50^0 and 130^0

7. If $x\in(-\pi,\pi)$ **such that**

$y=1+|\cos x|+|\cos^2 x|+|\cos^3 x|+........$

and $8^y=64$, **then no.of values of** x **is**

1) 1 2) 2 3) 3 4) 4

8. The equation (Cosp - 1) x^2 **+ (Cosp) x + Sinp = 0, where x is a variable with real roots. then the interval of p may be any one of the following. (Eamcet- 2019)**

1) $(0,2\pi)$ 2) $(-\pi,0)$ 3) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ 4) $(0,\pi]$

9. Number of solutions of the equation Tan x+Secx = 2 Cos x in the interval $[0,2\pi]$ **is**

1) 0 2) 1 3) 2 4) 3

10. The number of integral values of k for which the equation 7 cosx+ 5 sinx=2k +1 has a solution is

1) 4 2) 8 3) 10 4) 12

11. $Cos^3\alpha+Cos^3(120^0+\alpha)+Cos^3(120^0-\alpha)=\frac{3\sqrt{3}}{4}$ **then the general solution of** α **is**

1) ϕ 2) $2n\pi\pm\frac{\pi}{3},\forall n\in Z$

3) $(2n+1)\frac{\pi}{2}$, $\forall n \in Z$ 4) $n\pi, \forall n \in Z$

12. The general solution of $\sin^{100} x - \cos^{100} x = 1$ **is**

1) $2n\pi + \frac{\pi}{3}, n \in I$ 2) $n\pi + \frac{\pi}{4}, n \in I$

3) $n\pi + \frac{\pi}{2}, n \in I$ 4) $2n\pi - \frac{\pi}{3}, n \in I$

13. The equation 4Sin^{2}x + 4Sinx + a^2 - 3 = 0 has a solution if (Eamcet- 2016)

1) $-2 \le a \le 2$ 2) $-1 \le a \le 1$

3) $-3 \le a \le 3$ 4) $-4 \le a \le 4$

14. If 1/6 sinx, cosx, tan x are in G.P. then x =

1) $n\pi \pm \frac{\pi}{3}, n \in z$ 2) $2n\pi \pm \frac{\pi}{3}, n \in z$

3) $n\pi \pm (-1)^n \frac{\pi}{3}, n \in z$ 4) $n\pi \pm \frac{\pi}{6}, n \in z$

15. The sum of the solutions of the equaiton $\tan x . \tan 4x = 1$ **for 0 < x < π is**

1) 10π 2) $\frac{3\pi}{2}$ 3) $\frac{5\pi}{2}$ 4) 2π

16. If $Tan^2 2\theta = Cot^2\alpha$ **then the general solution is (Eamcet-2019)**

1) $\theta = \left(\frac{1}{4}\right)\left\{n\pi \pm \left(\frac{\pi}{2} - \alpha\right)\right\}$, $\forall n \in Z$

2) $\theta = \left(\frac{1}{2}\right)\left\{n\pi \pm \left(\frac{\pi}{4} - \alpha\right)\right\}$, $\forall n \in Z$

3) $\theta = \left(\frac{1}{2}\right)\left\{n\pi \pm \left(\frac{\pi}{2} - \alpha\right)\right\}$, $\forall n \in Z$

4) $\theta = \left(\frac{1}{4}\right)\left\{n\pi \pm \left(\frac{\pi}{2} + \alpha\right)\right\}$, $\forall n \in Z$

17. If Sinx + Cosx = 1 + Sinx.Cosx then x = ---

1) $n\pi + \frac{\pi}{3}$ 2) $n\pi + (-1)^n \frac{\pi}{6}$

3) $n\pi + (-1)^n \frac{\pi}{2} \cup 2n\pi$ 4) $n\pi$

18. The inequation $3^{Sin^2\theta} + 3^{Cos^2\theta} \ge 2\sqrt{3}$ **is true**

1) for all real values of θ
2) some real values of θ
3) for imaginary values of θ
4) for no value of θ

19. If the complex numbers Sinx + i Cos2x and Cosx - i Sin2x are conjugate to each other, then the set of values of x =

1) $n\pi$ 2) $(2n+1)\frac{\pi}{2}$ 3) $\{0\}$ 4) ϕ

20. If the equation $k\cos x - 3\sin x = k+1$ **has a solution for 'x' then (Eamcet-2018)**

1) $K \le 4$ 2) $K \ge 4$

3) $1 \le k \le 6$ 4) $0 \le k \le 8$

KEY

01) 2	02) 3	03) 2	04) 2	05) 2	06) 3
07) 4	08) 4	09) 3	10) 2	11) 1	12) 3
13) 1	14) 2	15) 4	16) 3	17) 3	18) 1
19) 4	20) 1				

SOLUTIONS

1. $\cos 6\theta + \cos 4\theta + \cos 2\theta + 1 = 0 \quad 0 \le \theta \le \pi$

$\Rightarrow (\cos 6\theta + \cos 2\theta) + (1 + \cos 4\theta) = 0$

$\Rightarrow 2\cos 4\theta \cos 2\theta + 2\cos^2 2\theta = 0$

$\Rightarrow 2\cos 2\theta (\cos 4\theta + \cos 2\theta) = 0$

$\Rightarrow 4\cos 2\theta \cos 3\theta \cos\theta = 0$

$\theta = (2n+1)\frac{\pi}{2}$ or $2\theta = (2n+1)\frac{\pi}{2}$ or $3\theta = (2n+1)\frac{\pi}{2}$

$\theta = \frac{\pi}{2}; \theta = \frac{\pi}{4}, \frac{3\pi}{4}; \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

2. $4\cos 2\theta \cos 3\theta \cos\theta = 1 \quad 0 < \theta < \pi$

$2\cos 2\theta (\cos 4\theta + \cos 2\theta) - 1 = 0$

$\Rightarrow 2\cos 4\theta \cos 2\theta + 2\cos^2 2\theta - 1 = 0$

$\Rightarrow 2\cos 4\theta \cos 2\theta + \cos 4\theta = 0$

$\Rightarrow \cos 4\theta = 0$ or $\cos 2\theta = -\frac{1}{2}$

$4\theta = (2n+1)\frac{\pi}{2}, 2\theta = 2n\pi \pm \frac{2\pi}{3}$

$\Rightarrow \theta = (2n+1)\frac{\pi}{8}, \theta = n\pi \pm \frac{\pi}{3}$

$\therefore \theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}; \theta = \frac{\pi}{3}, \frac{2\pi}{3}$

3. $\dfrac{1+\tan\theta}{1-\tan\theta}=3\left(\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$

$\Rightarrow 3\tan^4\theta-6\tan^2\theta+8\tan\theta-1=0$

$\tan\alpha+\tan\beta+\tan\gamma+\tan\delta=0$

4. $\sin x\sin y=3\cos x\cos y$

$\cos x.\cos y=\dfrac{1}{4}$; $\sin x.\sin y=\dfrac{3}{4}$

$\cos(x+y)=\dfrac{1}{4}-\dfrac{3}{4}=-\dfrac{1}{2}$

$\cos(x-y)=\dfrac{4}{4}=1$

$x-y=0, x+y=\dfrac{2\pi}{3}$, $\quad x=y=\dfrac{\pi}{3}$

5. $3\cos 2\theta=1\Rightarrow\dfrac{3\left(1-\tan^2\theta\right)}{1+\tan^2\theta}=1\Rightarrow\tan^2\theta=\dfrac{1}{2}$

$32\left(\dfrac{1}{2}\right)^4=2\cos^2\alpha-3\cos\alpha$

$2\cos^2\alpha-3\cos\alpha-2=0$

$\Rightarrow(2\cos\alpha+1)(\cos\alpha-2)=0\Rightarrow\cos\alpha=\dfrac{-1}{2}$

$\alpha=2\mathrm{n}\pi\pm\dfrac{2\pi}{3}$

6. $\cos 20^0=\mathrm{K};\cos\mathrm{x}=2\mathrm{K}^2-1$

$=2\cos^2 20^0-1=\cos 40^0$

on verification $\mathrm{x}=40^0$ & 320^0 satisfies

7. $y=\dfrac{1}{1-|\cos x|}$, $8^{\frac{1}{1-|\cos x|}}=64=8^2$

$1-|\cos x|=\dfrac{1}{2}$, $|\cos x|=\dfrac{1}{2}\Rightarrow\cos x=\pm\dfrac{1}{2}$

8. $\Delta\geq 0\Rightarrow\cos^2\mathrm{p}-4\sin\mathrm{p}(\cos\mathrm{p}-1)\geq 0$

$\Rightarrow\cos^2\mathrm{p}+4\sin\mathrm{p}(1-\cos\mathrm{p})\geq 0$

$\cos^2\mathrm{p}\geq 0$ and $1-\cos\mathrm{p}\geq 0$ for all values of p and $\sin\mathrm{p}\geq 0$ for $\mathrm{p}\in(0,\pi]$.

9. $\dfrac{\sin x+1}{\cos x}=2\cos x \qquad (\cos x\neq 0)$

$\Rightarrow 2\sin^2 x+\sin x-1=0$

$\Rightarrow(2\sin x-1)(\sin x+1)=0$, $\Rightarrow\sin x=\dfrac{1}{2}$

but $\sin x+1\neq 0\ \because\cos x\neq 0)$

$\Rightarrow x=\dfrac{\pi}{6},\dfrac{5\pi}{6}$ in $[0,2\pi]$

10. $-\sqrt{49+25}\leq 2k+1\leq\sqrt{49+25}$

$-1-\sqrt{74}\leq 2k\leq -1+\sqrt{74}$

$\dfrac{-1-\sqrt{74}}{2}\leq k\leq\dfrac{-1+\sqrt{74}}{2}$

k=-4,-3,-2,-1,0,1,2,3.

$\dfrac{3}{4}\cos 3\alpha=\dfrac{3\sqrt{3}}{4}\Rightarrow\cos 3\alpha=\sqrt{3}\,(>1)$

$\therefore$ There exists no solution

12. $\sin^{100}\mathrm{x}=1+\cos^{100}\mathrm{x}$

$\sin^{100}\mathrm{x}$ cannot be > 1

13. $\sin x=\dfrac{-4\pm\sqrt{16-16\left(a^2-3\right)}}{8}$

$-1\leq\dfrac{-1\pm\sqrt{4-a^2}}{2}\leq 1$

$-2\leq -1\pm\sqrt{4-a^2}\leq 2$

$0\leq\sqrt{4-a^2}\leq 3$

$4-a^2\geq 0\Rightarrow a\in[-2,2]$

14. $6\cos^2 x=\dfrac{\sin^2 x}{\cos x}$

$6\cos^3 x+\cos^2 x-1=0$

$\cos x=\dfrac{1}{2}$, $\qquad x=2n\pi\pm\dfrac{\pi}{3}$

15. $\sin\mathrm{x}\sin 4\mathrm{x}-\cos\mathrm{x}\cos 4\mathrm{x}=0$

$\Rightarrow\cos(\mathrm{x}+4\mathrm{x})=0\Rightarrow\cos 5\mathrm{x}=0$

$\Rightarrow 5\mathrm{x}=(2\mathrm{n}+1)\dfrac{\pi}{2}$

$\Rightarrow\mathrm{x}=\dfrac{\pi}{10},\dfrac{3\pi}{10},\dfrac{7\pi}{10},\dfrac{9\pi}{10}$ but $\mathrm{x}\neq\dfrac{5\pi}{10}$

$\therefore$ sum of solutions 2π

16. $\tan^2 2\theta = \cot^2 \alpha = \tan^2\left(\frac{\pi}{2}-\alpha\right)$

$\therefore 2\theta = n\pi \pm \left\{\frac{\pi}{2}-\alpha\right\} \Rightarrow \theta = \frac{1}{2}\left\{n\pi \pm \left(\frac{\pi}{2}-\alpha\right)\right\}$

17. $\sin x + \cos x - \sin x \cos x - 1 = 0$

$(\sin x - 1)(1-\cos x) = 0$

$\sin x = 1$ or $\cos x = 1$

$x = n\pi + (-1)^n \frac{\pi}{2}$ or $x = 2n\pi$

18. $3^{\sin^2\theta} = a$, $A.M \geq G.M$

$\frac{a+\frac{3}{a}}{2} \geq \sqrt{a \times \frac{3}{a}}$, $a + \frac{3}{a} \geq 2\sqrt{3}$

19. $\sin x - i\cos 2x = \cos x - i \sin 2x$

$\Rightarrow \sin x = \cos x$ and $\cos 2x = \sin 2x$

No 'x'satisfies both the above equations simultaneously

20. Given $k\cos - 3\sin x = k+1$ has a solution

of $\frac{|k+1|}{\sqrt{k^2+1}} \leq 1$ squaring on both sides

$(k+1)^2 \leq k^2 + 9$

$k^2 + 2k + 1 \leq k^2 + 9$

$2k \leq 8$

$k \leq 4$

EXERCISE - III

1. **The number of points of intersection of the two curves** $y = 2\sin x$ and $y = 5x^2 + 2x + 3$ **is**

1) 0 2) 1 3) 2 4) ∞

2. **The equation**

$2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + \frac{1}{x^2}, 0 \leq x \leq \frac{\pi}{2}$ **has**

1) one real solution 2) no solution
3) more than one real solution
4) Two solutions

3. **Let** $2\sin^2 x + 3\sin x - 2 > 0$ and $x^2 - x - 2 < 0$ **(x is measured in radians). Then x lies in the interval**

1) $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ 2) $\left(-1, \frac{5\pi}{6}\right)$ 3) (-1, 2) 4) $\left(\frac{\pi}{6}, 2\right)$

4. **The solution set of equation** $\cos^5 x = 1 + \sin^4 x$ **is**

1) $\{n\pi, n \in I\}$ 2) $\{2n\pi, n \in I\}$

3) $\{4n\pi, n \in I\}$ 4) $\{n\pi/2, n \in I\}$

5. **The roots of the equation** $\cos^7 x + \sin^4 x = 1$ **in the interval** $(-\pi, \pi)$ **are**

1) $\left\{-\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$ 2) $\left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$ 3) $\left\{\frac{\pi}{2}\right\}$ 4) $\{\pi\}$

6. If $\sin^4 x + \cos^4 y + 2 = 4 \quad \sin x \cos y$ **and** $0 \leq x, y \leq \frac{\pi}{2}$ **then** $\sin x + \cos y$ **is equal to**

1) - 2 2) 0 3) 2 4) 5

7. **Number of ordered pairs (a,x) satisfying the equation** $\sec^2(a+2)x + a^2 - 1 = 0;$ $-\pi < x < \pi$ is

1) 2 2) 1 3) 3 4) infinite

8. **The number of the solutions of the equation** $\cos\left(\pi\sqrt{x-4}\right)\cos\left(\pi\sqrt{x}\right) = 1$ is

1) > 2 2) 2 3) 1 4) 0

9. **The most general values of** θ **for which** $\sin\theta - \cos\theta = \min_{a\in R}(1, a^2 - 6a + 11)$ **are given by**

1) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}, n \in I$

2) $n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{4}, n \in I$

3) $2n\pi + \frac{\pi}{4}, n \in I$ 4) $n\pi + \frac{\pi}{2}, n \in I$

10. **The number of values of x in** $[0, 2\pi]$ **satisfying**

the equation $\left|\cos x - \sin x\right| \geq \sqrt{2}$, **is**

1) 0 2) 1 3) 2 4) 3

11. Let [x] = the greatest integer less than or equal to x and let $f(x) = \sin x + \cos x$. **Then the most general solution of**

$f(x) = \left[f\left(\frac{\pi}{10}\right) \right]$ **are**

1) $2n\pi \pm \frac{\pi}{2}, n \in Z$ 2) $n\pi, n \in Z$

3) $2n\pi - \frac{\pi}{2}, n \in Z$ 4) $2n\pi$ or $2n\pi + \frac{\pi}{2}$

12. If $0 \leq x \leq 2\pi$ and $\left|\cos x\right| \leq \sin x$, **then**

1) $x \in \left[0, \frac{\pi}{4}\right]$ 2) $x \in \left[\frac{\pi}{4}, 2\pi\right]$

3) $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ 4) $[0, \pi]$

13. The equation $1 + \sin^2 ax = \cos x$ **has a unique solution then a is**

1) rational 2) irrational

3) integer 4) whole number

14. If α, β **are solutions of** $\sin^2 x + a\sin x + b = 0$ **and** $\cos^2 x + c\cos x + d = 0$ **then** $\sin(\alpha + \beta)$ **equals**

1) $\frac{2ac}{a^2 + c^2}$ 2) $\frac{a^2 + c^2}{2ac}$ 3) $\frac{2bd}{b^2 + d^2}$ 4) $\frac{b^2 + d^2}{2bd}$

15. If n be the number of solutions of the equation

$\left|\cot x\right| = \cot x + \frac{1}{\sin x}(0 < x < 2\pi)$, **then n =**

1) 1 2) 2 3) 3 4) 4

16. If $\frac{1 - \tan x}{1 + \tan x}$ **= tan y and x -y =** $\frac{\pi}{6}$, **then x,y are respectively**

1) $\frac{5\pi}{24}, \frac{\pi}{24}$ 2) $-\frac{7\pi}{24}, -\frac{11\pi}{24}$

3) $-\frac{115\pi}{24}, -\frac{119\pi}{24}$ 4) All the above

17. If $\theta \in [0, 5\pi]$ and $r \in R$ **such that** $2\sin\theta = r^4 - 2r^2 + 3$ **then the maximum number of values of the pair** (r, θ) **is**

1) 8 2) 10 3) 6 4) 4

18. The general solution of the equation

$\frac{1 - \sin x + ... + (-1)^n \sin^n x + ...}{1 + \sin x + ... + \sin^n x + ...} = \frac{1 - \cos 2x}{1 + \cos 2x}$ *is*

1) $(-1)^n \left(\frac{\pi}{3}\right) + n\pi, \forall n \in I$

2) $(-1)^n \left(\frac{\pi}{6}\right) + n\pi, \forall n \in I$

3) $(-1)^{n+1} \left(\frac{\pi}{6}\right) + n\pi, \forall n \in I$

4) $(-1)^{n-1} \left(\frac{\pi}{3}\right) + n\pi, \forall n \in I$

19. The least difference between the roots, in the first quadrant $\left(0 \leq x \leq \frac{\pi}{2}\right)$**,of the equation**

$4\cos x\ (2 - 3\sin^2 x) + (\cos 2x + 1)$ **= 0, is**

1) $\frac{\pi}{6}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{3}$ 4) $\frac{\pi}{2}$

20. The set of all x in $(-\pi, \pi)$ **satisfying** $\left|4\sin x - 1\right| < \sqrt{5}$ **is given by**

1) $\left(-\frac{3\pi}{10}, \frac{3\pi}{10}\right)$ 2) $\left(-\frac{\pi}{10}, \pi\right)$

3) $(-\pi, \pi)$ 4) $\left(-\pi, \frac{3\pi}{10}\right)$

21. The no. of values of x in the interval $[0, 3\pi]$ **satisfying the equation** $2\sin^2 x + 5\sin x - 3 = 0$ **is**

1) 1 2) 4 3) 6 4)2

22. The solution set satisfying tan x > 1 is

1) $\left(n\pi+\frac{\pi}{4}, n\pi+\frac{\pi}{2}\right)$ 2) $\left(n\pi+\frac{\pi}{4}, n\pi\right)$

3) $\left(n\pi+\frac{\pi}{4}, \infty\right)$ 4) ϕ

23. The number of solution of $\sum_{r=1}^{5}\cos rx = 5$ in the interval $[0, 2\pi]$ is

1) 0 2) 1 3) 5 4) 2

24. If the equation $\cot^4 x - 2\operatorname{cosec}^2 x + a^2 = 0$ has at least one solution, then the sum of all possible integral values of a is equal to

1) 4 2) 3 3) 2 4) 0

25. The equation $\sin^4 x + \cos^4 x + \sin 2x + \alpha = 0$ is solvable for

1) $-\frac{5}{2} \le \alpha \le \frac{1}{2}$ 2) $-3 \le \alpha \le 1$

3) $-\frac{3}{2} \le \alpha \le \frac{1}{2}$ 4) $-1 \le \alpha \le 1$

26. $0 \le a \le 3, 0 \le b \le 3$ **and the equation, $x^2+4+3\cos(ax+b) = 2x$ has atleast one solution then the value of a + b**

1) $\frac{\pi}{2}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{3}$ 4) π

27. The number of solution of $x \in [0, 2\pi]$ for which $[\sin x + \cos x] =$ $3 + [-\sin x] + [-\cos x]$ (where [.] denotes the greatest integer function) is

1) 0 2) 4 3) infinite 4) 1

KEY

01) 1	02) 2	03) 4	04) 2	05) 1	06) 3
07) 3	08) 3	09) 2	10) 3	11) 4	12) 3
13) 2	14) 1	15) 1	16) 4	17) 3	18) 2
19) 1	20) 1	21) 2	22) 1	23) 4	24) 4
25) 3	26) 4	27) 3			

SOLUTIONS

1. We have,

$$y = 5x^2+2x+3 = 5\left[x^2+\frac{2}{5}x+\frac{3}{5}\right]$$

$$= 5\left[\left(x+\frac{1}{5}\right)^2+\frac{3}{5}-\frac{1}{25}\right] = 5\left(x+\frac{1}{5}\right)^2+\frac{14}{5} > 2$$

since $y = 2\sin x \le 2$, so there cannot be any point of intersection.

2. Since $x^2+x^{-2} = \left(x-x^{-1}\right)^2+2 \ge 2$ *and*

$2\cos^2\frac{x}{2}\sin^2 x \le 2.$

$\therefore$ the given equation is valid only if

$2\cos^2\frac{x}{2}\sin^2 x = 2$

$\Rightarrow \cos\frac{x}{2} = \sin x = 1,$ which cannot be true

3. we have $2\sin^2 x + 3\sin x - 2 > 0$

$\Rightarrow (2\sin x - 1)(\sin x + 2) > 0$

$\Rightarrow 2\sin x - 1 > 0\ [\because \sin x + 2 > 0\ \forall x \in R]$

$\Rightarrow \sin x > 1/2 \Rightarrow x \in (\pi/6, 5\pi/6)$ *Also* \

$x^2 - x - 2 < 0$

$\Rightarrow (x-2)(x+1) < 0 \Rightarrow -1 < x < 2.$

As $2 < \frac{5\pi}{6}$, we obtain that x must lie in $\left(\frac{\pi}{6}, 2\right)$

4. $\cos^5 x = 1+\sin^4 x$ Minimum value of R.H.S. is 1 and maximum value of L.H.S. is 1. So equality holds only when both sides are simultaneously 1. i.e., cos x = 1 and sin x = 0

cos x = 1 $\Rightarrow x = 2n\pi$ and sin x = 0 $\Rightarrow$ $x = n\pi$

So, solution set x = $2n\pi$.

5. $\cos^7 x + \sin^4 x = 1$ in the interval $(-\pi, \pi)$. In L.H.S. one term has to be one and one term is 0, otherwise it is not possible.

$\cos^7 x = 0,\ \sin^4 x = 1 \Rightarrow x = \pm\frac{\pi}{2}$

or $\cos^7 x = 1,\ \sin x = 0 \Rightarrow x = 0$

6. We have $\sin^4 x+\cos^4 y+2=4\sin x\cos y$

$\Rightarrow \sin^4 x+\cos^4 y+2-4\sin x\cos y=0$

$\Rightarrow \left(\sin^2 x-1\right)^2+\left(\cos^2 y-1\right)^2+2\sin^2 x$
$+2\cos^2 y-4\sin x\cos y=0$

$\Rightarrow \left(\sin^2 x-1\right)^2+\left(\cos^2 y-1\right)^2+2\left(\sin x-\cos y\right)^2=0$

Which is true if $\sin^2 x=1, \cos^2 y=1$

sin x = cos y. $As,\ 0\le x,\ y\le \pi/2.$ we get

sin x = cos y = 1 $\Rightarrow \sin x+\cos y=2$

7. Given equation is $\sec^2(a+2)x+a^2-1=0$

$\Rightarrow \tan^2(a+2)x+a^2=0$

$\Rightarrow \tan^2(a+2)x=0\ and\ a=0$

$\Rightarrow \tan^2 2x=0\Rightarrow x=0,\frac{\pi}{2},-\frac{\pi}{2}$

$\therefore (0,0),(0,\pi/2),(0,-\pi/2)$ are ordered pairs satisfying the equation.

8. C learly, $x\ge 4\left(Since\ \sqrt{x-4}\ is\ real\right)$

$so\ that\ \sqrt{x}\ is\ also\ real$ .Again,

if $\cos\left(\pi\sqrt{x}\right)<1$ then

$\cos\left(\pi\sqrt{x-4}\right)>1$ (since their product =1) But both of these are not possible (since $\cos\theta$ cannot be greater than 1) $\therefore \cos\left(\pi\sqrt{x-4}\right)=1\ and$

$\cos\left(\pi\sqrt{x}\right)=1$

$\therefore x-4=0\ and\ \ x=0$

But x =0 is not possible $\therefore x=4$ is the only solution

9 . We have $\sin\theta-\cos\theta=\min_{a\in R}\left\{1,\ a^2-6a+11\right\}$

Since $a^2-6a+11=(a-3)^2+2\ge 2$ for all a

$\therefore \sin\theta-\cos\theta=1$

$\Rightarrow \sin\left(\theta-\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}=\sin\frac{\pi}{4}$

$\Rightarrow \theta-\frac{\pi}{4}=n\pi+(-1)^n\frac{\pi}{4}$

$\therefore \theta=n\pi+(-1)^n\frac{\pi}{4}+\frac{\pi}{4}\ where\ n\in Z$.

10. Given equation is $\left|\cos x-\sin x\right|\ge\sqrt{2}$

Since $\left|\cos x-\sin x\right|\le\sqrt{1+1}=\sqrt{2}$

$\therefore$ we must have $\left|\cos x-\sin x\right|=\sqrt{2}$

$\Rightarrow \left|\cos\left(x+\frac{\pi}{4}\right)\right|=1\Rightarrow \cos\left(x+\frac{\pi}{4}\right)=1,-1$

$\therefore x=\frac{3\pi}{4},\frac{7\pi}{4}$

11. Here $f\left(\frac{\pi}{10}\right)=\sin 18^0+\cos 18^0$

$=\sqrt{2}\sin\left(45^0+18^0\right)=\sqrt{2}\sin 63^0$

Since $\sin 63^0>\sin 45^0=\frac{1}{\sqrt{2}}\ and\ \sqrt{2}\sin 63^0>1,$

$\therefore 1<f\left(\frac{\pi}{10}\right)<\sqrt{2}\Rightarrow\left[f\left(\frac{\pi}{10}\right)\right]=1$

$\therefore\ The\ equation\ is\ \sin x+\cos x=1$

$\Rightarrow \cos\left(x-\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}=\cos\frac{\pi}{4}$

$\Rightarrow x-\frac{\pi}{4}=2n\pi\pm\frac{\pi}{4}$

12. We have,

$\left|\cos x\right|\le\sin x\Rightarrow \sin x\ge 0\left(\because \left|\cos x\right|\ge 0\right)$

$\Rightarrow x\notin(\pi,2\pi)$ If $x=2\pi, \left|\cos 2\pi\right|\le\sin 2\pi$

which is not possible $\therefore x\in\left[0,\frac{\pi}{2}\right], then$

$\left|\cos x\right|\le\sin x\Rightarrow x\in\left[\frac{\pi}{4},\frac{\pi}{2}\right]$

If $x\in\left(\frac{\pi}{2},\pi\right)$, then $\left|\cos x\right|\le\sin x$

$\Rightarrow -\cos x\le\sin x\ \left(\because \cos x<0\right)$

$\Rightarrow \tan x\le -1\Rightarrow x\in\left(\frac{\pi}{2},\frac{3\pi}{4}\right]$

$$\Rightarrow x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right]$$

13. $1+\sin^2 ax = \cos x$
Minimum value of L.H.S is 1 and maximum value of R.H.S is 1 , equality exist only when both sides are1 $\Rightarrow \sin^2 ax = 0$ and $\cos x = 1$ $\Rightarrow ax = n\pi$ and $x = 2n\pi$
If a is rational, there are infinite or no common solution but when a is irrational only one common solution i.e., x =0

14. $\sin\alpha + \sin\beta$ = - a, $\cos\alpha + \cos\beta$ = -c
Apply transformations and on dividing

$$\tan\frac{\alpha+\beta}{2} = \frac{a}{c}, \quad \sin(\alpha+\beta) = \frac{2ac}{a^2+c^2}$$

15. $|\cot x| = \begin{cases} -\cot x, & \cot x < 0 \\ \cot x, & \cot x \geq 0 \end{cases}$

CASE I : $\cot\ x \geq 0$

$$\cot\ x = \cot x + \frac{1}{\sin x} \Rightarrow \frac{1}{\sin x} = 0$$

$\Rightarrow \sin x = \infty$, This is not possible.

CASE II : $\cot x < 0$

$$\Rightarrow -\cot x = \cot x + \frac{1}{\sin x}$$

$$\therefore \cos x = -\frac{1}{2} = -\cos\frac{\pi}{3}$$

$$\Rightarrow \cos x = \cos\left(\pi + \frac{\pi}{3}\right) or\ \cos\left(\pi - \frac{\pi}{3}\right)$$

$$\therefore x = \frac{2\pi}{3}, \frac{4\pi}{3}$$

But, $x = \frac{4\pi}{3} \Rightarrow \cot x > 0$, $\therefore n = 1$

16. $\tan\left(\frac{\pi}{4} - x\right) = \tan y = \tan\left(x - \frac{\pi}{6}\right)$

$$\left\{\because x - y = \frac{\pi}{6}\right\}$$

$$\therefore \frac{\pi}{4} - x = n\pi + x - \frac{\pi}{6} \Rightarrow \left(\frac{5\pi}{12} - n\pi\right) = 2x$$

$\Rightarrow x = (5-12n)\frac{\pi}{24}$ *and* $y = x - \frac{\pi}{6}$ Options (1)(2),(3) correspond to n =0,1, 10 respectively.

17. But max value of $2\sin\theta$ is 2 and it is attained at r =1, -1 $\Rightarrow \theta = n\pi + (-1)^n \frac{\pi}{2}$

$$\therefore \theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2} \{\because \theta \in [0, 5\pi]\}$$

$\therefore$ The number of values of the pair

$(r, \theta) = 2 \times 3 = 6$

18. The equation

$$\frac{1-\sin x + ... + (-1)^n \sin^n x + ...}{1+\sin x + ... + \sin^n x + ...} = \frac{1-\cos 2x}{1+\cos 2x}$$

$$\Rightarrow \frac{1}{1+\sin x} . \frac{1-\sin x}{1} = \frac{2\sin^2 x}{2\cos^2 x}$$

$$\Rightarrow 2\sin^2 x + \sin x - 1 = 0$$

$$\Rightarrow \sin x = \frac{-1 \pm \sqrt{1+8}}{4} = \frac{-1 \pm 3}{4}$$

$$\Rightarrow \sin x = -1 \text{ or } \sin x = \frac{1}{2}$$

Since , $\sin x \neq -1$, we have

$$\sin x = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right),$$

$$\therefore x = n\pi + (-1)^n \frac{\pi}{6}$$

19. we have $4\cos x(3\cos^2 x - 1) + 2\cos^2 x = 0$

$2\cos x(3\cos x + 2)(2\cos x - 1) = 0$

either $\cos x = 0$ which gives

$x = \pi/2$ or $\cos x = -2/3$

Which gives no values of x for which

$0 \leq x \leq \pi/2$ or $\cos x = 1/2$

which gives x $= \pi/3$

So, the required difference $= \frac{\pi}{6}$

20. We have , $|4\sin x - 1| < \sqrt{5}$

$$\Rightarrow -\left(\frac{\sqrt{5}+1}{4}\right) < \sin x < \frac{\sqrt{5}+1}{4}$$

$$\Rightarrow \sin\left(\frac{-3\pi}{10}\right) < \sin x < \sin\left(\frac{3\pi}{10}\right)$$

21. Given equation is

$2\sin^2 x + 5\sin x - 3 = 0$

$\Rightarrow (\sin x + 3)(2\sin x - 1) = 0$

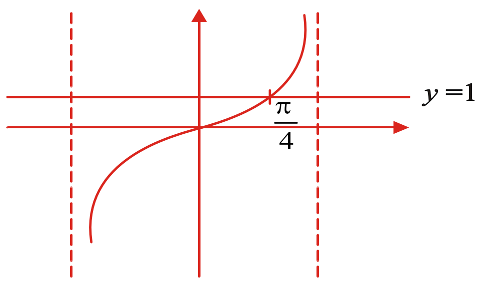
$\Rightarrow \sin x \neq -3$ or $\sin x = \frac{1}{2}$

given $0^0 \le x \le 540^0$, $\therefore$ no. of values are 4.

i.e., 30^0, 150^0, 390^0, 510^0

22. Graph of $y=$ tanx is as shown

$\therefore$ For $\tan x > 1$



$n\pi + \frac{\pi}{4} < x < n\pi + \frac{\pi}{2}$.

23. $\sum_{r=1}^{5} \cos rx = 5 \Rightarrow \cos x + \cos 2x + \cos 3x$

$+\cos 4x + \cos 5x = 5$

Which is possible only when

$\cos x = \cos 2x = \cos 3x$

$= \cos 4x = \cos 5x = 1$ and is satisfied by x=0 and

$x = 2\pi$.

24. $\cot^4 x - 2\cot^2 x - 2 + \alpha = 0$

$(\cot^2 x - 1)^2 = 3 - \alpha$, $3 - \alpha \ge 0$, $\alpha = -1, 0, 1$

25. $\sin^4 x + \cos^4 x + \sin 2x + \alpha = 0$

Let $\sin 2x = y$. then the given equation becomes

$y^2 - 2y - 2(1+\alpha) = 0$ where $-1 \le y \le 1$

$(\because -1 \le \sin 2x \le 1)$, for real, discriminant ≥ 0

26.. $x^2 - 2x + 4 = -3\cos(a\alpha + b)$

$\Rightarrow (x-1)^2 + 3 = -3\cos(ax+b)$

As $-1 \le \cos(ax+b) \le 1$ and $(x-1)^2 \ge 0$

$\therefore$ equation (i) is only possible if,

$\cos(ax+b) = -1$ and $(x-1) = 0$ so

$a + b = \pi$, (where $a + b \le 6$)

27. $[\sin x + \cos x] = 3 + [-\sin x] + [-\cos x]$

Maximum value of left-hand side is 1 and minimum of right hand side is also 1

$\Rightarrow [\sin x + \cos x] = 3 + [\sin x] + [-\cos x] = 1$

$\Rightarrow [\sin x + \cos x] = 1,$

$[-\sin x] = -1, [-\cos x] = -1$

EXAMPLES

1. If $\cos\theta + \sqrt{3}\sin\theta = 2$ then $\theta =$

1) $\frac{\pi}{3}$ 2) $\frac{2\pi}{3}$ 3) $\frac{4\pi}{3}$ 4) $\frac{5\pi}{3}$

Key:- 1

Sol:- Given $\cos\theta + \sqrt{3}\sin\theta = 2$

$a = 1 \quad b = \sqrt{3}$

Divided $\sqrt{a^2 + b^2} = 2$ on both sides

$\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta = 1$

$\cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3} = 1$

$\cos\left(\theta - \frac{\pi}{3}\right) = 1$

$\cos\left(\theta - \pi/3\right) = \cos 0^o$

$\theta - \pi/3 = 0$

$\theta = \pi/3$

2. If $\cos ec\theta + \cot\theta = \sqrt{3}$ then $\theta =$

1) $\frac{\pi}{6}$ 2) $\frac{\pi}{3}$ 3) $\frac{-\pi}{6}$ 4) $\frac{-\pi}{3}$

Key:-2

Sol:- Given $\cos ec\theta + \cot\theta = \sqrt{3}$

$$\frac{1+\cos\theta}{\sin\theta}=\sqrt{3}$$

$$1+\cos\theta=\sqrt{3}\sin\theta$$

$$-\cos\theta+\sqrt{3}\sin\theta=1$$

Here $a=-1,\ b=\sqrt{3}$

Divide $\sqrt{a^2+b^2}=2$ on bothsides

$$-\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta=\frac{1}{2}$$

$$\sin\theta\cos\frac{\pi}{6}-\cos\theta\sin\frac{\pi}{6}=\frac{1}{2}$$

$$\sin\left(\theta-\theta/6\right)=\sin\pi/6 \qquad \theta=\pi/3$$

3. The number of does not solutions of the equation $\log|\sin x|=2-\log_{\frac{1}{2}}|\cos x|$ in the interval $[0,2\pi]$

Key:- 8

Sol:- Given $\log_{\frac{1}{2}}|\sin x|=2-\log_{\frac{1}{2}}|\cos x|$

$$\log_{\frac{1}{2}}|\sin x|+\log_{\frac{1}{2}}|\cos x|=2$$

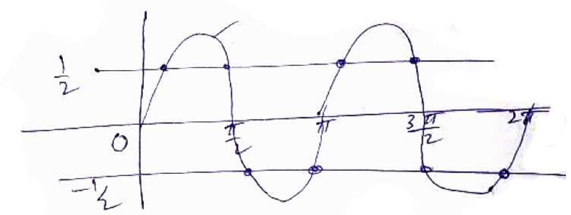
$$\Rightarrow \log_{\frac{1}{2}}|\sin x|\cos x|=2$$

$$|\sin x\cos x|=\left(\frac{1}{2}\right)^2$$

$$\Rightarrow|2\sin x\cos x|=\frac{1}{2}$$

$$\sin 2x=\ \pm\frac{1}{2}$$

graph of $y=\sin 2x$



Hence, total number of solutions=8

4. Let $s=\{\theta\in[-2\pi,2\pi]:2\cos^2\theta+3\sin\theta=0\}$ then the sum of the elements of S is

1) $\frac{13\pi}{6}$ 2) $\frac{5\pi}{3}$ 3) 2π 4) π

Key:- 3

Sol:- $2\cos^2\theta+3\sin\theta=0$

$$2\left(1-\sin^2\theta\right)+3\sin\theta=0$$

$$2\sin^2\theta-3\sin\theta-2=0$$

$$2\sin^2\theta-4\sin\theta+\sin\theta-2=0$$

$$2\sin\theta(\sin\theta-2)+1(\sin\theta-2)$$

$$(2\sin\theta+1)(\sin\theta-2)=0$$

$$\sin\theta=\frac{-1}{2}(\text{or})\sin\theta=2(\text{not possible})$$

$$\sin\theta=\sin\left(-\frac{\pi}{6}\right)$$

General solution $\theta=n\pi+(-1)^n\left(-\frac{\pi}{6}\right)$

$put\quad n=0$

Then $\theta=-\frac{\pi}{6}$

Then $\theta=\pi+\frac{\pi}{6}=\frac{7\pi}{6}$

$put\ \ n=-1$

Then $\theta=-\pi+\frac{\pi}{6}=\frac{-5\pi}{6}$

$put\ \ n=2$

Then $\theta=2\pi-\frac{\pi}{6}=\frac{11\pi}{6}\in[-2\pi,2\pi]$

1) $\left(\frac{-5}{4},-1\right)$ 2) $\left[-1,\frac{-1}{2}\right]$

3) $\left(\frac{-1}{2},\frac{-1}{4}\right]$ 4) $\left[-\frac{3}{2},\frac{-5}{4}\right]$

Sum of solutions $= \frac{-\pi}{6}+\frac{7\pi}{6}-\frac{5\pi}{6}+\frac{11\pi}{6}$

$= \frac{2^2\pi}{6} = 2\pi$

5. If the equation $\cos^4\theta+\sin^4\theta+\lambda=0$ has real solutions for θ, then λ lies in the interval

Key:- 2

Sol:- Given $\sin^4\theta+\cos^4\theta=-\lambda$

$\left(\sin^2\theta\right)^2+\left(\cos^2\theta\right)^2=-\lambda$

$\left(\sin^2\theta+\cos^2\theta\right)^2-2\sin^2\theta\cos^2\theta=-\lambda$

$1-2\sin^2\theta\cos^2=-\lambda$

$\lambda=\frac{4\sin^2\theta\cos^2\theta}{2}-1$

$\lambda=\frac{\sin^2 2\theta}{2}-1$

since $0\le\sin^2 2\theta\le 1$

$0\le\frac{\sin^2 2\theta}{2}\le 1/\varepsilon$

$-1\le\frac{\sin^2 2\theta}{2}\le\frac{-1}{2}$

$\lambda\in\left[-1,\frac{-1}{2}\right]$

6. If $\cos 3x+\cos 2x=\sin\frac{3x}{2}+\sin\frac{x}{2}\quad 0\le x\le 2\pi$

then k

1) $\frac{\pi}{7},\frac{5\pi}{7},\pi,\frac{9\pi}{7},\frac{13\pi}{7}$ 2) $\frac{\pi}{6},\frac{5\pi}{6},\pi,\frac{7\pi}{6},\frac{13\pi}{6}$

3) $\frac{\pi}{5},\frac{2\pi}{5},\pi,\frac{2\pi}{5},\frac{13\pi}{5}$ 4) $\frac{\pi}{3},\frac{2\pi}{3},\pi,\frac{7\pi}{3},\frac{5\pi}{3}$

Key:- 1

Sol;- Given $\cos 3x+\cos 2x=\sin\frac{3x}{2}+\sin x/2$

$2\cos\frac{5x}{2}\cos x/2 = 2\sin x\cos x/2$

$\cos x/2\left(\cos\frac{5x}{2}-\sin x\right)=0$

$\cos\frac{x}{2}=0\ (\text{or})\ \cos\frac{5x}{2}-\sin x=0$

$\cos\frac{x}{2}=0\ (\text{or})\ \cos\frac{5x}{2}=\sin x=\cos(90-x)$

* If $\cos x/\varepsilon=0$ then G.S $\frac{x}{2}=(2n+1)\frac{\pi}{2}$

$x=(2n+1)\pi$

$put\ n=0$

Then $x=\pi$

$x=\frac{\pi}{7}$

$put\ n=1$

$x=\frac{5\pi}{7}$

$n=2$ Then $x=\frac{9\pi}{7}$

ADVANCED LEVEL QUESTIONS

SINGLE CORRECT

1)**The greatest possible value of the expression**

$$\tan\left(x+\frac{2\pi}{3}\right)-\tan\left(x+\frac{\pi}{6}\right)+\cos\left(x+\frac{\pi}{6}\right)$$

on the interval $\left[-5\pi/12,-\pi/3\right]$ **is**

A) $\frac{12}{5}\sqrt{2}$ B) $\frac{11}{6}\sqrt{2}$

C) $\frac{12}{5}\sqrt{3}$ D) $\frac{11}{6}\sqrt{3}$

KEY- D

Let $u=-x-\pi/6$ then $u\in\left[\pi/6,\pi/4\right]$

and then $2u\in\left[\pi/3,\pi/2\right]$

$\tan\left(x+2\pi/3\right)=-\cot\left(x+\pi/6\right)=\cot u$

now

$\tan\left(x+2\pi/3\right)-\tan\left(x+\pi/6\right)+\cos\left(x+\pi/6\right)$

$=\cot u+\tan u+\cos u$

$=\dfrac{2}{\sin 2u}+\cos u$

both $\dfrac{2}{\sin 2u}$ and $\cos u$ monotonic

decreasing on $\left[\pi/6,\pi/4\right]$ and thus

the greatest value occurs at

$u=\pi/6$

i.e

$$\frac{2}{\sin\pi/3}+\cos\pi/6=\frac{4}{\sqrt{3}}+\frac{\sqrt{3}}{2}=\frac{11}{2\sqrt{3}}=\frac{11\sqrt{3}}{6}.$$

2

If m and $n(>m)$ are positive integers, the number of solutions of the equation $n|\sin x|=m|\cos x| in[0,2\pi]$ is

A) m B) n C) mn D) 4

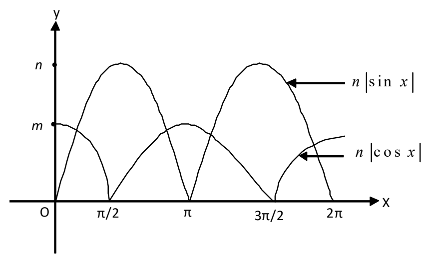
KEY- B

Let $y=n|\sin x|=m|\cos x|$

The curve $y=n|\sin x|$

and $y=m|\cos x|$.

Intersect at 4 points in $\left[0,2\pi\right]$.



3

If $\left[\sin x\right]+\left[\sqrt{2}\cos x\right]=-3, x\in\left[0,2\pi\right]$($[.]$ denotes the greatest integer function), then x belongs to

A) $\left(\pi,\frac{5\pi}{4}\right)$ B) $\left[\pi,\frac{5\pi}{4}\right]$

C) $\left(\frac{5\pi}{4},2\pi\right)$ D) $\left[\frac{5\pi}{4},2\pi\right]$

KEY- A

$[\sin x]+\left[\sqrt{2}\cos x\right]=-3$

$\Rightarrow\left[\sin x\right]=-1$ *and* $\left[\sqrt{2}\cos x\right]=-2$

or $-1\le\sin x<0$

and $-2\le\sqrt{2}\cos x<-1$

or $-1\le\sin x<0$

and $-\sqrt{2}<\cos x<-\dfrac{1}{\sqrt{2}}$

or $-1\le\sin x<0$

and $-1 \le \cos x < -\frac{1}{\sqrt{2}}$

$\therefore\ x \in (\pi, 2\pi)\, and\ x \in \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$

$\therefore\ x \in (\pi, 2\pi) \cap \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$

$\therefore\ x \in \left(\pi, \frac{5\pi}{4}\right)$

4

Values of x and y satisfying the equation

$$\sin^7 y = \left|x^3 - x^2 - 9x + 9\right| + \left|x^3 - x^2 - 4x + 4\right| + \sec^2 2y + \cos^4 y$$

are

A) $x = 1, y = n\pi, n \in I$

B) $x = 1, y = 2n\pi + \frac{\pi}{2}, n \in I$

C) $x = 1, y = 2n\pi, n \in I$

D) none of the above

KEY- B

for $x = 1$

$\sin^7 y = \sec^2 2y + \cos^4 y$

$\Rightarrow \quad \sin^7 y \cos^2 2y = 1 + \cos^4 y \cos^2 2y$

Since, LHS ≤ 1.

and RHS ≥ 1

which is possible only when

$\therefore\ \sin^7 y \cos^2 2y = 1$

$\Rightarrow \quad \sin^7 y = 1\, and \cos^2 2y = 1$

$y = \frac{\pi}{2}$

General value of y is $2n\pi + \frac{\pi}{2}$

Hence, $x = 1\, and\ y = 2n\pi + \frac{\pi}{2}, n \in I$

5

If $[y] = [\sin x]\, and\ y = \cos x$ are two given equations, then the number of solutions is (where $[.]$ denotes the greatest integer function)

A) 2 B) 3 C) 4

D) infinitely many solutions

KEY- B

$\therefore [y] = [\sin x] = -1, 0, 1$

$\because -1 \le \sin x \le 1$

$\Rightarrow\ -1 \le y < 0$

$\therefore [\sin x] = -1, 0, 1$

$[\sin x] = -1$

$\Rightarrow \quad -1 \le \sin x < 0$

Then, $x \in \{(2n+1)\pi, (2n+2)\pi\}, n \in I$

and $[y] = 0$

$\Rightarrow \quad 0 \le y < 1$

$[\sin x] = 0$

$\therefore\ 0 \le \sin x < 1$

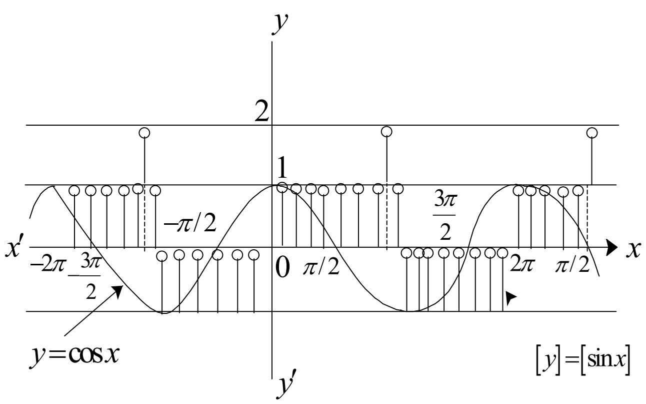
Then,

$x \in [2n\pi, (2n+1)\pi] \sim \left\{2n\pi + \frac{\pi}{2}\right\}, n \in 1$

and $[y] = 1, \sin x = 1$

$x = 2n\pi + \frac{\pi}{2}$

$1 \le y < 2$



$n=3 \qquad Then\ x=\frac{13\pi}{7}$

Solutions are $\frac{\pi}{7},\frac{5\pi}{7},\pi,\frac{9\pi}{7},\frac{13\pi}{7}$

$\frac{5x}{2}=2n\pi=(90-x)$

$5x=4n\pi=(180-2x)$

$x=\frac{4n\pi}{5}=\left(\frac{180-2x}{5}\right)$

Hence, $[y]=[\sin x]\, and\ y=\cos x$ infinite points. Hence, number of solutions = infinite.

MULTIPLE ANSWER TYPE QUESTIONS

1. For the smallest positive values of 'x' and 'y' the equation $2(\sin x+\sin y)-2\cos(x-y)=3$ has a solution then which of the following is/are true

A) $\sin\left(\frac{x+y}{2}\right)=1$

B) $\cos\left(\frac{x-y}{2}\right)=\frac{1}{2}$

C) number of ordered pairs (x,y) is 2

D) number of ordered pairs (x,y) is 3

KEY. A,B,C

$2(\sin x+\sin y)-2\cos(x-y)=3$

$2\left[2\sin\frac{x+y}{2}\cos\frac{x-y}{2}\right]-2\left[2\cos^2\frac{x-y}{2}-1\right]=3$

Let $\cos\frac{x-y}{2}=t$

For $\sin\frac{x+y}{2}=1, 4t^2-4t+1=0$

$\Rightarrow(2t-1)^2=0\Rightarrow t=\frac{1}{2}$

$\Rightarrow\cos\left(\frac{x-y}{2}\right)=\frac{1}{2}$

$\frac{x+y}{2}=\frac{\pi}{2},\frac{x-y}{2}=\frac{\pi}{3}$

$\Rightarrow x+y=\pi\ \& x-y=\frac{2\pi}{3}$

$\Rightarrow 2x=\frac{5\pi}{3},2y=\frac{\pi}{3}\Rightarrow x=\frac{5\pi}{6},y=\frac{\pi}{6}$

For

$\sin\frac{x+y}{2}=-1\Rightarrow\frac{x+y}{2}=\frac{3\pi}{2}\Rightarrow x+y=3\pi$

$x-y=\frac{2\pi}{3}\Rightarrow 2x=\frac{11\pi}{3}\ \& 2y=\frac{7\pi}{3}$

$\Rightarrow x=\frac{11\pi}{6}\ \& y=\frac{7\pi}{6}$ Ordered pairs are

$\left(\frac{5\pi}{6},\frac{\pi}{6}\right)\ \&\ \left(\frac{11\pi}{6},\frac{7\pi}{6}\right)$

2

Which of the following inequalities hold true in any $\Delta\ ABC$?

A) $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\le\frac{1}{8}$

B) $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\le\frac{3\sqrt{3}}{8}$

C) $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}<\frac{3}{4}$

D) $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\le\frac{9}{4}$

KEY. A,B,D

Let $k = \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

$\Rightarrow 2k = \left(2\sin\frac{A}{2}\sin\frac{B}{2}\right)\sin\frac{C}{2}$

$\Rightarrow 2k = \left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right)\sin\frac{C}{2}$

$\Rightarrow 2k = \cos\frac{A-B}{2}\sin\frac{C}{2} - \sin^2\frac{C}{2}$

$\Rightarrow \sin^2\frac{C}{2} - \sin\frac{C}{2}\cos\frac{A-B}{2} + 2k = 0$

$\therefore \sin\frac{C}{2}$ is real

$D \geq 0$

$\cos^2\frac{A-B}{2} - 8k \geq 1$

$8k \leq \cos^2\frac{A-B}{2} \leq 1$

$k \leq \frac{1}{8}$

Maximum occurs for equilateral triangle

Similarly for other choice has occurs for equilateral triangle.

3)**The equation** $\sin x = [1+\sin x] + [1-\cos x]$ **has (where [x] is the greatest integer less than or equal to x)**

A) no solution in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

B) no solution in $\left[\frac{\pi}{2}, \pi\right]$

C) no solution in $\left[\pi, \frac{3\pi}{2}\right]$ D) no solution for $x \in R$

KEY. A,B,C,D

$\because \sin x = [1+\sin x] + [1-\cos x]$

$\Rightarrow \quad \sin x = 2 + [\sin x] + [-\cos x]$

At $x = -\frac{\pi}{2}$ $\quad \sin x = 2 - 1$

$\therefore \sin x = 1$ $\quad$ -1 = 1 (impossible)

At $x = \frac{\pi}{2}$ $\quad$ 1 = 2 + 1- 0 $\quad$ 1 = 3 (impossible)

At $x = \pi, 0 = 1$ $\quad$ (impossible)

In $x \in \left(-\frac{\pi}{2}, 0\right)$, $\sin x = 2 + (-1) + 0 = 1$ (impossible)

In $x \in \left(0, \frac{\pi}{2}\right)$, $\sin x = 2 + 0 - 1 = 1$ (impossible)

In $x \in \left(\frac{\pi}{2}, \pi\right)$, $\sin x = 2 + 0 + 0 = 2$ (impossible)

In $x \in (\pi, 3\pi/2)$, $\sin x = 2 - 1 + 0 = 1$(impossible)

4)The solution of the equation $9\cos^2 x + \cos^2 2x + 1 = 6\cos^6 x\cos 2x + 6\cos^6 x - 2\cos 2x$ is/are

A) $x = n\pi + \frac{\pi}{2}, n \in I$

B) $x = n\pi + \cos^{-1}\left(\sqrt[4]{\frac{2}{3}}\right), n \in I$

C) $x = n\pi - \cos^{-1}\left(\sqrt[4]{\frac{2}{3}}\right), n \in I$

D) none of the above

KEY. A,B,C

$$9\cos^2 x + \cos^2 2x + 1 - 6\cos^6 x\cos 2x - 6\cos^6 x + 2\cos 2x = 0$$

$$\Rightarrow \quad \left(3\cos^6 x - 1 - \cos 2x\right)^2 = 0$$

$$\Rightarrow \quad \cos^2 x\left(3\cos^4 x - 2\right) = 0$$

$$\Rightarrow \quad \cos x = 0$$

$\Rightarrow \quad x = n\pi + \frac{\pi}{2}, n \in I$

and $\quad \cos^4 x = \frac{2}{3}$

$\Rightarrow \quad \cos^2 x = \pm\sqrt{\frac{2}{3}}$

$\therefore \quad \cos x = \pm\sqrt[4]{\left(\frac{2}{3}\right)}$

$\Rightarrow \quad x = n\pi \pm \cos^{-1}\sqrt[4]{\left(\frac{2}{3}\right)}, n \in I$

5) $\sqrt{(\cos 2x)} + \sqrt{(1+\sin 2x)} = 2\sqrt{(\sin x + \cos x)}$, If

A) $\sin x + \cos x = 0$

B) $x = 2n\pi, n \in I$

C) $x = n\pi - \frac{\pi}{4}, n \in I$

D) $x = 2n\pi \pm \cos^{-1}\left(-\frac{1}{5}\right), n \in I$

KEY. A,B,C

The given equation can be written as

$$\sqrt{(\cos^2 x - \sin^2 x)} + \sqrt{(\cos x + \sin x)^2}$$
$$= 2\sqrt{(\cos x + \sin x)}$$

$$\Rightarrow \sqrt{(\cos x+\sin x)}\sqrt{(\cos x-\sin x)} + \sqrt{(\cos x+\sin x)}\} = 2\sqrt{(\cos x+\sin x)}$$

$\Rightarrow \quad Either\ \cos x + \sin x = 0$

$\Rightarrow \quad \tan x = -1$

$\Rightarrow \quad x = n\pi - \frac{\pi}{4}(n \in I)$

or $\sqrt{(\cos x - \sin x)} + \sqrt{(\cos x + \sin x)} = 2$

$\Rightarrow \quad 2\cos x + 2\sqrt{(\cos^2 x - \sin^2 x)} = 4$

$\Rightarrow \quad \cos^2 x - \sin^2 x = (2 - \cos x)^2$

$\Rightarrow \quad \cos^2 x + 4\cos x - 5 = 0$

$\Rightarrow \quad \cos x = \frac{-4 \pm \sqrt{(16+20)}}{2} = -5 or 1$

But $\quad \cos x \neq -5\, or \cos x = 1$

$\Rightarrow \quad x = 2n\pi, n \in I$.

COMPREHENSION TYPE QUESTIONS

Passage - 1

Consider the cubic equation

$$x^3 - (1+\cos\theta+\sin\theta)x^2 + (\cos\theta\sin\theta+\cos\theta+\sin\theta)x - \sin\theta\cos\theta = 0$$

whose roots are $x_1, x_2, and\ x_3$.

1. **The value of** $x_1^2 + x_2^2 + x_3^2$ **equals**

A) 1 B) 2

C) $2\cos\theta$ D) $\sin\theta(\sin\theta + \cos\theta)$

2. **Number of values of** θ **in** $[0, 2\pi]$ **for which at least two roots are equal**

A) 3 B) 4 C) 5 D) 6

3. **Greatest possible difference between two of the roots if** $\theta \in [0, 2\pi]$ **is**

A) 2 B) 1 C) $\sqrt{2}$ D) $2\sqrt{2}$ K

KEY- 1)B 2)C 3)A

1

$$x^3-(1+\cos\theta+\sin\theta)x^2+(\cos\theta\sin\theta+\cos\theta+\sin\theta)x$$
$$-\sin\theta\cos\theta=0$$

Given cubic function is

$f(x)=(x-1)(x-\cos\theta)(x-\sin\theta)$.

Therefore, roots are 1, $\sin\theta$, *and* $\cos\theta$.

Hence,

$x_1^2+x_2^2+x_3^2=1+\sin^2\theta+\cos^2\theta=2$

2 Now if $1=\sin\theta\Rightarrow\theta=\dfrac{\pi}{2}$

if $1=\cos\theta\Rightarrow\theta=0,2\pi$ and if

$\sin\theta=\cos\theta\Rightarrow\tan\theta=1$

$\therefore\theta=\dfrac{\pi}{4},\dfrac{5\pi}{4}$

3. Again the maximum possible difference between the two roots is 2

$1-\sin\theta=2$ *when* $\theta=3\pi/2$

or $1-\cos\theta=2$ *when* $\theta=\pi$

MATRIX MATCHING TYPE QUESTIONS

1. Observe the following columns:

Column-I

A) If α,β are the solutions of $\sin x=-\dfrac{1}{2}\,in\,[0,2\pi]$ *and* α,γ are solutions of $\cos x=\dfrac{-\sqrt{3}}{2}$ in $[0,2\pi]$ then

B) If α,β are the solutions of $\cot x=-\sqrt{3}$ in $[0,2\pi]$ and α,γ are the solutions of $\operatorname{cosec} x=-2$ in $[0,2\pi]$, then

C) If α,β are the solutions of $\sin x=\dfrac{1}{2}\,in\,[0,2\pi]$ *and* α,γ are the solutions of $\tan x=\dfrac{1}{\sqrt{3}}\,in\,[0,2\pi]$, *then*

Column-II

P) $\alpha-\beta=\pi$

Q) $\beta-\gamma=\pi$

R) $\alpha-\gamma=\pi$

1 S) $\alpha+\beta=3\pi$

T) $\beta+\gamma=2\pi$

KEY

01) A-Q,S, B-P,T, C-R,S,T

(A) $\because\sin x=-\dfrac{1}{2}$

$=-\sin\dfrac{\pi}{6}$

$=\sin\left(\pi+\dfrac{\pi}{6}\right),\sin\left(2\pi-\dfrac{\pi}{6}\right)$

$\therefore x=\dfrac{7\pi}{6},\dfrac{11\pi}{6}$ ....(i) and $\cos x = -\frac{\sqrt{3}}{2} = -\cos\frac{\pi}{6}$

$= \cos\left(\pi - \frac{\pi}{6}\right), \cos\left(\pi + \frac{\pi}{6}\right)$

$\therefore x = \frac{5\pi}{6}, \frac{7\pi}{6}$ ....(ii)

From Eqs. (i) and (ii). It is clear tht

$\alpha = \frac{7\pi}{6}, \beta\frac{11\pi}{6}, \gamma = \frac{5\pi}{6}$

$\Rightarrow \alpha + \beta = 3\pi(S), \beta - \gamma = \pi(Q)$

(B) $\therefore \cot x = -\sqrt{3}$

$= -\cot\frac{\pi}{6}$

$= \cot\left(\pi - \frac{\pi}{6}\right), \cot\left(2\pi - \frac{\pi}{6}\right)$

$\therefore x = \frac{5\pi}{6}, \frac{11\pi}{6}$ ....(i)

and $\operatorname{cosec} x = -2 = -\operatorname{cosec}\frac{\pi}{6}$

$= \operatorname{cosec}\left(\pi + \frac{\pi}{6}\right), \operatorname{cosec}\left(2\pi - \frac{\pi}{6}\right)$

$\therefore\ x = \frac{7\pi}{6}, \frac{11\pi}{6}$ (C) $\because \sin x = -\frac{1}{2}$

$= -\sin\frac{\pi}{6}$

$= \sin\left(\pi + \frac{\pi}{6}\right), \sin\left(2\pi - \frac{\pi}{6}\right)$

$\therefore x = \frac{7\pi}{6}, \frac{11\pi}{6}$ ....(i)

and $\tan x = \frac{1}{\sqrt{3}}$

$= \tan\frac{\pi}{6}$

$= \tan\frac{\pi}{6}, \tan\left(\pi + \frac{\pi}{6}\right)$

$\therefore\ x = \frac{\pi}{6}, \frac{7\pi}{6}$ ....(ii)

From Eqs. (i) and (ii), it is clear that

$\alpha = \frac{7\pi}{6}, \beta = \frac{11\pi}{6}, \gamma = \frac{\pi}{6}$

$\Rightarrow \alpha + \beta = 3\pi, (S), \beta + \gamma = 2\pi(T), \alpha - \gamma = \pi(R)$

......(ii)

From Eqs. (i) and (ii), it is clear that

$\alpha = \frac{11\pi}{6}, \beta = \frac{5\pi}{6}, \gamma = \frac{7\pi}{6}$

$\Rightarrow\ \beta + \gamma = 2\pi(T), \alpha - \beta = \pi(P)$

JEE MAINS , EAMCET QUESTIONS

1. If $0 \le x < 2\pi$, then the number of real values of x, which satisfy the equation $\cos x + \cos 2x + \cos 3x + \cos 4x = 0$, **is**
1) 5 2) 7 3) 9 4) 3

2. The number of $x \in [0, 2\pi]$ for which $\left|\sqrt{2\sin^4 x + 18\cos^2 x} - \sqrt{2\cos^4 x + 18\sin^2 x}\right|$ **equals 1 is: [JEE Mains -2016]**
1) 2 2) 4 3) 6 4) 8

3. The number of solutions of sin 3x= cos 2x, in the interval $\left(\frac{\pi}{2}, \pi\right)$ is:{MAINS-2018]
1) 1 2) 2 3) 3 4) 4

4. If $0 \le x < \frac{\pi}{2}$, thenthe number of values of x for which $\sin x - \sin 2x + \sin 3x = 0$ **is [JEE Mains 2019]**
1) 2 2) 4 3) 3 4) 1

5. The sum of all values of $\theta \in \left(0, \frac{\pi}{2}\right)$ satisfying $\sin^2 2\theta + \cos^4 2\theta = \frac{3}{4}$ **is** 1) $\frac{5\pi}{4}$ 2) $\frac{\pi}{2}$ 3) π 4) $\frac{3\pi}{8}$

6. The number of solutions of the equation
$$\left|\sin \frac{\pi}{2}(1-x) + \cos \frac{\pi}{2}(1-x)\right| = \sqrt{\left|\log_e |x|\right|^3 + 1}$$
is are
1) 4 2) 6 3) 8 4) 10

7. All solutions of $[\sin x + \cos x] = [\sin 2x]$, where $[\ ]$ represents greatest integer function & $x \in [0, 2\pi]$ is
1) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ 2) $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right) - \left[\pi, \frac{3\pi}{2}\right]$
3) $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right) - \left\{\frac{5\pi}{4}\right\}$ 4) $\left(\frac{\pi}{4}, \frac{3\pi}{2}\right]$

8. If $x\sin a + y\sin 2a + z\sin 3a = \sin 4a$, $x\sin b + y\sin 2b + z\sin 3b = \sin 4b$, $x\sin c + y\sin 2c + z\sin 3c = \sin 4c$ then the roots of the equation
$$t^3 - \left(\frac{z}{2}\right)t^2 - \left(\frac{y+2}{4}\right)t + \left(\frac{z-x}{8}\right) = 0,$$
a,b,c $\ne n\pi$, are
1) $\sin a, \sin b, \sin c$
2) $\sin 2a, \sin 2b, \sin 2c$
3) $\cos a, \cos b, \cos c$
4) $\cos 2a, \cos 2b, \cos 2c$

9. The value of 'b' such that the equation $\frac{b\cos x}{2\cos 2x - 1} = \frac{b + \sin x}{(\cos^2 x - 3\sin^2 x)\tan x}$ **possess solutions, belong to the set**
1) $\left(-\infty, \frac{1}{2}\right)$ 2) $\left[-\infty, -\frac{1}{2}\right)$
3) $(-\infty, \infty)$ 4) $\left(-\infty, \frac{1}{2}\right) \cup (1, \infty)$

KEY

1) 2	2) 4	3)1	4)1
5)2	6) 2	7) 4	8) 3
9) 4			

1. $\cos 4x + \cos x + \cos 3x + \cos 2x = 0$

$\cos\frac{5x}{2}\left[\cos\frac{3x}{2} + \cos\frac{x}{2}\right] = 0$ $\cos\frac{5x}{2}\left(2\cos x \cos\frac{x}{2}\right) = 0$

$\therefore x = \frac{\pi}{5}, \frac{\pi}{2}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{3\pi}{2}, \frac{9\pi}{5}$

$\therefore$ Number of solutions $= 7$

2. Use $a^2 - b^2 = (a+b)(a-b)$

3. Given $\sin 3x = \cos 2x \Rightarrow 3sinx - 4\sin^2 x = 1 - 2\sin^2 x$

$\Rightarrow 4\sin^3 x - 2\sin^2 x - 3\sin x + 1 = 0$

$\Rightarrow 4\sin^3 x(\sin x - 1) + 2\sin x(\sin x - 1) - (\sin x - 1) = 0$

$\Rightarrow (\sin x - 1)(4\sin^2 x + 2\sin x - 1)$

$\Rightarrow \sin x - 1$ or $4\sin 2x + 2\sin x - 1 = 0$

$4\sin^2 x + 2\sin x - 1 = 0, \quad \sin x = \frac{1 \pm \sqrt{5}}{4}$

We know that, $\sin x > 0 \forall x \in (\pi/2, \pi)$

and $\sin x \ne 1$ $\sin x = \frac{\sqrt{5} - 1}{4}$

Therefore, number of solution is 1

4. $\sin x-\sin 2x+\sin 3x=0 \quad x\in\left[0,\frac{\pi}{2}\right)$

$\sin 2x=0$ and $\cos x=\frac{1}{2}$

$x=0$ and $x=\frac{\pi}{3}$ two solutions

5. $1-\cos^2 2\theta+\cos^4 2\theta=\frac{3}{4}\Rightarrow\left(\cos^2 2\theta-\frac{1}{2}\right)^2=0$

$\cos^2 2\theta=\frac{1}{2}=\cos^2\frac{\pi}{4}\Rightarrow\quad 2\theta=n\pi+\frac{\pi}{4}$

n=0 $\theta=\frac{\pi}{8}\Rightarrow$ n=1 $\theta=\frac{\pi}{2}+\frac{\pi}{8}=\frac{5\pi}{8}$

$\theta=\frac{\pi}{2}-\frac{\pi}{8}=\frac{3\pi}{8}$

Sum of angles $=\frac{\pi}{8}+\frac{3\pi}{8}=\frac{\pi}{2}$

6. On squaring both sides, we get

$1+\sin\pi(1-x)=\left|\log|x|\right|^3+1$

$\Rightarrow\sin\pi x=\left(\left|l\,n|x|\right|\right)^3$

There are 6 solutions 3 right side of $y-$axis and 3 left side of y-axis

7. $\sin x+\cos x=t,\ \Rightarrow-\sqrt{2}\le t\le\sqrt{2}$

$\Rightarrow[t]=\left[t^2-1\right]$

Solve to get $-1<t<0\cup\{\sqrt{2}\}$

$\Rightarrow-1<\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)<0$ or $x=\frac{\pi}{4}$

Ans is $\left(\frac{3\pi}{4},\frac{7\pi}{4}\right)-\left[\pi,\frac{3\pi}{2}\right]\cup\left\{\frac{\pi}{4}\right\}$

8. $x\sin a+y\sin 2a+z\sin 3a=\sin 4a$

$\Rightarrow x\sin a+2y\sin a\cos a+3\sin a\left(3-4\sin^2 a\right)$

$=2\sin a\cos a\left(2-\cos^2 a-1\right)$

$\Rightarrow 8\cos^3 a-4z\cos^2 a-(2y+4)\cos a+(z-x)=0$

$\cos a$ is a root of the given equation

$8t^2-4zt^2-(2y+4)t+(z-x)=0$

Similarty cos b, cos c are roots of this equation.

9. Let us find domain of given equation

i) $2\cos 2x-1\ne 0\Rightarrow x\ne n\pi\pm\frac{\pi}{6}$

ii) $\tan x\ne 0\Rightarrow x\ne\frac{n\pi}{2}$

iii) $\cos^2 x-3\sin^2 x\ne 0\Rightarrow x\ne n\pi\pm\frac{\pi}{6}$

Also, $2\cos 2x-1=$

$2\left(\cos^2 x-\sin^2 x\right)-\left(\cos^2 x+\sin^2 x\right)=\cos^2 x-3\sin^2 x$

Now given equation reduces to

$b\sin x=b+\sin x$

$\Rightarrow\sin x=\frac{b}{b-1},\sin ce-1\le\sin x\le 1$

$\Rightarrow-1\le\frac{b}{b-1}\le 1\Rightarrow b\le\frac{1}{2}$ or $b>1$, when

$b=\frac{1}{2},\sin x=1$ which is not possible

$\therefore b<\frac{1}{2}$ or $b>1$

JEE ADVANCED

1. **The number of solutions of the pair equations** $2\sin^2\theta - \cos 2\theta = 0$, $2\cos^2\theta - 3\sin\theta = 0$, **in the interval [0, 2 π] is** **[ADV-2018]**
A) zero B) one C) two D) four

2. **If** $\sin\theta = \cos\phi$, **then the possible values of** $\frac{1}{\pi}\left(\theta \pm \phi - \frac{\pi}{2}\right)$ **are** **[IIT 2008]**
(A) 0 (B) 1 (C) 2 (D) 3

3. **Let** $\theta, \phi \in [0, 2\pi]$ **be such that**
$2\cos\theta(1-\sin\phi) = \sin^2\theta\left(\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right)\cos\phi - 1,$
$\tan(2\pi - \theta) > 0$ **and** $-1 < \sin\theta < -\frac{\sqrt{3}}{2}$. **Then** ϕ **cannot satisfy** **[ADV- 2019]**
(A) $0 < \phi < \frac{\pi}{2}$ (B) $\frac{\pi}{2} < \phi < \frac{4\pi}{3}$
(C) $\frac{4\pi}{3} < \phi < \frac{3\pi}{2}$ (D) $\frac{3\pi}{2} < \phi < 2\pi$

4. **The number of all possible values of** θ **where** $0 < \theta < \pi$ **for which the system of equations**
$(y+z)\cos 3\theta = (xyz)\sin 3\theta$
; $x\sin 3\theta = \frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z}$
$(xyz)\sin 3\theta = (y+2z)\cos 3\theta + y\sin 3\theta$ **have a solution** (x_0, y_0, z_0) **with** $y_0 z_0 \neq 0$, **is**
[IIT 2010]

5. **The number of values of** θ **in the interval** $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **such that** $\theta \neq \frac{n\pi}{5}$, **for n = 0,** $\pm$ **1,** $\pm$ **2 and** $\tan\theta = \cot 5\theta$ **as well as** $\sin 2\theta = \cos 4\theta$ **is** **[IIT 2010]**

6. **The number of solutions of the equation** $2\cos\left(\frac{x}{2}\right) = 3^x + 3^{-x}$ **is**

7. **The number of solutions satisfying** $[\sin x] + \left[\frac{x}{2\pi}\right] + \left[\frac{2x}{5\pi}\right] = \frac{9x}{10\pi}$ **in the interval** $(30, 40)$ **is (where [.] is GIF)45. The number of solutions of the equation** $2\cos\left(\frac{x}{2}\right) = 3^x + 3^{-x}$ **is**

8. **Let** a, b, c **are three non-zero real numbers such that the equation** $\sqrt{3}a\cos x + 2b\sin x = c, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, **has two distinct real roots** α **and** β **with** $\alpha + \beta = \frac{\pi}{3}$. **Then the value of** $\frac{b}{a}$ **is**

KEY

1)C	2) AC	3) ACD	4)3
5) 3	6)1	7)1	8)0.5

1. From the given equations $\sin^2\theta = \frac{1}{4}$ and $\sin\theta = \frac{1}{2}$ which has two solutions in [0, 2 π]

2. $\sin\theta = \cos\phi \Rightarrow \cos\left(\frac{\pi}{2} - \theta\right) = \cos\phi$
$\frac{\pi}{2} - \theta = 2n\pi \pm \phi$; $\frac{1}{\pi}\left(\theta \pm \phi - \frac{\pi}{2}\right) = -2n$
$\Rightarrow$ 0 and 2 are possible

3. $2\cos\theta(1 - \sin\varphi) = \frac{2\sin^2\theta}{\sin\theta}\cos\varphi - 1$
$= 2\sin\theta\cos\varphi - 1$
$2\cos\theta - 2\cos\theta\sin\varphi = 2\sin\theta\cos\varphi - 1$
$2\cos\theta + 1 = 2\sin(\theta + \varphi)$
$\tan(2\pi - \theta) > 0 \Rightarrow \tan\theta < 0$ and
$-1 < \sin\theta < \frac{\sqrt{3}}{2} \Rightarrow \theta \in \left(\frac{3\pi}{2}, \frac{5\pi}{3}\right)$
$\frac{1}{2} < \sin(\theta + \varphi) < 1$,
$\left(\frac{3\pi}{2} < \theta < \frac{5\pi}{3}, \frac{1}{2} < \frac{2\cos\theta + 1}{2} < 1\right)$

$$\Rightarrow 2\pi+\frac{\pi}{6}<\theta+\varphi<\frac{5\pi}{6}+2\pi$$

$$2\pi+\frac{\pi}{6}-\theta_{\max}<\varphi<2\pi+\frac{5\pi}{6}-\theta_{\min} \quad ;$$

$$\frac{\pi}{2}<\varphi<\frac{4\pi}{3}$$

4. $(y+z)\cos 3\theta-(xyz)\sin 3\theta=0$ ....(1)

$xyz\sin 3\theta=2(\cos 3\theta)z+(2\sin 3\theta)y$ ....(2)

$\therefore (y+z)\cos 3\theta=(2\cos 3\theta)z$

$+(2\sin 3\theta)y=(y+2z)\cos 3\theta+y\sin 3\theta$

$y(\cos 3\theta-2\sin 3\theta)=z\cos 3\theta and$

$y(\sin 3\theta-\cos 3\theta)=0$

$\Rightarrow \sin 3\theta-\cos 3\theta=0\Rightarrow \sin 3\theta=\cos 3\theta$

$\therefore 3\theta=n\pi+\pi/4$

5. $\tan\theta=\cot 5\theta \quad \Rightarrow \cos 6\theta=\theta$

$4\cos^3 2\theta-3\cos 2\theta=0$

$\Rightarrow \cos 2\theta=0 \; or \; \pm\frac{\sqrt{3}}{2}, \; \sin 2\theta=\cos 4\theta$

$\Rightarrow 2\sin^2 2\theta+\sin 2\theta-1=0$

$\therefore \theta=-\frac{\pi}{4},\frac{\pi}{12},\frac{5\pi}{12}$

6. $\because AM\ge GM \; \because 3^x+3^{-x}\ge 2\Rightarrow RHS\ge 2$

now $2\cos\left(\frac{x}{2}\right)\ge 2$

Equation will have a solution if
LHS = RHS = 2 for the same value of x = 0
is the only solution.

7. $[\sin x]=\frac{x}{2\pi}-\left[\frac{x}{2\pi}\right]+\frac{2x}{5\pi}-\left[\frac{2x}{5\pi}\right]=\left\{\frac{x}{2\pi}\right\}+\left\{\frac{2x}{5\pi}\right\}$

$RHS\in[0,2)$

now $LHS=0,1(-1, \; not \quad possible)$

case-(i): $[\sin x]=0$

then $\left\{n+\frac{1}{4}\right\}+\left\{\frac{4n+1}{10}\right\}=\frac{1}{4}+\left\{\frac{4n+1}{10}\right\}\ne 1$

for $30<x<40.3<k\pi<4\Rightarrow\frac{3}{\pi}<k<\frac{4}{\pi}\Rightarrow k=1$

$\therefore x=10\pi$

case(ii): $[\sin x]=1\Rightarrow x=(4n+1)\frac{\pi}{2}$

now $\left\{\frac{x}{2\pi}\right\}+\left\{\frac{x}{5\pi}\right\}$

$=\left\{n+\frac{1}{4}\right\}+\left\{\frac{4n+1}{10}\right\}=\frac{1}{4}+\left\{\frac{4n+1}{10}\right\}\ne 1$

for any n.

8. $\sqrt{3}a\cos x+2b\sin x=c$

$\sqrt{3}a\cos\left(\frac{\pi}{3}-x\right)+2b\sin\left(\frac{\pi}{3}-x\right)=c$

$\sqrt{3}a\left(\cos x.\frac{1}{2}+\frac{\sin x\sqrt{3}}{2}\right)+2b\left(\frac{\sqrt{3}}{2}\cos x-\frac{1}{2}\sin x\right)=c$

$\Rightarrow\left(\frac{\sqrt{3}}{2}a+\sqrt{3}b\right)\cos x+\left(\frac{3}{2}a-b\right)\sin x=c$

$\left(\sqrt{3}b-\frac{\sqrt{3}}{2}a\right)\cos x+\left(\frac{3}{2}a-3b\right)\sin x=0$

$\Rightarrow\frac{b}{a}=\frac{1}{2}$

TRIGONOMETRIC RATIOS

SYNOPSIS

ANGLE

An angle is the union of two rays with common end point 'O' together with the rotation in a plane necessary to bring one ray (the initial side) into the position of the other ray (the terminal side). The end point O is calle the vertex of the angle.

If A and B are any two points such that A lies on the initial side and B lies on the terminal side of an angle θ then the angle is denoted by $\angle AOB$

If the rotation of an angle θ is of anti clock wise, then the angle' θ' is regarded as positive and if the rotation is of clock wise then the angle θ is regarded as negative.

✈ **Measurement of an angle :**

✈ **Sexagesimal system:**

(i) One right angle = $\frac{\pi}{2}$ radian = 90º.

(ii) π radian = 2 right angles = 180º.

(iii) $1° = 60$ minutes$(60')$

(iv) $1' = 60$ seconds$(60'')$

(v) $1º = 0.001745$ radian

(vi) 1 radian $\cong 57^0 17^1 45^{11}$ (approx)

✈ **Centisimal system:**

(i) 1 right angle = 100 grades written as 100^g

(ii) 1 grade or 1^g = 100 minutes (100')

(iii) 1 minute or 1' = 100 seconds (100")

✈ **Circular system :**

✈ **Radian:** A radian is the angle subtended at the centre of a circle by an arc equal in length to the radius of the circle. The length of arc $l = r\theta$.

(i) 1 revolution = 2π radians $= 360^\circ$

(ii) π radians = 2right angles $= 2\times 90^\circ = 180^\circ$

(iii) 1 degree $(1^\circ) = \frac{\pi}{180}$ rad $\cong 0.01745$ rad

(iv) 1 rad $(1^C) = \frac{180}{\pi}$ degrees $\cong 57^\circ 17'46''$

✈ **Note :**(i)Value of $\pi = \frac{22}{7}(or)\frac{355}{113}(or)3.1416$

(ii) π is an irrational number

(iii) $\pi = \frac{\text{circumference of the circle}}{\text{diameter of the circle}}$

✈ **Trigonometric Identities :**

i) $(\sin\theta)(\cos ec\,\theta) = 1, \theta \neq n\pi, n \in Z$

ii) $(\cos\theta)(\sec\theta) = 1, \theta \neq (2n+1)\frac{\pi}{2}, n \in Z$

iii) $(\tan\theta)(\cot\theta) = 1, \theta \neq (2n+1)\frac{\pi}{2}, \theta \neq n\pi$

iv) $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \sin^2\theta = 1 - \cos^2\theta$

v) $\sec^2\theta - \tan^2\theta = 1 \Rightarrow \sec^2\theta = 1 + \tan^2\theta$

$$\left(\theta \neq (2n+1)\frac{\pi}{2}, n \in Z\right).$$

(vi) $\cos ec^2\theta - \cot^2\theta = 1$

$$\Rightarrow \cos ec^2\theta = 1 + \cot^2\theta \quad (\theta \neq n\pi, n \in Z)$$

✈ **Note:** (i) If $\theta \neq (2n+1)\frac{\pi}{2}, n \in Z$, then

$$\sec^2\theta - \tan^2\theta = 1 \Rightarrow (\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$$

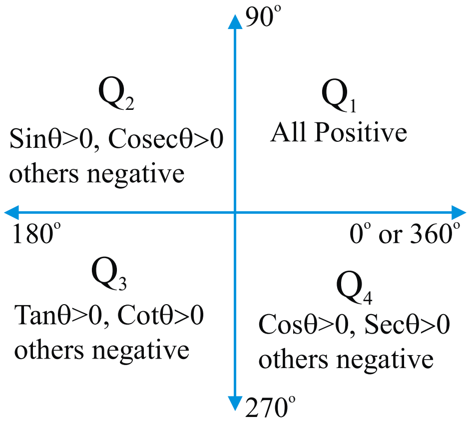
$$\Rightarrow \sec\theta + \tan\theta = \frac{1}{\sec\theta - \tan\theta}$$

(ii) If $\theta \neq n\pi, n \in Z$ then

$$\cos ec\theta - \cot\theta = \frac{1}{\cos ec\theta + \cot\theta}$$

Trigonometric ratios of various angles:

Trig. Ratio	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
$\mathrm{cosec}\,\theta$	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
$\cot\theta$	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0



Note : i) $\sin n\pi = \tan n\pi = \cos(2n+1)\frac{\pi}{2} = 0, \forall n \in Z$

ii) $\sin(2n+1)\frac{\pi}{2} = (-1)^n$ *and* $\cos n\pi = (-1)^n, \forall n \in Z$

✈ Domain and range of trigonometric functions :

Trig. function	Domain	Range
sin	$\mathbb{R}$	[-1, 1]
cos	$\mathbb{R}$	[-1, 1]
tan	$\mathbb{R}-\left\{(2n+1)\frac{\pi}{2}, n\in z\right\}$	$\mathbb{R}$
cot	$\mathbb{R}-\{n\pi, n\in z\}$	$\mathbb{R}$
cosec	$\mathbb{R}-\{n\pi, n\in z\}$	$(-\infty,-1]U[1,\infty)$
sec	$\mathbb{R}-\left\{(2n+1)\frac{\pi}{2}, n\in z\right\}$	$(-\infty,-1]U[1,\infty)$

EXERCISE - I

1. $\cos 23^\circ \,\mathrm{cosec}\,67^\circ - \sin 23^\circ \sec 67^\circ =$

1) 0 2) 2 3) 1 4) -1

2. $\sin 48^\circ \sec 42^\circ + \cos 48^\circ \,\mathrm{cosec}\,42^\circ =$

1) 0 2) 2 3) 1 4) -1

3. $\log\tan 17^\circ + \log\tan 37^\circ + \log\tan 53^\circ + \log\tan 73^\circ =$

1) 0 2) 1 3) 2 4) 3

4. $\cos 5^\circ + \cos 24^\circ + \cos 175^\circ + \cos 204^\circ + \cos 300^\circ =$

1) 1/2 2) 1 3) 3/2 4) 2

5. $\sec\theta - \tan\theta = 3 \Rightarrow \theta$ **lies in the quadrant**

1) I 2) II 3) III 4) IV

6. $\sin 10^\circ + \sin 20^\circ + \sin 30^\circ + \ldots\ldots + \sin 360^\circ =$

1) 0 2) 1 3) – 1 4) 2

7. $3[\sin x - \cos x]^4 + 6[\sin x + \cos x]^2 + 4[\sin^6 x + \cos^6 x] =$

1) 3 2) 6 3) 4 4) 13

8. $\frac{\sin^2\alpha}{1+\cot^2\alpha} + \frac{\tan^2\alpha}{(1+\tan^2\alpha)^2} + \cos^2\alpha =$

1) – 1 2) 0 3) 1 4) 2

9. If A, B, C are the angles of a triangle ABC then $\cos\left(\frac{3A+2B+C}{2}\right) + \cos\left(\frac{A-C}{2}\right) =$

1) 0 2) 1 3) cosA 4) cosC

10. $\frac{1+\cot\alpha+\mathrm{cosec}\,\alpha}{1-\cot\alpha+\mathrm{cosec}\,\alpha} =$

1) $\frac{\sin\alpha}{1+\cos\alpha}$ 2) $\frac{\sin\alpha}{1-\cos\alpha}$

3) $\frac{1+\cos\alpha}{\sin\alpha}$ 4) $\frac{1-\sin\alpha}{\cos\alpha}$

11. $\sin^2\frac{\pi}{18} + \sin^2\frac{2\pi}{18} + \sin^2\frac{4\pi}{18} + \sin^2\frac{8\pi}{18} + \sin^2\frac{7\pi}{18} + \sin^2\frac{5\pi}{18} =$

1) 1 2) 2 3) 3 4) 4

12. If A B C D is a quadrilateral then

$\tan\left(\frac{A+B}{4}\right)=$

1) $\cos\left(\frac{C-D}{4}\right)$ 2) $\cot\left(\frac{C-D}{4}\right)$

3) $\cos\left(\frac{C+D}{4}\right)$ 4) $\cot\left(\frac{C+D}{4}\right)$

13. If $\cos\theta_1+\cos\theta_2+\cos\theta_3+\cos\theta_4+\cos\theta_5=5$ **then** $\sin\theta_1+\sin\theta_2+\sin\theta_3+\sin\theta_4+\sin\theta_5=$

1) 3 2) 2 3) 1 4) 0

14. $9\cos^2 x+4\sin^2 x=5\Rightarrow \tan x=$

1) ±1 2) ±2 3) ±3 4) ±4

15. $\cos x+\cos^2 x=1\Rightarrow \sin^8 x+2\sin^6 x+\sin^4 x=$

1) 0 2) 1 3) 2 4) -1

16. If $\tan\alpha+\cot\alpha=2$ then $\sqrt{\tan\alpha}+\sqrt{\cot\alpha}=$

1) $\sqrt{2}$ 2) $2\sqrt{2}$ 3) 2 4) $4\sqrt{2}$

17. $(1+\sin A)(1+\sin B)(1+\sin C)=(1-\sin A)(1-\sin B)(1-\sin C)=k\Rightarrow k=$

1) $+\sin A\sin B\sin C$

2) $\pm\cos A\cos B\cos C$

3) $\pm\sec A\sec B\sec C$

4) $\pm$ cosec A cosecB cosecC

18. $\tan^2\theta+\sec\theta=5\Rightarrow\sec\theta=$

1) 3 2) 2 3) 1 4) -1

19. $\log\sin 1^\circ.\log\sin 2^\circ........\log\sin 179^\circ$

1) 1 2) 0 3) – 1 4) 2

20. If $\tan(\alpha+\beta)=\sqrt{3}, \tan(\alpha-\beta)=1$ **then** $\tan 6\beta=$

1) -1 2) 0 3) 1 4) 2

21. $\sin^2 A\cos^2 B+\cos^2 A\sin^2 B+\sin^2 A\sin^2 B+\cos^2 A\cos^2 B=$

1) -1 2) 0 3) 1 4) 2

22. If $x=\sin 130^\circ+\cos 130^\circ$ **then**

1) $x<0$ 2) $x=0$ 3) $x>0$ 4) $x\geq 0$

23. $\cot\frac{\pi}{20}.\cot\frac{3\pi}{20}.\cot\frac{5\pi}{20}.\cot\frac{7\pi}{20}.\cot\frac{9\pi}{20}.\cot\frac{15\pi}{20}=$

1) 1 2) -1 3) $\sqrt{3}$ 4) $-\sqrt{3}$

24. $x=a\sec^3\theta\tan\theta$, $y=b\tan^3\theta\sec\theta$ **then** $\sin^2\theta=$

1) $\frac{x}{a}-\frac{y}{b}$ 2) $\frac{x}{a}+\frac{y}{b}$ 3) $\frac{xy}{ab}$ 4) $\frac{ay}{bx}$

25. If $x=a\sec^n\theta; y=b\tan^n\theta$ **then**

$\left(\frac{x}{a}\right)^{2/n}-\left(\frac{y}{b}\right)^{2/n}=$

1) 0 2) -1 3) 1 4) 2

26. If A, B, C are angles of a triangle such that A is obtuse then

1) tanA tan B > 1 2) tan B tan C <1

3) tan C tan A > 1 4) tanA tan B tanC>1

27. $4a^2\sin^2\left(\frac{3\pi}{4}\right)-3\left[a\tan 225^\circ\right]^2+\left[2a\cos 315^\circ\right]^2=$

1) 0 2) a 3) $\sqrt{2a}$ 4) a^2

28. If $\tan\theta=\frac{p}{q}$ **then** $\frac{p\sin\theta-q\cos\theta}{p\sin\theta+q\cos\theta}=$

1) $\frac{2p}{p^2+q^2}$ 2) $\frac{2pq}{p^2+q^2}$

3) $\frac{p^2-q^2}{p^2+q^2}$ 4) $\frac{q^2-p^2}{p^2+q^2}$

29. If A,B,C are in arcthmetic progression and $B=\frac{\pi}{4}$ Then $\tan A\tan B\tan C=$

30. If the value of $(1+\tan 1)(1+\tan 2)(1+\tan 3)-----(1+\tan 44)(1+\tan 45)$ is 2^λ. Then the sum of the digites of the number $\lambda=$

KEY

01) 1	02) 2	03) 1	04) 1	05) 4	06) 1
07) 4	08) 3	09) 1	10) 3	11) 3	12) 4
13) 4	14) 2	15) 2	16) 3	17) 2	18) 2
19) 2	20) 3	21) 3	22) 3	23) 2	24) 4

25) 3 26) 2 27) 4 28) 3 29) 1 30) 5

SOLUTIONS

1. $\cos 23^{o} \cos ec\left(90^{o}-23^{o}\right)-\sin 23^{o} \sec\left(90^{o}-23^{o}\right)$

$=1-1=0$

2. $\sin 48^{0} \cos ec 48^{0}+\cos 48^{0} \sec 48^{0}=1+1=2$

3. $\log\left(\tan(7)^{o}\right)+\log\left(\tan 37^{o}\right)+\log\left(\tan 53\right)+\log\left(\tan 73^{o}\right)$

$=\log\left(\tan 17^{o} \tan 73^{o} \tan 53^{o} \tan 37^{o}\right)$

We know $A+B=90^{o}$ Then $\tan A \tan B=1$

$=\log(1\times 1)=\log 1=0$

4. $\cos 5^{o}+\cos 24^{o}+\cos 175^{o}+\cos 204^{o}+\cos 300^{o}$

$=\cos 5^{o}+\cos 24^{o}+\cos(180-5)$

$+\cos\left(180+24^{o}\right)+\cos\left(360-60^{o}\right)$

$=\cos 5^{o}+\cos 24^{o}-\cos 5^{o}-\cos 24^{o}+\cos 60^{o}$

$=\frac{1}{2}$

5. Given $seeA-\tan A=3$ (1)

Using $\sec^{2} A-\tan^{2} A=1$

$seeA+\tan A=\frac{1}{seeA-\tan A}$

$seeA+\tan A={}^{1}\!/\!_{3}$ (2)

Adding (1) and (2) $2seeA=3+{}^{1}\!/\!_{3}=\frac{10}{3}$

$seeA={}^{5}\!/\!_{3}>0$

$(1)-(2)\Rightarrow -2\tan A=3-\frac{1}{3}=\frac{8}{3}$

$\tan A=\frac{-4}{3}<0$

A lies IV Quadrant

6. $A+B=360°\Rightarrow \sin A+\sin B=0$

7. Put $x=90^{o}$

8. Put $\alpha=45^{o}$

9. A = B = C = 60^{0}

10. $\frac{\cos ec\alpha+\cot\alpha+1}{1-\cot\alpha+\cos ec\alpha}$

We know $\cos ec^{2}\alpha-\cot^{2}\alpha=1$

$=\frac{\cos ec\alpha+\cot\alpha+\cos ec^{2}\alpha-\cot^{2}\alpha}{1-\cot-\alpha+\cos ec\alpha}$

$=\frac{\cos ec\alpha+\cot\alpha(1+\cos ec\alpha-\cot\alpha)}{1-\cot\alpha+\cos ec\alpha}$

$=\frac{1+\cos\alpha}{\sin\alpha}$

11. If $(A+B)=\frac{\pi}{2}$ then $\sin^{2} A+\sin^{2} B=1$

12. $(A+B+C+D)=360^{o}\Rightarrow\frac{A+B}{4}=\frac{\pi}{2}-\frac{C+D}{4}$

13. Put $\theta_1=\theta_2=\theta_3=0^{o}$

14. Given $9\cos^{2} x+4\sin^{2} x=5$

Divide by $\cos^{2} x$ on both sides

$9+4\tan^{2} x=5\sec^{2} x$

$9+4\tan^{2} x=5\left(1+\tan^{2} x\right)$

$4=\tan^{2} x$

$\tan x=\pm 2$

15. Given $\cos x+\cos^{2} x=1$

$\cos x=1-\cos^{2} x$

$\cos x=\sin^{2} x$

$\cos^{2} x=\sin^{4} x$

$\sin^{4} x=1-\sin^{2} x$

$\sin^{4} x+\sin^{2} x=1$

squaring on both sides

$\left(\sin^{4} x+\sin^{2} x\right)^{2}=1$

$\sin^{8} x+2\sin^{6} x+\sin^{4} x=1$

16. Put $\alpha=\frac{\pi}{4}$

17. $k^{2}=(1+\sin A)(1+\sin B)(1+\sin C)(1-\sin A)$

$(1-\sin B)(1-\sin C)$

$$K^2=(1-\sin^2 A)(1-\sin^2 B)(1-\sin^2 C)$$
$$K^2=Cos^2A.Cos^2B.Cos^2C$$
$$K=\pm CosA\,CosB\,CosC$$

18. $\tan^2\theta+\sec\theta=5$, $\Rightarrow (Sec^2\theta-1)+\sec\theta=5$, $\Rightarrow Sec^2\theta+\sec\theta-6=0$, $\sec\theta=2\ or\ -3$

19. $\log(\sin 90^0)=0$

20. Let $(\alpha+\beta)=60^0$
$(\alpha-\beta)=45^0 \Rightarrow 2\beta=15^0 \Rightarrow 6\beta=45^0$

21. $\sin^2 A(\cos^2 B+\sin^2 B)+\cos^2 A(\sin^2 B+\cos^2 B)\ or$
$Put\ A=B=0^0$

22. $x=\sin 50^o-\cos 50^o$

23. $A+B=90^o \Rightarrow \cot A\cot B=1$

24. $\frac{y}{x}=\frac{b\tan^3\theta\sec\theta}{a\sec^3\theta\tan\theta}=\frac{b}{a}\sin^2\theta$

25. Given $x=a\sec^n\theta, y=b\tan^n\theta$

$x=\sec^n\theta,\frac{y}{b}=\tan^n\theta$

$\sec\theta=\left(\frac{x}{a}\right)^{1/n},\tan\theta=\left(\frac{y}{b}\right)^{1/n}$

$\sec^2\theta=\left(\frac{x}{a}\right)^{2/n},\tan^2\theta=\left(\frac{y}{b}\right)^{2/n}$

We know $\sec^2\theta-\tan^2\theta=1$

$$\left(\frac{x}{a}\right)^{2/n}-\left(\frac{y}{b}\right)^{2/n}=1$$

26. A is obtuse $\Rightarrow B+C<90^o \Rightarrow \tan B\tan C<1$

27. Given $=4a^2\left(\frac{1}{\sqrt{2}}\right)^2-3[a\times 1]^2+\left[2a\times\frac{1}{\sqrt{2}}\right]^2$

28. Given $\tan\theta=\frac{p}{q}$

Now $\frac{p\sin\theta-q\cos\theta}{p\sin\theta+q\cos\theta}$ divide by $\cos\theta$ Nr and Dr

$$\frac{p\tan\theta-q}{p\tan\theta+q}=\frac{p^2-q^2}{p^2+q^2}$$

29. Given angles A,B,C are in A.P and $B=45^o$

Then $A=\frac{\pi}{4}-\theta, C=\frac{\pi}{4}+\theta$

Hence $\tan(A)\tan B\tan C=\tan\left(\frac{\pi}{4}-0\right)\tan\frac{\pi}{4}\tan\left(\frac{\pi}{4}+\theta\right)$

$=\frac{1-\tan\theta}{1+\tan\theta}\ 1\ \frac{1+\tan\theta}{1-\tan\theta}$

$=1$

30. If $A+B=45^o$

$\tan(A+B)=\tan 45^0$

$\frac{\tan A+\tan B}{1-\tan A\tan B}=1$

$\tan A+\tan B=1-\tan A\tan B$

$\tan A+\tan B+\tan A\tan B=1$

$1+\tan A+\tan B+\tan A\tan B=2$

$(1+\tan A)(1+\tan B)=2$

L.H.S=$(1+\tan 1)(1+\tan 44)(1+\tan 2)(1+\tan 43)-(1+\tan 45)$

$=2^{22}(1+1)$

$=2^{23}\ =2^{\lambda}$

$=\lambda=23$

Sum of the digits of λ is=2+3=5

EXERCISE II

1. $f(x)=x^3-2x^2+3x-5$

$\Rightarrow f\left[\sin\left(\frac{5\pi}{2}\right)\right]+f\left[\sin\left(\frac{3\pi}{2}\right)\right]=$

1) 10 2) -10 11) 14 4) -14

2. cosA, sinA, cotA are in GP then

$\tan^6 A-\tan^2 A=$

1) –1 2) 0 3) 1 4) 2

3. If *ABCD* is a cydic quadrilateral such that $12\tan A - 5 = 0$ **and** $5\cos B + 3 = 0$**, then** $\cos C \tan D =$

1) $\frac{-16}{13}$ 2) $\frac{16}{13}$ 3) $\frac{-13}{16}$ 4) $\frac{23}{16}$

4. If $\tan\theta, 2\tan\theta + 2$**,** $3\tan\theta + 3$ **are in G.P then the value of** $\frac{7 - 5\cot\theta}{9 - 4\sqrt{\sec^2\theta - 1}}$ **is**

1) $\frac{12}{5}$ 2) $\frac{-33}{28}$ 3) $\frac{33}{100}$ 4) $\frac{12}{13}$

5. Which of the following is correct?

1) $\sin 1^0 > \sin 1$ 2) $\sin 1^0 < \sin 1$

3) $\sin 1^0 = \sin 1$ 4) $\sin 1^0 = \frac{\pi}{180}\sin 1$

6. $\operatorname{cosec} A = 4p + \frac{1}{16p} \Rightarrow \operatorname{cosec} A + \cot A =$

1) $8p$ 2) $\frac{1}{8p}$ 3) $-8p$ (or) $\frac{1}{8p}$ 4) $8p$ (or) $\frac{1}{8p}$

7. If $1 + \sin x + \sin^2 x + ...$ to $\infty = 4 + 2\sqrt{3}, 0 < x < \pi$ **and** $x \neq \frac{\pi}{2}$ **then** $x =$

1) 30º, 60º 2) 60º, 120º

3) 90º, 120º 4) 30º, 45º

8. $\cos^2 5^\circ + \cos^2 10^\circ + \cos^2 15^\circ + ... + \cos^2 360^\circ =$

1) 18 2) 27 3) 36 4) 45

9. If $\frac{\cos^2\theta}{a} = \frac{\sin^2\theta}{b}$ **then** $\frac{\cos^4\theta}{a} + \frac{\sin^4\theta}{b} =$

1) $\frac{1}{a+b}$ 2) $\frac{1}{(a+b)^2}$

3) $\frac{1}{a^2} + \frac{1}{b^2}$ 4) $a + b$

10. Eliminate θ **from** $x = 1 + \tan\theta$**,** $y = 2 + \cot\theta$

1) $xy + 1 = x + y$ 2) $xy + 2 = 2x + y$

3) $xy + 1 = 2x + y$ 4) $xy + 1 = 2y + x$

11. If $x = a\cos^3\theta\sin^2\theta, y = a\sin^3\theta\cos^2\theta$ **and** $\frac{(x^2+y^2)^p}{(xy)^q}, (p, q \in N)$ **is independent of** θ **then**

1) $p + q = 6$ 2) $4p = 5q$

3) $4q = 5p$ 4) $pq = 16$

12. If $\frac{\sin x}{a} = \frac{\cos x}{b} = \frac{\tan x}{c} = k$ **then** $bc + \frac{1}{ck} + \frac{ak}{1+bk}$ **is**

1) $k\left(a + \frac{1}{a}\right)$ 2) $\frac{1}{k}\left(a + \frac{1}{a}\right)$ 3) $\frac{1}{k^2}$ 4) $\frac{a}{k}$

13. If $a\sin^3 x + b\cos^3 x = \sin x \cos x$ **and** $a\sin x = b\cos x$ **then** $a^2 + b^2 =$

1) 0 2) 1 3) 2 4) 3

14. $f(x) = \sin^2 x + \operatorname{cosec}^2 x \Rightarrow$

1) $f(x) < 1$ 2) $f(x) = 1$

3) $1 < f(x) < 2$ 4) $f(x) \geq 2$

15. Which of the following is not possible ?

1) $\sin\theta = \frac{5}{7}$ 2) $\cos\theta = \frac{1+a^2}{1-a^2}, |a| \neq 1$

3) $\tan\theta = 100$ 4) $\sec\theta = \frac{5}{2}$

16. If $e^{\left(\sin^2 x + \sin^4 x + \sin^6 x + ...\infty\right)\log 2} = 8$ and $0 < x < \frac{\pi}{2}$ then $\frac{\cos x}{\cos x + \sin x} =$

1) $\frac{\sqrt{3}+1}{2}$ 2) $\frac{\sqrt{3}-1}{2}$ 3) $\frac{2}{\sqrt{3}+1}$ 4) $\frac{2}{\sqrt{3}-1}$

17. If $0 \leq x \leq \pi, 81^{\sin^2 x} + 81^{\cos^2 x} = 30$ **then x =**

1) $\frac{\pi}{6}$ 2) $\frac{\pi}{4}$ 3) $\frac{\pi}{15}$ 4) $\frac{\pi}{8}$

18. If $2\sin x + 5\cos y + 7\sin z = 14$ then $7\tan\frac{x}{2} + 4\cos y - 6\cos z =$

1) 4 2) -3 3) 11 4) 5

19. If $\sin\theta$ and $\cos\theta$ are the roots of $px^2 + qx + r = 0$ then $q^2 - p^2 =$

1) 0 2) –2pr 3) 2qr 4) 2rp

KEY

01) 4 02) 3 03) 1 04) 2 05) 2
06) 4 07) 2 08) 3 09) 1
10) 3 11) 2 12) 2
13) 2 14) 4 15) 2 16) 2
17) 1 18) 3 19) 4

SOLUTIONS

1. $\sin\frac{3\pi}{2} = -1, \sin\frac{5\pi}{2} = 1$

2. Given $\cos A, \sin A, \cot A$ are in G.P

$\sin^2 A = \cos A.\cot A$

$\sin^2 A = \cos A\frac{\cos A}{\sin A}$

$\tan^2 A = \cos ecA$

Now $\tan^6 A - \tan^2 A = \cos ec^3 A - \cos ecA$

$= \cos ecA\left(\cos ec^2 A - 1\right)$

$= \cos ecA.\cot^2 A$

$= \cos ecA.\frac{1}{\cos ecA}\left(\tan^2 A = \cos ec\right)$

$=1$

3. $A + C = 180^o, B + D = 180^o$

Given $\tan A = \frac{5}{12} > 0 \therefore 0 < A < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < C < \pi$

$\cos B = \frac{-3}{5} < 0 \therefore \frac{\pi}{2} < B < \pi \Rightarrow 0 < D < \frac{\pi}{2}$

$\therefore \cos C = \frac{-12}{13}; \tan D = \frac{4}{3}$

4. $(2\tan\theta + 2)^2 = \tan\theta(3\tan\theta + 3)$

$\Rightarrow \tan^2\theta + 5\tan\theta + 4 = 0 \Rightarrow \tan\theta = -1, -4$

but $\tan\theta = -4$ only satisfies above condition

5. $1 > 1^0$

6. Given $\cos ecA = 4P + \frac{1}{16P}$

We know $\cos ec^2 A - \cot^2 A = 1$

$\cot^2 A = \cos ec^2 A - 1$

$= \left(4P + \frac{1}{16p}\right)^2 - 1$

$\cot^2 A = \left(4P - \frac{1}{16p}\right)^2$

$\cot A = \pm\left(4p - \frac{1}{16p}\right)$

Now $\cos ecA + \cot A = 8P(or)\frac{1}{8P}$

7. Use formula $S_\infty = \frac{a}{1-r}$ for G.P , $a = 1$

$r = \sin x$

$\Rightarrow \frac{a}{1-\gamma} = 4 + 2\sqrt{3}$

$\frac{1}{1-\sin x} = 4 + 2\sqrt{3}$

$\Rightarrow 1 - \sin x = \frac{1}{4 + 2\sqrt{3}}$

$1 - \sin x = \frac{4 - 2\sqrt{3}}{4}$

$\sin x = 1 - \left(\frac{4 - 2\sqrt{3}}{4}\right) = \frac{\sqrt{3}}{2}$

$x = 60^o, 120^o$

8. If $(A+B) = 360^0$ or $(A+B) = 180^0$

then $\cos^2 A = \cos^2 B$ and $(A+B) = 90^0$ then

$\left(\cos^2 A + \cos^2 B\right) = 1$

9. Given $\frac{\cos^2\theta}{a} = \frac{\sin^2\theta}{b} = \tan^2\theta = \frac{b}{a}$

Now

$$\frac{\cos^4\theta}{a}+\frac{\sin^4\theta}{b}==\frac{a}{(a+b)^2}+\frac{b}{(a+b)^2}=\frac{1}{a+b}$$

$$\tan\theta=\frac{\sqrt{b}}{\sqrt{a}}$$

$$\cos\theta=\frac{\sqrt{a}}{\sqrt{a+b}}$$

$$\sin\theta=\frac{\sqrt{b}}{\sqrt{a+b}}$$

10. $x-1=\tan\theta, y-2=\cot\theta$
Multiply two equations

11. $\frac{(x^2+y^2)^p}{(xy)^q}=\frac{(a^2\sin^4\theta\cos^4\theta)^p}{(a^2\sin^5\theta\cos^5\theta)^q}$

$$=\frac{a^{2p}(\sin\theta\cos\theta)^{4p}}{a^{2q}(\sin\theta\cos\theta)^{5q}}$$

which is independent of θ, if 4p=5q

12. $bc+\frac{1}{ck}+\frac{ak}{1+bk}$

$$=\frac{\sin x}{k^2}+\frac{\cos x(1+\cos x)+\sin^2 x}{\sin x(1+\cos x)}$$

$$=\frac{a}{k}+\frac{1}{\sin x}=\frac{a}{k}+\frac{1}{ak}=\frac{1}{k}\left(a+\frac{1}{a}\right)$$

13. Put $\sin x=b$ and $\cos x=a$

$$\therefore a^2+b^2=\cos^2 x+\sin^2 x=1$$

14. $f(x)=\left(\sin^2 x+\frac{1}{\sin^2 x}\right)\geq 2$

$\because \left(a+\frac{1}{a}\right)\geq 2$ for any real number $a>0$

15. for $a=2$, $\cos\theta=\frac{5}{-3}$ not possible

16. $\sin^2 x+\sin^4 x+\ldots\infty=\frac{\sin^2 x}{1-\sin^2 x}=\tan^2 x$

$$\therefore \left(e^{\log_e 2}\right)^{\tan^2 x}=8\Rightarrow 2^{\tan^2 x}=2^3\Rightarrow \tan x=\sqrt{3}$$

$$\therefore \text{G.E}=\frac{1}{1+\tan x}=\frac{1}{1+\sqrt{3}}=\frac{\sqrt{3}-1}{2}$$

17. If $81^{\sin^2 x}=y\Rightarrow 81^{\cos^2 x}=\frac{81}{y}$

$$\therefore y+\frac{81}{y}=30\Rightarrow y^2-30y+81=0\Rightarrow y=27,3$$

18. put $x=\frac{\pi}{2}$, $y=0$, $z=\frac{\pi}{2}$.

19. $\sin\theta+\cos\theta=\frac{-q}{p},\sin\theta\cos\theta=\frac{r}{p}$

now $(\sin\theta+\cos\theta)^2=1+2\sin\theta\cos\theta$

$\sin^4\theta+\sin^2\theta=1$
cubing on both sides
$(\sin^4+\sin^2\theta)=1$
$\sin^{12}\theta\sin^6\theta+3\sin^{10}\theta+3\sin^8\theta=1$
campassing with
$a\sin^{12}\theta+b\sin^{10}\theta+c\sin^8\theta+d\sin^6\theta=1$
$a=1,\ b=3,\ c=3,\ d=1$

Now $\frac{b+c}{a+d}=\frac{6}{2}=3$

EXERCISE - III

1. $\tan^2\alpha=1-p^2$, **then** $\sec\alpha+\tan^3\alpha\,\text{cosec}\alpha=$

1) $(2+p^2)^{\frac{3}{2}}$ 2) $(1+p^2)^{\frac{3}{2}}$

3) $(2-p^2)^{\frac{3}{2}}$ 4) $(1-p^2)^{\frac{3}{2}}$

2. $\frac{\pi}{2}<\alpha<\pi\Rightarrow\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}+\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}=$

1) $2\sec\alpha$ 2) $-2\sec\alpha$

3) $2\,\text{cosec}\alpha$ 4) $-2\,\text{cosec}\alpha$

3. If $\sin\beta$ **is the G.M. between** $\sin\alpha$ **and** $\cos\alpha$

then $(\cos\alpha-\sin\alpha)^2-2\cos^2\beta=$

1) 0 2) 1 3) 2 4) -1

4. If $\sin^2\theta=\dfrac{x^2+y^2+1}{2x}$ then x must be

1) -3 2) -2 3) 1 4) 3

5. $\sin x+\sin^2 x+\sin^3 x=1\Rightarrow\cos^6 x-4\cos^4 x+8\cos^2 x=$

1) 4 2) 2 3) 1 4) 0

6. $\cos\theta+\cos^2\theta=1$, $a\sin^{12}\theta+b\sin^{10}\theta+c\sin^8\theta+d\sin^6\theta=1\Rightarrow\dfrac{b+c}{a+d}=$

1) 2 2) 3 3) 4 4) 6

7. $a\sin x=b\cos x=\dfrac{2c\tan x}{1-\tan^2 x}$ and $\left(a^2-b^2\right)^2=kc^2\left(a^2+b^2\right)\Rightarrow k=$

1) 1 2) 2 3) 3 4) 4

8. The greatest among $(\sin 1+\cos 1)$, $\left(\sqrt{\sin 1}+\sqrt{\cos 1}\right),(\sin 1-\cos 1)$ and 1 is

1) $\sin 1+\cos 1$ 2) $\sqrt{\sin 1}+\sqrt{\cos 1}$
3) $\sin 1-\cos 1$ 4) 1

9. If $a\sin^2 x+b\cos^2 x=c$, $b\sin^2 y+a\cos^2 y=d$ and $a\tan x=b\tan y$ then $\dfrac{a^2}{b^2}=$

1) $\dfrac{(a-d)(c-a)}{(b-c)(d-b)}$ 2) $\dfrac{(b-c)(b-d)}{(a-c)(a-d)}$

3) $\dfrac{(b-c)(d-b)}{(a-d)(c-a)}$ 4) $\dfrac{(d-a)(c-a)}{(b-c)(d-b)}$

10. If $\tan^2\alpha\tan^2\beta+\tan^2\beta\tan^2\gamma+\tan^2\gamma\tan^2\alpha+2\tan^2\alpha\tan^2\beta\tan^2\gamma=1$

then the value of $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=$

1) 0 2) -1 3) 1 4) 2

11. Let $\theta\in\left(0,\dfrac{\pi}{4}\right)$ and $t_1=(\tan\theta)^{\tan\theta}$, $t_2=(\tan\theta)^{\cot\theta}$, $t_3=(\cot\theta)^{\tan\theta}$, $t_4=(\cot\theta)^{\cot\theta}$ then

1) $t_1>t_2>t_3>t_4$ 2) $t_4>t_3>t_1>t_2$
3) $t_3>t_1>t_2>t_4$ 4) $t_2>t_3>t_1>t_4$

12 Two arcs of same length of two different circles subtended angles of 25° and 75° at their centres respectively. Then the ratio of the radii of the circles is

1) 3 : 1 2) 1 : 3 3) 1 : 2 4) 2 : 1

13. If $f_k(x)=\dfrac{1}{k}\left(\sin^k x+\cos^k x\right)$ where $x\in R,k\geq 1$ then $f_4(x)-f_6(x)=$ (MAINS-2014)

1) $\dfrac{1}{6}$ 2) $\dfrac{1}{3}$ 3) $\dfrac{1}{9}$ 4) $\dfrac{1}{12}$

14. The expression $\dfrac{\tan A}{1-\cot A}+\dfrac{\cot A}{1-\tan A}$ can be written as (MAINS-2013)

1) $\sin A\cos A+1$ 2) $\sec A\,\mathrm{cosec}A+1$
3) $\tan A+\cot A$ 4) $\sec A+\mathrm{cosec}A$

15. In a ΔPQR, if 3 sin P + 4 cos Q = 6 and 4 sin Q + 3 cos P = 1 then the acute angle 'R' is equal to [AIEEE-2012]

1) $\dfrac{\pi}{4}$ 2) $\dfrac{3\pi}{4}$ 3) $\dfrac{5\pi}{6}$ 4) $\dfrac{5\pi}{6}$

KEY

01) 3 02) 3 03) 4 04) 3 05) 1 06) 2
07) 4 08) 2 09) 1 10) 3 11) 2 12) 1 13) 4
14) 2 15) 4

HINTS

1. $\sec\alpha\left[1+\dfrac{\sin^3\alpha}{\cos^3\alpha}\dfrac{\cos\alpha}{\sin\alpha}\right]$

$=\sec\alpha\left[1+\tan^2\alpha\right]=\sec^3\alpha$

$=\left(\sec^2\alpha\right)^{3/2}=\left(2-P^2\right)^{3/2}$

2. Simplifying we get $\dfrac{2}{\sqrt{\sin^2\alpha}}=2\cos ec\,\alpha$

$\because \sin\alpha$ is positive in II quadrant.

3. $\sin^2\beta=\sin\alpha\cos\alpha$

4. $\sin^2\theta\le 1$

5. $\sin x\left(1+\sin^2 x\right)=\cos^2 x$

Squaring both sides we get

$\left(1-\cos^2 x\right)\left(2-\cos^2 x\right)^2=\cos^4 x$

Simplify further

6. $\sin^4\theta+\sin^2\theta=1$

cubing on both sides

$\left(\sin^4+\sin^2\theta\right)=1$

$\sin^{12}\theta\sin^6\theta+3\sin^{10}\theta+3\sin^8\theta=1$

campassing with

$a\sin^{12}\theta+b\sin^{10}\theta+c\sin^8\theta+d\sin^6\theta=1$

$a=1,\ b=3,\ c=3,\ d=1$

Now $\dfrac{b+c}{a+d}=\dfrac{6}{2}=3$

7. Take $a\ \sin\ x=b\cos x=c\tan 2x=l$

$\Rightarrow \sin x=\dfrac{l}{a},\cos x=\dfrac{l}{b},\tan 2x=\dfrac{l}{c}$

Elimination l we get

$\left(a^2-b^2\right)^2=4c^2\left(a^2+b^2\right)\Rightarrow K=4$

8. $\sqrt{\sin 1}>\sin 1>\sin^2 1$

$\sqrt{\cos 1}>\cos 1>\cos^2 1$

Adding $\left(\sqrt{\sin 1}+\sqrt{\cos 1}\right)>\left(\sin 1+\cos 1\right)$

$>\left(\sin^2 1+\cos^2 1\right)$

$\Rightarrow\left(\sqrt{\sin 1}+\sqrt{\cos 1}\right)>\left(\sin 1+\cos 1\right)>1$

9. $a\sin^2 x+b\cos^2 x=c$

$\Rightarrow a\tan^2 x+b=c\sec^2 x$

$\Rightarrow (a-c)\tan^2 x=c-b \quad \Rightarrow \tan^2 x=\dfrac{c-b}{a-c}$

similarly $\tan^2 y=\dfrac{d-a}{b-d}$

but $a\tan x=b\tan y\Rightarrow\dfrac{a^2}{b^2}=\dfrac{\tan^2 y}{\tan^2 x}$

10. Let $\tan^2\alpha=x,\tan^2\beta=y,\tan^2\gamma=z$

$\sin^2\alpha+\sin^2\beta+\sin^2\gamma=\sum\dfrac{\tan^2\alpha}{1+\tan^2\alpha}=\sum\dfrac{x}{1+x}$

$=\dfrac{(x+y+z)+(xy+yz+zx+2xyz)+xy+yz+zx+xyz}{(1+x)(1+y)(1+z)}$

$=\dfrac{1+x+y+z+xy+yz+zx+xyz}{(1+x)(1+y)(1+z)}=1$

11. In $\left(0,\dfrac{\pi}{4}\right)$, $\tan\theta<1$ and $\cot\theta>1$

$\left(\cot\theta\right)^{\cot\theta}>\left(\cot\theta\right)^{\tan\theta}>\left(\tan\theta\right)^{\tan\theta}>\left(\tan\theta\right)^{\cot\theta}$

$\therefore t_4>t_3>t_1>t_2$

12. Length of arc, $l=r\theta$

$\therefore\ \dfrac{r_1}{r_2}=\dfrac{l/\theta_1}{l/\theta_2}=\dfrac{\theta_2}{\theta_1}=\dfrac{75}{25}=\dfrac{3}{1}$

13. $f_4(x)=1-2\sin^2\theta\cos^2\theta$ and

$f_6(x)=1-3\sin^2\theta\cos^2\theta$

14. $\dfrac{\dfrac{\sin A}{\cos A}}{1-\dfrac{\cos A}{\sin A}}+\dfrac{\dfrac{\cos A}{\sin A}}{1-\dfrac{\sin A}{\cos A}}=\dfrac{\sin^2 A}{\cos A(\sin A-\cos A)}+\dfrac{\cos^2 A}{\sin A(\cos A-\sin A)}$

$=\dfrac{\sin^3 A-\cos^3 A}{\sin A\cos A(\sin A-\cos A)}=\dfrac{\sin^2 A+\sin A\cos A+\cos^2 A}{\sin A\cos A}$

$=\dfrac{1+\sin A\cos A}{\sin A\cos A}=\sec A\cos ecA+1$

15. Squaring both the equations and adding both the given equations then we get the angle p and q as the sum of angles in triangle is 180

JEE MAINS, EAMCET QUESTIONS

1. **If A, B, C are the angles of a triangle ABC then** $\cos\left(\frac{3A+2B+C}{2}\right)+\cos\left(\frac{A-C}{2}\right)=$

2. **If ABCD is a cyclic quadrilateral, then** $\cos(180^\circ+A)+\cos(180^\circ-B)+\cos(180^\circ-C)-\sin(90^\circ-D)=$

3. **If** $\sin x+\operatorname{cosec} x=2$ then $\sin^8 x+\operatorname{cosec}^8 x=$

4. **In a** ΔABC**, if** $\cot A\cot B\cot C>0$ **then the triangle is**
   1) acute angled 2) right angled
   3) obtuse angled 4) can't be decided

5. **If** $\tan\theta, 2\tan\theta+2, 3\tan\theta+3$ **are in G.P then the value of** $\frac{7-5\cot\theta}{9-4\sqrt{\sec^2\theta-1}}$ **is**
   1) $\frac{12}{5}$ 2) $\frac{-33}{28}$ 3) $\frac{33}{100}$ 4) $\frac{12}{13}$

6. **In a triangle ABC, C = 90^0, then the equation whose roots are tanA, tanB is**
   1) $ab\,x^2+c^2x+ab=0$ 2) $ab\,x^2+c^2x-ab=0$
   3) $ab\,x^2-c^2x-ab=0$ 4) $ab\,x^2-c^2x+ab=0$

7. If $1+\sin x+\sin^2 x+.....\text{to}\,\infty=4+2\sqrt{3}, 0<x<\pi$ **and** $x\neq\frac{\pi}{2}$ **then** $x=$
   1) 30°, 60° 2) 60°,120° 3) 90°, 120° 4) 30°, 45°

8. $f(x)=\sin^2 x+\operatorname{cosec}^2 x\Rightarrow$
   1) $f(x)<1$ 2) $f(x)=1$
   3) $1<f(x)<2$ 4) $f(x)\geq 2$

9. If $e^{\left(\sin^2 x+\sin^4 x+\sin^6 x+.....\infty\right)\log 2}=8$ and $0<x<\frac{\pi}{2}$ then $\frac{\cos x}{\cos x+\sin x}=$
   1) $\frac{\sqrt{3}+1}{2}$ 2) $\frac{\sqrt{3}-1}{2}$ 3) $\frac{2}{\sqrt{3}+1}$ 4) $\frac{2}{\sqrt{3}-1}$

10. **If** $\sin\beta$ **is the G.M. between** $\sin\alpha$ **and** $\cos\alpha$ **then** $(\cos\alpha-\sin\alpha)^2-2\cos^2\beta=$
    1) 0 2) 1 3) 2 4) -1

11. **If** $\tan^2\alpha\tan^2\beta+\tan^2\beta\tan^2\gamma+\tan^2\gamma\tan^2\alpha+2\tan^2\alpha\tan^2\beta\tan^2\gamma=1$ **then the value of** $\sin^2\alpha+sin^2\beta+\sin^2\gamma=$
    1) 0 2) -1 3) 1 4) 2

12. **Let** $\theta\in\left(0,\frac{\pi}{4}\right)$ **and** $t_1=(\tan\theta)^{\tan\theta}$, $t_2=(\tan\theta)^{\cot\theta}$, $t_3=(\cot\theta)^{\tan\theta}$, $t_4=(\cot\theta)^{\cot\theta}$ **then**
    1) $t_1>t_2>t_3>t_4$ 2) $t_4>t_3>t_1>t_2$
    3) $t_3>t_1>t_2>t_4$ 4) $t_2>t_3>t_1>t_4$

13 **Two arcs of same length of two different circles** subtended angles of 25° **and 75° at their centres respectively. Then the ratio of the radii of the circles is**
   1) 3 : 1 2) 1 : 3 3) 1 : 2 4) 2 : 1

14. **If for real values of x,** $\cos\theta=x+\frac{1}{x}$**, then**
    1) θ is an acute angle 2) θ is right angle
    3) θ is an obtuse angle
    4) no value of θ is possible

15. **In** $\Delta ABC, \tan(A-B-C)=$
    1) sin 2A 2) 1 3) tan 2A 4) 0

16. **If ABCD is a quadrilateral then** $\tan\left(\frac{A+B}{4}\right)=$
    1) $\cos\left(\frac{C-D}{4}\right)$ 2) $\cot\left(\frac{C-D}{4}\right)$
    3) $\cos\left(\frac{C+D}{4}\right)$ 4) $\cot\left(\frac{C+D}{4}\right)$

17. If $\tan\alpha+\cot\alpha=2$ then $\sqrt{\tan\alpha}+\sqrt{\cot\alpha}=$
    1) $\sqrt{2}$ 2) $2\sqrt{2}$ 3) 2 4) $4\sqrt{2}$

18. **If A, B, C are angles of a triangle such that A is obtuse then**
    1) tanA tan B > 1 2) tan B tan C <1
    3) tan C tan A > 1 4) tanA tan B tanC>1

19. **If cosA, sinA, cotA are in GP then** $\tan^6 A-\tan^2 A=$
    1) -1 2) 0 3) 1 4) 2

20. If $ABCD$ **is a cyclic quadrilateral such that** $12\tan A-5=0$ **and** $5\cos B+3=0$**, then** $\cos C\tan D=$
    1) $\frac{-16}{13}$ 2) $\frac{16}{13}$ 3) $\frac{-13}{16}$ 4) $\frac{23}{16}$

21. $1+\cos x+\cos^2 x+...$ **to** $\infty=4+2\sqrt{3}$ **then x =**
    1) 30° 2) 60° 3) 45° 4) 90°

22. **If** $e^{\left(1+\sin^2 x+\sin^4 x+\ldots\infty\right)\log 2}=16$**, then** $\tan^2 x=$
1) 1 2) 2 3) 3 4) 4

23. **If** $\sin\theta$ **and** $\cos\theta$ **are the roots of** $px^2+qx+r=0$ **then** $q^2-p^2=$
1) 0 2) $-2pr$ 3) $2qr$ 4) $2rp$

24. **If** $f_k(x)=\frac{1}{k}\left(\sin^k x+\cos^k x\right)$ **where** $x\in R, k\geq 1$ **then** $f_4(x)-f_6(x)=$
(JEE MAINS-2014)
1) $\frac{1}{6}$ 2) $\frac{1}{3}$ 3) $\frac{1}{9}$ 4) $\frac{1}{12}$

25. **The expression** $\frac{\tan A}{1-\cot A}+\frac{\cot A}{1-\tan A}$ **can be written as (JEE MAINS-2013)**
1) $\sin A\cos A+1$ 2) $\sec A\operatorname{cosec} A+1$
3) $\tan A+\cot A$ 4) $\sec A+\operatorname{cosec} A$

26. **In a** ΔPQR **, if 3 sin P + 4 cos Q = 6 and 4 sin Q + 3 cos P = 1 then the acute angle 'R' is equal to (AIEEE-2012)**
1) $\frac{\pi}{4}$ 2) $\frac{3\pi}{4}$ 3) $\frac{5\pi}{6}$ 4) $\frac{\pi}{6}$

27. $\sum_{r=1}^{10}\cos^3\frac{r\pi}{3}=$
A) $-\frac{1}{8}$ B) $-\frac{7}{8}$ C) $-\frac{9}{8}$ D) $\frac{1}{8}$

28. **Let** $P=\left\{\theta:\sin\theta-\cos\theta=\sqrt{2}\cos\theta\right\}$ **and** $Q=\left\{\theta:\sin\theta+\cos\theta=\sqrt{2}\sin\theta\right\}$ **be two sets. Then (IIT-2011)(JEE Mains-2016)**
(A) $P\subset Q$ and $P\neq\varnothing$ (B) $Q\not\subset P$
(C) $P\not\subset Q$ (D) $P=Q$

29. **For any** $\theta\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$**, the expression** $3(\sin\theta-\cos\theta)^4+6(\sin\theta+\cos\theta)^2+4\sin^6\theta$ **equals (JEE MAIN-2019)**
1) $13-4\cos^4\theta+2\sin^2\theta\cos^2\theta$
2) $13-4\cos^2\theta+6\sin^2\theta\cos^2\theta$
3) $13-4\cos^2\theta+6\cos^4\theta$
4) $13-4\cos^6\theta$

30. **.Let** $f_k(x)=\frac{1}{k}(\sin^k x+\cos^k x)$ **for k= 1,2,3.... Then for all** $x\in R$, **the value of** $f_4(x)-f_6(x)$ **is equal to [MAINS-2019]**

KEY

01)0.00	02) 0.00	03) 2.00	04) 1	05) 2
06) 4	07) 2	08) 4	09) 2	10) 4
11) 3	12) 2	13) 1	14) 4	15) 3
16) 4	17) 3	18) 2	19) 3	20) 1
21) 1	22) 3	23) 4	24) 4	25) 2
26) 4	27) 3	28) 4	29)4	30) 4

SOLUTIONS

1. $A=B=C=60^0$
2. $A+C=180^0$, $B+D=180^0$
3. Put $x=90^0$
4. $\cot A\cot B\cot C>0$
$\Rightarrow \cot A,\cot B,\cot C$ are positive
$\Rightarrow \Delta ABC$ is acute angled
5. $(2\tan\theta+2)^2=\tan\theta(3\tan\theta+3)$
$\Rightarrow \tan^2\theta+5\tan\theta+4=0\Rightarrow \tan\theta=-1,-4$
but $\tan\theta=-4$ only satisfies above condition
6. $\tan A=\frac{a}{b}$, $\tan B=\frac{b}{a}$ and $\left(a^2+b^2\right)=c^2$
Find sum and product of roots
7. Use formula $S_\infty=\frac{a}{1-r}$ for G.P
8. $f(x)=\left(\sin^2 x+\frac{1}{\sin^2 x}\right)\geq 2$
$\because\left(a+\frac{1}{a}\right)\geq 2$ for any real number a > 0
9. $\sin^2 x+\sin^4 x+\ldots\infty=\frac{\sin^2 x}{1-\sin^2 x}=\tan^2 x$
$\therefore\left(e^{\log_e 2}\right)^{\tan^2 x}=8\Rightarrow 2^{\tan^2 x}=2^3\Rightarrow \tan x=\sqrt{3}$
$\therefore$ G.E $=\frac{1}{1+\tan x}=\frac{1}{1+\sqrt{3}}=\frac{\sqrt{3}-1}{2}$
10. $\sin^2\beta=\sin\alpha\cos\alpha$
Cubing both sides we get the answer
11. Let $\tan^2\alpha=x,\tan^2\beta=y,\tan^2\gamma=z$
$\sin^2\alpha+\sin^2\beta+\sin^2\gamma=\sum\frac{\tan^2\alpha}{1+\tan^2\alpha}=\sum\frac{x}{1+x}$
$=\frac{(x+y+z)+(xy+yz+zx+2xyz)+xy+yz+zx+xyz}{(1+x)(1+y)(1+z)}$
$=\frac{1+x+y+z+xy+yz+zx+xyz}{(1+x)(1+y)(1+z)}=1$
12. In $\left(0,\frac{\pi}{4}\right)$, $\tan\theta<1$ and $\cot\theta>1$
$(\cot\theta)^{\cot\theta}>(\cot\theta)^{\tan\theta}>(\tan\theta)^{\tan\theta}>(\tan\theta)^{\cot\theta}$
$\therefore t_4>t_3>t_1>t_2$

13. Length of arc, $l = r\theta$;

$$\therefore \frac{r_1}{r_2} = \frac{l/\theta_1}{l/\theta_2} = \frac{\theta_2}{\theta_1} = \frac{75}{25} = \frac{3}{1}$$

14. $x + \frac{1}{x} \geq 2$ no value of θ is possible

15. $B + C = 180^\circ - A \Rightarrow \tan\left(A + A - 180^\circ\right) = \tan 2A$

16. $\left(A+B+C+D\right) = 360^\circ \Rightarrow \frac{A+B}{4} = \frac{\pi}{2} - \frac{C+D}{4}$

17. Put $\alpha = \frac{\pi}{4}$

18. A is obtuse $\Rightarrow B + C < 90^\circ \Rightarrow \tan B \tan C < 1$

19.. $\sin^2 A = \cos A.\cot A \quad \Rightarrow \sin^3 A = \cos^2 A$

$\Rightarrow \tan^2 A = \cos ecA$ and simplify

20. $A + C = 180^\circ, B + D = 180^\circ$ Given

$\tan A = \frac{5}{12} > 0 \therefore 0 < A < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < C < \pi$

$\cos B = \frac{-3}{5} < 0 \therefore \frac{\pi}{2} < B < \pi \Rightarrow 0 < D < \frac{\pi}{2}$

$\therefore \cos C = \frac{-12}{13}; \tan D = \frac{4}{3}$

21. Use formula $S_\infty = \frac{a}{1-r}$ for G.P

22. $\left(e^{\log 2}\right)^{\frac{1}{1-\sin^2 x}} = 16 \Rightarrow 2^{\frac{1}{\cos^2 x}} = 2^4$

$\Rightarrow \sec^2 x = 4 \Rightarrow \tan^2 x = 3$

23. $\sin\theta + \cos\theta = \frac{-q}{p}, \sin\theta\cos\theta = \frac{r}{p}$

now $\left(\sin\theta + \cos\theta\right)^2 = 1 + 2\sin\theta\cos\theta$

24. $f_4\left(x\right) = \frac{1}{4}\left[1 - 2\sin^2 x\cos^2 x\right]$ and

$f_6\left(x\right) = \frac{1}{6}\left[1 - 3\sin^2 x\cos^2 x\right]$

25. $$\frac{\frac{\sin A}{\cos A}}{1-\frac{\cos A}{\sin A}} + \frac{\frac{\cos A}{\sin A}}{1-\frac{\sin A}{\cos A}} = \frac{\sin^2 A}{\cos A\left(\sin A - \cos A\right)} + \frac{\cos^2 A}{\sin A\left(\cos A - \sin A\right)}$$

$$= \frac{1 + \sin A\cos A}{\sin A\cos A} = \sec A\cos ecA + 1$$

26. Squaring both the equations and adding
27. Use Standard angles.
28. In set P,

$\sin\theta = \left(\sqrt{2}+1\right)\cos\theta \Rightarrow \tan\theta = \sqrt{2}+1$

In set Q, $\left(\sqrt{2}-1\right)\sin\theta = \cos\theta$

$\Rightarrow \tan\theta = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1 \Rightarrow \mathrm{P} = \mathrm{Q}$

29. $3(\sin\theta - \cos\theta)^4 + 6(\sin\theta + \cos\theta)^2 + 4\sin^6\theta$

$= 3(1 - 2\sin\theta\cos\theta)^2 + 6(1 + 2\sin\theta\cos\theta) + 4(1 - \cos^2\theta)^3$

$= 13 - 4\cos^6\theta$

30. $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$

$f_4(x) - f_6(x) = \frac{1}{4}(\sin^4 x + \cos^4 x) - \frac{1}{6}(\sin^6 x + \cos^6 x)$

$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$

******

COMPOUND ANGLES

SYNOPSIS

✈ **Definitions and Formulae :**

The algebraic sum of two or more angles is called a compound angle. i.e., $A+B, A-B, A+B+C, A+B-C$, ...etc., are called compound angles.

I If A and B are any two angles then

i) sin (A+B) = sin A cos B + cos A sin B

ii) sin (A-B) = sin A cos B - cos A sin B

iii) cos (A+B) = cos A cos B - sin A sin B

iv) cos (A-B) = cos A cos B + sin A sin B

v) $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\,\tan B}$

vi) $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

vii) $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot B+\cot A}$

viii) $\cot(A-B)=\dfrac{\cot A\cot B+1}{\cot B-\cot A}$

II i) $\sin(A+B)+\sin(A-B)=2\sin A\cos B$

ii) $\sin(A+B)-\sin(A-B)=2\cos A\sin B$

iii) $\cos(A+B)+\cos(A-B)=2\cos A\cos B$

iv) $\cos(A+B)-\cos(A-B)=-2\sin A\sin B$

v) $\cos(A-B)-\cos(A+B)=2\sin A\sin B$

III i) sin (A+B) sin (A-B) = $\sin^2$A - $\sin^2$B

= $\cos^2$B - $\cos^2$ A

ii) cos (A+B) cos (A-B) = $\cos^2$A - $\sin^2$B

= $\cos^2$B - $\sin^2$A

iii) $\tan(A+B)\tan(A-B)=\dfrac{\tan^2 A-\tan^2 B}{1-\tan^2 A\tan^2 B}$

IV i) $\tan(45^o+\theta)=\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$

$=\cot(45^o-\theta)=\dfrac{1+\tan\theta}{1-\tan\theta}$

ii) $\tan(45^o-\theta)=\dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}$

$=\cot(45^o+\theta)=\dfrac{1-\tan\theta}{1+\tan\theta}$ and

iii) $\tan(45^o+\theta).\tan(45^o-\theta)=1$

EXERCISE - I

1. $\sin 11^o 19^| \cos 18^o 41^| + \cos 11^o 19^| \sin 18^o 41^| =$

1) 1 2) $\dfrac{\sqrt{3}}{2}$ 3) $\dfrac{1}{2}$ 4) 0

2. If $0<\theta<\frac{\pi}{2}$ **and** $2\sin\theta=\sqrt{3}\cos 10^o+\sin 10^o$ **then** $\theta=$

1) 70^o 2) 50^o 3) 60^o 4) 40^o

3. If $\tan 22^o+\tan 38^o-\sqrt{3}=k\tan 22^o\tan 38^o$ **then k =**

1) -1 2) $-\sqrt{3}$ 3) 0 4) 1

4. If $\dfrac{1-\tan 2^o\cot 62^o}{\tan 152^o-\cot 88^o}=k\sqrt{3}$ **then k =**

1) -1 2) $\dfrac{1}{2}$ 3) 1 4) $-\dfrac{1}{2}$

5. If $\tan(A+B)=p, \tan(A-B)=q$ **then** $\cot 2B=$

1) $\dfrac{1+pq}{p-q}$ 2) $\dfrac{1+pq}{p+q}$ 3) $\dfrac{1-pq}{p-q}$ 4) $\dfrac{1-pq}{p+q}$

6. If $\sin(\theta+\alpha)=\cos(\theta+\alpha)$ **then** $\tan\theta=$

1) $\frac{1+\tan\alpha}{1-\tan\alpha}$ 2) $\frac{1+\sin\alpha}{1-\sin\alpha}$ 3) $\frac{1-\tan\alpha}{1+\tan\alpha}$ 4) $\frac{1+\cos\alpha}{1-\cos\alpha}$

7. $\cos^2 52\frac{1}{2}^\circ-\sin^2 22\frac{1}{2}^\circ=$

1) $\frac{\sqrt{3}+1}{4\sqrt{2}}$ 2) $\frac{\sqrt{3}-1}{4\sqrt{2}}$ 3) $\frac{3+\sqrt{3}}{4\sqrt{2}}$ 4) $\frac{3-\sqrt{3}}{4\sqrt{2}}$

8. In $\Delta ABC,\ \sum\frac{\sin(A+B).\sin(A-B)}{\cos^2 A\cos^2 B}=$

1) 0 2) 1 3) 2 4) $\frac{1}{2}$

9. $\frac{\tan 80^\circ-\tan 10^\circ}{\tan 70^\circ}=$

1) 0 2) 1 3) 2 4) 3

10. if $\tan 20^\circ=p$ **then** $\frac{\tan 160^\circ-\tan 110^\circ}{1+\tan 160^\circ\tan 110^\circ}$

1) $\frac{2p}{1+p^2}$ 2) $\frac{1-p^2}{2p}$ 3) $\frac{1+p^2}{2p}$ 4) $\frac{2p}{1-p^2}$

11. $\frac{(1+\tan 33^\circ)(1+\tan 12^\circ)}{(1+\tan 18^\circ)(1+\tan 27^\circ)}=$

1) 1 2) 2 3) 4 4) $\sqrt{3}$

12. $\sin 20^\circ+\sin 40^\circ-\sin 80^\circ=$

1) -1 2) 1 3) 2 4) 0

13. $\cos 20^\circ+\cos 100^\circ+\cos 140^\circ=$

1) 0 2) -1 3) 1 4) 2

14. $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\frac{a+b}{a-b}\Rightarrow\frac{\tan\alpha}{\tan\beta}=$

1) a/b 2) b/a 3) $a/2b$ 4) $-a/b$

15. If $\sin A=\frac{1}{\sqrt{10}},\sin B,\ 0<A<B<\frac{\pi}{4}$ then $A+B=$

1) $\frac{\pi}{2}$ 2) $\frac{\pi}{3}$ 3) $\frac{\pi}{4}$ 4) π

16. if ABC are acute angles of triangle and if $\tan A=\frac{1}{2},\tan B=\frac{1}{5}\ \tan C=\frac{1}{8}$.Then $A+B+C=$

1) $\frac{\pi}{12}$ 2) $\frac{\pi}{6}$ 3) $\frac{\pi}{4}$ 4) $\frac{\pi}{2}$

KEY

01) 3 02) 1 03) 2 04) 1 05) 1 06) 3
07) 4 08) 1 09) 3 10) 2 11) 1 12) 4
13) 1 14) 1 15) 3 16) 3

SOLUTIONS

1. $\sin 11^\circ 19' \cos 18^\circ 41' + \cos 11^\circ 19' \sin 18^\circ 41'$

$=\sin(11^\circ 19'+18^\circ 41')\quad=\sin 30^\circ=\frac{1}{2}$

2. $\sin\theta=\frac{\sqrt{3}}{2}\cos 10^\circ+\frac{1}{2}\sin 10^\circ$

$=\sin(60^\circ+10^\circ)\quad=\sin 70^\circ$

3. $22^\circ+38^\circ=60^\circ$

$\Rightarrow\tan 60^\circ-\tan 22^\circ-\tan 38^\circ=\tan 60^\circ\tan 22^\circ\tan 38^\circ$

$\Rightarrow-\sqrt{3}\tan 22^\circ\tan 38^\circ=\tan 22^\circ+\tan 38^\circ-\sqrt{3}$

4. $\frac{1-\cot 88^\circ\cot 62^\circ}{-\cot 62^\circ-\cot 88^\circ}=\frac{\cot 88^\circ\cot 62^\circ-1}{\cot 88^\circ+\cot 62^\circ}$

$=\cot 150^\circ=-\sqrt{3}$

5. $\cot 2B=\cot[(A+B)-(A-B)]$

$=\frac{\frac{1}{p}\frac{1}{q}+1}{\frac{1}{q}-\frac{1}{p}}=\frac{1+pq}{p-q}$

6. $\tan(\theta+\alpha)=1\Rightarrow\tan\theta+\tan\alpha=1-\tan\theta\tan\alpha$

$\Rightarrow\tan\theta(1+\tan\alpha)=1-\tan\alpha$

$\Rightarrow\tan\theta=\frac{1-\tan\alpha}{1+\tan\alpha}$

7. $\cos\left(52\frac{1}{2}^\circ+22\frac{1}{2}^\circ\right)\cos\left(52\frac{1}{2}^\circ-22\frac{1}{2}^\circ\right)$

$= \cos 75^\circ \cos 30^\circ = \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\frac{\sqrt{3}}{2}$

8. $\sum \frac{\cos^2 B - \cos^2 A}{\cos^2 A . \cos^2 B} = 0$

9. $70^\circ + 10^\circ = 80^\circ$

$\Rightarrow \tan 80^\circ - \tan 70^\circ - \tan 10^\circ = \tan 10^\circ \tan 70^\circ \tan 80^\circ$

$\Rightarrow \tan 80^\circ - \tan 10^\circ = 2\tan 70^\circ$

10. $\frac{-\tan 20^\circ + \cot 20^\circ}{1+\tan 20^\circ \cot 20^\circ} = \frac{-p+\frac{1}{p}}{2} = \frac{1-p^2}{2p}$

11. $12^\circ + 33^\circ = 45^\circ \Rightarrow (1+\tan 12^\circ)(1+\tan 33^\circ) = 2$

12. $\sin 20^\circ + \sin(60^\circ - 20^\circ) - \sin(60^\circ + 20^\circ) = 0$

13. $\cos 20^\circ + \cos(120^\circ - 20^\circ) + \cos(120^\circ + 20^\circ) = 0$

14. $\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\sin(\alpha+\beta)-\sin(\alpha-\beta)} = \frac{a+b+a-b}{a+b-a+b}$

$\Rightarrow \frac{2\sin\alpha\cos\beta}{2\cos\alpha\sin\beta} = \frac{a}{b} \Rightarrow \frac{\tan\alpha}{\tan\beta} = \frac{a}{b}$

15. Given $\sin A = \frac{1}{\sqrt{10}}, \sin B = \frac{1}{\sqrt{5}}$

A,Bare lies in Q, and also A+B lies in Q

$\cos A = \sqrt{1-\sin^2 A}$ $\quad$ $\cos B = \sqrt{1-\sin^2 B}$

$= \frac{3}{\sqrt{10}}$ $\quad$ $= \frac{2}{\sqrt{5}}$

$A + B = \frac{\pi}{4}$

16. Given $\tan A = \frac{1}{2}, \tan\frac{B}{2} = \frac{1}{5}, \tan C = \frac{1}{8}$

Now

$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

$= \frac{\frac{1}{2}+\frac{1}{5}+\frac{1}{8}-\frac{1}{2}\frac{1}{5}\frac{1}{8}}{1-\frac{1}{2}-\frac{1}{5}-\frac{1}{5}\frac{1}{8}-\frac{1}{8}\frac{1}{2}} = 1$

$= \quad A + B + C = \frac{\pi}{4}$

EXERCISE II

1. If A, B are acute angles, $\sin A = \frac{4}{5}$, $\tan B = \frac{5}{12}$ **then** $\sin(A+B) =$

1) $\frac{36}{65}$ 2) $\frac{65}{56}$ 3) $\frac{65}{63}$ 4) $\frac{63}{65}$

2. In a ΔABC, **A is obtuse,** $\sin A = \frac{3}{5}$, $\sin B = \frac{5}{13}$ **then** $\sin C =$

1) $\frac{33}{65}$ 2) $\frac{16}{65}$ 3) $\frac{4}{5}$ 4) $\frac{12}{13}$

3. If $B, A+B$ **are acute angles,** $\sin(A+B) = \frac{12}{13}, \sin B = \frac{5}{13}$ **then** $\sin A =$

1) $-\frac{119}{169}$ 2) $\frac{119}{169}$ 3) $\frac{169}{119}$ 4) $-\frac{169}{119}$

4. If $\cos A = \frac{5}{13}, \tan B = \frac{-15}{8}$, $270^\circ < A < 360^\circ, 90^\circ < B < 180^\circ$, **then the quadrant to which A+B belongs is**

1) IV 2) III 3) II 4) I

5. If $x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right) = z\cos\left(\theta+\frac{4\pi}{3}\right)$ **then the value of** $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} =$

1) 1 2) 2 3) 0 4) $3\cos\theta$

6. If $\cos(A-B) = 3/5$ **and** $\tan A \tan B = 2$ **then which one of the following is true**

1) $\sin(A+B) = 1/5$ 2) $\sin(A+B) = -1/5$

3) $\cos(A-B) = 1/5$ 4) $\cos(A+B) = -1/5$

7. **If** $\cos\alpha+\cos\beta=0=\sin\alpha+\sin\beta$ **then** $\cos(\alpha-\beta)=$
1) 0 2) 1 3) -1 4) 2

8. $\cot A\cot B=2, \cos(A+B)=3/5 \Rightarrow \sin A.\sin B=$
1) $2/5$ 2) $1/5$ 3) $4/5$ 4) $3/5$

9. **If** $\sin x\cos y=\frac{1}{4}$ **and** $3\tan x=4\tan y$ **then** $\sin(x-y)=$
1) $\frac{1}{16}$ 2) $\frac{7}{16}$ 3) $\frac{3}{4}$ 4) $\frac{3}{16}$

10. **If** $\sin A+\sin B=\sqrt{3}(\cos B-\cos A)$ **then** $\sin 3A+\sin 3B=$
1) 0 2) 2 3) 1 4) -1

11. **If** $2\tan A+\cot A=\tan B$ **then** $\cot A+2\tan(A-B)=$
1) -1 2) 0 3) 1 4) $1/2$

12. **If** $\sin B=\frac{1}{5}.\sin(2A+B)$ **then** $\frac{\tan(A+B)}{\tan A}=$
1) $5/3$ 2) $2/3$ 3) $3/2$ 4) $3/5$

13. **If** $\tan\alpha, \tan\beta$ **are the roots of the equation** $x^2+px+q=0$ $(p\neq 0)$ **then**
1) $\cos(\alpha+\beta)=1-q$ 2) $\sin(\alpha+\beta)=-p$
3) $\tan(\alpha+\beta)=\frac{p}{q-1}$ 4) $\cot(\alpha+\beta)=\frac{p}{q-1}$

14. **If** $\alpha+\beta+\gamma=\frac{\pi}{2}$ **and** $\cot\alpha, \cot\beta, \cot\gamma$ **are in A.P then** $\cot\alpha.\cot\gamma=$
1) 1 2) 2 3) 3 4) 4

15. **In a** ΔABC **if** $\tan B=\frac{2\sin A\sin C}{\sin(A+C)}$ **then** $\tan A, \tan B, \tan C$ **are in**
1) A.P 2) G.P 3) H.P 4) A.G.P.

16. **If** $x^2+y^2+z^2=r^2$ **and** $\tan\alpha=\frac{xy}{zr}$, $\tan\beta=\frac{yz}{xr}, \tan\gamma=\frac{zx}{yr}$ **then** $\alpha+\beta+\gamma=$
1) $\frac{\pi}{4}$ 2) π 3) $\pi/2$ 4) $\pi/3$

17. **In** ΔABC **if** $\cot A+\cot B+\cot C=\sqrt{3}$ **then** ΔABC **is**
1) equilateral triangle 2) right angled triangle
3) isosceles 4) scalene triangle.

18. $\tan A+\tan B+\tan C-\frac{\sin(A+B+C)}{\cos A\cos B\cos C}=$
1) $\tan A\tan B\tan C$ 2) $\sin A\sin B\sin C$
3) $\cos A\cos B\cos C$ 4) $\sum \tan A\tan B$

19. **In a** ΔABC **, if** $\cos A\cos B\cos C=\frac{1}{3}$**, then the value of** $\tan A\tan B+\tan B\tan C+\tan C\tan A$ **is**
1) 1 2) 4/3 3) 4 4) 3

20. **If** $\tan A=1, \tan B=2, \tan C=3$ **then A+B+C =**
1) $\frac{n\pi}{2}, n\in z$ 2) $n\pi, n\in z$
3) $\frac{n\pi}{4}, n\in z$ 4) $\frac{2n\pi}{3}, n\in z$

21. **In a** ΔABC**,** $\tan A+\tan B+\tan C=6$ **and** $\tan A\ \tan B=2$ **then the triangle ABC is**
1) right angled isosceles 2) acute angled isosceles
3) acute angled 4) equilateral

22. $\cos(x-y)+\cos(y-z)+\cos(z-x)=-\frac{3}{2}$ $\Rightarrow \sum(\cos x)=$
1) 0 2) 1 3) 2 4) 3

23. **If** $\tan A=\frac{x\sin B}{1-x\cos B}$ **and** $\tan B=\frac{y\sin A}{1-y\cos A}$ **then** $\frac{\sin A}{\sin B}=$
1) x / y 2) y / x 3) $x+y$ 4) $x-y$

KEY

01) 4	02) 2	03) 2	04) 4	
05) 3	06) 4	07) 3	08) 4	09) 1
10)	11) 1	12) 2	13) 3	14) 3
15) 3	16) 3	17)	18) 3	19) 1
20) 1	21) 3	22) 2	23) 3	
31) 1	32) 1			

SOLUTIONS

1. $\sin(A+B)=\left(\frac{4}{5}\right)\left(\frac{12}{13}\right)+\left(\frac{3}{5}\right)\left(\frac{5}{13}\right)=\frac{63}{65}$

2. $A+B+C=180^\circ \Rightarrow A+B=180^\circ-C$

$\Rightarrow \sin C=\left(\frac{3}{5}\right)\left(\frac{12}{13}\right)+\left(\frac{-4}{5}\right)\left(\frac{5}{13}\right)=\frac{16}{65}$

3. $\sin A=\sin(A+B-B)$

$=\sin(A+B)\cos B-\cos(A+B)\sin B$

$=\frac{12}{13}\frac{12}{13}-\frac{5}{13}\frac{5}{13}=\frac{119}{169}$

4. Given $\cos A=\frac{5}{13}, \tan B=\frac{-15}{8}$

$27^o < A < 360, 90 < B < 180^0$

$\cos B=\frac{-8}{17}$

$\sin B=\frac{15}{17}$

$\sin A=\frac{-12}{13}$

Now $\sin(A+B)=\sin A\cos B+\cos A\sin B$

$=\frac{-12}{13}\frac{-8}{17}+\frac{5}{13}\frac{5}{17}=\frac{96+75}{221}>0$

$\cos(A+B)=\cos A\cos B-\sin A\sin B$

$=\frac{5}{13}\left(\frac{-8}{17}\right)-\frac{-12}{13}\frac{15}{17}=\frac{-40+180}{221}>0$

$A+B\in Q$

5. $x\cos\theta=y\cos\left(\theta+\frac{2\pi}{3}\right)=z\cos\left(\theta+\frac{4\pi}{3}\right)=k$

$\frac{k}{x}=\cos\theta.\frac{k}{y}=\cos\left(\theta+\frac{2\pi}{3}\right).\frac{k}{z}=\cos\left(\theta+\frac{4\pi}{3}\right)$

$\therefore k\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\cos\theta+\cos\left(\frac{2\pi}{3}+\theta\right)+\cos\left(\frac{4\pi}{3}+\theta\right)$

$=\cos\theta+\cos(120^\circ+\theta)+\cos(120^\circ-\theta)=0$

6. $\cos A\cos B+\sin A\sin B=3/5$ and

$\sin A\sin B=2\cos A\cos B$

$\Rightarrow \cos A\cos B=1/5, \sin A\sin B=2/5$

$\therefore \cos(A+B)=\frac{1}{5}-\frac{2}{5}=-1/5$

7. $\cos\alpha+\cos\beta=0$ ..... (1)

$\sin\alpha+\sin\beta=0$ ..... (2)

$(1)^2+(2)^2 \Rightarrow 2+2\cos(\alpha-\beta)=0$

$\therefore \cos(\alpha-\beta)=-1$

8. $\cos A\cos B=2\sin A\sin B$ and

$\cos(A+B)=\frac{3}{5}$

$\Rightarrow \cos A\cos B-\sin A\sin B=\frac{3}{5}$

$\Rightarrow 2\sin A\sin B-\sin A\sin B=3/5$

9. $\sin x\cos y=1/4$, $\frac{3\sin x}{\cos x}=\frac{4\sin y}{\cos y}$

$\Rightarrow 3\sin x\cos y=4\cos x\sin y$

$\Rightarrow \frac{3}{16}=\cos x\sin y$

$\therefore \sin(x-y)=\frac{1}{4}-\frac{3}{16}=\frac{1}{16}$

10. $\sin A+\sqrt{3}\cos A=\sqrt{3}\cos B-\sin B$

$\Rightarrow \frac{1}{2}\sin A+\frac{\sqrt{3}}{2}\cos A=\frac{\sqrt{3}}{2}\cos B-\frac{1}{2}\sin B$

$\Rightarrow \sin(A+60^\circ)=\sin(60^\circ-B)$

$A+60^\circ=60^\circ-B$, $\therefore \sin 3(-B)+\sin 3B=0$

11. $2\tan A+\cot A=\tan B$

$\therefore \cot A+2\left(\frac{\tan A-\tan B}{1+\tan A\tan B}\right)$

$$= \frac{\cot A + \tan B + 2\tan A - 2\tan B}{1 + \tan A \tan B}$$

$$= \frac{\cot A - \cot A}{1 + \tan A \tan B} = 0$$

12. Given $\sin B = \frac{1}{5}\sin(2A+B)$

$$\frac{\sin(2A+B)}{\sin B} = \frac{5}{1}$$

Apply component and divided to rule

$$\frac{\sin(2A+B)\sin B}{\sin(2A+B) - \sin B} = \frac{5+1}{5-1}$$

$$\Rightarrow \frac{2\sin(A+B)\cos A}{2\cos(A+B)\sin A} = \frac{3}{2}$$

$$\frac{\tan(A+B)}{\tan A} = \frac{3}{2}$$

$$\sin C + \sin B = 2\sin\frac{(C+D)}{2}\cos\frac{(C-B)}{2}$$

$$\sin C - \sin D = 2\cos\frac{(C+1)}{2}\sin\frac{(C-1)}{2}$$

13. $\tan\alpha + \tan\beta = -p, \tan\alpha\tan\beta = q$

$$\tan(\alpha+\beta) = \frac{-p}{1-q} = \frac{p}{q-1}$$

14. $\cot\alpha, \cot\beta, \cot\gamma$ are in A.P.

$\Rightarrow 2\cot\beta = \cot\alpha + \cot\gamma$

since $\alpha + \beta + \gamma = \pi/2 \Rightarrow \sum\cot\alpha = \Pi\cot\alpha$

$\Rightarrow \cot\alpha + \cot\beta + \cot\gamma = \cot\alpha\cot\beta\cot\gamma$

$\Rightarrow 3\cot\beta = \cot\alpha\cot\beta\cot\gamma$

15. $\dfrac{\sin A\cos C + \cos A\sin C}{\sin A\sin C} = \dfrac{2}{\tan B}$

$\Rightarrow \cot C + \cot A = 2\cot B$

$\therefore \cot A, \cot B, \cot C$ are in A.P

$\Rightarrow \tan A, \tan B, \tan C$ are in H.P.

16. put x = y = z

17. Take $A = B = C = \pi/3$

18. Take $A = B = C = \pi/3$

19. $\tan A\tan B + \tan B\tan C + \tan C\tan A$

$$= \frac{\sin A\sin B\cos C + \sin B\sin C\cos A + \sin A\sin C\cos B}{\cos A\cos B\cos C}$$

$$= \frac{-\cos(A+B+C) + \cos A\cos B\cos C}{\cos A\cos B\cos C}$$

$$= \frac{1+\frac{1}{3}}{\frac{1}{3}} = 4$$

20. $\tan(A+B+C) = \dfrac{1+2+3-1(2)(3)}{1-1.2-2.3-3.1} = 0$

$\therefore A+B+C = n\pi / n \in z$

21. Take $\tan A = 1,\ \tan B = 2,\ \tan C = 3$

$A = \pi/4,\ B < \pi/2,\ C < \pi/2$

$\therefore \Delta ABC$ is acute angled triangle.

22. $x - y = y - z = z - x = 120^\circ$

$y = x - 120^\circ,\ z = x + 120^\circ$

$\therefore \sum\cos x = \cos x + \cos(x - 120^\circ) + \cos(x + 120^\circ) = 0$

23. $\dfrac{\sin A}{\cos A} = \dfrac{x\sin B}{1 - x\cos B}$

$\sin A - x\sin A\cos B = x\cos A\sin B$

$\sin A = x\sin(A+B)$

similary $\sin B = y\sin(A+B)$

EXERCISE III

1. If $\tan\beta=\dfrac{n\tan\alpha}{1+(1-n)\tan^2\alpha}$ **then** $\tan(\alpha-\beta)=$

1) (1+n) $\tan\alpha$ 2) (1-n) $\tan\alpha$
3) - (1+n) $\tan\alpha$ 4) -(1-n) $\tan\alpha$

2. If $\sec\theta+\tan\theta=1,$ **then root of the equation** $(a-2b+c)x^2+(b-2c+a)x+(c-2a+b)=0$ **is**

1) $\sec\theta$ 2) $\tan\theta$ 3) $\sin\theta$ 4) $\cot\theta$

3. If $\tan\alpha,\tan\beta$ **are the roots of the equation** $x^2+px+q=0\left(p\neq 0\right)$ **then**

$\sin^2(\alpha+\beta)+p\sin(\alpha+\beta)\cos(\alpha+\beta)+q\cos^2(\alpha+\beta)=$

1) 0 2) 1 3) p 4) q

4. If $\tan\beta=2\sin\alpha\sin\gamma\, cosec(\alpha+\gamma)$, **then** $\cot\alpha,\cot\beta$ **and** $\cot\gamma$ **are in**

1) A.P. 2) G.P. 3) H.P 4) A.G.P

5. If an angle α **is divided into two parts A & B such that A - B = x and tan A : tan B = K : 1, then the value of sinx is**

1) $\dfrac{k+1}{k-1}\sin\alpha$ 2) $\dfrac{k}{k+1}\sin\alpha$

3) $\dfrac{k-1}{k+1}\sin\alpha$ 4) $\dfrac{k}{k-1}\sin\alpha$

6. If $\tan\left(\dfrac{\pi}{4}+\theta\right)+\tan\left(\dfrac{\pi}{4}-\theta\right)=a$

then $\tan^3\left(\dfrac{\pi}{4}+\theta\right)+\tan^3\left(\dfrac{\pi}{4}-\theta\right)=$

1) 0 2) a 3) $3a$ 4) a^3-3a

7. If $\dfrac{\cos x}{\cos y}=2$ **and** $\cos(x-y)=\dfrac{\sqrt{3}}{2}$ **then** $\tan y=$

1) $\sqrt{3}+4$ 2) $\sqrt{3}+1$ 3) $\sqrt{3}-4$ 4) $\sqrt{3}-1$

8. If $\dfrac{\cos x-\cos\alpha}{\cos x-\cos\beta}=\dfrac{\sin^2\alpha\cos\beta}{\sin^2\beta\cos\alpha}$ **then cos x =**

1) $\dfrac{\cos\alpha-\cos\beta}{1+\cos\alpha\cos\beta}$ 2) $\dfrac{\cos\alpha-\cos\beta}{1-\cos\alpha\cos\beta}$

3) $\dfrac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}$ 4) $\dfrac{\cos\alpha+\cos\beta}{1-\cos\alpha\cos\beta}$

KEY

01) 2 02) 1 03) 4
04) 1 05) 3 06) 4 17) 3 08) 3

SOLUTIONS

1. $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

$$=\frac{\tan\alpha-\dfrac{n\tan\alpha}{1+(1-n)\tan^2\alpha}}{1+\tan\alpha\dfrac{n\tan\alpha}{1+(1-n)\tan^2\alpha}}$$

$$=\frac{\tan\alpha+(1-n)\tan^3\alpha-n\tan\alpha}{1+(1-n)\tan^2\alpha+n\tan^2\alpha}=(1-n)\tan\alpha$$

2. $\sec\theta+\tan\theta=1$

$\theta=0$ is the solution of the above equation.

$\therefore \sec 0^{o}=1$ is a root of the equation

$(a-2b+c)x^2+(b-x+a)x+(c-2a+b)=0$

3. $\tan\alpha+\tan\beta=-p,\tan\alpha\tan B=q$

$\tan(\alpha+\beta)=\dfrac{p}{q-1}$

$\therefore \sin^2(\alpha+\beta)+p\sin(\alpha+\beta)\cos(\alpha+\beta)+2\cos^2(\alpha+\beta)$

$$=\frac{1}{\sec^2(\alpha+\beta)}\left[\tan^2(\alpha+\beta)+p\tan(\alpha+\beta)+q\right]$$

$$=\frac{1}{1+\dfrac{p^2}{(q-1)^2}}\left[\frac{p^2}{(q-1)^2}+\frac{p^2}{q-1}+q\right]=q$$

4. $\dfrac{\sin(\alpha+\gamma)}{\sin\alpha\sin\gamma}=\dfrac{2}{\tan\beta}$; $\cot\gamma+\cot\alpha=2\cot\beta$

$\therefore \cot\alpha,\cot\beta\cot\gamma$ are in A.P.

$\therefore \sec\theta$ in a root of the given equation.

5. $A+B=\alpha, A-B=x$

$\frac{\tan A}{\tan B}=\frac{k}{1} \Rightarrow \frac{\tan A+\tan B}{\tan A-\tan B}=\frac{k+1}{k-1}$

$\Rightarrow \frac{\sin(A+B)}{\sin(A-B)}=\frac{k+1}{k-1} \quad \Rightarrow \frac{\sin\alpha}{\sin x}=\frac{k+1}{k-1}$

6. $\tan^3\left(\frac{\pi}{4}+\theta\right)+\tan^3\left(\frac{\pi}{4}-\theta\right)$

$=\left[\tan\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right)\right]^3$

$-3\left[\tan\left(\frac{\pi}{4}+\theta\right)\tan\left(\frac{\pi}{4}-\theta\right)\right]\left[\tan\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right)\right]$

$=a^3-3(1)a=a^3-3a$

7. $\cos(x-y)=\frac{\sqrt{3}}{2} \Rightarrow x-y=\frac{\pi}{6}$

$\Rightarrow x=\frac{\pi}{6}+y \quad \therefore \cos x=2\cos y$

$\Rightarrow \frac{\sqrt{3}}{2}\cos y-\frac{1}{2}\sin y=2\cos y$

$\Rightarrow \left(\frac{\sqrt{3}-4}{2}\right)\cos y=\frac{1}{2}\sin y \Rightarrow \tan y=\sqrt{3}-4$

8. The given relation can be written as

$\cos x\left(\sin^2\beta\cos\alpha-\sin^2\alpha\cos\beta\right)$

$=\cos^2\alpha\sin^2\beta-\sin^2\alpha\cos^2\beta$

$\Rightarrow \cos x=\frac{\cos^2\alpha\left(1-\cos^2\beta\right)-\left(1-\cos^2\alpha\right)\cos^2\beta}{\cos\alpha\left(1-\cos^2\beta\right)-\cos\beta\left(1-\cos^2\alpha\right)}$

$=\frac{\cos^2\alpha-\cos^2\beta}{(\cos\alpha-\cos\beta)(1+\cos\alpha\cos\beta)}=\frac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}$

JEE MAINS, EAMCET QUESTIONS

1. **In** ΔABC, $\sum\frac{\sin(A+B).\sin(A-B)}{\cos^2 A\cos^2 B}=$

1) 0 2) 1 3) 2 4) 1/2

2. $A+B+C=180^{\circ}$ **then the value of** $\operatorname{cosec}A(\sin B\cos C+\cos B\sin C)=$

1) 1 2) -1 3) 0 4) 2

3. **In a** ΔABC, **A is obtuse,** $\sin A=\frac{3}{5}$, $\sin B=\frac{5}{13}$ **then** $\sin C=$

1) $\frac{33}{65}$ 2) $\frac{16}{65}$ 3) $\frac{4}{5}$ 4) $\frac{12}{13}$

4. **If** $\alpha+\beta+\gamma=\frac{\pi}{2}$ **and** $\cot\alpha,\cot\beta,\cot\gamma$ **are in A.P then** $\cot\alpha.\cot\gamma=$

1) 1 2) 2 3) 3 4) 4

5. **In a** ΔABC, $\sum\frac{\cos(B-C)}{\sin B\sin C}=$

6. **In a** ΔABC, $\tan A+\tan B+\tan C=6$ **and** $\tan A\tan B=2$ **then the triangle ABC is**

1) right angled isosceles

2) acute angled isosceles

3) acute angled 4) equilateral

7. $\tan A+\tan B+\tan C-\frac{\sin(A+B+C)}{\cos A\cos B\cos C}=$

1) $\tan A\tan B\tan C$ 2) $\sin A\sin B\sin C$

3) $\cos A\cos B\cos C$ 4) $\sum\tan A\tan B$

8. **In a** ΔABC, **if** $\sin A\sin B\sin C=\frac{1}{3}$ **then the valueof** $\cot A\cot B+\cot B\cot C+\cot C\cot A$ **is**

1) 1 2) 4/3 3) 4 4) 3

9. **Match the following:**

List -I	**List -II**
1. $\cot\left(\frac{\pi}{4}+\theta\right).\cot\left(\frac{\pi}{4}-\theta\right)$	a. 0
2. $\sin\left(45^0+\theta\right)-\cos\left(45^0-\theta\right)$	b. $\tan 56^0$
3. $\frac{\cos 11^0+\sin 11^0}{\cos 11^0-\sin 11^0}$	c. $\frac{\sqrt{3}}{2}$

1) 1-d, 2-a, 3-b,4-c 2) 1-a,2-b,3-c,4-d
3) 1-c,2-b,3-d,4-c 4) 1-b,2-c,3-a,4-d

10. Statement (I) : If $0 < \alpha, \beta < \frac{\pi}{4}$,

$\sin\alpha = \frac{a}{\sqrt{1+a^2}}, \quad \cos\beta = \frac{b}{\sqrt{1+b^2}}$

then $\tan(\alpha+\beta) = \frac{a+b}{a-b}$.

Statement (II) : If $\tan(A+B) = m$,

$\tan(A-B) = n$ **then** $\tan 2B = \frac{m-n}{m+n}$

1) Only I is true 2) Only II is true
3) Both I and II are true
4) Neither I nor II are true

11. Statement (I): If $A+B+C=\pi$ $(A,B,C>0)$ **and the angle C is obtuse then** $\tan A\tan B < 1$.

Statement (II): If A,B,C are acute positive angles such that $A+B+C=\pi$ **and** $\cot A\cot B\cot C = K$ **then** $K \le \frac{1}{3\sqrt{3}}$

Which of the above statements is correct?
1) Only I 2) Only II
3) Both I and II 4) Neither I nor II

12. If $0<\alpha,\beta<\frac{\pi}{4}$, $\cos(\alpha+\beta)=\frac{4}{5}$,

$\sin(\alpha-\beta)=\frac{5}{13}$ **then** $\tan 2\alpha =$

(AIEEE-2010)

1) $\frac{33}{56}$ 2) $\frac{56}{33}$ 3) $\frac{16}{63}$ 4) $\frac{14}{63}$

13. If tanA and tan B are the roots of the quadratic equation, $3x^2-10x-25=0$**, then the value of** $3\sin^2(A+B)-10\sin(A+B)$.
$\cos(A+B)-25\cos^2(A+B)$ **is:**
1) -10 2) 10 3) -25 4) 25

14. If $\sin^4\alpha+\cos^4\beta+2=4\sqrt{2}\sin\alpha\cos\beta$;
$\alpha,\beta\in[0,\pi]$**, then** $\cos(\alpha+\beta)-\cos(\alpha-\beta)$ **is equal to:**
(JEE MAIN-2019)
1) $-\sqrt{2}$ 2) 0 3) -1 4) $\sqrt{2}$

KEY

01) 1	02) 1	03) 2	04) 3	05) 4.00	
06) 3	07) 1	08) 1	09) 1	10) 4	11)
3	12) 2	13) 3	14) 1		

SOLUTIONS

1. $\sum\frac{\cos^2 B-\cos^2 A}{\cos^2 A.\cos^2 B}=\sum(\sec^2 A-\sec^2 B)=0$
2. $\operatorname{cosec}A\sin(B+C)=\operatorname{cosec}A\sin(\pi-A)=1$
3. $A+B+C=180^o \Rightarrow A+B=180^o-C$

   $\Rightarrow \sin C=\left(\frac{3}{5}\right)\left(\frac{12}{13}\right)+\left(\frac{-4}{5}\right)\left(\frac{5}{13}\right)=\frac{16}{65}$
4. $\cot\alpha,\cot\beta,\cot\gamma$ are in A.P.

   $\Rightarrow 2\cot\beta=\cot\alpha+\cot\gamma$

   since $\alpha+\beta+\gamma=\pi/2 \Rightarrow \sum\cot\alpha=\Pi\cot\alpha$

   $\Rightarrow \cot\alpha+\cot\beta+\cot\gamma=\cot\alpha\cot\beta\cot\gamma$

   $\Rightarrow 3\cot\beta=\cot\alpha\cot\beta\cot\gamma$
5. $\sum\frac{\cos B\cos C+\sin B\sin C}{\sin B\sin C}$

   $\sum\cot B\cot C+\sum 1 = 1+3=4$
6. Take $\tan A=1,\ \tan B=2,\ \tan C=3$

   $A=\pi/4,\ B<\pi/2,\ C<\pi/2$

   $\therefore \Delta ABC$ is acute angled triangle.
7. Take $A=B=C=\pi/3$
8. $\sum\frac{\cos A\cos B\sin C}{\sin A\sin B\sin C}=\frac{\sin(A+B+C)+\sin A\sin B\sin C}{\sin A\sin B\sin C}$

   $=\frac{0+\frac{1}{3}}{\frac{1}{3}}=1$
9. $\cot\left(\frac{\pi}{4}-\theta\right)=\tan\left(\frac{\pi}{4}+\theta\right)\ \cos(45^0-\theta)=\sin(45^0+\theta)$

   $\frac{\cos 11^0+\sin 11^0}{\cos 11^0-\sin 11^0}=\frac{1+\tan 11^0}{1-\tan 11^0}$
10. (I) : $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{a+b}{1-ab}$

    (II) : $\tan 2B=\tan[(A+B)-(A-B)]$

    $=\frac{\tan(A+B)-\tan(A-B)}{1+\tan(A+B)\tan(A-B)}=\frac{m-n}{1+mn}$

MULTIPLE & SUBMULTIPLE ANGLES

SYNOPSIS

- If A is an angle, then its integral multiples 2A,3A,4A,.... are called "Multiple angles of A.
- The multiples of A by fractions like $\frac{A}{2},\frac{A}{3},\ldots$ are called submultiple angles of A
- i) $\sin 2A = 2\sin A\cos A$

  ii) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$
  $= 1 - 2\sin^2 A$

  iii) $\tan 2A = \frac{2\tan A}{1-\tan^2 A}\left(A, 2A \neq (2n+1)\frac{\pi}{2}, n \in Z\right)$

  iv) $\cot 2A = \frac{\cot^2 A - 1}{2\cot A}\left(A \neq \frac{n\pi}{2}, n \in Z\right)$
- i)

  $\sin 2A = \frac{2\tan A}{1+\tan^2 A}\left(A \neq (2n+1)\frac{\pi}{2}, n \in Z\right)$

  ii)

  $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}\left(A \neq (2n+1)\frac{\pi}{2}, n \in Z\right)$
- i) $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

  ii) $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$
  $= 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2}$

  iii) $\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}\left(\frac{A}{2}, A \neq (2n+1)\frac{\pi}{2}, n \in Z\right)$

  iv) $\cot A = \frac{\cot^2\frac{A}{2} - 1}{2\cot\frac{A}{2}}\left(A \neq n\pi, n \in Z\right)$
- i) $\sin A = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}}\left(A \neq (2n+1)\pi, n \in Z\right)$

  ii) $\cos A = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}\left(A \neq (2n+1)\pi, n \in Z\right)$
- i) $\sin 3A = 3\sin A - 4\sin^3 A$

  ii) $\cos 3A = 4\cos^3 A - 3\cos A$

  iii) $\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$,

  $\left(A, 3A \neq (2n+1)\frac{\pi}{2}, n \in Z\right)$

  iv) $\cot 3A = \frac{3\cot A - \cot^3 A}{1 - 3\cot^2 A}, \left(A \neq \frac{n\pi}{3}, n \in Z\right)$
- i) $\sin A = \pm\sqrt{\frac{1-\cos 2A}{2}}$

  ii) $\cos A = \pm\sqrt{\frac{1+\cos 2A}{2}}$

  iii) $\tan A = \pm\sqrt{\frac{1-\cos 2A}{1+\cos 2A}}\left(A \neq (2n+1)\frac{\pi}{2}, n \in Z\right)$

  In above formulae, sign is based on quadrant.
- i) $\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$

  ii) $\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$

  iii) $\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}\left(A \neq (2n+1)\pi, n \in Z\right)$

  In above formulae, sign is based on quadrant.

✈ i) $\tan\left(\frac{\pi}{4}+\frac{A}{2}\right)=\dfrac{1+\tan\frac{A}{2}}{1-\tan\frac{A}{2}}$

$$=\frac{\cos\frac{A}{2}+\sin\frac{A}{2}}{\cos\frac{A}{2}-\sin\frac{A}{2}} = \sqrt{\frac{1+\sin A}{1-\sin A}}$$

$$=\frac{1+\sin A}{\cos A} = \frac{\cos A}{1-\sin A}$$

where $A \neq (4n+1)\frac{\pi}{2}, n \in Z$

ii) $\tan\left(\frac{\pi}{4}-\frac{A}{2}\right)=\dfrac{1-\tan\frac{A}{2}}{1+\tan\frac{A}{2}}$

$$=\frac{\cos\frac{A}{2}-\sin\frac{A}{2}}{\cos\frac{A}{2}+\sin\frac{A}{2}} = \sqrt{\frac{1-\sin A}{1+\sin A}}$$

$$=\frac{1-\sin A}{\cos A} = \frac{\cos A}{1+\sin A}$$

Where $A \neq (4n-1)\frac{\pi}{2}, n \in Z$

✈ **Standard results :**

i) $\sin^4\theta+\cos^4\theta = 1-2\sin^2\theta\cos^2\theta$

$$=1-\frac{1}{2}\sin^2 2\theta$$

ii) $\sin^6\theta+\cos^6\theta = 1-3\sin^2\theta\cos^2\theta$

$$=1-\frac{3}{4}\sin^2 2\theta$$

iii) $\cot A+\tan A = 2\operatorname{cosec} 2A \left(A \neq \frac{n\pi}{2}, n \in Z\right)$

iv) $\cot A - \tan A = 2\cot 2A \left(A \neq \frac{n\pi}{2}, n \in Z\right)$

v) $\tan\left(\frac{\pi}{4}+A\right)+\tan\left(\frac{\pi}{4}-A\right) = 2\sec 2A$

vi) $\tan\left(\frac{\pi}{4}+A\right)-\tan\left(\frac{\pi}{4}-A\right) = 2\tan 2A$

vii) $\tan A + 2\tan 2A + \ldots\ldots\ldots + 2^{n-1}\tan 2^{n-1}A + 2^n\cot 2^n A = \cot A$

(or)

$\cot A - \tan A - 2\tan 2A \ldots\ldots\ldots - 2^{n-1}\tan 2^{n-1}A$

$$= 2^n\cot 2^n A \left(A \neq \frac{n\pi}{2}, n \in Z\right)$$

✈ If $\alpha =$ 60°(or)120°(or)240°(or) 300° then

i) $\sin\theta.\sin(\alpha-\theta).\sin(\alpha+\theta) = \frac{1}{4}\sin 3\theta$

ii) $\cos\theta.\cos(\alpha-\theta).\cos(\alpha+\theta) = \frac{1}{4}\cos 3\theta$

iii) $\tan\theta.\tan(\alpha-\theta).\tan(\alpha+\theta) = \tan 3\theta$

where $\theta, 3\theta \neq (2n+1)\frac{\pi}{2}, n \in Z$

iv) $\cot\theta.\cot(\alpha-\theta).\cot(\alpha+\theta) = \cot 3\theta$

where $\theta \neq \frac{n\pi}{3}, n \in Z$

v) $\tan\theta + \tan(\theta-\alpha) + \tan(\theta+\alpha) = 3\tan 3\theta$

where $\theta, 3\theta \neq (2n+1)\frac{\pi}{2}, n \in Z$

vi) $\cot\theta + \cot(\theta-\alpha) + \cot(\theta+\alpha) = 3\cot 3\theta$

where $\theta \neq \frac{n\pi}{3}, n \in Z$

✈
$$\left.\begin{array}{l}\tan\theta + \tan(60^\circ+\theta) + \tan(120^\circ+\theta)\\ \tan\theta + \tan(120^\circ+\theta) + \tan(240^\circ+\theta)\\ \tan\theta + \tan(240^\circ+\theta) + \tan(300^\circ+\theta)\\ \tan\theta + \tan(300^\circ+\theta) + \tan(60^\circ+\theta)\end{array}\right\} = 3\tan 3\theta$$

where $\theta, 3\theta \neq (2n+1)\frac{\pi}{2}, n \in Z$

✈
$$\left.\begin{array}{l}\cot\theta + \cot(60^\circ+\theta) + \cot(120^\circ+\theta)\\ \cot\theta + \cot(120^\circ+\theta) + \cot(240^\circ+\theta)\\ \cot\theta + \cot(240^\circ+\theta) + \cot(300^\circ+\theta)\\ \cot\theta + \cot(60^\circ+\theta) + \cot(300^\circ+\theta)\end{array}\right\}$$

$$= 3\cot 3\theta$$

where $\theta \neq \frac{n\pi}{3}, n \in Z$

- $$\left.\begin{array}{l}\sin^2\theta+\sin^2(\alpha-\theta)+\sin^2(\alpha+\theta)\\ \cos^2\theta+\cos^2(\alpha-\theta)+\cos^2(\alpha+\theta)\end{array}\right\}=\frac{3}{2}$$
  where $\alpha = 60^\circ$ (or) 120° (or) 240° (or) 300°

- If $A+B=60^{\circ}$ then

  i) $\sin^2 A+\sin^2 B+\sin A\sin B=\frac{3}{4}$

  ii) $\cos^2 A+\cos^2 B-\cos A\cos B=\frac{3}{4}$

- If $A-B=60^{\circ}$ then

  i) $\sin^2 A+\sin^2 B-\sin A\sin B=\frac{3}{4}$

  ii) $\cos^2 A+\cos^2 B-\cos A\cos B=\frac{3}{4}$

- $$\left.\begin{array}{l}\sin^3\theta+\sin^3(60^\circ-\theta)-\sin^3(60^\circ+\theta)\\ \sin^3\theta-\sin^3(120^\circ-\theta)+\sin^3(120^\circ+\theta)\\ \sin^3\theta-\sin^3(240^\circ-\theta)+\sin^3(240^\circ+\theta)\\ \sin^3\theta+\sin^3(300^\circ-\theta)-\sin^3(300^\circ+\theta)\end{array}\right\}$$
  $$=-\frac{3}{4}\sin 3\theta$$

- $$\left.\begin{array}{l}\cos^3\theta-\cos^3(60^\circ-\theta)-\cos^3(60^\circ+\theta)\\ \cos^3\theta+\cos^3(120^\circ-\theta)+\cos^3(120^\circ+\theta)\\ \cos^3\theta+\cos^3(240^\circ-\theta)+\cos^3(240^\circ+\theta)\\ \cos^3\theta-\cos^3(300^\circ-\theta)-\cos^3(300^\circ+\theta)\end{array}\right\}$$
  $$=\frac{3}{4}\cos 3\theta$$

- $$\left.\begin{array}{l}\sin^3\theta-\sin^3(60^\circ+\theta)+\sin^3(120^\circ+\theta)\\ \sin^3\theta+\sin^3(120^\circ+\theta)+\sin^3(240^\circ+\theta)\\ \sin^3\theta+\sin^3(240^\circ+\theta)-\sin^3(300^\circ+\theta)\\ \sin^3\theta-\sin^3(60^\circ+\theta)-\sin^3(300^\circ+\theta)\end{array}\right\}$$
  $$=-\frac{3}{4}\sin 3\theta$$

- $$\left.\begin{array}{l}\cos^3\theta-\cos^3(60^\circ+\theta)+\cos^3(120^\circ+\theta)\\ \cos^3\theta+\cos^3(120^\circ+\theta)+\cos^3(240^\circ+\theta)\\ \cos^3\theta+\cos^3(240^\circ-\theta)-\cos^3(300^\circ+\theta)\\ \cos^3\theta-\cos^3(60^\circ+\theta)-\cos^3(300^\circ+\theta)\end{array}\right\}$$
  $$=\frac{3}{4}\cos 3\theta$$

- $$\sqrt{2+\sqrt{2+\sqrt{2+\ldots\ldots\sqrt{2+2\cos\theta}}}}=2\cos\left(\frac{\theta}{2^n}\right)$$
  Where 'n' is the number of square roots and $0<\theta<\pi$

- $$\sqrt{2+\sqrt{2+\sqrt{2+\ldots\ldots}}}=2\cos\left(\frac{\pi}{2^{n+1}}\right)$$
  Where 'n' is the number of square roots

- $$\cos\theta.\cos 2\theta.\cos 2^2\theta\ldots\ldots\cos 2^{n-1}\theta=\frac{\sin 2^n\theta}{2^n\sin\theta}$$
  Where $\theta\neq n\pi$

- $$\cos\left(\frac{\pi}{2n+1}\right)\cos\left(\frac{2\pi}{2n+1}\right)\cos\left(\frac{3\pi}{2n+1}\right)\cos\left(\frac{4\pi}{2n+1}\right)\ldots\ldots\cos\left(\frac{n\pi}{2n+1}\right)=\frac{1}{2^n}$$

- $$(1+\sec 2\theta)(1+\sec 2^2\theta)\ldots\ldots(1+\sec 2^n\theta)=\frac{\tan(2^n\theta)}{\tan\theta}$$

- i) $\sqrt{1+\sin 2A}=|\cos A+\sin A|$

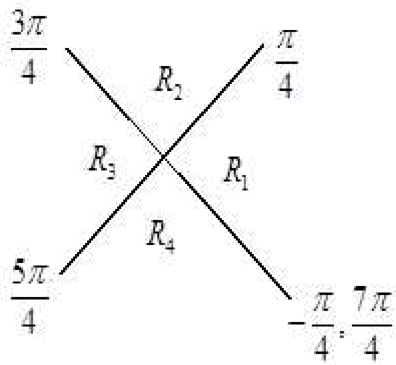
  ii) $\sqrt{1+\sin A}=\left|\cos\frac{A}{2}+\sin\frac{A}{2}\right|$

  iii) $\sqrt{1-\sin 2A}=|\cos A-\sin A|$

  iv) $\sqrt{1-\sin A}=\left|\cos\frac{A}{2}-\sin\frac{A}{2}\right|$

- Let $C=\cos\frac{A}{2}$; $S=\sin\frac{A}{2}$, then

  $C+S=\pm\sqrt{1+\sin A}$, $C-S=\pm\sqrt{1-\sin A}$



	+ve Regions	-ve Regions
C + s	R_1, R_2	R_3, R_4
C-s	R_1, R_4	R_2, R_3

EXERCISE - I

1. $\frac{\sin\theta - \sin 2\theta}{1-\cos\theta+\cos 2\theta} =$

1) $\tan\frac{\theta}{2}$ 2) $\cot\frac{\theta}{2}$ 3) $\tan\theta$ 4) $-\tan\theta$

2. $\frac{1-\tan^2 7\frac{1}{2}^{\circ}}{2\tan 7\frac{1}{2}^{\circ}} =$

1) $2-\sqrt{3}$ 2) $2+\sqrt{3}$ 3) $\sqrt{2}-1$ 4) $\sqrt{2}+1$

3. $\frac{\sin 12A}{\sin 4A} - \frac{\cos 12A}{\cos 4A} =$

1) 6 2) 4 3) 2 4) 1

4. **$4\cos^3 40^0 - 3\sin 50^0 =$**

1) $\frac{1}{2}$ 2) $\frac{1}{\sqrt{2}}$ 3) $\frac{-\sqrt{3}}{2}$ 4) $\frac{-1}{2}$

5. $\cos\theta = \frac{1}{2}\left(a+\frac{1}{a}\right)$ **then** $\cos 3\theta = K\left(a^3+\frac{1}{a^3}\right)$

where K is equal to

1) $\frac{1}{2}$ 2) $-\frac{1}{2}$ 3) 1 4) $\frac{3}{2}$

6. **If** $180^0 < \theta < 270^0, \sin\theta = -\frac{3}{5}$**, then** $\cos\frac{\theta}{2} =$

1) $\frac{-1}{\sqrt{10}}$ 2) $\frac{1}{\sqrt{10}}$ 3) $\frac{1}{10}$ 4) 10

7. $\sin^4\theta + \cos^4\theta = 1 - \frac{1}{2}f(\theta)$ then $f\left(\frac{\pi}{4}\right) =$

1) 1 2) 0 3) $\frac{1}{2}$ 4) $\frac{1}{4}$

8. $\tan A = \frac{1-\cos B}{\sin B} \Rightarrow \tan 2A - \tan B =$

1) 0 2) 1 3) 1/2 4) 1/4

9. $\cos^6 A + \sin^6 A = 1 - k\sin^2(2A) \Rightarrow k =$

1) $\frac{1}{4}$ 2) $\frac{1}{2}$ 3) $\frac{3}{4}$ 4) 1

10. $\sin 12^0 \sin 48^0 \sin 54^0 =$

1) $\frac{1}{2}$ 2) $\frac{1}{4}$ 3) $\frac{1}{8}$ 4) $\frac{1}{16}$

11. $\cos^2 25^0 + \cos^2 95^0 + \cos^2 145^0 =$

1) $\frac{1}{2}$ 2) $\frac{3}{2}$ 3) $\frac{3}{4}$ 4) $\frac{1}{\sqrt{2}}$

12. **The value of**

$\sin^2 46^0 + \sin^2 14^0 + \sin 46^0 \sin 14^0 =$

1) $\frac{1}{4}$ 2) $\frac{3}{4}$ 3) $\frac{5}{4}$ 4) $\frac{1}{2}$

13. $\cos^3 10^0 + \cos^3 110^0 + \cos^3 130^0 =$

1) $\frac{3}{4}$ 2) $\frac{3}{8}$ 3) $\frac{3\sqrt{3}}{8}$ 4) $\frac{3\sqrt{3}}{4}$

14. $\sin^3 10^\circ + \sin^3 250^\circ - \sin^3 310^\circ =$

1) $-\frac{3}{8}$ 2) $\frac{3}{8}$ 3) $\frac{1}{8}$ 4) $\frac{3}{4}$

15. $\theta < \frac{\pi}{16}$, $\sqrt{2+\sqrt{2+\sqrt{2+2\cos 8\theta}}} = k\cos\theta \Rightarrow k =$

1) 2 2) 4 3) 8 4) 16

16. $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}} =$

1) $\cos\frac{\pi}{32}$ 2) $2\cos\frac{\pi}{32}$ 3) $2\cos\frac{\pi}{64}$ 4) $\cos\frac{\pi}{64}$

17. $\frac{\sin x}{\sin x/8} =$

1) $8\cos\frac{x}{8}\cos\frac{x}{4}\cos\frac{x}{2}$ 2) $8\cos\frac{x}{8}\sin\frac{x}{4}\sin\frac{x}{2}$

3) $8\sin\frac{x}{8}\sin\frac{x}{4}\sin\frac{x}{2}$ 4) $8\sin\frac{x}{8}\sin\frac{x}{4}\cos\frac{x}{2}$

18. $\cos 36^\circ - \cos 72^\circ =$

1) 1 2) 0.5 3) $\frac{1}{4}$ 4) $\frac{1}{8}$

19. If $\pi < x < 2\pi$ then $\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}$ is

1) $\tan\left(\frac{\pi}{4}+\frac{1}{2}\right)$ 2) $\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$

3) $\tan\left(\frac{\pi}{4}+x\right)$ 4) $\tan\left(\frac{\pi}{4}-x\right)$

KEY

01) 4	02) 2	03) 3	04) 4	05) 1	06) 1
07) 1	08) 1	09) 3	10) 3	11) 2	12) 2
13) 3	14) 1	15) 1	16) 3	17) 1	18) 2
19) 2					

SOLUTIONS

1. Use, $\sin 2\theta = 2\sin\theta\cos\theta$ and

$\cos 2\theta = 2\cos^2\theta - 1$

$$\frac{\sin\theta - \sin 2\theta}{1-\cos\theta+\cos 2\theta} = \frac{\sin\theta - 2\sin\theta\cos\theta}{1-\cos\theta+2\cos^2\theta-1}$$

$$= \frac{\sin\theta(1-2\cos\theta)}{\cos\theta(2\cos\theta-1)}$$

$$= -\frac{\sin\theta(2\cos\theta-1)}{\cos\theta(2\cos\theta-1)} = -\tan\theta$$

2. $\frac{1-\tan^2\theta}{2\tan\theta} = \cot 2\theta$.

.

3. $$\frac{\sin 12A}{\sin 4A} - \frac{\cos 12A}{\cos 4A} = \frac{\sin 3(4A)}{\sin 4A} - \frac{\cos 3(4A)}{\cos 4A}$$

$$= \frac{3\sin 4A - 4\sin^3 4A}{\sin 4A} - \frac{(4\cos^3 4A - 3\cos 4A)}{\cos 4A}$$

$= 3 - 4\sin^2 4A - 4\cos^2 4A + 3$

$= 6 - 4(\sin^2 4A + \cos^2 4A)$

$= 6 - 4(1) = 2$

4. $4\cos^3 40^\circ - 3\sin 50^\circ \quad = 4\cos^3 40^\circ - 3\cos 40^\circ$

$= \cos 3(40^\circ) = \cos 120^\circ = -\frac{1}{2}$

5. Given $\cos\theta = \frac{1}{2}\left(a+\frac{1}{a}\right)$

$$\cos^3\theta = \frac{1}{8}\left(a+\frac{1}{a}\right)^3$$

Now $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$$= 4\frac{1}{8}\left(a+\frac{1}{a}\right)^3 - 3\frac{1}{2}\left(a+\frac{1}{a}\right)$$

$$= \frac{1}{2}\left(a+\frac{1}{a}\right)^3 - 3\left(a+\frac{1}{a}\right)$$

$$= \frac{1}{2}\left(a^3+\frac{1}{a^3}\right)$$

$k = \frac{1}{2}$

6. $\cos\theta = -\frac{4}{5}$; Use, $2\cos^2\frac{\theta}{2} = 1+\cos\theta$

7. $\sin^4\theta + \cos^4\theta = 1 - \frac{1}{2}.\sin^2 2\theta$

8. $$\tan A = \frac{1-\cos B}{\sin B} = \frac{2\sin^2 B/2}{2\sin\frac{b}{2}\cos\frac{b}{2}}$$

$\tan A = \tan\frac{B}{2}$

$A = \frac{B}{2}$

$2A = B$

Now $\tan 2A - \tan B = \tan B - \tan B = 0$

9. $\cos^6 A + \sin^6 A = 1 - \frac{3}{4}\sin^2 2A$

10. $\sin 12^0 \sin 48^o \sin 54^o$

multiply and divided by $\sin 72$

$$= \frac{\sin 12^o \sin 48^o \sin 72^o \sin 54^o}{\sin 72^o}$$

$$= \frac{\sin 12^o \sin(60-12^o)\sin(60+12^o)\sin 54^o}{\sin 72^o}$$

$$= \frac{1}{4}\frac{\sin 3(12^o)\sin 54^o}{\sin 72^o}$$

$$= \frac{1}{8}\frac{2\sin 36^o\cos 36^o}{\sin 72^o} = \frac{1}{8}\frac{\sin 72^o}{\sin 72}$$

$$= \frac{1}{8}$$

11. $\cos^2\theta + \cos^2(120^0-\theta) + \cos^2(120^0+\theta) = \frac{3}{2}$

12. $A+B=46^{o}+14^{o}=60^{o}$ then

$\sin^2 A+\sin^2 B+\sin A\sin B=\frac{3}{4}$

13. $\cos^3\theta+\cos^3(120^o-\theta)+\cos^3(120^o+\theta)=\frac{3}{4}\cos 3\theta$

14. Use, $\sin^3\theta+\sin^3(240^o+\theta)-\sin^3(300^o+\theta)=\frac{-3}{4}\sin 3\theta$

15. $n=3$, given $=2\cos\left(\frac{8\theta}{2^3}\right)=2\cos\theta$

16. $\sqrt{2+\sqrt{2+\sqrt{2+....nterms}}}=2\cos\left(\frac{\pi}{2^{n+1}}\right)$

17. $\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}\quad=4\sin\frac{x}{4}.\cos\frac{x}{4}.\cos\frac{x}{2}$

$=8\sin\frac{x}{8}.\cos\frac{x}{8}.\cos\frac{x}{4}\cos\frac{x}{2}$

18. $\cos 36^o-\cos 72^o$

$=\frac{\sqrt{5}+1}{4}-\frac{(\sqrt{5}-1)}{4}=\frac{\sqrt{5}+1-\sqrt{5}+1}{4}=\frac{2}{4}=\frac{1}{2}$

19. Given $\pi<x<2\pi$

Now $\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}=$

$=\frac{-\sqrt{2}\cos\frac{1}{2}+\sqrt{2}\sin\frac{x}{\varepsilon}}{-\sqrt{2}\cos\frac{x}{\varepsilon}-\sqrt{2}\sin\frac{x}{\varepsilon}}$

$=\frac{\cos\frac{x}{\varepsilon}-\sin\frac{x}{\in}}{\cos\frac{x}{\in}+\sin\frac{x}{\varepsilon}}=\frac{1-\tan\frac{x}{2}}{1+\tan\frac{2}{\varepsilon}}$

$=\tan\left(\frac{\pi}{4}-\frac{x}{\varepsilon}\right)$

EXERCISE - II

1. $\sqrt{3}\operatorname{cos}ec20^0-\sec 20^0=$ **(EAM-2008)**

1) 2 2) 3 3) 1 4) 4

2. $\tan 9^0-\tan 27^0-\tan 63^0+\tan 81^0=$ **(EAM-2014, 08)**

1) 0 2) 2 3) 1 4) 4

3. $\tan\theta=\frac{b}{a}\left(0<\theta<\frac{\pi}{4}\right)\Rightarrow\sqrt{\frac{a+b}{a-b}}+\sqrt{\frac{a-b}{a+b}}=$

1) $\frac{2\sin\theta}{\sqrt{\sin 2\theta}}$ 2) $\frac{2\cos\theta}{\sqrt{\cos 2\theta}}$

3) $\frac{2\cos\theta}{\sqrt{\sin 2\theta}}$ 4) $\frac{2\sin\theta}{\sqrt{\cos 2\theta}}$

4. The value of the expression

$\tan^6 20^o-33\tan^4 20^o+27\tan^2 20^o$ **is**

1) 0 2) 1 3) 2 4) 3

5. If $\theta\in Q_3$, then

$\sqrt{4\sin^4\theta+\sin^2 2\theta}+4\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)=$

1) 2 2) -2 3) 0 4) 1

6. If $\cos\alpha+\cos\beta=a$ **,** $\sin\alpha+\sin\beta=b$ **and**

$\alpha-\beta=2\theta$**, then** $\frac{\cos 3\theta}{\cos\theta}=$

1) a^2+b^2-2 2) a^2+b^2-3

3) $3-a^2-b^2$ 4) none

7. If $\sec\theta-\cos\theta=1$**, then** $\tan^2\frac{\theta}{2}=$

1) $\sqrt{5}+2$ 2) $\sqrt{5}-2$ 3) $2-\sqrt{5}$ 4) 0

8. $\cos^4\alpha-\sin^4\alpha=a$ **then** $\frac{1-a}{1+a}=$

1) $\tan^2\alpha$ 2) $\cot^2\alpha$ 3) $-\tan^2\alpha$ 4) $-\cot^2\alpha$

9. If $\tan\frac{\theta}{2}=\operatorname{cos}ec\theta-\sin\theta$**, then** $\cos^2\frac{\theta}{2}=$

1) $\frac{\sqrt{3}-1}{4}$ 2) $\frac{\sqrt{5}+1}{4}$ 3) $\frac{\sqrt{3}+1}{4}$ 4) $\frac{\sqrt{5}-1}{4}$

10. **If** $2\tan\frac{\alpha}{2}=\tan\frac{\beta}{2}$ **then** $\frac{3+5\cos\beta}{5+3\cos\beta}=$

1) $\cos\alpha$ 2) $\sin\alpha$ 3) $\tan\alpha$ 4) $\cot\alpha$

11. $\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}=$

1) $\frac{1}{2}$ 2) $\frac{1}{4}$ 3) $\frac{3}{2}$ 4) $\frac{3}{4}$

12. $\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11}=$

1) $\frac{1}{4}$ 2) $\frac{1}{8}$ 3) $\frac{1}{16}$ 4) $\frac{1}{32}$

13. **The quadratic equation whose roots are**

$\sin^2 18^0, \cos^2 36^0$

1) $16x^2-12x+1=0$ 2) $x^2-12x+1=0$

3) $16x^2-12x-1=0$ 4) $16x^2+12x+1=0$

14. If $\cos x=\tan y, \cot y=\tan z, \cot z=\tan x$ then $\sin x=$ (Eamcet - 2017)

KEY

01) 4 02) 4 03) 2 04) 4 05) 1 06) 2

07) 2 08) 1 09) 2 10) 1 11) 3 12) 4

13) 1 14) $2\sin 18^o$

SOLUTIONS

1. Given $\sqrt{3}\cos ec 20^o \sec 20^o$

$$=\frac{\sqrt{3}}{\sin 20}-\frac{1}{\cos 20}$$

$$=\frac{\sqrt{3}\cos 20^o-\sin 20^o}{\sin 20^o\cos 20^o}$$

$$=\frac{2\left(\frac{\sqrt{3}}{2}\cos 20-\frac{1}{2}\sin 20\right)}{\sin 20\cos 20}$$

$$=\frac{4(\sin 60\cos 20-\cos 60\sin 20)}{2\sin 20\cos 20}$$

$$=\frac{4\sin(60-20)}{\sin 2(20)}=\frac{4\sin 40}{\sin 40}=4$$

2. $\tan 9^o+\tan 81-\left(\tan 27^o+\tan 63^o\right)$

$$=\tan 9+\cot 9-(\tan 27+\cot 27)$$

$$=\frac{\sin 9}{\cos 9}+\frac{\cos 9}{\sin 9}-\left(\frac{\sin 27}{\cos 27}+\frac{\cos 27}{\sin 27}\right)$$

$$=\frac{1}{\sin 9\cos 9}-\left(\frac{1}{\sin 27\cos 27}\right)$$

$$=\frac{2}{\sin 8}-\frac{2}{\sin 54}$$

$$=\frac{8}{15-1}-\frac{8}{\sqrt{5+1}}=8\left(\frac{\sqrt{5}+1-\sqrt{5+1}}{4}\right)$$

$$=2(2)=4$$

3. $\sqrt{\frac{a+b}{a-b}}+\sqrt{\frac{a-b}{a+b}}=\frac{2a}{\sqrt{a^2-b^2}}=\frac{2}{\sqrt{1-\frac{b^2}{a^2}}}$

4. $\tan 60^o=\tan 3(20^o)=\frac{3\tan 20^o-\tan^3 20^o}{1-3\tan^2 20^o}$

5. Given $\theta\in Q_3$

$$\sqrt{4\sin^4\theta+\sin^2 2\theta}+4\cos^2\left(\frac{\pi}{4}-\theta/2\right)$$

$$=\sqrt{4\sin^4\theta+4\sin^2\theta\cos^2\theta}+4^2\left(\frac{1+\cos 2\left(\frac{\pi}{4}{}^{\theta}_{\omega}\right)}{2}\right)$$

$$\therefore \cos^2\theta=\frac{1+\cos 2\theta}{2}$$

$$=2\sqrt{\sin^2\theta\left(\sin^2\theta+\cos^2\theta\right)}+2+2\cos\left(\frac{\pi}{2}-\theta\right)$$

$$=2|\sin\theta|+2+2\sin\theta$$

$$=-2\sin\theta+2+2\sin\theta \qquad \therefore \theta\in Q_3$$

$=2$

6. Given $\cos\alpha+\cos\beta=a$ (1)

$$\sin\alpha+\sin\beta=b \qquad (2)$$

Squaring and adding

$$(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2=a^2+b^2$$

$$\cos^2\alpha+\cos^2\beta+2\cos\alpha\cos\beta$$
$$+\sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta$$
$$=a^2+b^2$$

$$1+12\cos(\alpha-\beta)=a^2+b^2$$

$$2(1+\cos 2\theta)=a^2+b^2$$

$$4\cos^2\theta=a^2+b^2$$

Now $\dfrac{\cos 3\theta}{\cos\theta}=\dfrac{4\cos^3\theta-3\cos\theta}{\cos\theta}=4\cos^2\theta-3$

$$=a^2+b^2-3$$

7. Given $\sec\theta-\cos\theta=1$

$$\frac{1-\cos^2\theta}{\cos\theta}=1$$

$$1-\cos^2\theta=\cos\theta$$

$$\cos^2\theta+\cos\theta-1=0$$

$$\cos\theta=\frac{-1\pm\sqrt{+4}}{2}=\frac{\sqrt5-1}{2}$$

$$\tan^2\theta\varepsilon=\frac{1-\cos\theta}{1+\cos\theta}=\frac{1-\left(\dfrac{\sqrt5-1}{2}\right)}{1+\dfrac{\sqrt5}{2}-1}=\frac{3-\sqrt5}{1+\sqrt5}$$

$$\tan^2\theta\varepsilon=\frac{(3-\sqrt5)}{1+\sqrt5}\times\frac{(1+\sqrt5)}{1-\sqrt5}$$

$$=\frac{3-3\sqrt5-\sqrt5+5}{-4}$$

$$=\frac{8-4\sqrt5}{-4}=\sqrt5-2$$

8 $\cos^4\alpha-\sin^4\alpha=a$

$$(\cos^2\alpha-\sin^2\alpha)(\cos^2\alpha+\sin^2\alpha)=a$$

Now $\dfrac{1-a}{1+a}=\dfrac{1-\cos 2\alpha}{1+\cos 2\alpha}=\dfrac{2\sin^2\alpha}{2\cos^2\alpha}=\tan^2\alpha$

9. $\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}=\dfrac{1-\sin^2\theta}{\sin\theta} \Rightarrow \quad 2\sin^2\dfrac{\theta}{2}=\cos^2\theta$

$$\Rightarrow 2\left(1-\cos^2\frac{\theta}{2}\right)=\left(2\cos^2\frac{\theta}{2}-1\right)^2$$

$$\Rightarrow 4\cos^4\frac{\theta}{2}-2\cos^2\frac{\theta}{2}-1=0 \Rightarrow \cos\frac{\theta}{2}=\frac{\sqrt5+1}{4}$$

10. $4\tan^2\dfrac{\alpha}{2}=\tan^2\dfrac{\beta}{2}$, use $\cos\beta=\dfrac{1-\tan^2\beta/2}{1+\tan^2\beta/2}$

19.1Use, $\sin^4\theta+\cos^4\theta=1-\dfrac{1}{2}\sin^2 2\theta$

12. $\cos\theta.\cos 2\theta.\cos 2^2\theta..........\cos 2^{n-1}\theta=\dfrac{\sin 2^n\theta}{2^n\sin\theta}$

13. $\alpha=\sin^2 18^\circ$, $\beta=\cos^2 36^\circ$

$x^2-(\alpha+\beta)x+\alpha\beta=0$; α,β are roots

.

14. Gien $\cos x=\tan y\cot y=\tan z,\cot z=\tan x$

$$\cos x=\frac{1}{\cot y}$$

$$\cos x=\frac{1}{\tan z} \quad =\cot z$$

$$\cos x=\tan x$$

$$\cos^2 x=\sin x$$

$1-\sin^2 x=\sin x \qquad \sin^2 x+\sin x-1=0$

$$\sin x=\frac{-1\pm\sqrt{1+4}}{2(1)}=\frac{\sqrt5-1}{2}=2\frac{(\sqrt5-1)}{4}$$

$$=2\sin 18^o$$

EXERCISE - III

1. If $f\left(\frac{2\tan x}{1+\tan^2 x}\right)=$

$\frac{(\cos 2x+1)(\sec^2 x+2\tan x)}{2}$ **then** $f(4)=$

1) 1 2) 3 3) 0 4) 5

2. If $\sin\beta$ **is geometric mean between** $\sin\alpha$ **and** $\cos\alpha$**, then** $\cos 2\beta=$

1) $2\sin^2\left(\frac{\pi}{4}-\alpha\right)$ *or* $2\cos^2\left(\frac{\pi}{4}+\alpha\right)$

2) $2\sin^2\left(\frac{\pi}{3}-\alpha\right)$ *or* $2\cos^2\left(\frac{\pi}{3}+\alpha\right)$

3) $\sin^2\left(\frac{\pi}{4}-\alpha\right)$ *or* $\cos^2\left(\frac{\pi}{4}+\alpha\right)$

4) $\sin^2\left(\frac{\pi}{3}-\alpha\right)$ *or* $\cos^2\left(\frac{\pi}{3}+\alpha\right)$

3. If $\cos\alpha+\cos\beta=0=\sin\alpha+\sin\beta$, **then** $\cos 2\alpha+\cos 2\beta$ **is equal to**

1) $-2\sin(\alpha+\beta)$ 2) $-2\cos(\alpha+\beta)$

3) $2\sin(\alpha+\beta)$ 4) $2\cos(\alpha+\beta)$

4. If $\alpha\in Q_4$ **and** $\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}=\frac{-1}{2}$

then $\sin 2\alpha=$

1) $\frac{3\sqrt{7}}{8}$ 2) $\frac{3\sqrt{7}}{16}$ 3) $\frac{-3\sqrt{7}}{4}$ 4) $\frac{-3\sqrt{7}}{8}$

5. If $\frac{\tan 3A}{\tan A}=k$**, then** $\frac{\sin 3A}{\sin A}=$

1) $\frac{3k}{k-1}, k\in R$ 2) $\frac{2k}{k-1}, k\in\left(\frac{1}{3},3\right)$

3) $\frac{2k}{k-1}, k\notin\left(\frac{1}{3},3\right)$ 4) $\frac{k-1}{2k}, k\notin\left(\frac{1}{3},3\right)$

6. If $\alpha,\beta,\gamma,\delta$ **are the smallest positive angles in ascending order of magnitude which have their sines equal to the positive quantity K the value of**

$4\sin\left(\frac{\alpha}{2}\right)+3\sin\left(\frac{\beta}{2}\right)+2\sin\left(\frac{\gamma}{2}\right)+\sin\left(\frac{\delta}{2}\right)=$

1) $2\sqrt{1-K}$ 2) $2\sqrt{1+K}$ 3) $2\sqrt{K}$ 4) $\sqrt{K+1}$

7. If $\cos(\theta-\alpha),\cos\theta,\cos(\theta+\alpha)$ **are in H.P,**

then $\cos\theta\sec\frac{\alpha}{2}=$

1) $\pm\frac{1}{\sqrt{2}}$ 2) $\pm\sqrt{2}$ 3) ± 1 4) $\pm\frac{1}{2}$

8. If $\cos\theta_1=2\cos\theta_2$, **then**

$\tan\frac{\theta_1-\theta_2}{2}\tan\frac{\theta_1+\theta_2}{2}$ **is equal to**

1) $\frac{1}{3}$ 2) $-\frac{1}{3}$ 3) 1 4) -1

9. If $\cos\theta=\cos\alpha\cos\beta$, **then**

$\tan\frac{\theta+\alpha}{2}\tan\frac{\theta-\alpha}{2}$ **is equal to**

1) $\tan^2(\alpha/2)$ 2) $\tan^2(\beta/2)$

3) $\tan^2(\theta/2)$ 4) $\cot^2(\beta/2)$

10. The value of $\tan\frac{\pi}{16}+2\tan\frac{\pi}{8}+4$ is equal to

1) $\cot\frac{\pi}{8}$ 2) $\cot\frac{\pi}{16}$ 3) $\cot\frac{\pi}{16}-4$ 4) 0

11. $\cos\left\{\frac{2\pi}{2^{64}-1}\right\}\cos\left\{\frac{2^2\pi}{2^{64}-1}\right\}......\cos\left\{\frac{2^{64}\pi}{2^{64}-1}\right\}=$

1) $\frac{1}{16^{16}}$ 2) $\frac{1}{8^8}$ 3) $\frac{1}{32^{32}}$ 4) 0

KEY

01) 4 02) 1 03) 2 04) 4 05) 3 06) 2
07) 2 08) 2 09) 2 10) 2 11) 1

SOLUTIONS

1. $f\left(\frac{2\tan x}{1+\tan^2 x}\right)=\frac{2\cos^2 x\left(1+\tan x\right)^2}{2}$

$=\frac{1+\tan^2 x+2\tan x}{1+\tan^2 x}=1+\frac{2\tan x}{1+\tan^2 x}$

$\Rightarrow f\left(x\right)=1+x,$ whenever defined

$\therefore f\left(4\right)=5$

2. $\sin^2\beta=\sin\alpha\cos\alpha$,

$\cos 2\beta=1-2\sin\alpha\cos\alpha=(\sin\alpha-\cos\alpha)^2$

3. $\left(\cos\alpha+\cos\beta\right)^2-\left(\sin\alpha+\sin\beta\right)^2=0$

$\Rightarrow\left(\cos^2\alpha+\cos^2\beta+2\cos\alpha\cos\beta\right)-$

$\left(\sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta\right)=0$

$\Rightarrow\cos 2\alpha+\cos\beta=-2\cos\left(\alpha+\beta\right)$

4. Squaring on both sides

$\Rightarrow 1+\sin\alpha=\frac{1}{4}\Rightarrow\sin\alpha=\frac{-3}{4},\cos\alpha=\frac{\sqrt{7}}{4}$

5. $\frac{\tan 3A}{\tan A}=k\qquad\Rightarrow\tan^2 A=\frac{k-3}{3k-1}>0$

and $\frac{\sin 3A}{\sin A}=3-4\sin^2 A=3-\frac{4\tan^2 A}{1+\tan^2 A}$

6. $\beta=180^0-\alpha$, $\gamma=360^0+\alpha$, $\delta=540^0-\alpha$

then $4\sin\frac{\alpha}{2}+3\sin\frac{\beta}{2}+2\sin\frac{\gamma}{2}+\sin\frac{\delta}{2}$

$=4\sin\frac{\alpha}{2}+3\cos\frac{\alpha}{2}-2\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}$

$=2\sqrt{1+\sin\alpha}=2\sqrt{1+k}$

7. $\frac{1}{\cos(\theta-\alpha)},\frac{1}{\cos\theta},\frac{1}{\cos(\theta+\alpha)}$ are in A.P

$\frac{2}{\cos\theta}=\frac{1}{\cos(\theta-\alpha)}+\frac{1}{\cos(\theta+\alpha)}$ and simplify

8. $\tan\left(\frac{\theta_1-\theta_2}{2}\right)\tan\left(\frac{\theta_1+\theta_2}{2}\right)=$

$\frac{\sin\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right)}=\frac{\sin^2\frac{\theta_1}{2}-\sin^2\frac{\theta_2}{2}}{\cos^2\frac{\theta_1}{2}-\sin^2\frac{\theta_2}{2}}$

$=\frac{1-\cos\theta_1-\left(1-\cos\theta_2\right)}{1+\cos\theta_1-\left(1-\cos\theta_2\right)}=-\frac{1}{3}$

9. $\tan\frac{\theta+\alpha}{2}\tan\frac{\theta-\alpha}{2}$

$=\frac{\tan^2\left(\theta/2\right)-\tan^2\left(\alpha/2\right)}{1-\tan^2\left(\theta/2\right)\tan^2\left(\alpha/2\right)}$

$=\frac{\frac{1-\cos\theta}{1+\cos\theta}-\frac{1-\cos\alpha}{1+\cos\alpha}}{1-\frac{1-\cos\theta}{1+\cos\theta}.\frac{1-\cos\alpha}{1+\cos\alpha}}$

10. $\tan\theta=\cot\theta-2\cot 2\theta$, put $\theta=\frac{\pi}{16},\frac{\pi}{8}$

11. $\cos\theta.\cos 2\theta.\cos 2^2\theta....\cos 2^{n-1}\theta=\frac{\sin 2^n\theta}{2^n\sin\theta}$

JEEMAINS, EAMCET QUESTIONS

01. If $f_n(x)=$
$$\frac{\sin x}{\cos 3x}+\frac{\sin 3x}{\cos 3^2 x}+\frac{\sin 3^2 x}{\cos 3^3 x}+....+\frac{\sin 3^{n-1}x}{\cos 3^n x}$$
Then $f_2\left(\pi/4\right)+f_3\left(\pi/4\right)=$

02. Let $f_n(\theta)=\tan\frac{\theta}{2}(1+\sec\theta)(1+\sec 2\theta)(1+\sec 4\theta)$ $.....\left(1+\sec 2^n\theta\right)$**, then**
$$f_2\left(\frac{\pi}{16}\right)+f_3\left(\frac{\pi}{32}\right)+f_4\left(\frac{\pi}{64}\right)+f_5\left(\frac{\pi}{128}\right)=$$
1) 0 2) 2 3) 4 4) 8

03. $\cos 9^\circ-\sin 9^\circ=\frac{1}{2}\sqrt{5-\sqrt{k}}$**. Then** $k=$

04. $4\cos 36^0+\cot 7\frac{1}{2}^0=$

1) $1+\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{5}+\sqrt{6}$
2) $1+\sqrt{2}-\sqrt{3}+\sqrt{4}-\sqrt{5}+\sqrt{6}$
3) $1-\sqrt{2}+\sqrt{3}-\sqrt{4}+\sqrt{5}-\sqrt{6}$
4) $1+\sqrt{2}-\sqrt{3}+\sqrt{4}+\sqrt{5}-\sqrt{6}$

05. The acute angle of a rhombus whose side is a mean proportional between its diagonals is
1) 15° 2) 20° 3) 30° 4) 80°

06. $1+\operatorname{cosec}\frac{\pi}{4}+\operatorname{cosec}\frac{\pi}{8}+\operatorname{cosec}\frac{\pi}{16}=$

1) $\cot\frac{\pi}{8}$ 2) $\cot\frac{\pi}{16}$ 3) $\cot\frac{\pi}{32}$ 4) $\operatorname{cosec}^2\frac{\pi}{16}$

07. $\cos\left\{\frac{2\pi}{2^{64}-1}\right\}\cos\left\{\frac{2^2\pi}{2^{64}-1}\right\}......\cos\left\{\frac{2^{64}\pi}{2^{64}-1}\right\}=$

1) $\frac{1}{16^{16}}$ 2) $\frac{1}{8^8}$ 3) $\frac{1}{32^{32}}$ 4) $\frac{1}{64^{64}}$

08. Consider the following two statements:
Statement p: The value of sin 120^0 can be derived by taking $\theta=240^0$ **in the equation**
$$2\sin\frac{\theta}{2}=\sqrt{1+\sin\theta}-\sqrt{1-\sin\theta}$$
Statement q: The angles A,B,C and D of any quadrilateral ABCD satisfy the equation
$$\cos\left(\frac{1}{2}(A+C)\right)+\cos\left(\frac{1}{2}(B+D)\right)=0$$
Then the truth values of p and q are respectively
1) F,T 2) T,F 3) T,T 4) F,F

09. The value of
$\cos\frac{\pi}{2^2}.\cos\frac{\pi}{2^3}..........\cos\frac{\pi}{2^{10}}.\sin\frac{\pi}{2^{10}}$ **is**

1) $\frac{1}{256}$ 2) $\frac{1}{2}$ 3) $\frac{1}{1024}$ 4) $\frac{1}{512}$

10. $\frac{\sqrt{2}\sin\alpha}{\sqrt{1+\cos 2\alpha}}=\frac{1}{7}$ and $\sqrt{\frac{1-\cos 2\beta}{2}}=\frac{1}{\sqrt{10}},\alpha,\beta\in\left(0,\frac{\pi}{2}\right)$
then tan($\alpha+2\beta$) is equal to :(jee main-2020)

11. The value of
$\sin 10^0\sin 30^0\sin 50^0\sin 70^0$ **is**
(JEE MAIN - 2019)

1) $\frac{1}{36}$ 2) $\frac{1}{32}$ 3) $\frac{1}{16}$ 4) $\frac{1}{18}$

12. The value of
$\cos^2 10^0-\cos 10^0\cos 50^0+\cos^2 50^0$ **is**
(JEE MAIN - 2019)

1) $\frac{3}{2}\left(1+\cos 20^0\right)$ 2) $\frac{3}{4}+\cos 20^0$
3) $\frac{3}{2}$ 4) $\frac{3}{4}$

13. If $\cos(\alpha+\beta)=\frac{3}{5},\sin(\alpha-\beta)=\frac{5}{13}$ **and**
$0<\alpha,\beta<\frac{\pi}{4}$**, then** $\tan(2\alpha)$ **is equal to**
(JEE MAIN - 2019)

1) $\frac{63}{52}$ 2) $\frac{63}{16}$ 3) $\frac{21}{16}$ 4) $\frac{33}{52}$

KEY

01)-1.00	02) 3	03) 5.00	04) 1	05) 3
06) 3	07) 1	08) 1	09) 4	10) 1.00
11) 3	12) 4	13) 2		

SOLUTIONS

01. $f_2(x)+f_3(x)$

$$=\frac{1}{2}(\tan 9x-\tan x)+\frac{1}{2}(\tan 27x-\tan x)$$

$$f_2\left(\pi/4\right)+f_3\left(\pi/4\right)=\frac{1}{2}(1-1)+\frac{1}{2}(-1-1)=-1$$

02. $f_n(\theta)=\tan 2^n\theta$

03. $(\cos 9^o-\sin 9^o)^2=1-\sin 18^o$

04. use $\cot\frac{\theta}{2}=\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$, put $\theta=15^o$

05. Let $BC=a$ and $\angle ABC=\alpha$ given that $BC=\sqrt{AC\times BD}$.

06. Use, $\operatorname{cosec}\theta=\cot\frac{\theta}{2}-\cot\theta$

07. $\cos\theta.\cos 2\theta.\cos 2^2\theta....\cos 2^{n-1}\theta=\frac{\sin 2^n\theta}{2^n\sin\theta}$

08. For statement p:

$\theta=240^0$

$$2\sin\left(\frac{240^0}{2}\right)=\sqrt{1+\sin 240^0}-\sqrt{1-\sin 240^0}$$

$$2\sin 120^0=\sqrt{1-\frac{\sqrt{3}}{2}}-\sqrt{1+\frac{\sqrt{3}}{2}}$$

$$2\frac{\sqrt{3}}{2}=\sqrt{\frac{4-2\sqrt{3}}{4}}-\sqrt{\frac{4+2\sqrt{3}}{4}}$$

$$\sqrt{3}=\sqrt{\frac{(\sqrt{3}-1)^2}{4}}-\sqrt{\frac{(\sqrt{3}+1)^2}{4}}$$

$$\sqrt{3}=\frac{\sqrt{3}-1}{2}-\frac{\sqrt{3}+1}{2}$$ (Not equal value)

$\sqrt{3}\neq -1$ Therefore, statement p is false.

For statement q:

$$\cos\left(\frac{1}{2}(A+C)\right)+\cos\left(\frac{1}{2}(B+D)\right)=0$$

$$A+B+C+D=2\pi\Rightarrow\frac{A+C}{2}=\pi-\left(\frac{B+D}{2}\right)$$

$$\cos\frac{A+C}{2}+\cos\frac{B+D}{2}\cos\left(\pi-\frac{B+D}{2}\right)$$

$$+\cos\left(\frac{B+D}{2}\right)=0$$

Therefore, statement q is true.

09.. $\left(\left(\cos\theta\cos(2^1\theta).\cos(2^2\theta)\right)....... \cos(2^{n-1}\theta)=\frac{\sin 2^n\theta}{2^n\sin\theta}\right)$

$$=\left(\cos\frac{\pi}{2^2}\cos\frac{\pi}{2^3}.\cos\frac{\pi}{2^4}..........\cos\frac{\pi}{2^{10}}\right)\sin\frac{\pi}{2^{10}}$$

$$=\left(\cos\frac{\pi}{2^{10}}\cos\frac{\pi}{2^9}.\cos\frac{\pi}{2^8}..........\cos\frac{\pi}{2^2}\right)\sin\frac{\pi}{2^{10}}$$

$$=\frac{\sin\left(2^9.\frac{\pi}{2^{10}}\right)}{2^9\sin\left(\frac{\pi}{2^{10}}\right)}.\sin\left(\frac{\pi}{2^{10}}\right)=\frac{\sin\left(\frac{\pi}{2}\right)}{2^9}=\frac{1}{512}$$

10. $$\frac{\sqrt{2}\sin\alpha}{\sqrt{2}\cos\alpha}=\frac{1}{7}\Rightarrow\tan\alpha=\frac{1}{7}$$

$$\sin\beta=\frac{1}{\sqrt{10}}\Rightarrow\tan\beta=\frac{1}{3}\Rightarrow\tan 2\beta=\frac{3}{4}$$

$$\tan(\alpha+2\beta)=\frac{\tan\alpha+\tan 2\beta}{1-\tan\alpha\tan 2\beta}=1$$

11. $\frac{1}{2}\left[\sin(10^0)\sin(60^0-10^0)\sin(60^0+10^0)\right]$

12. If $A-B=60^0$ *then* $\cos^2 A+\cos^2 B-\cos A\cos B=\frac{3}{4}$

13. $\tan(2\alpha)=\tan\left[(\alpha+\beta)+(\alpha-\beta)\right]$

TRANSFORMATIONS

SYNOPSIS

- The sum or difference of the trigonometric ratios are transforms into their products is said to be transformation between trigonometric ratios and vice versa.
  i) Sin (A + B) + Sin (A - B)= 2 SinA CosB
  ii) Sin (A + B) - Sin (A - B) = 2 CosA Sin B
  iii) Cos (A + B) + Cos (A - B) = 2 CosA CosB
  iv) Cos (A - B) - Cos (A + B) = 2 Sin A Sin B

- i) $\text{Sin}C + \text{Sin}\,D = 2\,\text{Sin}\left(\frac{C+D}{2}\right)\text{Cos}\left(\frac{C-D}{2}\right)$

  ii) $\text{Sin}C - \text{Sin}\,D = 2\,\text{Cos}\left(\frac{C+D}{2}\right)\text{Sin}\left(\frac{C-D}{2}\right)$

  iii) $\text{Cos}C+\text{Cos}D=2\,\text{Cos}\left(\frac{C+D}{2}\right)\text{Cos}\left(\frac{C-D}{2}\right)$

  iv) $\text{Cos}C-\text{Cos}D = -2\text{Sin}\left(\frac{C+D}{2}\right)\text{Sin}\left(\frac{C-D}{2}\right)$

  $= 2\text{Sin}\left(\frac{C+D}{2}\right)\text{Sin}\left(\frac{D-C}{2}\right)$

- If $\cos x+\cos y=a, \sin x+\sin y=b$ then
  i) $\tan\frac{x+y}{2}=\frac{b}{a}$
  ii) $\sin(x+y)=\frac{2ab}{a^2+b^2}$
  iii) $\cos(x+y)=\frac{a^2-b^2}{a^2+b^2}$
  iv) $\tan(x+y)=\frac{2ab}{a^2-b^2}$

- If $\cos x-\cos y=a, \sin x-\sin y=b$ then
  i) $\tan\frac{x+y}{2}=-\frac{a}{b}$
  ii) $\sin(x+y)=\frac{-2ab}{a^2+b^2}$
  iii) $\cos(x+y)=\frac{b^2-a^2}{b^2+a^2}$
  iv) $\tan(x+y)=-\frac{2ab}{a^2-b^2}$

- If $\cos x-\cos y=a, \sin x+\sin y=b$ then
  i) $\tan\frac{x-y}{2}=-\frac{a}{b}$
  ii) $\sin(x-y)=-\frac{2ab}{a^2+b^2}$
  iii) $\cos(x-y)=\frac{b^2-a^2}{b^2+a^2}$
  iv) $\tan(x-y)=-\frac{2ab}{a^2-b^2}$

- If $\cos x+\cos y=a, \sin x-\sin y=b$ then
  i) $\tan\frac{x-y}{2}=\frac{b}{a}$
  ii) $\sin(x-y)=\frac{2ab}{a^2+b^2}$
  iii) $\cos(x-y)=\frac{a^2-b^2}{a^2+b^2}$
  iv) $\tan(x-y)=\frac{2ab}{a^2-b^2}$

- $\sin 9^\circ = \frac{1}{4}\left[\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}\right]$

  $= \sqrt{\frac{4-\sqrt{10+2\sqrt{5}}}{8}}$

  $= \frac{1}{4}\sqrt{8-2\sqrt{10+2\sqrt{5}}}$

  $= \cos 81^0$

- $\cos 9^\circ = \frac{1}{4}\left[\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}\right]$

  $= \sqrt{\frac{4+\sqrt{10+2\sqrt{5}}}{8}}$

  $= \frac{1}{4}\sqrt{8+2\sqrt{10+2\sqrt{5}}}$

  $= \sin 81^0$

✈ $\cos x.\cos 2x.\cos 4x.........\cos(2^n x)$

$$=\frac{1}{2^{n+1}}\frac{\sin\left(2^{n+1}x\right)}{\sin x}$$

✈ i) $\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+....\sin\left[\alpha+(n-1)\beta\right]$

$$=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left\{\alpha+\left(\frac{n-1}{2}\right)\beta\right\}$$

ii) $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+....\cos\left[\alpha+(n-1)\beta\right]$

$$=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left\{\alpha+\left(\frac{n-1}{2}\right)\beta\right\}$$

✈ $\forall x\in R$, $\tan x+\frac{1}{2}\tan\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}+\ ....$

$.....+\frac{1}{2^{n-1}}\tan\left(\frac{x}{2^{n-1}}\right)$

$=\frac{1}{2^{n-1}}\cot\left(\frac{x}{2^{n-1}}\right)-2\cot 2x$

✈ α, β are the solutions of $a\cos\theta+b\sin\theta=c$, then

i) $\tan\left(\frac{\alpha+\beta}{2}\right)=\frac{b}{a}$

ii) $\sin(\alpha+\beta)=\frac{2ab}{a^2+b^2}$

iii) $\cos(\alpha+\beta)=\frac{a^2-b^2}{a^2+b^2}$

iv) $\tan(\alpha+\beta)=\frac{2ab}{a^2-b^2}$

EXAMPLES

1. In ΔABC, $\tan A+\tan B+\tan C\geq 3\sqrt{3}$, where A,B,C are acute angles.

Sol.: In ΔABC,

$\tan A+\tan B+\tan C=\tan A\tan B\tan C$,

$A.M\geq G.M$

$$\Rightarrow\frac{\tan A+\tan B+\tan C}{3}\geq\sqrt[3]{\tan A\tan B\tan C}$$

$$\Rightarrow\tan A\tan B\tan C\geq 3\sqrt[3]{\tan A\tan B\tan C}$$

$$\Rightarrow\tan^2 A\tan^2 B\tan^2 C\geq 27$$

$$\Rightarrow\tan A\tan B\tan C\geq 3\sqrt{3}$$

$$\Rightarrow\tan A+\tan B+\tan C\geq 3\sqrt{3}$$

:

2. In ΔABC, prove that

$\cos A+\cos B+\cos C\leq\frac{3}{2}$.

Sol: Let $\cos A+\cos B+\cos C=x$

$$\Rightarrow 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+1-2\sin^2\frac{C}{2}=x$$

$$\Rightarrow 2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)+1-2\sin^2\frac{C}{2}=x$$

$$\Rightarrow 2\sin^2\frac{C}{2}-2\cos\left(\frac{A-B}{2}\right)\sin\frac{C}{2}+x-1=0$$

this is quadratic in $\sin\frac{C}{2}$ which is real. So, discriminant $D\geq 0$

$$4\cos^2\left(\frac{A-B}{2}\right)-4\times 2(x-1)\geq 0$$

$$\Rightarrow 2(x-1)\leq\cos^2\left(\frac{A-B}{2}\right)$$

$$\Rightarrow 2(x-1)\leq 1\ \Rightarrow x\leq\frac{3}{2}$$

Thus, $\cos A+\cos B+\cos C\leq\frac{3}{2}$

3. Find the least value of $\sec A+\sec B+\sec C$ in an acute angle triangle.

Sol.: In an acute angle triangle, $\sec A,\sec B,\sec C$ are positive.

Now $A.M.\geq H.M.$

$$\Rightarrow\frac{\sec A+\sec B+\sec C}{3}\geq\frac{3}{\cos A+\cos B+\cos C}$$

but in $\Delta ABC,\cos A+\cos B+\cos C\leq\frac{3}{2}$

$$\Rightarrow \frac{\sec A+\sec B+\sec C}{3} \geq 2$$

$$\Rightarrow \sec A+\sec B+\sec C \geq 6$$

4. In ΔABC, prove that

$$\operatorname{cosec}\frac{A}{2}+\operatorname{cosec}\frac{B}{2}+\operatorname{cosec}\frac{C}{2} \geq 6.$$

Sol: In ΔABC we know that

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leq \frac{1}{8}$$

Now $A.M \geq G.M$

$$\Rightarrow \frac{\operatorname{cosec}\frac{A}{2}+\operatorname{cosec}\frac{B}{2}+\operatorname{cosec}\frac{C}{2}}{3} \geq$$

$$\left(\operatorname{cosec}\frac{A}{2}\operatorname{cosec}\frac{B}{2}\operatorname{cosec}\frac{C}{2}\right)^{\frac{1}{3}}$$

$$\Rightarrow \frac{\operatorname{cosec}\frac{A}{2}+\operatorname{cosec}\frac{B}{2}+\operatorname{cosec}\frac{C}{2}}{3} \geq$$

$$\left(\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\right)^{\frac{1}{3}}$$

$$\Rightarrow \frac{\operatorname{cosec}\frac{A}{2}+\operatorname{cosec}\frac{B}{2}\operatorname{cosec}\frac{C}{2}}{3} \geq (8)^{\frac{1}{3}}$$

$$\Rightarrow \operatorname{cosec}\frac{A}{2}+\operatorname{cosec}\frac{B}{2}+\operatorname{cosec}\frac{C}{2} \geq 6$$

5. The value of $\cot 70^0 + 4\cos 70^0$ **is**

Sol: $\cot 70^0 + 4\cos 70^0 =$

$$\frac{\cos 70^0 + 4\sin 70^0 \cos 70^0}{\sin 70^0}$$

$$= \frac{\cos 70^0 + 2\sin 140^0}{\sin 70^0}$$

$$= \frac{\cos 70^0 + 2\sin\left(180^0 - 40^0\right)}{\sin 70^0}$$

$$= \frac{\sin 20^0 + \sin 40^0 + \sin 40^0}{\sin 70^0}$$

$$= \frac{2\sin 30^0 \cos 10^0 + \sin 40^0}{\sin 70^0}$$

$$= \frac{\sin 80^0 + \sin 40^0}{\sin 70^0} = \frac{2\sin 60^0 \cos 20^0}{\sin 70^0} = \sqrt{3}$$

6. The absolute value of the expression

$$\tan\frac{\pi}{16}+\tan\frac{5\pi}{16}+\tan\frac{9\pi}{16}+\tan\frac{13\pi}{16}$$ **is ___**

Sol: Let $\theta = \frac{\pi}{16} \Rightarrow 8\theta = \frac{\pi}{2}$

$y = \tan\theta + \tan 5\theta + \tan 9\theta + \tan 13\theta$

$\therefore y = \left(\tan\theta - \cot\theta\right) + \left(\tan 5\theta - \cot 5\theta\right)$ [as

$\tan 13\theta = \tan\left(8\theta + 5\theta\right) = -\cot 5\theta$ and

$\tan 9\theta = \tan\left(8\theta + \theta\right) = -\cot\theta$]

$= \left(\tan\theta - \cot\theta\right) + \left(\cot 3\theta - \tan 3\theta\right)$

$= 2\left(\cot 6\theta - \cot 2\theta\right)$

$$\Rightarrow y = 2\left[\frac{\cos 6\theta}{\sin 6\theta} - \frac{\cos 2\theta}{\sin 2\theta}\right]$$

$$= 2\left[\frac{\sin 2\theta\cos 6\theta - \cos 2\theta \sin 6\theta}{\sin 6\theta \sin 2\theta}\right]$$

$$= -2\left[\frac{\sin 4\theta}{\cos 2\theta \sin 2\theta}\right] = -4 \left(\because 6\theta = \frac{\pi}{2} - 2\theta\right)$$

Hence, absolute value $= 4$.

7. If $\alpha = \frac{2\pi}{7}$**, then**

$\tan\alpha\tan 2\alpha + \tan 2\alpha\tan 4\alpha + \tan 4\alpha\tan\alpha =$

Sol: $\alpha = \frac{2\pi}{7} \Rightarrow 7\alpha = 2\pi \Rightarrow \cos 7\alpha = 1$

$\tan\alpha\tan 2\alpha + \tan 2\alpha\tan 4\alpha + \tan 4\alpha\tan\alpha$

$$=\frac{\sin\alpha\sin 2\alpha}{\cos\alpha\cos 2\alpha}+\frac{\sin 2\alpha\sin 4\alpha}{\cos 2\alpha\cos 4\alpha}+\frac{\sin 4\alpha\sin\alpha}{\cos 4\alpha\cos\alpha}$$

$$=\frac{\cos\alpha\sin 2\alpha\sin 4\alpha+\sin\alpha\cos 2\alpha\sin 4\alpha+\sin\alpha\sin 2\alpha\cos 4\alpha}{\cos\alpha\cos 2\alpha\cos 4\alpha}$$

$$=\frac{\cos\alpha\cos 2\alpha\cos 4\alpha-\cos(\alpha+2\alpha+4\alpha)}{\cos\alpha\cos 2\alpha\cos 4\alpha}$$

$$=1-\frac{1}{\cos\alpha\cos 2\alpha\cos 4\alpha}=1-\frac{1}{\left(\frac{1}{8}\right)}=-7$$

EXERCISE - I

1. $\cos 48^0.\cos 12^0=$

1) $\frac{1-\sqrt{5}}{8}$ 2) $\frac{\sqrt{5}+3}{8}$

3) $\frac{\sqrt{5}-1}{8}$ 4) $\frac{\sqrt{5}+1}{8}$

2. $\cos 66^0+\sin 84^0=$

1) $\frac{\sqrt{15}-\sqrt{3}}{4}$ 2) $\frac{\sqrt{15}-3}{4}$

3) $\frac{\sqrt{15}+\sqrt{3}}{4}$ 4) $\frac{\sqrt{15}+3}{4}$

3. $\text{Sin}\frac{\theta}{2}.\text{Sin}\frac{7\theta}{2}+\text{Sin}\frac{3\theta}{2}.\text{Sin}\frac{11\theta}{2}-\sin 2\theta.\sin 5\theta$

1) 0 2) 1 3) -1 4) 2

4. $4\sin(420^0-\alpha)\cos(60^0+\alpha)=$

1) $\sqrt{3}-2\sin 2\alpha$ 2) $\sqrt{3}+2\sin 2\alpha$

3) $\sqrt{3}-2\cos 2\alpha$ 4) $\sqrt{3}+2\cos 2\alpha$

5. $\cos^2(45^0-\alpha)+\cos^2(15^0+\alpha)-\cos^2(15^0-\alpha)=$

1) 0 2) 1 3) $1/2$ 4) 2

6. $\cos 20^0.\cos 40^0.\cos 60^0.\cos 80^0=$

1) 3/16 2) 1/32 3) 1/16 4) 1/8

7. $\frac{\sin\alpha.\sin 3\alpha+\sin 3\alpha.\sin 7\alpha+\sin 5\alpha.\sin 15\alpha}{\sin\alpha.\cos 3\alpha+\sin 3\alpha\cos 7\alpha+\sin 5\alpha.\cos 15\alpha}=$

1) $\sin(11\alpha)$ 2) $\cot(11\alpha)$

3) $\cos(11\alpha)$ 4) $\tan(11\alpha)$

8. $\pi<\alpha-\beta<3\pi$, $\sin\alpha+\sin\beta=\frac{-21}{65}$,

$\cos\alpha+\cos\beta=\frac{-27}{65}\Rightarrow\cos\left(\frac{\alpha-\beta}{2}\right)=$

1) $\frac{-6}{65}$ 2) $\frac{-3}{\sqrt{130}}$ 3) $\frac{3}{\sqrt{130}}$ 4) $\frac{6}{65}$

9. $\cos x+\cos y=\frac{4}{5}$, $\cos x-\cos y=\frac{2}{7}$

$\Rightarrow 14\tan\left(\frac{x-y}{2}\right)+5\cot\left(\frac{x+y}{2}\right)=$

1) 0 2) 1/4 3) 5/4 4) 3/4

10. $\frac{\sin(x+y)}{\sin(x-y)}=\frac{a+b}{a-b}\Rightarrow\frac{\tan x}{\tan y}=$

1) b/a 2) a/b 3) 1 4) 0

11. The value of

$\left(\frac{\cos A+\cos B}{\sin A-\sin B}\right)^{2015}+\left(\frac{\sin A+\sin B}{\cos A-\cos B}\right)^{2015}=$

1) 0 2) $\cot^{2015}\left(\frac{A+B}{2}\right)$

3) $\cot^{2015}\left(\frac{A-B}{2}\right)$ 4) $2\tan^{2015}\left(\frac{A+B}{2}\right)$

12. $\cos 2B=\frac{\cos(A+C)}{\cos(A-C)}\Rightarrow\tan A,\tan B,\tan C$ are in

1) A.P. 2) H.P 3) G.P. 4) A.G.P

13. $A+B+C=180^0$

$\Rightarrow\cos 2A+\cos 2B+\cos 2C=$

1) $1-4\sin A\sin B\sin C$

2) $1+4\sin A\sin B\sin C$

3) $1+4\cos A\cos B\cos C$

4) $-1-4\cos A\cos B\cos C$

14. $A+B+C=180^0\Rightarrow$

$\cos^2 A+\cos^2 B+\cos^2 C=$

1) $1+2\cos A\cos B\cos C$

2) $1+2\sin A\sin B\sin C$

3) $1-2\cos A\cos B\cos C$

4) $1-2\sin A\sin B\sin C$

15. $A+B+C=0^0 \Rightarrow \sin A+\sin B+\sin C=$

1) $2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ 2) $-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

3) $4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ 4) $-4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

KEY

01) 2 02) 3 03) 1 04) 1 05) 3 06) 3

07) 4 08) 2 09) 1 10) 2 11) 1 12) 3

13) 4 14) 3 15) 4

SOLUTIONS

1. Multiply and divide by 2 and 2 CosA. CosB = Cos(A+B) + Cos(A-B)
2. Write Sin84^0=Cos6^0 and Apply CosC+CosD
3. $\sin\theta/2\sin\frac{7\theta}{2}+\sin\frac{3\theta}{2}\sin\frac{11\theta}{2}-\sin 2\theta\sin 5\theta$

   multiply and divided by 2

   $$=\frac{2\sin\theta\varepsilon\sin\frac{7\theta}{2}+2\sin\frac{3\theta}{2}\sin\frac{11\theta}{2}-2\sin 2\theta\sin 5\theta}{2}$$

   Apply $\therefore 2\sin A\sin B=\cos(A-B)-\cos(A+B)$

   $=0$

4. $4\sin(420^o-\alpha)\cos(60^o+\alpha)$

   $=2(2\sin(420-\alpha)\cos(60+\alpha))$

   $=2(2\sin(60-\alpha)\cos(60+\alpha))$

   $=2(\sin(60-\alpha+60+\alpha)+\sin(60-\alpha-60-\alpha))$

   $=2(\sin 120^o+\sin(-2\alpha))$

   $=2(\sin(180-60^o)-\sin 2\alpha)$

   $=2\left(\frac{\sqrt{3}}{2}-\sin 2\alpha\right)$

   $=\sqrt{3}-2\sin 2\alpha$

5. Put $\alpha=0^0$ and verify
6. Use Cosθ.Cos(60-θ).Cos(60+θ) = $\frac{1}{4}\cos 3\theta$
7. $\frac{\sin\alpha\sin 3\alpha}{\sin\alpha\cos 3\alpha}\quad\frac{3\alpha\sin 7\alpha+\sin 5\alpha\sin 15\alpha}{\sin 3\alpha\cos 7\alpha+\sin 5\alpha\cos 15\alpha}$

   multiply and divide by 2

   $$\frac{2\sin\alpha\sin 3\alpha+2\sin 3\alpha\sin 7\alpha+2\sin 5\alpha\sin 15\alpha}{2\sin\alpha\cos 3\alpha+2\sin 3\alpha\cos 7\alpha+2\sin 5\alpha\cos 15\alpha}$$

   apply $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

   $2\sin A\cos B=\sin(A+B)+\sin(A-B)$

   $$=\frac{\cos(2\alpha)-\cos 4\alpha+\cos 4\alpha-\cos 10\alpha+\cos 10\alpha-\cos 20\alpha}{\sin 4\alpha-\sin 2\alpha+\sin 10\alpha-\sin 4\alpha+\sin(20\alpha)-\sin 10\alpha}$$

   $$=\frac{\cos 2\alpha-\cos 20\alpha}{\sin 2\alpha-\sin 2\alpha}$$

   $$=\frac{-2\sin(11\alpha)\sin(-9\alpha)}{2\cos 11\alpha\quad\sin 9\alpha}$$

   $$=\frac{\sin 11\alpha\quad\sin 9\alpha}{\cos 11\alpha\quad\sin 9\alpha}=\tan 11\alpha$$

8. Given $\pi<\alpha-<3\pi$, $\sin\alpha+\sin\beta=\frac{-21}{65}$

   $\cos\alpha+\cos\beta=\frac{-27}{65}$

   Squaring and adding

   $$(\sin\alpha+\sin\beta)^2+(\cos\alpha+\beta)^2=\frac{(-21)^2+(-27)^2}{(65)^2}$$

   $\sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta+\cos^2$

   $\alpha+\cos^2\beta+2\cos\alpha\cos\beta$

   $=\frac{441+729}{65\times 65}\qquad=\frac{1170}{65\times 65}$

   $1+1+2\cos(\alpha-\beta)\qquad=\frac{234}{65\times 13}$

$$\left(1+\cos(\alpha+\beta)\right)=\frac{117}{65\times 13}$$

$$2\cos^2\frac{(\alpha-\beta)}{2}=\frac{9}{65} \qquad \cos^2\left(\frac{\alpha-\beta}{2}\right)=\frac{9}{130}$$

$$\cos\left(\frac{\alpha-\beta}{2}\right)=\frac{-3}{\sqrt{130}} \qquad \because \frac{\pi}{2}<\frac{\alpha-\beta}{2}<\frac{3\pi}{2}$$

9. Given $\cos x+\cos y=\frac{4}{5}, \cos x-\cos y=\frac{2}{7}$

$$2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)=\frac{4}{5} \qquad (1)$$

$$-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)=2/7 \qquad (2)$$

$$1/2 \Rightarrow -\cot\left(\frac{x+y}{2}\right)\cot\left(\frac{x-y}{2}\right)=\frac{4}{5}\times 7$$

$$=\frac{-\cot\left(\frac{x+y}{2}\right)}{\tan\left(\frac{x-y}{2}\right)}=14$$

$$\Rightarrow 5\cot\left(\frac{x+y}{2}\right)+14\tan\left(\frac{x-y}{2}\right)=0$$

10. Apply componendo and dividendo

11. If n is odd then $\left(\frac{\cos A+\cos B}{\sin A-\sin B}\right)^n+\left(\frac{\sin A+\sin B}{\cos A-\cos B}\right)^n=0$

12. Given $\frac{\cos 2\beta}{1}=\frac{\cos(A+C)}{\cos(A-C)}$

$$\frac{1}{\cos 2B}=\frac{\cos(A-C)}{\cos(A+B)}$$

Apply componend and divided rule

$$\frac{1+\cos 2B}{1-\cos 2B}=\frac{\cos(A-C)+\cos(A+C)}{\cos(A-C)-\cos(A+C)}$$

$$\Rightarrow\frac{2\cos^2\beta}{2\sin^1\beta}=\frac{2\cos A\cos C}{2\sin A\sin C}$$

$$\cot^2\beta=\cot A\cot C$$

$$\tan A\tan C=\tan^2 B$$

$\therefore \tan A, \tan B, \tan C$ are in G.P

Apply componendo and dividendo

13. Put A = B = C= 60^0 and verify

14. Put A= B = C= 60^0 and verify

15. Put A = B = 60^0 , C= -120^0 and verify

EXERCISE - II

1. $\frac{\cos 6\theta+6\cos 3\theta+15\cos 2\theta+10}{\cos 5\theta+5\cos 3\theta+10}=$

1) $\sin\theta$ 2) $\cos\theta$ 3) $2\sin\theta$ 4) $2\cos\theta$

2. $\frac{(\cos\alpha-\cos 3\alpha)(\sin 8\alpha+\sin 2\alpha)}{(\sin 5\alpha-\sin\alpha)(\cos 4\alpha-\cos 6\alpha)}=$

1) 1 2) -1 3) 2 4) -2

3. **In a Quadrilateral ABCD,**

$\cos A.\cos B+\sin C\sin D=$

1) $\cos C\cos D+\sin A\sin B$

2) $\cos C\cos D-\sin A\sin B$

3) $\sin C\sin D-\cos A\cos B$

4) $\sin A+\sin B+\sin C+\sin D$

4. $K=\sin\left(\frac{\pi}{18}\right)\sin\left(\frac{5\pi}{18}\right)\sin\left(\frac{7\pi}{18}\right) \Rightarrow K=$

1) 1/4 2) 1/6 3)1/8 4) 1/2

5. $\frac{\sin 3\theta-\sin\theta.\sin^2(2\theta)}{\sin\theta+\sin 2\theta.\cos\theta}=\cos x \quad \Rightarrow x=$

1) 4θ 2) 2θ 3) θ 4) 3θ

6. $\sin\alpha+\cos\alpha=m \Rightarrow \sin^6\alpha+\cos^6\alpha=$

1) $\frac{4+3(m^2-1)^2}{4}$ 2) $\frac{4-3(m^2-1)^2}{4}$

3) $\frac{3+4(m^2-1)^2}{4}$ 4) $\frac{4-3(m^2+1)^2}{4}$

7. If $\cos x+\cos y+\cos z=0$

$=\sin x+\sin y+\sin z$ then $\tan(x-y)=$

1) $\sqrt{3}$ 2) $-\sqrt{3}$ 3) $\frac{1}{\sqrt{3}}$ 4) $-\frac{1}{\sqrt{3}}$

8. $x\cos\theta = y\cos\left(\theta+\dfrac{2\pi}{3}\right) = z\cos\left(\theta+\dfrac{4\pi}{3}\right)$

$\Rightarrow xy+yz+zx=$

1) -1 2) 1 3) 0 4) 2

9. $x\tan\left(\theta-\dfrac{\pi}{6}\right) = y\tan\left(\theta+\dfrac{2\pi}{3}\right) \Rightarrow \dfrac{x+y}{x-y} =$

1) $\cos 2\theta$ 2) $2\cos 2\theta$ 3) $\sin 2\theta$ 4) $2\sin 2\theta$

10. If $E=\cos^2 7i+\cos^2 49^o+\cos 7i\cos 49^o$
Then the value of loE is equal to

1)7.5 2)2.5 3)1.5 4)10.5

KEY

01) 4 02) 1 03) 1 04) 3 05) 2 06) 2
07) 2 08) 3 09) 2 10) 1

SOLUTIONS

1. $\theta = 0^\circ$ and verify

2. $\dfrac{2\sin 2\alpha\sin\alpha 2\sin 5\alpha\cos 3\alpha}{2\cos 3\alpha\sin 2\alpha 2\sin 5\alpha\sin\alpha}=1$

3. Use A+B+C+D =360, (A+B) = [360-(C+D)]
cos(A+B) = cos(C+D)

4. $k=\sin 10^\circ\sin\left(60^\circ-10^\circ\right)\sin\left(60^\circ+10^\circ\right)$

$=\dfrac{1}{4}\sin 30^\circ=\dfrac{1}{8}$

5. $\dfrac{\sin 3\theta-\sin\theta\sin^2 2\theta}{\sin\theta+\sin 2\theta\cos\theta}=\cos x$

$\Rightarrow \dfrac{3\sin\theta-4\sin^3\theta-\sin\theta\left(4\sin^2\theta\cos\theta\right)}{\sin\theta+2\sin\theta\cos^2\theta}=\cos x$

$\dfrac{\sin\theta\left(3-4\sin^2\theta-4\sin^2\theta\cos^2\theta\right)}{\sin\theta\left(1+2\cos^2\theta\right)}=\cos x$

$\Rightarrow \dfrac{3-6\sin^2\theta+2\sin^2\theta-4\sin^2\theta\cos^2\theta}{1+2\left(1-\sin^2\theta\right)}=\cos x$

$\Rightarrow \dfrac{3\left(1-2\sin^2\theta\right)-2\sin^2\theta\left(2\cos^2\theta-1\right)}{3-2\sin^2\theta}=\cos x$

$\dfrac{\cos 2\theta\left(3-2\sin^2\theta\right)}{3-2\sin^2\theta}=\cos x$

$\cos x=\cos 2\theta$

$x=2\theta$

6. $\left(\sin\alpha+\cos\alpha\right)^2=m^2 \Rightarrow \sin\alpha\cos\alpha=\dfrac{m^2-1}{2}$

Write $\sin^6\alpha+\cos^6\alpha=\left(\sin^2\alpha\right)^3+\left(\cos^2\alpha\right)^3$

$=1-3\sin^2\alpha\cos^2\alpha$

7. Given $\cos x+\cos y+\cos z=0$

$\cos x+\cos y\pm-\cos z$ (1)

$\sin x+\sin y+\sin z=0$

$\sin x+\sin y=-\sin z$ (2)

(1) and (2) squaring and adding

$\left(\cos x+\cos y\right)^2+\left(\sin x+\sin y\right)^2=\cos^2 z+\sin^2 z$

$\cos^2 x+\cos^2 y+2\cos x\cos y+$
$\sin^2 x+\sin^2 y+2\sin x\sin y=1$

$1+1+2\left(\cos x\cos y+\sin x\sin y\right)=1$

$2\left(1+\cos(x-y)\right)=1$

$2(2\cos^2\left(\dfrac{x-y}{2}\right)=1$

$\cos^2\left(\dfrac{x-y}{2}\right)=\dfrac{1}{4}$

$\cos\left(\dfrac{x-y}{2}\right)=\dfrac{1}{2}$

$\dfrac{x-y}{2}=60^o$

Now $\tan\left(x-y\right)=\tan\left(120^o\right)=\tan\left(180-60^o\right)$

$=-\tan 60^o=-\sqrt{3}$

8. Given

$$x\cos\theta = y\cos\left(\theta+\frac{2\pi}{3}\right)$$

$$= z\cos\left(\theta+\frac{4\pi}{3}\right) = k(say)$$

Put $\theta = 0$

$$x = k, y\cos\left(\frac{2\pi}{3}\right) = K, Z\cos\left(\frac{4\pi}{3}\right) = k$$

$$z = \cos\left(\frac{4\pi}{3}\right) = k \quad z\cos\theta\frac{4\pi}{3} = k$$

$$xy + yz + zx = k(-2k) + (-2k)(-2k) + (-2k)K$$

$$= -2k^2 + 4k^2 - 2^2 \qquad =0$$

9. Given $x\tan\left(\theta-\frac{\pi}{6}\right) = y\tan\left(\theta+\frac{2\pi}{3}\right)$

$$\frac{x}{y} = \frac{\tan\left(\theta+\frac{2\pi}{3}\right)}{\tan\left(\theta-\frac{\pi}{6}\right)}$$

apply component divide do rule

$$\frac{x+y}{x-y} = \frac{\tan\left(\theta+2\pi/3\right)+\tan\left(\theta-\pi/6\right)}{\tan\left(\theta+\frac{2\pi}{3}\right)-\tan\left(\theta-\pi/6\right)}$$

$$\frac{x+y}{x-y} = \frac{\frac{\sin\left(\theta+2\pi/3\right)}{\cos\left(\theta+2\pi/3\right)}+\frac{\sin\left(\theta-\pi/6\right)}{\cos\left(\theta-\pi/6\right)}}{\frac{\sin\left(\theta+2\pi/3\right)}{\cos\left(\theta+2\pi/3\right)}-\frac{\sin\left(\theta-\pi/6\right)}{\cos\left(\theta-\pi/6\right)}}$$

$$\frac{x+y}{x-y} = \frac{\sin\left(\theta+2\pi/3\right)\cos\left(\theta-\pi/6\right)+\cos\left(\theta+\frac{2\pi}{3}\right)\sin\left(\theta-\pi/6\right)}{\sin\left(\theta+2\pi/3\right)\cos\left(\theta-\pi/6\right)-\cos\left(\theta+\frac{2\pi}{3}\right)\sin\left(\theta-\pi/6\right)}$$

$$= \frac{\sin\left(\theta+\frac{2\pi}{3}\theta-\pi/6\right)}{\sin\left(\theta+\frac{2\pi}{3}-\theta+4/6\right)}$$

$$= \frac{\sin(90+2\theta)}{\sin(150^o)} = \frac{\cos 2\theta}{\sin 30} = 2\cos 2\theta$$

$$\text{Now}\frac{x+y}{x-y} = 2\cos 2\theta$$

10:- Given $= \cos^2 7i + \cos^2 49 + \cos 7i\cos 49^o$

$$= \frac{1+\cos 142}{2} + \frac{1+\cos 98}{2} + \frac{1}{2}\left(2\cos 7i\cos 49^o\right)$$

$$= 1 + \frac{1}{2}\left(\cos 142 + \cos 98^o\right) + \frac{1}{2}\left(\cos 120^o + \cos 22\right)$$

$$= 1 + \frac{1}{2}(2\cos 120\cos 22) + \frac{1}{2}\left(-\frac{1}{2}+\cos 22\right)$$

$$= 1 - \frac{1}{2}\cos 22 - \frac{1}{4} + \frac{1}{2}\cos 22$$

$$E = \frac{3}{4}$$

$$10E = \frac{15}{2} = 7.5$$

EXERCISE - III

1. The value of

$\sin 47^0 + \sin 61^0 - \sin 11^0 - \sin 25^0$ **is**

1) $\sin 7^0$ 2) $\cos 7^0$ 3) $\sin 14^0$ 4) $\cos 14^0$

2. If A, B, C, D are four angles then

$\sin(A-B).\cos(A+B)+\sin(B-C)$

$.\cos(B+C)+\sin(C-D).\cos(C+D)$

$+\sin(D-A).\cos(D+A)=$

1) 0 2) 1

3) 4sinA.sinBsinCsinD

4) 4cosAcosBcosCcosD

3. The value of the expression

$$\frac{1-4\sin 10^0 \sin 70^0}{2\sin 10^0}$$

1) 1/2 2) 1 3) 2 4) 1/3

4. If $\theta_1, \theta_2, \theta_3, ...\theta_n$ **are in A.P. then**

$$\frac{\sin\theta_1 + \sin\theta_2 + ... + \sin\theta_n}{\cos\theta_1 + \cos\theta_2 + ... + \cos\theta_n} =$$

1) 0 2) $\tan(\theta_1 + \theta_n)$

3) $\tan\left(\frac{\theta_1 + \theta_n}{2}\right)$ 4) $\tan\left(\frac{\theta_1 - \theta_n}{2}\right)$

5. In a triangle ABC, $\cos A + \cos B + \cos C$

1) < 1 2) > 1

3) > 1but not > 2 4) >1 but not >3/2

6. The smallest positive value of x **(in degrees) for which**

$$\tan\left(x+100^0\right) = \tan\left(x+50^0\right).\tan x.\tan\left(x-50^0\right)$$

is

1) 30^0 2) 45^0 3) 55^0 4) 15^0

7. Let $f(n) = 2\cos nx, \forall n \in N,$ **then**

$f(1)f(n+1)-f(n)$ **is equal to**

1) $f(n+3)$ 2) $f(n+2)$

3) $f(n+1)f(2)$ 4) $f(n+2)f(2)$

KEY

01) 2 02) 1 03) 2 04) 5) 4 6) 1

7) 2

SOLUTIONS

1. The given expression

$= \left(\sin 47^0 + \sin 61^0\right) - \left(\sin 11^0 + \sin 25^0\right)$

$= 2\sin 54^0 \cos 7^0 - 2\sin 18^0 \cos 7^0$

$= 2\cos 7^0\left[\sin 54^0 - \sin 18^0\right]$

$= 2\cos 7^0.2\cos 36^0.\sin 18^0$

$= 2\cos 7^0.2.\frac{\sqrt{5}+1}{4} \times \frac{\sqrt{5}-1}{4} = \cos 7^0$

2. $\frac{1}{2}[2\sin(A-B)\cos(A+B)+2\sin(B-C)$

$\cos(B+C)+2\sin(C-D)\cos(C+D)$

$+2\sin(D-A)\cos(D+A)]=$

$\frac{1}{2}[\sin 2A - \sin 2B + \sin 2B - \sin 2C$

$+\sin 2C - \sin 2D + \sin 2D - \sin 2A] = 0$

3. $\frac{1-4\sin 10^0 \cos 20^0}{2\sin 10^0}$

$= \frac{1-2\left[\sin 30^0 - \sin 10^0\right]}{2\sin 10^0} = 1$

4. $\sin\theta_1 + \sin\theta_2 + ... + \sin\theta_n =$

$$\frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}.\sin\left(\frac{\theta_1+\theta_n}{2}\right)$$

$\cos\theta_1 + \cos\theta_2 + ... + \cos\theta_n =$

$$\frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}.\cos\left(\frac{\theta_1+\theta_n}{2}\right)$$

where β is the common difference of angle

5. $\cos A+\cos B+\cos C$

$= 1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} > 1$

as neither of of $\sin\frac{A}{2},\sin\frac{B}{2},\sin\frac{C}{2}$ is $-$ive or zero

again $\cos A+\cos B+\cos C$

$$=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$+1-2\sin^2\frac{C}{2}\le 2\sin\frac{C}{2}.1+1-2\sin^2\frac{C}{2}$$

$$1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\le\frac{3}{2}$$

6. Given

$$\tan(x+100)=\tan(x+50)\tan x\tan(x-50)$$

$$\Rightarrow\frac{\tan(x+100)}{\tan(x-50)}=\tan(x+50)\tan x$$

$$\Rightarrow\frac{\sin(x+100)\cos(x-50)}{\cos(x+100)\sin(x-50)}$$

$$=\frac{\sin(x+50)}{\cos(x+50)}-\frac{\sin x}{\cos x}$$

Apply componendo and dividendo rule

$$\frac{\sin(x+100)\cos(x-50^o)+\cos(x+100)\sin(x-50^0)}{\sin(x+100)\cos(x-50^o)-\cos(x+100)\sin(x-50^o)}$$

$$=\frac{\sin(x+50)\sin x+\cos(x+50)\cos x}{\sin(x+50)\sin x-\cos(x+50^o)\cos x}$$

$$\Rightarrow\frac{\sin(x+100+x-50^o)}{\sin(x+100-x+50^o)}=\frac{\cos(x+50-x)}{-\cos(x+50+x)}$$

$$\Rightarrow\frac{\sin(2x+50^o)}{\sin(180-30^o)}=\frac{\cos 50^o}{-\cos(2x+50)}$$

$$\sin(2x+50^o)\cos(2x+50^o)=-\cos 50^o\sin 30^o$$

$$2\sin(2x+50^o)\cos(2x+50^o)=-\cos 50^o$$

$$\sin 2(2x+50^o)=-\sin 40^o$$

$$\Rightarrow\sin(4x+100^o)=\sin(-40^o)(or)\sin 220^o$$

$$4x+100=220^o$$

$$4x=120^o$$

$$x=30^o$$

$$=\frac{\pi}{6}$$

7. $f(n)=2\cos nx\Rightarrow f(1)f(n+1)-f(n)$

$$=4\cos x\cos(n+1)x-2\cos n\pi$$

$$=2\cos(n+2)x=f(n+2)$$

ADVANCED QUESTIONS

SINGLE ANSWER TYPE QUESTIONS

1. $\sin 20^o(4+\sec 20^o)=$

A) $\frac{1}{2}$ B) $\sqrt{2}$ C) $\sqrt{3}$ D) 1

KEY : C

SOL : $\frac{\sin 20^o}{\cos 20^o}(4\cos 20^o+1)=\frac{2\sin 40^o+\sin 20^o}{\cos 20^o}$

$=\frac{\sin 40^o+\sin 40^o+\sin 20^o}{\cos 20^o}$

$=\frac{\sin 40^o+\cos 10^o}{\cos 20^o}=\frac{\sin 40^o+\sin 80^0}{\cos 20^o}$

$=2\sin 60^o=\sqrt{3}$

2. **$0\le a\le 3, 0\le b\le 3$ and the equation $x^2+4+3\cos(ax+b)=2x$ has at least one solution then the value of (a+b)**

A) $\frac{\pi}{2}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{3}$ D) π

KEY : D

SOL :

$x^2-2x+4=-3\cos(ax+b)$

$\Rightarrow (x-1)^2+3=-3\cos(ax+b)$

As $-1\le\cos(ax+b)\le 1$ and $(x-1)^2\ge 0$

$\therefore$ equation (i) is only possible if, $\cos(ax+b)=-1$ *and* $(x-1)=0$

so $a+b=\pi, 3\pi, 5\pi, ...$

and $3\pi>6$

$\Rightarrow a+b=\pi$ where $a+b\le 6$

3 . If $\sin^3 x\sin 3x=\sum_{m=0}^{6}c_m\cos^m x,$ where $c_0, c_1, ..., c_6$ are constants, then

A) $c_0+c_2+c_4+c_6=0$ B) $c_1+c_3+c_5=6$

C) $2c_2+3c_6=0$ D) $c_4+2c_6=0$

KEY . A

SOL . $\sin^3 x\sin 3x=\sin^3 x\left(3\sin x-4\sin^3 x\right)$

$=\sin^4 x\left(3-4\sin^2 x\right)$

$=\left(1-\cos^2 x\right)^2\left(3-4\left(1-\cos^2 x\right)\right)$

$=\left(1-2\cos^2 x+\cos^4 x\right)\left(4\cos^2 x-1\right)$

$=4\cos^6 x-9\cos^4 x+6\cos^2 x-1$

So that

$c_0=-1, c_2=6, c_4=-9, c_6=4$

and $c_1=c_3=c_5=0$

$\Rightarrow c_0+c_2+c_4+c_6=0$

4. $\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}=$

A) $\frac{\sqrt{7}}{8}$ B) $\frac{\sqrt{7}}{4}$ C) $\frac{\sqrt{7}}{2}$ D) $\sqrt{7}$

KEY . A

SOL . $\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{4\pi}{7}$

$=\frac{1}{2}\left(\cos\frac{\pi}{7}-\cos\frac{3\pi}{7}\right)\sin\frac{4\pi}{7}$

$=\frac{1}{4}\left[\sin\frac{5\pi}{7}+\sin\frac{3\pi}{7}-\left(\sin\pi+\sin\frac{\pi}{7}\right)\right]$

$=\frac{1}{4}\left[\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}\right]=\frac{\sqrt{7}}{8}$

5. **If the mapping** $f(x)=ax+b, a<0$ **maps [-1,1] onto [0,2] then for all values of** $\theta, A=\cos^2\theta+\sin^4\theta$ **is such that**

A) $f\left(\frac{1}{4}\right)\le A\le f(0)$ B) $f(0)\le A\le f(-1)$

C) $f\left(\frac{1}{3}\right)\le A\le f(0)$ D) $f(1)<A\le f(-1)$

KEY . A

SOL. Given, f(x) = ax + b

$\therefore f'(x)=a$

since, a< 0 f(x) is a decreasing fucntion

$\therefore$ f(-1) = 2 and f(1) = 0

$\Rightarrow$ -a + b = 2 and a+b=0

$\therefore$ a=-1 and b=1

Thus, f(x) =-x + 1

clearly, f(0) =1, $f\left(\frac{1}{4}\right)=\frac{3}{4}, f(-2)=3$

$f\left(\frac{1}{3}\right)=\frac{2}{3}, f(-1)=2$

Also, $A=\frac{1+\cos 2\theta}{2}+\left(\frac{1-\cos 2\theta}{2}\right)^2$

$=\frac{1}{2}+\frac{1}{2}\cos 2\theta+\frac{1}{4}-\frac{1}{2}\cos 2\theta+\frac{1}{4}\cos^2 2\theta$

$=\frac{3}{4}+\frac{1}{4}\left(\frac{1+\cos 4\theta}{2}\right)=\frac{7}{8}+\frac{1}{8}\cos 4\theta$

$\therefore \frac{3}{4}\le A\le 1$

$\Rightarrow f\left(\frac{1}{4}\right)\le A\le f(0)$

6. **If** $\tan\theta_1, \tan\theta_2, \tan\theta_3$ *and* $\tan\theta_4$ **are the roots of the equation** $x^4-x^3\sin 2\beta$
$+x^2\cos 2\beta - x\cos\beta-\sin\beta=0$
then $\tan(\theta_1+\theta_2+\theta_3+\theta_4)$ **is equal to**

A) $\sin\beta$ B) $\cos\beta$ C) $\tan\beta$ D) $\cot\beta$

KEY. D

SOL . rom the given equation we get

$S_1=\tan\theta_1+\tan\theta_2+\tan\theta_3+\tan\theta_4=\sin 2\beta$

$S_2=\sum\tan\theta_1\tan\theta_2=\cos 2\beta$

$S_3=\sum\tan\theta_1\ \tan\theta_2\ \tan\theta_3=\cos\beta$

and $S_4=\tan\theta_1\tan\theta_2\tan\theta_3\tan\theta_4=-\sin\beta$

Now, $\tan(\theta_1+\theta_2+\theta_3+\theta_4)=\frac{S_1-S_3}{1-S_2+S_4}$

$=\frac{\sin 2\beta-\cos\beta}{1-\cos 2\beta-\sin\beta}=\frac{\cos\beta(2\sin\beta-1)}{\sin\beta(2\sin\beta-1)}=\cot\beta$

7. **The maximum value of the expression** $\left|\sqrt{\left(\sin^2 x+2a^2\right)}-\sqrt{(2a^2-1-\cos^2 x)}\right|$ **where** a **and** x **are real numbers is** (ADV- 2016)

A) $\sqrt{3}$ B) $\sqrt{2}$ C) 1 D) $\sqrt{5}$

KEY .B

SOL. For maximum value

$2a^2-1-\cos^2 x=0$

$\therefore\ \cos^2 x=2a^2-1$

$104\Rightarrow \sin^2 x=1-\cos^2 x=(2-2a^2)$

$\Rightarrow 2a^2+\sin^2 x=2$

$\therefore$ Maximum value of

$\left|\sqrt{\left(\sin^2 x+2a^2\right)}-\sqrt{(2a^2-1-\cos^2 x}\right|$

$=|\sqrt{2}-0|=\sqrt{2}$

MULTI - CORRECT

1. **For $0<\phi<\pi/2$, if**

$$x=\sum_{n=0}^{\infty}\cos^{2n}\phi,\ y=\sum_{n=0}^{\infty}\sin^{2n}\phi \text{ and}$$

$$z=\sum_{n=0}^{\infty}\cos^{2n}\phi\sin^{2n}\phi \text{ then } xyz=$$

(A) $xy+z$ (B) $xz+y$
(C) $x+y+z$ (D) $yz+x$

KEY . A,C

SOL. $x=\sum_{n=0}^{\infty}\cos^{2n}\phi=\cos ec^{2}\phi$

$$y=\sum_{n=0}^{\infty}\sin^{2n}\phi=\sec^{2}\phi$$

$$z=\sum_{n=0}^{\infty}\cos^{2n}\phi\sin^{2n}\phi=\frac{1}{1-\cos^{2}\phi\sin^{2}\phi}$$

$$=\frac{1}{1-\frac{1}{xy}}=\frac{xy}{xy-1}$$

so $xyz=xy+z$ or $xyz=x+y+z$ as $xy=x+y$.

2. If $\frac{\tan 3A}{\tan A}=k\ (k\neq 1)$ then

A) $\frac{\cos A}{\cos 3A}=\frac{k^{2}-1}{2k}$ B) $\frac{\sin 3A}{\sin A}=\frac{2k}{k-1}$

C) $k<\frac{1}{3}$ D) $k>3$

KEY: B,C,D

SOL.Given, $\frac{\tan 3A}{\tan A}=k$

$$\Rightarrow \frac{\tan 3A-\tan A}{\tan A}=k-1$$

$$\Rightarrow \frac{\sin 2A}{\cos 3A\sin A}=k-1$$

$$\Rightarrow \frac{2\cos A}{\cos 3A}=k-1$$

$$\Rightarrow \frac{\cos A}{\cos 3A}=\frac{k-1}{2}$$

$\Rightarrow (a)$ is incorrect.

Again, $\frac{\tan 3A}{\tan A}=k$

$$\Rightarrow \frac{\sin 3A}{\cos 3A}.\frac{\cos A}{\sin A}=k$$

$$\Rightarrow \frac{\sin 3A}{\sin A}=k.\frac{2}{k-1}=\frac{2k}{k-1}$$

$\Rightarrow (b)$ is correct.

$$\Rightarrow \frac{3\sin A-4\sin^{2}A}{\sin A}=\frac{2k}{k-1}$$

$$\Rightarrow 3-4\sin^{2}A=\frac{2k}{k-1}\, or\, 4\sin^{2}A=\frac{k-3}{k-1}$$

$$\Rightarrow\ 0<\frac{k-3}{k-1}<4\left[\sin A\neq 0\, or\, 1\right]$$

Now, $\frac{k-3}{k-1}>0$

$\therefore\ k\in(-\infty,1)\cup(3,\infty)$

and $\frac{k-3}{k-1}<4$

$$\Rightarrow \frac{3k-1}{k-1}>0$$

$\therefore k\in(-\infty,1/3)\cup(1,\infty)$

From Eqs. (ii) and (iii), we get

$k<\frac{1}{3}\, and\, k>3$.

3.**If the equation** $\sin x(\sin x+\cos x)=k$ **has real solutions then k may lie in the interval**

A) $\left[0,\frac{\sqrt{2}+1}{2}\right]$ B) $\left[2-\sqrt{3},2+\sqrt{3}\right]$

C) $\left[0,2-\sqrt{3}\right]$ D) $\left[\frac{1-\sqrt{2}}{2},\frac{1+\sqrt{2}}{2}\right]$

KEY . A,C,D

SOL . Given expression is 1-cos2x + sin 2x = 2k

$\Rightarrow \sin 2x-\cos 2x=2k-1$

$\Rightarrow \sin\left(2x-\frac{\pi}{4}\right)=\frac{2k-1}{\sqrt{2}}$

$\Rightarrow -1\le\frac{2k-1}{\sqrt{2}}\le 1$

$\Rightarrow \frac{1-\sqrt{2}}{2}\le k\le\frac{1+\sqrt{2}}{2}$

4. **The equation** $x^3-\frac{3}{4}x=-\frac{\sqrt{3}}{8}$ **is satisfied by**

(A) $x=\cos\left(\frac{5\pi}{18}\right)$ (B) $x=\cos\left(\frac{7\pi}{18}\right)$

(C) $x=\cos\left(\frac{23\pi}{18}\right)$ (D) $x=-\sin\left(\frac{7\pi}{9}\right)$

KEY . A,B,C,D

SOL . Let x = cos q
[12th, 13th 07-01-2007]

$4\cos^3 q-3\cos q=-\frac{\sqrt{3}}{2}$

$\cos 3q=\cos\frac{5\pi}{6}$ Þ 3q = 2np ±

$\frac{5\pi}{6}$ Þ $q=\frac{2n\pi}{3}\pm\frac{5\pi}{18}$

put n = 0, $q=\frac{5\pi}{18}$

n = 1, $q=\frac{2\pi}{3}+\frac{5\pi}{18}=\frac{17\pi}{18}$;

$q=\frac{2\pi}{3}-\frac{5\pi}{18}=\frac{7\pi}{18}$

5. $x=\sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha}$
$+\sqrt{a^2\sin^2\alpha+b^2\cos^2\alpha}$,

then $x^2=a^2+b^2+2\sqrt{p\left(a^2+b^2\right)-p^2}$ **,**

where p is equal to

A) $a^2\cos^2\alpha+b^2\sin^2\alpha$

B) $a^2\sin^2\alpha+b^2\cos^2\alpha$

C) $\frac{1}{2}\left[a^2+b^2+\left(a^2-b^2\right)\cos 2\alpha\right]$

D) $\frac{1}{2}\left[a^2+b^2-\left(a^2-b^2\right)\cos 2\alpha\right]$

KEY . A,B,C,D

$x=\sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha}$
$+\sqrt{a^2\sin^2\alpha+b^2\cos^2\alpha}$

$$\Rightarrow x^2=a^2+b^2+2\sqrt{\begin{matrix}(a^2\cos^2\alpha+b^2\sin^2\alpha)\\(a^2\sin^2\alpha+b^2\cos^2\alpha)\end{matrix}}$$

$=a^2+b^2+k.$ where

$k=\sqrt{\left[\left(a^2+b^2\right)-(a^2\sin^2\alpha+b^2\cos^2\alpha)\right]\times a^2\sin^2\alpha+b^2\cos^2\alpha}$

$\therefore x=a^2+b^2+2\sqrt{(a^2+b^2)P-P^2}$

where $P=a^2\sin^2\alpha+b^2\cos^2\alpha$

or $P=\frac{a^2}{2}(1-\cos 2\alpha)+\frac{b^2}{2}(1+\cos 2\alpha)$

$=\frac{1}{2}\left[a^2+b^2-(a^2-b^2)\cos 2\alpha\right]$

also,

$$x^2=\frac{a^2+b^2+2\sqrt{(a^2\cos\alpha+b^2\sin^2\alpha)}}{\left[\left(a^2+b^2\right)-\left(a^2\cos^2\alpha+b^2\sin^2\alpha\right)\right]}$$

$a^2+b^2+2\sqrt{\left(a^2+b^2\right)P-P^2}$

where , $P=a^2\cos^2\alpha+b^2\sin^2\alpha$

COMPREHENSION TYPE QUESTIONS

Passage - 1

A line OA of length r starts from its initial position OX and traces an angle AOB= α in the anticlockwise direction. It then traces back in the clockwise direction an angle BOC=3 θ (where $\alpha > 3\theta$). L is the foot of the perpendicular from C on OA.

$$\frac{\sin^3\theta}{CL} = \frac{\cos^3\theta}{OL} = 1$$

1. $\dfrac{1 - r\cos\alpha}{r\sin\alpha}$ **is equal to**

A) tan 2θ B) cot 2θ C) sin 2θ D) cos 2θ

2. $\dfrac{2r\sin\alpha}{1+2r\cos\alpha}$ **is equal to**

A) $\tan^2\theta$ B) $\cot^2\theta$ C) $\cot 2\theta$ D) $\tan 2\theta$

3. $\dfrac{2r^2 - 1}{r}$ **is equal to**

A) $\sin\alpha$ B) $\cos\alpha$ C) $\sin\theta$ D) $\cos\theta$

KEY . 1) A
2) D
3) B

SOL $OL = r\cos(\alpha - 3\theta)$

$CL = r\sin(\alpha - 3\theta)$

$$so\frac{\sin^3\theta}{r\sin(\alpha-3\theta)} = \frac{\cos^3\theta}{r\cos(\alpha-3\theta)} = 1$$

$$\Rightarrow \frac{\sin^3\theta}{\sin(\alpha-3\theta)} = \frac{\cos^3\theta}{\cos(\alpha-3\theta)} = r$$

Now $\cos^4\theta - \sin^4\theta = \cos\theta(r\cos(\alpha-3\theta))$
$-\sin\theta(r\sin(\alpha-3\theta))$

$\Rightarrow \cos 2\theta = r\cos(\alpha - 2\theta)$

$\Rightarrow \cos 2\theta = r(\cos\alpha\cos 2\theta + \sin\alpha\sin 2\theta)$

$\Rightarrow (1 - r\cos\alpha)\cos 2\theta = r\sin\alpha\sin 2\theta$

$$\Rightarrow \frac{1-r\cos\alpha}{r\sin\alpha} = \tan 2\theta$$

Next, $\cos^3\theta\sin\theta + \sin^3\theta\cos\theta$

$= r[\cos(\alpha-3\theta)\sin\theta + \sin(\alpha-3\theta)\cos\theta]$

$\Rightarrow \sin\theta\cos\theta = r\sin(\alpha - 2\theta)$

$\Rightarrow \sin 2\theta = 2r[\sin\alpha\cos 2\theta - \cos\alpha\sin 2\theta]$

$\Rightarrow (1 + 2r\cos\alpha)\sin 2\theta = 2r\sin\alpha\cos 2\theta$

$$\Rightarrow \frac{1+2r\cos\alpha}{2r\sin\alpha} = \cot 2\theta$$

from (A) and (B) we get

$$\frac{1-r\cos\alpha}{r\sin\alpha} = \frac{2r\sin\alpha}{1+2r\cos\alpha}$$

$\Rightarrow 1 - r\cos\alpha + 2r\cos\alpha - 2r^2\cos^2\alpha = 2r^2\sin^2\alpha$

$$\Rightarrow \frac{2r^2-1}{r} = \cos\alpha$$

Passage - 2

Given, $\cos 2^m\theta\cos 2^{m+1}\theta....\cos 2^n\theta$

$$= \frac{\sin 2^{n+1}\theta}{2^{n-m+1}\sin 2^m\theta}$$

where $2^m\theta \neq k\pi, n, m, k \in I$ **solve the following**

1. $\sin\dfrac{9\pi}{14}\sin\dfrac{11\pi}{14}\sin\dfrac{13\pi}{14}$ **is equal to**

A) $\dfrac{1}{64}$ B) $-\dfrac{1}{64}$ C) $\dfrac{1}{8}$ D) $-\dfrac{1}{8}$

2. $\cos 2^3\dfrac{\pi}{10}\cos 2^4\dfrac{\pi}{10}\cos 2^5\dfrac{\pi}{10}....\cos 2^{10}\dfrac{\pi}{10}$ **is equal to**

A) $\dfrac{1}{128}$ B) $\dfrac{1}{256}$

C) $\frac{1}{512}\sin\frac{\pi}{10}$ D) $\frac{\sqrt{5}-1}{512}\sin\frac{3\pi}{10}$

3. $\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}....\cos\frac{10\pi}{11}$ **is equal to**

A) $-\frac{1}{32}$ B) $\frac{1}{512}$ C) $\frac{1}{1024}$ D) $-\frac{1}{1024}$

KEY . 1)
2)
3)

SOL .

$$\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}=\sin\frac{5\pi}{14}\sin\frac{3\pi}{14}\sin\frac{\pi}{14}$$

$$=\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}=-\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$$

$$=-\frac{\sin\frac{8\pi}{7}}{8\sin\frac{\pi}{7}}=\frac{1}{8}$$

Hence, (c) is the correct answer

$$\cos 2^3\frac{\pi}{10}\cos 2^4\frac{\pi}{10}\cos 2^5\frac{\pi}{10}....\cos^{2^{10}}\frac{\pi}{10}$$

$$=\frac{\sin 2^{11}\frac{\pi}{10}}{256\sin 2^3\frac{\pi}{10}}=\frac{1}{256}$$

$$\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}...\cos\frac{10\pi}{11}$$

$$=\left(\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11}\right)^2$$

$$=\left(\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{4\pi}{11}\cos\frac{8\pi}{11}\cos\frac{5\pi}{11}\right)^2$$

$$=\left(\frac{\sin 16\frac{\pi}{11}}{16\sin\frac{\pi}{11}}.\cos\frac{5\pi}{11}\right)^2=\left(\frac{2\sin\frac{5\pi}{11}\cos\frac{5\pi}{11}}{32\sin\frac{\pi}{11}}\right)^2=\frac{1}{1024}$$

MATRIX MATCHING TYPE QUESTIONS

1.Match the following

COLUMN-I

A)When θ is fixed constant then the maximum value of $\{\cos(2A+\theta)+\cos(2B+\theta)\}$

B) The maximum value of $\{\cos 2A+\cos 2B\}$ where (A+B) is constant and A,B $\in(0,\ \pi/2)$, is

C) The minimum value of $\{\sec 2A+\sec 2B\}$, where (A+B) is constant and A,B $\in(0,\ \pi/4)$, is

D) The minimum value of $\sqrt{\{\tan\theta+\cot\theta-2\cos 2(A+B)\}}$, where A,B are constant and $\theta\in(0,\pi/2)$ is

COLUMN-II

p) $2\sin(A+B)$

q) 2 sec (A+B)

r) 2 cos (A+B)

s) 2 cos (A-B)

KEY . $A\to s; B\to r; C\to q; D\to p$

SOL . A) $\cos(2A+\theta)+\cos(2B+\theta)$

$=2\cos(A+B+\theta)\cos(A-B)$

Maximum value is 2cos(A-B) when

cos $(A+B+\theta)$=1

B) $\{\cos 2A+\cos 2B\}$

2cos(A+B)cos(A-B)

Maximum value is 2cos(A-B) when cos (A+B)=1

C) For y= sec x, $x\in(0,\pi/2)$, tangent drawn to it any point lies completely below the graph of $y=\sec x, thus\ \frac{\sec 2A+\sec 2B}{2}\geq\sec(A+B)$

$\Rightarrow\sec 2A+\sec 2B\geq 2\sec(A+B)$

Hence minimum value is A+B

D) $\sqrt{\{\tan\theta+\cot\theta-2\cos 2(A+B)\}}$

$$=\sqrt{\left(\sqrt{\tan\theta}-\sqrt{\cot\theta}\right)^2+2-2\cos 2(A+B)}$$

$$=\sqrt{\left(\sqrt{\tan\theta}-\sqrt{\cot\theta}\right)^2+4\sin^2(A+B)}$$

Minimum value occurs when $\sqrt{\tan\theta}=\sqrt{\cot\theta}$ and Minimum value is

$$\sqrt{4\sin^2(A+B)}=2\sin(A+B)$$

2 . **Let $f_n(\theta)$ =**

$$\frac{\cos\frac{\theta}{2}+\cos 2\theta+\cos\frac{7\theta}{2}+......+\cos(3n-2)\frac{\theta}{2}}{\sin\frac{\theta}{2}+\sin 2\theta+\sin\frac{7\theta}{2}+.......+\sin(3n-2)\frac{\theta}{2}}.$$

Then match the entries of column-I with their corresponding values given in column-II.

Column-I	Column-II
(A) $f_3\left(\frac{3\pi}{16}\right)$	(P) $2-\sqrt{3}$
(B) $f_5\left(\frac{\pi}{28}\right)$	(Q) $\sqrt{3+2\sqrt{2}}$
(C) $f_7\left(\frac{\pi}{60}\right)$	(R) $\sqrt{2}-1$
	(S) $\sqrt{7+4\sqrt{3}}$

KEY . (A) R; (B) Q; (C) S

SOL . e have $f_n(q)=\cot\left((3n-1)\frac{\theta}{4}\right)$

A) $f_3\left(\frac{3\pi}{16}\right)=\cot\left(8\times\frac{1}{4}\times\frac{3\pi}{16}\right)=\cot\frac{3\pi}{8}$

$=\sqrt{2}-1$

B) $f_5\left(\frac{\pi}{28}\right)=\cot\left(14\times\frac{1}{4}\times\frac{\pi}{28}\right)$

$=\cot\frac{\pi}{8}=\sqrt{2}+1=\sqrt{3+2\sqrt{2}}$

C) $f_7\left(\frac{\pi}{60}\right)=\cot\left(20\times\frac{1}{4}\times\frac{\pi}{60}\right)=\cot\frac{\pi}{12}$

$=\left(2+\sqrt{3}\right)=\sqrt{7+4\sqrt{3}}$]

INTEGER TYPE QUESTIONS

1. **If** $4\cos 36^o+\cot\left(7\frac{1^o}{2}\right)=$ $\sqrt{n_1}+\sqrt{n_2}+\sqrt{n_3}+\sqrt{n_4}+\sqrt{n_5}+\sqrt{n_6}$ **then the product of the digits in** $\sum_{i=1}^{6}n_i^2=$

KEY .91

SOL $\cot\left(7\frac{1^o}{2}\right)=\frac{1+\cos 15^o}{\sin 15^o}$

$$=\frac{1+\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}=\frac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1}$$

$$=\frac{\left(2\sqrt{2}+\sqrt{3}+1\right)\left(\sqrt{3}+1\right)}{2}$$

$$\frac{2\sqrt{6}+2\sqrt{2}+3+\sqrt{3}+\sqrt{3}+1}{2}$$

$$=\sqrt{6}+\sqrt{2}+2+\sqrt{3}$$

$= \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$

and

$$4\cos 36^o = 4\left(\frac{\sqrt{5}+1}{4}\right) = \sqrt{5}+1 = \sqrt{5}+\sqrt{1}$$

Hence, $4\cos 36^o + \cot\left(7\frac{1}{2}\right)^o$

$=\sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{5}+\sqrt{6}$

$\therefore\ n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 4$

$n_5 = 5\, and\ n_6 = 6$

$$\therefore \sum_{i=1}^{6} n_i^2 = n_2^2 + n_3^2 + n_4^2 + n_5^3 + n_6^2$$

$= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$

$= 91$

2. If $\tan^2\left(\frac{\pi}{16}\right)+\tan^2\left(\frac{2\pi}{16}\right)+\tan^2\left(\frac{3\pi}{16}\right)$

$+...+\tan^2\left(\frac{7\pi}{16}\right)=\lambda$ **and if**

$x^y + y^x = \lambda\left(x, y \in, Z^+\right)$ **then the sum of the digits in** $(x+y)$ **is**

KEY. 35

SOL . $\lambda = \left\{\tan^2\left(\frac{\pi}{16}\right)+\tan^2\left(\frac{7\pi}{16}\right)\right\}$

$+\left\{\tan^2\left(\frac{2\pi}{16}\right)+\tan^2\left(\frac{6\pi}{16}\right)\right\}$

$+\left\{\tan^2\left(\frac{3\pi}{16}\right)+\tan^2\left(\frac{5\pi}{16}\right)\right\}+\tan^2\left(\frac{4\pi}{16}\right)$

$$= \tan^2\left(\frac{\pi}{16}\right)+\cot^2\left(\frac{\pi}{2}-\frac{7\pi}{16}\right)$$

$$=\left\{\tan^2\left(\frac{\pi}{16}\right)+\cot^2\left(\frac{\pi}{16}\right)\right\}+\left\{\tan^2\left(\frac{2\pi}{16}\right)+\cot^2\left(\frac{2\pi}{16}\right)\right\}$$

$$+\left\{\tan^2\left(\frac{3\pi}{16}\right)+\cot^2\left(\frac{3\pi}{16}\right)\right\}+1$$

$$=\left\{\tan\left(\frac{\pi}{16}\right)+\cot\left(\frac{\pi}{16}\right)\right\}^2+\left\{\tan\left(\frac{2\pi}{16}\right)+\cot\left(\frac{2\pi}{16}\right)\right\}^2$$

$$+\left\{\tan\left(\frac{3\pi}{11}\right)+\cot\left(\frac{3\pi}{16}\right)\right\}^2-2-2-2+1$$

$$=\frac{1}{\left\{\sin\left(\frac{\pi}{16}\right)\cos\left(\frac{\pi}{16}\right)\right\}^2}+\frac{1}{\left\{\sin\left(\frac{2\pi}{16}\right)\cos\left(\frac{2\pi}{16}\right)\right\}^2}$$

$$+\frac{1}{\left\{\sin\left(\frac{3\pi}{16}\right)\cos\left(\frac{3\pi}{16}\right)\right\}^2}-5$$

$$=\frac{4}{\sin^2\frac{\pi}{8}}+\frac{4}{\sin^2\frac{\pi}{4}}+\frac{4}{\sin^2\left(\frac{3\pi}{8}\right)}-5$$

$$=4\left\{\frac{1}{\sin^2\left(\frac{\pi}{8}\right)}+\frac{1}{\sin^2\left(\frac{3\pi}{8}\right)}\right\}+4.2-5$$

$$=4\left\{\frac{1}{\sin^2\left(\frac{\pi}{8}\right)}+\frac{1}{\cos^2\left(\frac{\pi}{8}\right)}\right\}+3$$

$$=\frac{4}{\left\{\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)\right\}^2}+3$$

$$= \frac{16}{\left\{\sin\left(\frac{\pi}{4}\right)\right\}^2} + 3$$

$$\therefore \lambda = 35$$

.3. **The number of integral values of α such that $\sin x \cos 3x - \alpha \cos x \sin 3x = 0$ does not have any real root other than $(2n+1)\frac{\pi}{2}, n \in 1$ for any real value of x, is**

KEY 3

SOL. $\sin x \cos 3x - \alpha \cos x \sin 3x = 0$ ...(i)

$$\Rightarrow \alpha = \frac{\sin x \cos 3x}{\cos x \sin 3x}$$

$$\alpha = \frac{\tan x}{\tan 3x}$$

$$\alpha = \frac{\tan x(1 - 3\tan^2 x)}{3\tan x - \tan^3 x}$$

$$\alpha = \frac{1 - 3\tan^2 x}{3 - \tan^2 x} \quad \text{....(ii)}$$

For real vlaue of x, RHS of 2nd never lies between

$$\left[\frac{1}{3}, 3\right)$$

$\Rightarrow$ Number of integral points $\{1,2,3\}$

$\Rightarrow$ 3 points

JEE MAINS, EAMCET QUESTIONS

1. **ΔABC, minimum value of $\tan A + \tan B + \tan C$, where A, B, C are acute angles is**
2. **In ΔABC, maximum value of $\cos A + \cos B + \cos C$ is**
3. **The least value of $\sec A + \sec B + \sec C$ in an acute angle triangle is**
4. **In ΔABC, minimum value of $\operatorname{cosec}\frac{A}{2} + \operatorname{cosec}\frac{B}{2} + \operatorname{cosec}\frac{C}{2}$ is**
5. **The value of $\cot 70^0 + 4\cos 70^0$ is**
6. **The absolute value of the expression $\tan\frac{\pi}{16} + \tan\frac{5\pi}{16} + \tan\frac{9\pi}{16} + \tan\frac{13\pi}{16}$ is ___**
7. $\cos^2(45^0 - \alpha) + \cos^2(15^0 + \alpha) - \cos^2(15^0 - \alpha) =$
8. $\cos 20^0 . \cos 40^0 . \cos 60^0 . \cos 80^0 =$

KEY

01) 5.19	02) 1.50	03) 6.00	04) 6.00
05) 1.73	06) 4.00	07) 0.50	08) 0.062

SOLUTIONS

1. In ΔABC, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$, $A.M \geq G.M$

$$\frac{\tan A + \tan B + \tan C}{3} \geq \sqrt[3]{\tan A \tan B \tan C}$$

$$\Rightarrow \tan A \tan B \tan C \geq 3\sqrt[3]{\tan A \tan B \tan C}$$

$$\Rightarrow \tan^2 A \tan^2 B \tan^2 C \geq 27$$

$$\Rightarrow \tan A \tan B \tan C \geq 3\sqrt{3}$$

$$\Rightarrow \tan A + \tan B + \tan C \geq 3\sqrt{3}$$

2. Let $\cos A+\cos B+\cos C=x$

$$\Rightarrow 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+1-2\sin^2\frac{C}{2}=x$$

$$\Rightarrow 2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)+1-2\sin^2\frac{C}{2}=x$$

$$\Rightarrow 2\sin^2\frac{C}{2}-2\cos\left(\frac{A-B}{2}\right)\sin\frac{C}{2}+x-1=0$$ this is

quadratic in $\sin\frac{C}{2}$ which is real. So,

discriminant $D\geq 0$;

$$4\cos^2\left(\frac{A-B}{2}\right)-4\times 2(x-1)\geq 0$$

$$\Rightarrow 2(x-1)\leq \cos^2\left(\frac{A-B}{2}\right)$$

$$\Rightarrow 2(x-1)\leq 1\Rightarrow x\leq \frac{3}{2}$$

Thus, $\cos A+\cos B+\cos C\leq \frac{3}{2}$

3. In an acute angle triangle, $\sec A, \sec B, \sec C$ are positive.

Now $A.M.\geq H.M.$

$$\Rightarrow \frac{\sec A+\sec B+\sec C}{3}\geq \frac{3}{\cos A+\cos B+\cos C}$$

but in $\Delta ABC, \cos A+\cos B+\cos C\leq \frac{3}{2}$

$$\Rightarrow \frac{\sec A+\sec B+\sec C}{3}\geq 2$$

$$\Rightarrow \sec A+\sec B+\sec C\geq 6$$

. In ΔABC we know that $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\leq\frac{1}{8}$

Now $A.M\geq G.M$

$$\Rightarrow \frac{\operatorname{cos}ec\frac{A}{2}+\operatorname{cos}ec\frac{B}{2}+\operatorname{cos}ec\frac{C}{2}}{3}\geq$$

$$\left(\operatorname{cos}ec\frac{A}{2}\operatorname{cos}ec\frac{B}{2}\operatorname{cos}ec\frac{C}{2}\right)^{\frac{1}{3}}$$

$$\Rightarrow \frac{\operatorname{cos}ec\frac{A}{2}+\operatorname{cos}ec\frac{B}{2}+\operatorname{cos}ec\frac{C}{2}}{3}\geq$$

$$\left(\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\right)^{\frac{1}{3}}$$

$$\Rightarrow \frac{\operatorname{cos}ec\frac{A}{2}+\operatorname{cos}ec\frac{B}{2}\operatorname{cos}ec\frac{C}{2}}{3}\geq (8)^{\frac{1}{3}}$$

$$\Rightarrow \operatorname{cos}ec\frac{A}{2}+\operatorname{cos}ec\frac{B}{2}+\operatorname{cos}ec\frac{C}{2}\geq 6$$

5. $\cot 70^0+4\cos 70^0=$

$$\frac{\cos 70^0+4\sin 70^0\cos 70^0}{\sin 70^0}$$

$$=\frac{\cos 70^0+2\sin 140^0}{\sin 70^0}$$

$$=\frac{\cos 70^0+2\sin\left(180^0-40^0\right)}{\sin 70^0}$$

$$=\frac{\sin 20^0+\sin 40^0+\sin 40^0}{\sin 70^0}$$

$$=\frac{2\sin 30^0\cos 10^0+\sin 40^0}{\sin 70^0}$$

$$=\frac{\sin 80^0+\sin 40^0}{\sin 70^0}=\frac{2\sin 60^0\cos 20^0}{\sin 70^0}=\sqrt{3}$$

6. Let $\theta=\frac{\pi}{16}\Rightarrow 8\theta=\frac{\pi}{2}$

$$y=\tan\theta+\tan 5\theta+\tan 9\theta+\tan 13\theta$$

$$\therefore y=(\tan\theta-\cot\theta)+(\tan 5\theta-\cot 5\theta)$$

[as $\tan 13\theta=\tan(8\theta+5\theta)=-\cot 5\theta$ and $\tan 9\theta=\tan(8\theta+\theta)=-\cot\theta$]

$$=(\tan\theta-\cot\theta)+(\cot 3\theta-\tan 3\theta)$$

$$=2(\cot 6\theta-\cot 2\theta)$$

$$\Rightarrow y=2\left[\frac{\cos 6\theta}{\sin 6\theta}-\frac{\cos 2\theta}{\sin 2\theta}\right]$$

$$=2\left[\frac{\sin 2\theta\cos 6\theta-\cos 2\theta\sin 6\theta}{\sin 6\theta\sin 2\theta}\right]$$

$$=-2\left[\frac{\sin 4\theta}{\cos 2\theta\sin 2\theta}\right]=-4\left(\because 6\theta=\frac{\pi}{2}-2\theta\right)$$

Hence, absolute value $=4$.

7. Apply $\cos^2 A-\cos^2 B$ formulae

8. Use $\cos\theta.\cos(60-\theta).\cos(60+\theta)=\frac{1}{4}\cos 3\theta$

PERIODICITY & EXTREME VALUES

SYNOPSIS

- **Periodic function:** A real function $f: A \to R$ is said to be a periodic function if there exists a positive real number p such that $f(x+p) = f(x),\ \forall\ x \in A$. Least value of p is known as the fundamental period of f.
- The period of $f(ax+b) = \dfrac{p}{|a|}$
- Let $g(x) = \dfrac{C_1 f_1(x) \pm C_2 f_2(x)}{C_3 f_3(x) \pm C_4 f_4(x)}$ (where f_1, f_2, f_3, f_4 are trigonometric functions, and C_1, C_2, C_3. C_4 are constants).

  Then period of $g(x) =$ L.C.M. of period of f_1, f_2, f_3, f_4

  Note: i) Numerator and Denominator can be extended to sum of any number of trignometric functions.

  ii) If p and q are rational quantities.Then LCM of p and q exists

  iii) If p and q are irrational, then LCM of p and q does not exist unless they have same irrational surd.

  iv) LCM of rational and irrational is not possible.

  v) L.C. M $\left\{\dfrac{a}{b}, \dfrac{c}{d}, \dfrac{e}{f}\right\} = \dfrac{L.C.M\{a,c,e\}}{G.C.D\{b,d,f\}}$
- If 'f' is a periodic function with fundamental period 'p' then '$-f$' and $\dfrac{1}{f}$ are also a periodic functions with period p.
- A constant function is periodic but does not have a fundamental period.
- Periods of trignometric functions are given below:

Function	**Period**
1. $Sin\,(ax+b)$	$\dfrac{2\pi}{\lvert a\rvert}$
2. $Cos\,(ax+b)$	$\dfrac{2\pi}{\lvert a\rvert}$
3. $Tan\,(ax+b)$	$\dfrac{\pi}{\lvert a\rvert}$
4. $Co\sec\,(ax+b)$	$\dfrac{2\pi}{\lvert a\rvert}$
5. $Sec\,(ax+b)$	$\dfrac{2\pi}{\lvert a\rvert}$
6. $Cot\,(ax+b)$	$\dfrac{\pi}{\lvert a\rvert}$

- Period of $\{x\}$ is 1. where $\{.\}$ denotes the fractional part.
- Standard results :

  i) Period of $|\sin x|, |\cos x|, |\tan x|$, $|\cot x|, |\sec x|, |\operatorname{cosec} x|$ is π

  ii) Period of $|\sin x| + |\cos x|, |\tan x| + |\cot x|$ is $\dfrac{\pi}{2}$

  iii) Period of $|\sin x + \cos x|, |\sin x - \cos x|$ is π

  iv) Period of $|\tan x + \cot x|, |\tan x - \cot x|$ is $\dfrac{\pi}{2}$

  v) If $n \in z^+$, the period of $\sin^{2n} x + \cos^{2n} x$, $\sec^{2n} x + \operatorname{cosec}^{2n} x, \tan^{2n} x + \cot^{2n} x$ is $\dfrac{\pi}{2}$

  vi) Period of $a\sin^{2n} x + b\cos^{2m} x,\ a\sec^{2n} x + b\operatorname{cosec}^{2m} x$, $a\tan^{2n} x + b\cot^{2m} x$ is π $\quad (n, m \in z^+)$

✈ Maximum and minimum values :

i) Minimum value of $\sin x$ is -1

ii) Maximum value of $\sin x$ is 1

iii) Range of $\sin x$ is $[-1,1]$

✈ i) Minimum value of $\cos x$ is -1

ii) Maximum value of $\cos x = 1$

iii) Range of $\cos x$ is $[-1,1]$

✈ Minimum value of

$a\cos x + b\sin x + c$ *is* $c - \sqrt{a^2+b^2}$

✈ Maximum value of $a\cos x + b\sin x + c$ is $c + \sqrt{a^2+b^2}$

✈ Range of $a\cos x + b\sin x + c$ is $\left[c - \sqrt{a^2+b^2},\, c + \sqrt{a^2+b^2}\right]$

✈ The minimum value of $a^2\sin^2\theta + b^2\operatorname{cosec}^2\theta$, $a^2\cos^2\theta + b^2\sec^2\theta$, $a^2\tan\theta + b^2\cot\theta$ is 2ab.

EXERCISE - I

1. Period of *tan (x + 2x + 3x + ...+ nx)* is

1) $\frac{2\pi}{n(n+1)}$ 2) $\frac{4\pi}{n(n+1)}$

3) $\frac{\pi}{n(n+1)}$ 4) $\frac{6\pi}{n(n+1)}$

2. Period of *tan (x + 4x + 9x+...+ n²x)* is

1) $\frac{2\pi}{n(n+1)(2n+1)}$ 2) $\frac{4\pi}{n(n+1)(2n+1)}$

3) $\frac{6\pi}{n(n+1)(2n+1)}$ 4) $\frac{8\pi}{n(n+1)(2n+1)}$

3. Period of $\cos\left(x+\frac{x}{3}+\frac{x}{3^2}+........\right)$ **is**

1) $\frac{\pi}{3}$ 2) $\frac{2\pi}{3}$ 3) π 4) $\frac{4\pi}{3}$

4. Period of $\cos^6 x + \sin^6 x$ **is**

1) $\frac{\pi}{2}$ 2) π 3) $\frac{3\pi}{2}$ 4) 2π

5. Period of $6\cos^4 x - 7\sin^4 x$ **is**

1) $\frac{\pi}{2}$ 2) π 3) $\frac{3\pi}{2}$ 4) 2π

6. Period of $\cos x\cos\left(60^0-x\right)\cos\left(60^0+x\right)$ **is**

1) $\frac{\pi}{2}$ 2) $\frac{\pi}{3}$ 3) $\frac{2\pi}{3}$ 4) π

7. If $f(x)=\sin^2\left(\frac{\pi}{8}+\frac{x}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{x}{2}\right)$**, then the period of f is**

1) π 2) $\frac{\pi}{2}$ 3) $\frac{\pi}{3}$ 4) 2π

8. Sine function whose period is 6 is

1) $\sin\frac{2\pi x}{3}$ 2) $\sin\frac{\pi x}{3}$ 3) $\sin\frac{\pi x}{6}$ 4) $\sin\frac{3\pi x}{2}$

9. Period of $\frac{2\sin 2x - 5\cos 2x}{7\cos x - 8\sin x}$ **is**

1) π 2) 2π 3) $\frac{\pi}{2}$ 4) $\frac{\pi}{3}$

10. Period of $\frac{\sin(x+a)}{\cos x}$

1) $\frac{\pi}{2}$ 2) π 3) 2π 4) 3π

11. Period of $\sin\left(e^{\tan x}+e^{\cot x}\right)$ **is**

1) $\frac{\pi}{2}$ 2) π 3) $\frac{3\pi}{2}$ 4) 2π

12. The period of $\sin(\pi\sin\theta)$ **is**

1) π 2) $\frac{\pi}{2}$ 3) 2π 4) $\frac{\pi}{4}$

13. The minimum and maximum values of $2\sqrt{2}\cos x+\sin x$ **are**

1) - 1, 1 2) $-2\sqrt{2}, 2\sqrt{2}$

3) 1,8 4) -3,3

14. The minimum and maximum values of $8\cos 3x - 15\sin 3x$ **are**

1) -7,7 2) -23,23 3) -17,17 4) -15,8

15. The maximum value of $\frac{3}{5\sin x - 12\cos x + 19}$

1) 1 2) $\frac{1}{2}$ 3) $\frac{1}{3}$ 4) $\frac{1}{4}$

16. The range of $sin^6 x+cos^6 x$

1) $\left[-\frac{1}{4},\frac{1}{4}\right]$ 2) $\left[\frac{1}{4},\frac{1}{2}\right]$ 3) $\left[\frac{1}{4},1\right]$ 4) $\left[-\frac{1}{2},-\frac{1}{4}\right]$

17. $cos^2\left(60^0-x\right)+cos^2\left(60^0+x\right)\in$

1) $\left[-\frac{1}{2},\frac{1}{2}\right]$ 2) $\left[\frac{1}{2},1\right]$

3) $\left[\frac{1}{2},\frac{3}{2}\right]$ 4) $\left[\frac{3}{2},2\right]$

18. $cos^2\left(60^0+x\right)-cos^2\left(60^0-x\right)\in\left[-\frac{m}{2},\frac{m}{2}\right]$

$\Rightarrow$ m =

1) 1 2) 3 3) $-\sqrt{3}$ 4) $\sqrt{3}$

19. $sin^2\left(60^0+x\right)-sin^2\left(60^0-x\right)\in\left[-\frac{k}{2},\frac{k}{2}\right]$ $\Rightarrow$ **k =**

1) 3 2) 2 3) $\sqrt{3}$ 4) $\sqrt{2}$

20. $sin\theta sin\left(60^0-\theta\right)sin\left(60^0+\theta\right)\in[-k,k]\Rightarrow$ **k =**

1) $\frac{1}{2}$ 2) $\frac{1}{3}$ 3) $\frac{1}{4}$ 4) $\frac{3}{4}$

21. The maximum value of

$cosx\left(\frac{cos\ x}{1-sin\ x}+\frac{1-sin\ x}{cosx}\right)$ **is**

1) 4 2) 3 3) 2 4) 1

22. A + B = 90⁰ $\Rightarrow$ **cos A cos B** $\in$

1) $\left[-\frac{1}{3},\frac{1}{3}\right]$ 2) $\left[-\frac{1}{2},\frac{1}{2}\right]$

3) $\left[-\frac{1}{4},\frac{1}{4}\right]$ 4) $\left[-\frac{3}{4},\frac{3}{4}\right]$

23. A + B = 90⁰ $\Rightarrow$ **sin A + sin B** $\in$

1) [-1,1] 2) $\left[-\sqrt{2},\sqrt{2}\right]$

3) $\left[-\sqrt{3},\sqrt{3}\right]$ 4) $[-2,2]$

24. Minimum value of $9\sec^2\theta+4\cos ec^2\theta+7$ **is**

1) 32 2) 25 3) 20 4) 49

25. The minimum value of $27\tan^2\theta+3\cot^2\theta$ **is**

1) 15 2) 18 3) 24 4) 30

26. The period of $\sin\left(x+8x+27x+----n^3x\right)$

1) $\frac{2\pi}{n^2(nA)^2}$ **2)** $\frac{4\pi}{n^2(nA)^2}$

3) $\frac{6\pi}{n^2(nA)^2}$ **4)** $\frac{8\pi}{n^2(nB)^2}$

KEY

01) 1	02) 3	03) 4			
04) 1	05) 2	06) 3	07) 4	08) 2	09) 2
10) 2	11) 2	12) 3	13) 2	14) 3	15) 2
16) 3	17) 3	18) 4	19) 3	20) 3	21) 3
22) 2	23) 2	24) 1	25) 2	26) 4	

SOLUTIONS

1. $p=\frac{\pi}{|a|}=\frac{\pi}{\frac{n(n+1)}{2}}=\frac{2\pi}{n(n+1)}$

2. $p=\frac{\pi}{|a|}=\frac{\pi}{\frac{n(n+1)(2n+1)}{6}}=\frac{6\pi}{n(n+1)(2n+1)}$

3. $1+\frac{1}{3}+\frac{1}{3^2}+.....=\frac{1}{1-1/3}=\frac{3}{2}$

$\therefore$ Period of $\cos\left(\frac{3}{2}x\right)=\frac{2\pi}{3/2}=\frac{4\pi}{3}$

4. When coefficients are equal and powers are even then period is $\frac{\pi}{2}$

5. When coefficients are different and power are even period is π

6. $\cos x\cos\left(60°-x\right)\cos\left(60°+x\right)=\frac{1}{4}\cos 3x$

Period $=\frac{2\pi}{3}$

7. Given $f(x)=\sin^2\left(\frac{\pi}{8}+\frac{x}{2}\right)-\sin^2\left(\frac{\pi}{8}-x/2\right)$

$\therefore \sin^2 A-\sin^2 B=\sin(A+B)\sin(A-B)$

$f(x)=\sin\left(\frac{\pi}{8}+\frac{x}{2}+\frac{\pi}{8}\right)\sin\left(\frac{\pi}{8}+\frac{x}{2}\frac{-\pi}{8}+x/2\right)$

$=\sin\left(\frac{\pi}{4}\right)\sin(x)=\frac{1}{\sqrt{2}}\sin x$

$\therefore$ The period of $\sin x$ is $\frac{2\pi}{1}=2\pi$

$\therefore$ Period of $f(x)$ is 2π

8. $\frac{2\pi}{|a|}=6\Rightarrow|a|=\frac{\pi}{3}$

9. The period of $f(x)=\frac{2\sin 2x-5\cos 2x}{7\cos x-8\sin x}$

The period of $\sin 2x$ is $\frac{2\pi}{2}=\pi$

The period of $\cos\ 2x$ is $\frac{2\pi}{2}=\pi$

The period of $\cos x$ is 2π

The period of $\sin x$ is 2π

The period of $f(x)$ is L.C.M of

$[\pi,\pi,2\pi,2\pi]=2\pi$

10. Cosa. Tanx + Sina, $\therefore$ period is π

11. L.C.M of period of Tanx, Cotx is π

12. Period of $\sin\theta$

13. $\text{Min}=-\sqrt{a^2+b^2};\text{Max}=\sqrt{a^2+b^2}$

14. $\text{Min}=-\sqrt{a^2+b^2};\text{Max}=\sqrt{a^2+b^2}$

15. Given $f(x)=8\cos 3x-15\sin 3x$

comparing with $a\cos\theta+b\sin\theta+c$

$a=8, b=-15, c=0$

Now $\sqrt{a^2+b^2}=\sqrt{64+225}=17$

maximum $=c+\sqrt{a^2+b^2}=0+17=17$

maximum $=c-\sqrt{a^2+b^2}=0-17=-17$

minimum, maximum $=-17, 17$

16. $\sin^6 x+\cos^6 x=1-\frac{3}{4}\sin^2 2x$

17. If $f(x)=\cos^2(\alpha-x)+\cos^2(\alpha+x)$

for α = 60°, 120°, 240°, 300° , range is $\left[\frac{1}{2},\frac{3}{2}\right]$

18. If $f(x)=\cos^2(\alpha-x)-\cos^2(\alpha+x)$

for α = 60°, 120°, 240°, 300°

range is $\left[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right]$

19. If $f(x)=\sin^2(\alpha-x)-\sin^2(\alpha+x)$

for α = 60°, 120°, 240°, 300°

range is $\left[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right]$

20. If $f(\theta)=\sin\theta\sin(\alpha-\theta)\sin(\alpha+\theta)=\frac{1}{4}\sin 3\theta$

for $\alpha=60°,120°,240°,300°$, range is $\left[-\frac{1}{4},\frac{1}{4}\right]$

21. $\frac{\cos^2 x}{1-\sin x}+1-\sin x=2$

22. $A+B=90°\Rightarrow\cos A\cos B$

$=\frac{1}{2}\sin 2A\in\left[-\frac{1}{2},\frac{1}{2}\right]$

23. $(A+B)=90°\Rightarrow\sin A+\sin B=$

$\sin A+\cos A\in\left[-\sqrt{2},\sqrt{2}\right]$

24. A.M. $\geq$ G.M.

25. A.M $\geq$ G. M

26. **given,** $=\sin\left(1^3+2^3+3^3----n^3\right)x$

$$= \sin\left(\frac{n^2(n+1)^2}{4}\right)x$$

$\therefore$ **since the period** $\sin ax$ **is** $\frac{2\pi}{|a|}$

Here $\frac{2\pi}{\frac{n^2(n+1)^2}{4}} = \frac{8\pi}{n^2(n+1)^2}$

EXERCISE - II

1. **Period of** $sin\frac{3\pi x}{2}+cos\frac{\pi x}{2}$ **is**
   1) 1 2) 2 3) 3 4) 4

2. **The period of** $f(x)=sinx+\{x\}$**, where** $\{x\}$ **is fractional part of x is**
   1) 0 2) 1
   3) 2π 4) Does not exist

3. **The period of** $f(x)=sin\frac{\pi x}{n!}-cos\frac{\pi x}{(n+1)!}$ **is**
   1) $(n+1)!$ 2) $2(n+1)!$
   3) $2(n)!$ 4) $\frac{(n+1)}{2!}$

4. **The value of integer n for which the function** $f(x)=\frac{sinnx}{sin(x/n)}$ **has** 4π **as its period is**
   1) 2 2) 3 3) 4 4) 5

5. **Let** $f(x)=cos\sqrt{p}\ x$ **where** $p=[a]$ **where** $[x]$ **is the integral part of x. If the period of** $f(x)$ **is** π **, then** $a\in$
   1) $[4,5]$ 2) $[4,5)$ 3) $(4,5]$ 4) $(4,5)$

6. **If the period of** $\frac{\cos(\sin(nx))}{\tan(x/n)}, n\in N$ **is** 6π **then n =**
   1) 3 2) 2 3) 6 4) 1

7. **The period of the function** $\left|\sin^3\frac{x}{2}\right|+\left|\cos^5\frac{x}{5}\right|$ **is**
   1) 2π 2) 10π 3) 8π 4) 5π

8. **The function** $\sin(x^2)+\cos\sqrt{x}$ **is**
   1) Periodic with period π
   2) Periodic with period 2π
   3) Periodic with period 0
   4) Not a Periodic function

9. $sin^2x+cos^4x\in$
   1) $\left[\frac{1}{4},\frac{1}{2}\right]$ 2) $\left[\frac{3}{4},1\right]$ 3) $\left[-1,-\frac{3}{4}\right]$ 4) $\left[-\frac{3}{4},\frac{3}{4}\right]$

10. **The minimum and maximum values of** $cosx+3\sqrt{2}sin\left(x+\frac{\pi}{4}\right)+6$ **are**
    1) 1, 11 2) 11,-1 3) 6,5 4) 5, 6

11. $(sinx+cosx)^2+cos^2\left(\frac{\pi}{4}+x\right)\in$
    1) [0,1] 2) [0,2] 3) [1,2] 4) [0,3]

12. $A=\sin^8\theta+\cos^{14}\theta$**, then for all values of** θ
    1) $0<A\le 1$ 2) $1<2A\le 3$
    3) $A\ge 1$ 4) $0\le A\le\frac{1}{2}$

13. **For any real** θ **the maximum value of** $\cos^2(\cos\theta)+\sin^2(\sin\theta)$ **is**
    1) 1 2) $1+\sin^2 1$
    3) $1+\cos^2 1$ 4) $2+\cos^2 1$

14. **If** $5\cos x+12\cos y=13$**, then the maximum value of** $5\sin x+12\sin y$ **is**
    1) 12 2) $\sqrt{120}$ 3) $\sqrt{20}$ 4) 13

15. $cos\left(x+\frac{\pi}{6}\right)cos\left(x-\frac{\pi}{6}\right)\in$

1) $\left[-\frac{1}{4},\frac{1}{4}\right]$ 2) $\left[-\frac{3}{4},\frac{3}{4}\right]$

3) $\left[-\frac{1}{4},\frac{3}{4}\right]$ 4) $\left[-\frac{1}{2},\frac{1}{2}\right]$

KEY

01) 4 02) 4 03) 2 04) 1 05) 2 06) 3
07) 2 08) 4 09) 2 10) 1 11) 3 12) 1
13) 2 14) 2 15) 3

SOLUTIONS

1. L.C.M of $\left\{\frac{4}{3},4\right\} = 4$
2. Does not exists
3. L.C.M of $\{2n!, 2(n+1)!\} = 2(n+1)!$
4. Period of $\sin nx = \frac{2\pi}{n}$, Period of $\sin\frac{x}{n} = 2n\pi$

   $\therefore 2n\pi = 4\pi \Rightarrow n = 2$
5. Given $f(x) = \cos\sqrt{p}x$ where $p = [a]$

   since period of $\cos ax$ is $\frac{2\pi}{|a|}$

   $\sqrt{p} = 2$
   $p = 4$
   $[a] = 4$
   $a \in (4,5]$
6. The period of $\cos(\sin nx)$ is $\frac{\pi}{n}$

   period of $\tan\left(\frac{x}{n}\right)$ is $n\pi$.

   Thus, $6\pi = LCM\left(\frac{\pi}{n}, \pi n\right)$
   By checking $n = 6$
7. The period of $\left|\sin^3\frac{x}{2}\right| = 2\pi$

   The period of $\left|\cos^5\frac{x}{5}\right| = 5\pi$, LCM is 10π.
8. It is not a periodic function
9. $\sin^2 x + \cos^4 x = \sin^2 x + \cos^2 x(1 - \sin^2 x)$

   $= \sin^2 x + \cos^2 x - \sin^2 x\cos^2 x$

   $= 1 - \frac{1}{4}(2\sin x\cos x)^2$

   $= 1 - \frac{1}{4}\sin^2 2x$

   $= 1 - \frac{1}{4}\frac{(1-\cos 4x)}{2}$

   $= 1 - \frac{1}{8} + \frac{1}{8}\cos 4x$

   $= \frac{7}{8} + \frac{1}{8}\cos 4x$

   Here $a = \frac{1}{8} b = 0$ $c = \frac{7}{8}$

   minimum $c - \sqrt{a^2+b^2} = \frac{7}{8} - \frac{1}{8} = \frac{3}{4}$

   maximum $c + \sqrt{a^2+b^2} = \frac{7}{8} + \frac{1}{8} = 1$

   Range $= \left(\frac{3}{4}, 1\right)$

11. $\cos x + 3\sqrt{2}\left[\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right] + 6$

    $= 3\sin x + 4\cos x + 6$
12. $(\sin x + \cos x)^2 + \cos^2\left(\frac{\pi}{4} + x\right) =$

    $2\sin^2\left(\frac{\pi}{4} + x\right) + \cos^2\left(\frac{\pi}{4} + x\right)$
13. Clearly we know that $A > 0$.
    $\cos^{14}\theta \le \cos^2\theta \le 1$ and
    $\sin^8\theta \le \sin^2\theta \le 1$
    $\Rightarrow A \le 1$, Hence $0 < A \le 1$.
17. $\cos^2(\cos\theta) + \sin^2(\sin\theta)$
    $+\sin^2(\cos\theta) - \sin^2(\cos\theta)$
    $= 1 + \sin^2(\sin\theta) - \sin^2(\cos\theta)$, which is
    maximum when $\cos\theta = 0,\ \sin\theta = 1$
    Maximum value is $1 + \sin^2 1$.

14. Given $5\cos x+12\cos y=13$ (1)

$5\sin x+12\sin y=k$ (2)

squaring and adding (1) and (2)

$$(5\cos x+12\cos y)^2+(5\sin x+12\sin y)^2=13^2+6^2$$

$$25\cos^2 x+144\cos^2 y+12\cos x\cos y$$

$$+25\sin^2 x+144\sin^2 y$$

$$+120\sin x\sin y=13^2+k^2$$

$$25+144+120(\cos x\cos y+\sin x\sin y)=169+k^2$$

$$120\cos(x-y)=k^2$$

since maximum of $\cos(x-y)$ is 1

$$k^2=120$$

15. $$\cos\left(x+\frac{\pi}{6}\right)\cos\left(x-\frac{\pi}{6}\right)=\cos^2\frac{\pi}{6}-\sin^2 x$$

$$=\frac{3}{4}-\sin^2 x$$

EAMCET QUESTIONS

1 Let the maximum and minimum value of the expression $2\cos^2\theta+\cos\theta+1$ is M and m respectively then the value of $\left[\frac{M}{m}\right]$ is (where [.]is the greatest Integer function (eamcet - 2016)

key:- 4

Sol:- $2\cos^2\theta+\cos\theta+1=2\left(\cos^2\theta+\frac{\cos\theta}{2}+\frac{1}{2}\right)$

$$=\left(\left(\cos\theta+\frac{1}{4}\right)^2+\frac{7}{16}\right)$$

Given expression is maximum when $\cos\theta=1$ and minimum when

$$\cos\theta=\frac{-1}{4}$$

$$m=2\left(\left(\frac{5}{4}\right)^2+\frac{7}{16}\right)=2\left(\frac{32}{16}\right)=4$$

and $m=2\left(0+\frac{7}{16}\right)=\frac{7}{8}$

Hence $\left[\frac{M}{m}\right]=\left[\frac{4}{\frac{7}{8}}\right]=\left[\frac{32}{7}\right]=4$

* Let $f(x)=\cos x\left(\sin x+\sqrt{\sin^2 x+\sin^2\theta}\right)$ where θ is constant. The maximum value of is (eamcet -15)

1) $\sqrt{1+\cos^2\theta}$ 2) $|\cos\theta|$

3) $\sqrt{1+\sin^2\theta}$ 4) $|\sin\theta|$

Key:- 3

Given

$$f(x)=\cos x\left(\sin x+\sqrt{\sin^2 x+\sin^2\theta}\right)$$

$$f(x)\sin x=\sin x+\sqrt{\sin^2 x+\sin^2\theta}$$

$$\left(f(x)\sin x-\sin x\right)^2=\sin^2 x+\sin^2\theta$$

$$f(x)^2\sin^2 x+\sin^2 x-2f(x)\tan x=\sin^2 x+\sin^2\theta$$

$$f^2(x)\tan x-2f(x)\tan x+f^2(x)-\sin^2\theta=0$$

since $\tan x$ is rual

$$b^2-4ax\geq 0$$

$$4f^2(x)\geq 4f^2(x)\left(f^2(x)-\sin^2 a\right)$$

$$1\geq f^2(x)-\sin^2\theta$$

e

$$\Rightarrow f^2(x) \le 1+\sin^2\theta$$

$$|f(x)| \le \sqrt{1+\sin^2\theta}$$

maximum value of $f(x)$ is $\sqrt{1+\sin^2\theta}$

3. The maximum value of $3\cos\theta+5\sin\left(\theta\frac{-\pi}{6}\right)$ for any real value of (eamcet -15)

1) $\sqrt{34}$ 2) $\sqrt{31}$

3) $\sqrt{19}$ 4) $\frac{\sqrt{79}}{2}$

Key: 3

Sol: $f(\theta)=3\cos\theta+5\left(\sin\theta\cos{\pi/6}-\cos\theta\sin{\pi/6}\right)$

$$=3\cos\theta+\frac{5\sqrt{3}}{2}\sin\theta-\frac{5}{2}\cos\theta$$

$$=\frac{1}{2}\cos\theta+\frac{5\sqrt{3}}{2}\sin\theta$$

Here $a=\frac{1}{2} b=\frac{5\sqrt{3}}{2}\ C=0$

Maximum value of

$$=c+\sqrt{a^2+b^2}=\sqrt{\frac{1}{4}+\frac{75}{4}}=\sqrt{19}$$

4.If $f(x)\ f(x)=\lambda|\sin x|+\lambda^2|\cos x|+g(\gamma)$ **has the period is equal to** ${\pi/2}$ **. The period** $\lambda=$

sol. Given $f(x)=\lambda|\sin x|+\lambda^2|\cos x|+g(\lambda)$ **has the period** ${\pi/2}$

We know period of $|\sin x|+|\cos x|$ is ${\pi/2}$

Here $\lambda=\lambda^2$ $\Rightarrow \lambda=1$

5. The period of $f(x)=\frac{|\sin x|-|\cos x|}{|\sin x|+|\cos x|}$ *is*

1) $\frac{\pi}{2}$ **2)** 2π **3)** π **4)** 4π

key . 3

sol. **Now** $f(\pi+x)=\frac{|\sin(\pi+x)|-|\cos(\pi+x)|}{|\sin(\pi+x)|+|\cos(\pi+x)|}$

$$=\frac{|-\sin x|-|-\cos x|}{|-\sin x|+|-\cos x|}=\frac{|\sin x|-|\cos x|}{|\sin x|+|\cos x|}=f(x)$$

$$f(\pi+x)=f(x)$$

Period of $f(x)$ is π

JEE MAINS , EAMCET QUESTIONS

1. The range of the function $f(x)=\ln\left((\cos x)^{\cos x}+1\right),\quad x\in\left(0,\frac{\pi}{2}\right)$ **is** $\left[\left(\frac{1}{e}\right)^{\frac{1}{e}}+1,K\right]$ **then K is---**

2. Let f be a real valued function defined for all real numbers 'x' such that for some fixed $a>0, f(x+a)=\frac{1}{2}+\sqrt{f(x)-f(x)^2}$ $\forall x$ **.and** $f(x)\in[1,2]$**. the function** $f(x)$ **is periodic with period=--------------**

1) 2a 2) a 3) 3a 4) 5a

3. If $f:R\to R$ **is a function satisfying the property.** $f(2x+3)+f(2x+7)=2,\forall x\in R.$ **Then the period of** $f(x)$ **is---------**

1) 4 2) 3 3) 5 4) 6

4. If $f(a-x)=f(a+x)$ **and** $x\in\left(0,\frac{\pi}{2}\right)$ **for all real x, where a,b** $(a>b)$ **are constants then prove that f(x) is a periodic function.**

1) b 2) 2b 3) 3b 4) 4b

5. Let f be a real valued function with domain R. Now if for some positive constant a, the equation

$$f(x+a)=1+\left\{2-3f(x)+3(f(x))^2-(f(x))^3\right\}^{\frac{1}{3}}$$

holds good for $x\in R$, then prove that f(x) is a periodic fuction.

1) a 2) 2a 3) 3a 4) 4a

6. If the periodic function f(x) satisfies the equation $f(x+1)+f(x-1)=\sqrt{3}f(x)\ \forall x\in R$ **then find the period of f(x).**

1) 8 2) 4 3) 12 4)16

7. If a,b be two fixed positive integers.such that $f(a+x)=b+$

$$\left(b^3+1-3b^2f(x)+3b\{f(x)\}^2-\{f(x)\}^3\right)^{1/3}$$

$\forall x\in R$ **Then** $f(x)$ **is a periodic function with period .**

1) a 2) 2a 3) b 4) 2b

8. If $f(x)$ **satisfying the relation** $f(x)+f(x+4)=f(x+2)+f(x+6)\forall x,$ **then the period of** $f(x)$ **is----------**

KEY

1) 2.00 2)1 3) 2 4) 2 5) 3 6)3 7)2 8) 8.00

SOLUTIONS

1. Since $x\in\left(0,\frac{\pi}{2}\right)\Rightarrow 0<\cos x<1$

$\therefore f(x)=\ln\left((\cos x)^{\cos x}+1\right)=y(say)$

$\Rightarrow y=\ln\left((\cos x)^{\cos x}+1\right)$

Now range of $\ln\left((\cos x)^{\cos x}+1\right)$ in $\left(0,\frac{\pi}{2}\right)=$

Range of $\ln\left(x^x+1\right)$ in (0, 1)

$\Rightarrow y=\ln\left(x^x+1\right)$

Let $g(x)=x^x+1=e^{x\ln x}+1$

$\therefore g^{|}(x)=e^{x\ln x}(1+\ln x)$

$\Rightarrow g^{|}(x)>0$ for $x>\frac{1}{e}$ and $g^{|}(x)<0$ for $x<\frac{1}{e}$ Hence g(x) is minimum at $x=\frac{1}{e}$

$\therefore$ Minimum value of g(x) is $g\left(\frac{1}{e}\right)=\left(\frac{1}{e}\right)^{\frac{1}{e}}+1$

and maximum value of g(x) is g(1) =1 + 1 =2

2. $f(x+a+a)=\frac{1}{2}+\sqrt{f(x+a)-(f(x+a))^2}$

$=\frac{1}{2}+\sqrt{\frac{1}{2}+\sqrt{f(x)-(f(x))^2}}.\sqrt{\frac{1}{2}-\sqrt{f(x)-(f(x))^2}}$

$=\frac{1}{2}+\sqrt{\frac{1}{4}-f(x)+(f(x))^2}$

$=\frac{1}{2}+f(x)-\frac{1}{2}=f(x)$

3. Given $f\left(2x+3\right)+f\left(2x+7\right)=2\rightarrow\left(1\right)$
Replace x by $x+2$
$f\left(2x+7\right)+f\left(2x+11\right)=2\rightarrow\left(2\right)$
from $\left(1\right)-\left(2\right)$,we get.
$f\left(2x+3\right)-f\left(2x+11\right)=0.$
$f\left(2x+3\right)=f\left(2x+11\right)=\boxed{T=4}$

4. Given $f\left(b+x\right)=f\left(b-x\right)$
Replacing x by $x+b$
Then $f\left(x+2b\right)=f\left(b-\left(x+b\right)\right)=f\left(-x\right)$
Again replacing x by -x then
$f\left(x\right)=f\left(2b-x\right)=f\left(2b+x\right)$
$\left(\because f\left(a-x\right)=f\left(a+x\right) and\, a>b\ \ here\ \ a=2b\right)$
$\Rightarrow f\left(2b+x\right)=f\left(x\right)$
Hence f(x) is periodic with period 2b.

5. Given $f\left(x+a\right)=1+\left\{2-3f\left(x\right)+3\left(f\left(x\right)\right)^2-\left(f\left(x\right)\right)^3\right\}^{\frac{1}{3}}$
$=1+\left\{1+\left(1-f\left(x\right)\right)^3\right\}^{\frac{1}{3}}$ ....(1)
Replacing x by (x+a) then
$f\left(x+2a\right)=1+\left\{1+\left(1-f\left(x+a\right)\right)^3\right\}^{\frac{1}{3}}$
$=1+\left\{1-\left(f\left(x+a\right)-1\right)^3\right\}^{\frac{1}{3}}$
$=1+\left\{1-\left(1+\left(1-f\left(x\right)\right)\right)^3\right\}^{\frac{1}{3}}$ ....from (1)
$=1-\left\{\left(1-f\left(x\right)\right)^3\right\}^{\frac{1}{3}}$
$=1-\left(1-f\left(x\right)\right)=f\left(x\right)$

6. We have
$f\left(x+1\right)+f\left(x-1\right)=\sqrt{3}\,f\left(x\right)\ \forall x\in R$ ......(1)
Replacing x by x-1 and x+1 in eq(1) then
$f\left(x\right)+f\left(x-2\right)=\sqrt{3}f\left(x-1\right)$.......(2)
and $f\left(x+2\right)+f\left(x\right)=\sqrt{3}f\left(x+1\right)$...(3)
Adding (2) and (3), we get
$2f\left(x\right)+f\left(x-2\right)+f\left(x+2\right)$
$=\sqrt{3}\left(f\left(x-1\right)+f\left(x+1\right)\right)$
$\Rightarrow 2f\left(x\right)+f\left(x-2\right)+f\left(x+2\right)=\sqrt{3}.\sqrt{3}f\left(x\right)$
(from (1)) $\therefore$
$f\left(x+2\right)+f\left(x-2\right)=f\left(x\right)$.....(4)
Replacing x by x+2 is equation (4) then
$f\left(x+4\right)+f\left(x\right)=f\left(x+2\right)$ ....(5)
Adding (4) & (5), we get
$f\left(x+4\right)+f\left(x+2\right)=0$ ....(6)
Again replacing x by x+6 in eq(6) then
$f\left(x+10\right)+f\left(x+4\right)=0$ ....(7)
Subtracting (6) from (7), we get
$f\left(x+10\right)-f\left(x-2\right)=0$ ....(8)
Replacing x by x+2 in eq(8) then
$f\left(x+12\right)-f\left(x\right)=0$ (or) $f\left(x+12\right)=f\left(x\right)$
Hence f(x) is periodic function with period 12.

7. $f\left(a+x\right)=b+$
$\left[1+b^3-3b^2f\left(x\right)+3b\left\{f\left(x\right)\right\}^2-\left\{f\left(x\right)\right\}^3\right]^{1/3}$
$=b+\left[1+\left\{b-f\left(x\right)\right\}^3\right]^{1/3}$
$\Rightarrow f\left(a+x\right)-b=\left[1-\left\{f\left(x\right)-b\right\}^3\right]^{1/3}$
$\Rightarrow \phi\left(a+x\right)=$
$\left[1-\left\{\phi\left(x\right)\right\}^3\right]^{1/3} where\, \phi\left(x\right)=f\left(x\right)-b$
$\Rightarrow \phi\left(x+2a\right)=\left[1-\left\{\phi\left(x+a\right)\right\}^3\right]^{1/3}$
$\phi\left(x+2a\right)=\phi\left(x\right)$
$\Rightarrow f\left(x+2a\right)-b=f\left(x\right)-b$
$f\left(x+2a\right)=f\left(x\right)$
$f\left(x\right)$ is periodic with period $2a$.